Mathematics Extended Essay

What effect does damping have on a simple harmonic oscillator?

Topic : Calculus applied in real life scenarios.

Word count : 3999

1 Introduction

This essay explores the mathematical concepts behind the simple harmonic oscillator. A simple harmonic oscillator is an object that exhibits periodic[1] motion after being displaced from its initial position. In this investigation the object is a mass bound to a spring stretched by a vertical displacement ($-\Delta x$) and left alone to oscillate. This investigation, compares the difference between the dampening effect of a simple harmonic oscillator in air and water. The experiment was conducted by attaching a 300g mass onto a spring and recording its coordinates as it oscillated freely. The coordinates were generated using motion-tracking software. This data is then processed using a python script and a graph of displacement against time is generated for both the damped[2] and undamped oscillators.

2 Mathematical Analysis

A simple harmonic oscillator running in undamped conditions means that there are no resistive forces acting on it. So the motion of the oscillator can be described with $\Sigma F = -kx$ (Hooke's law)[3]. The dampening force $-bv$ or $-b\dot{x}$ will not exist in this case because the motion is undamped. Hence, the differential equation can be written as the following using Newton's[4] 2nd law.

$$F = -kx \rightarrow m\ddot{x} = -kx \tag{2.1}$$

The method to calculate the spring constant of a spring is shown below.

Method to calculate spring constant:

1. Set up a clamp stand and boss.

2. Measure the original length of the spring (l_0) using a ruler.

3. Hook a 100g mass onto the spring.

4. Guide the mass down until it is stable.

5. Measure the new length of the spring using a ruler (l_n).

6. Subtract new length with original length ($l_n - l_0$).

7. Repeat steps 1-6 using different masses increasing in 100g each time. (stop at 600g)

8. Plot the data on a graph.

9. Find the difference between the largest and smallest force. (Only after all points are plotted.)

10. Find the difference between the largest and smallest change in displacement. (Only after all points are plotted.)

11. Calculate the spring constant by finding the highest force divided by the highest change in displacement. (The gradient of the graph.)

The data generated by the experiment is displayed below.

Δx	0.000	0.045	0.087	0.139	0.180	0.220	0.270
F	0.000	0.981	1.962	2.96	3.92	4.91	5.89

The formula[5] to calculate spring constant is shown below.

$$k = \frac{F_f - F_i}{\Delta x_f - \Delta x_i} \tag{2.2}$$

Plotting these coordinates on a graph of force against change in displacement resembles a directly proportional relationship (Figure 1). Therefore, the average spring constant can be found by calculating the gradient of the graph using equation (2.2)

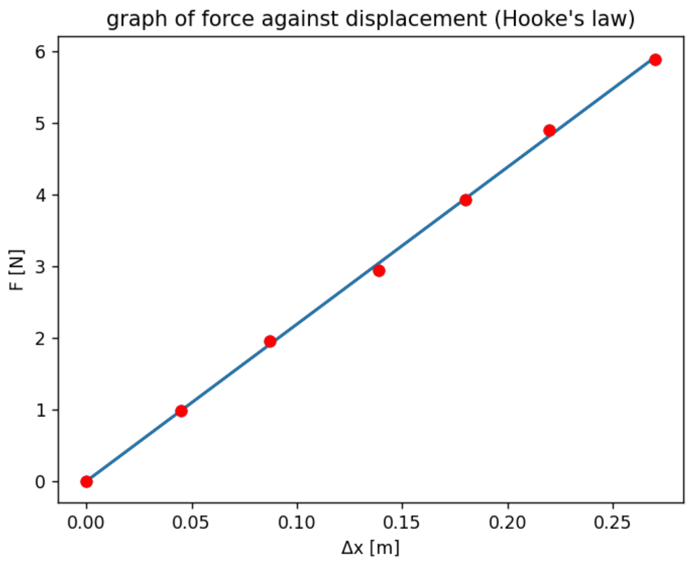

Figure 1: Relationship between Force and change in displacement.
Graph generated by candidate using Python version 3.12.8

Calculating the gradient of graph.

$$k = \frac{5.886}{0.27}$$

$$= 21.8$$

Gives the final value of k to be 21.8$\text{kgm}^2\text{s}^{-2}$ (to 3 s.f).

3 Undamped Simple Harmonic Oscillator

3.1 Experimental Method

To model the oscillations as a displacement-time graph, an experiment must be conducted where a mass (in this case 300g) hangs onto a hook and oscillates up and down while its movement is recorded in coordinates. There are two variations of simple harmonic motion, one with damping and one without damping. The experiment to model undamped oscillations is performed in air and the experiment to model damped oscillations is performed in water.

Apparatus:

- Clamp stand

  Clamp stand must be fastened to a flat and secure platform to minimise any experimental error and to keep if from falling and damaging itself.

- Boss

  Boss must be tightly fastened to the clamp stand to avoid the mass swinging out of control. Boss must be kept at the same height each time the experiment is conducted.

- Spring

  The same spring must be used to conduct each experiment. Otherwise the experiment is not fair, as the spring constant will be different.

- 3×100g weights

  3 weights will be used to model the oscillations but 6 will be used to calculate spring constant (Each weight increasing in increments of 100g).

- Meter rule

  To measure the initial height of the mass x_0.

- Camera

  Will be used to film the oscillating mass. The camera used in this experiment has a resolution of 720p at 30fps.

- Video-editing software

  Necessary for clipping parts of the video that are irrelevant to the experiment such as people moving in the background. Also can be used to immediately start the recording at a specific point (to make the plotting as accurate as possible).

- Motion-tracking software

  Main method of modeling the graph.

The apparatus stated above covers all the equipment needed to model the undamped oscillations. The method below details the instructions to model the mass-spring system.

Method :

1. Set up a clamp stand on a flat surface.
   (Surface must be elevated and kept away from any disturbance.)
2. Hold a ruler against the neck of the clamp stand.
   (Making sure it is ascending from it's lowest point.)
3. Pin the ruler onto the clamp stand by securing a boss onto the top of it.
4. Hang spring onto the long metal hook of the boss
5. Prepare a mass hook and hang 300g of mass onto it.
6. Secure the mass hook onto the spring
   (By tightening the boss to close the hook and secure the spring.)
7. Grab the neck of the mass hook and palace it so that the mass is held 0.34m above the base of the clamp stand.
8. Start recording.
9. Release the mass shortly after hitting record.
   (Ensure the camera always starts before the mass is dropped because the extra time at the front can easily be edited using video-editing software.)
10. Cut out irrelevant parts of the video using video editing software.
11. Run the video through Tracker Online[6] and generate data points.
12. Extract the data from Tracker Online
13. Clean the data using python's Pandas library[7].
    (Arranging data into arrays of time and displacement values.)
14. Plot the data on a graph using python's Matplotlib[8] library.
    (Shown in the evaluation section.)

The initial height of the mass is 0.34m (Figure 2).

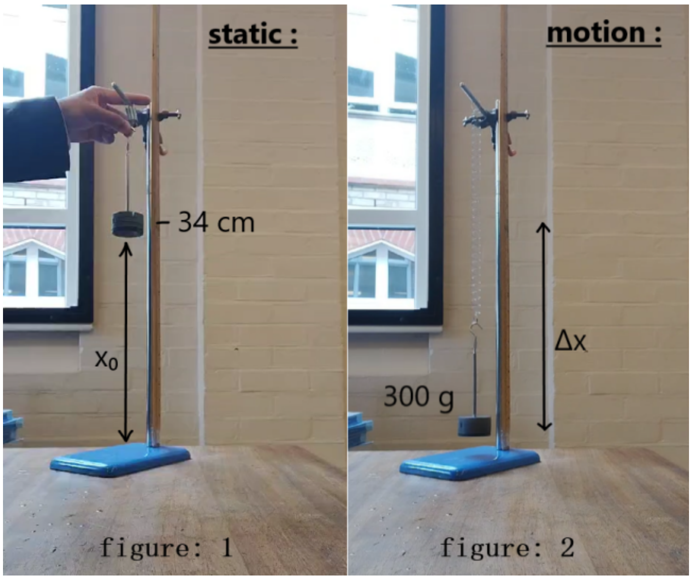

Figure 2: Undamped SHM system setup.
Photo taken by candidate.

Using this initial condition, it is possible to construct a model of the system

When the mass falls from its original position, it reaches a minimum position and returns back. These oscillations happen until the mass eventually stops moving due to drag forces acting on it. Since it is impossible to track the exact motion of this system using basic lab equipment, the experiment relies on an online motion-tracking software to track the motion of the mass.

3.2 Solution to Undamped SHM Oscillator

Now that the differential equation for Hooke's Law has been established, solving will give a solution in terms of displacement. Hooke's Law in it's differential form is stated below.

$$m\ddot{x} = -kx \tag{3.1}$$

The solution can be assumed to be an exponential function for displacement which can be differentiated twice and substituted into the differential equation.

Displacement can be represented as $\mathcal{X}(t)$.

$$\text{let } x = \mathcal{X}(t) \tag{3.2}$$

Substituting $\mathcal{X}(t)$ into equation (3.1) and rearranging to bring all the variables onto the left hand side.

$$m\mathcal{X}''(t) + k\mathcal{X}(t) = 0 \tag{3.3}$$

Then, dividing by m on both sides gives

$$\mathcal{X}''(t) + \frac{k}{m}\mathcal{X}(t) = 0 \tag{3.4}$$

For simplicity a substitution can be made for $\frac{k}{m}$.

$$\text{let } \omega^2 = \frac{k}{m} \tag{3.5}$$

Substituting this back into the differential equation gives the following equation.

$$\mathcal{X}''(t) + \omega^2\mathcal{X}(t) = 0 \tag{3.6}$$

The equation shown above is a second order homogeneous[9] differential equation. Meaning that the solution to the equation is an exponential in the form of $\psi e^{\lambda t}$. Where ψ, $t \in \mathbb{R}^+$ and $\lambda \in \mathbb{C}$.

$$\begin{cases} \mathcal{X}(t) & = \psi e^{\lambda t}, \\ \mathcal{X}'(t) & = \psi \lambda e^{\lambda t}, \\ \mathcal{X}''(t) & = \psi \lambda^2 e^{\lambda t} \end{cases}$$

Since, there is no velocity mentioned in the equation, there is no use for $\mathcal{X}'(t)$. Therefore $\mathcal{X}(t)$ and $\mathcal{X}''(t)$ can now be substituted into equation (3.6).

$$\lambda^2 \psi e^{\lambda t} + \omega^2 \psi e^{\lambda t} = 0 \tag{3.7}$$

Factorizing out $\psi e^{\lambda t}$ gives the auxiliary[10] equation for this differential equation. Solving this equation for λ, will give two complex roots as solutions (λ and λ^*).

$$\begin{aligned} \psi e^{\lambda t}(\lambda^2 + \omega^2) = 0 \\ \lambda^2 + \omega^2 = 0 \end{aligned} \tag{3.8}$$

Finally, using the quadratic formula to solve for λ will give a complex number and it's conjugate as values for λ and λ^* respectively.

$$\begin{aligned} \lambda, \lambda^* &= \pm \frac{\sqrt{-4\omega^2}}{2} \\ &= \pm \sqrt{-\omega^2} \\ &= \pm i\omega \end{aligned} \tag{3.9}$$

Now, λ and λ^* can be substituted into the solution. Since the two values for λ are not the same, (one is the conjugate of the other) different constants must be placed in front of the exponentials.

$$\mathcal{X}(t) = Ae^{\lambda t} + Be^{\lambda^* t} \tag{3.10}$$

Substituting in the values of λ and λ^* from equation (3.9), where $\lambda = i\omega$ and $\lambda^* = -i\omega$. The equation can be rewritten as the following.

$$\mathcal{X}(t) = Ae^{i\omega t} + Be^{-i\omega t} \tag{3.11}$$

Applying De Moivre's Theorem to equation (3.11), the equation splits into a sum of sines and cosines.

$$\begin{aligned} \mathcal{X}(t) &= Ae^{i\omega t} + Be^{-i\omega t} \\ &= A(\cos(\omega t) + i\sin(\omega t)) + B(cos(-\omega t) + i\sin(-\omega t)) \end{aligned} \tag{3.12}$$

Cosine is an even function, so $\cos(-\omega t)$ can also be written as $\cos(\omega t)$. Sine is an odd function, so $\sin(-\omega t)$ can be written as $-\sin(\omega t)$.

Editing equation (3.12) with these simplifications gives a new equation that can be rearranged to give the following.

$$\begin{aligned} \mathcal{X}(t) &= A(\cos(\omega t) + i\sin(\omega t)) + B(cos(\omega t) - i\sin(\omega t)) \\ &= A\cos(\omega t) + iA\sin(\omega t) + B\cos(\omega t) - iB\sin(\omega t) \end{aligned} \tag{3.13}$$

Here, the real component of the equation must be taken because the imaginary component has no effect on real phenomena.

$$\begin{aligned} \mathcal{X}(t) &= A\cos(\omega t) + iA\sin(\omega t) + B\cos(\omega t) - iB\sin(\omega t) \\ &= [A+B]\cos(\omega t) + i[A-B]\sin(\omega t) \\ \mathfrak{Re}\{\mathcal{X}(t)\} &= [A+B]cos(\omega t) \end{aligned} \tag{3.14}$$

The initial position of the mass (x_0) is where $t = 0$.

Therefore, substituting in $t = 0$ into equation (5.12) would give the following.

$$\mathcal{X}(t) = Ae^0 + Be^0$$

$$\mathcal{X}(0) = A + B$$

$$x_0 = A + B$$

Hence, substituting in x_0 into equation (3.14) will give the solution to the undamped oscillator.

$$\mathcal{X}(t) = x_0 \cos(\omega t) \tag{3.15}$$

Now, the variables must be replaced with numerical values. the value of omega ω can be found by using the previous substitution made in equation (3.5). The mass used in this experiment is 300g which converts into 0.3kg and the spring constant is 21.8kgs^{-1}. Substituting these two values into equation (3.5) gives the value of ω.

$$\omega = \sqrt{\frac{k}{m}}$$

$$\omega = \sqrt{\frac{21.8}{0.3}}$$

$$\omega = 8.5244745683629478857127218247877$$

$$\omega = 8.52 \text{ to } (3 \text{ s.f})$$

After observing equation (3.15), It's evident that the solution resembles a cosine curve. Implying that x_0 must be the amplitude of the function.

knowing that x_0 is 0.34m, therefore the graph's equilibrium is at the midpoint of the distance between the mass and the base of the clamp (Figure 3).

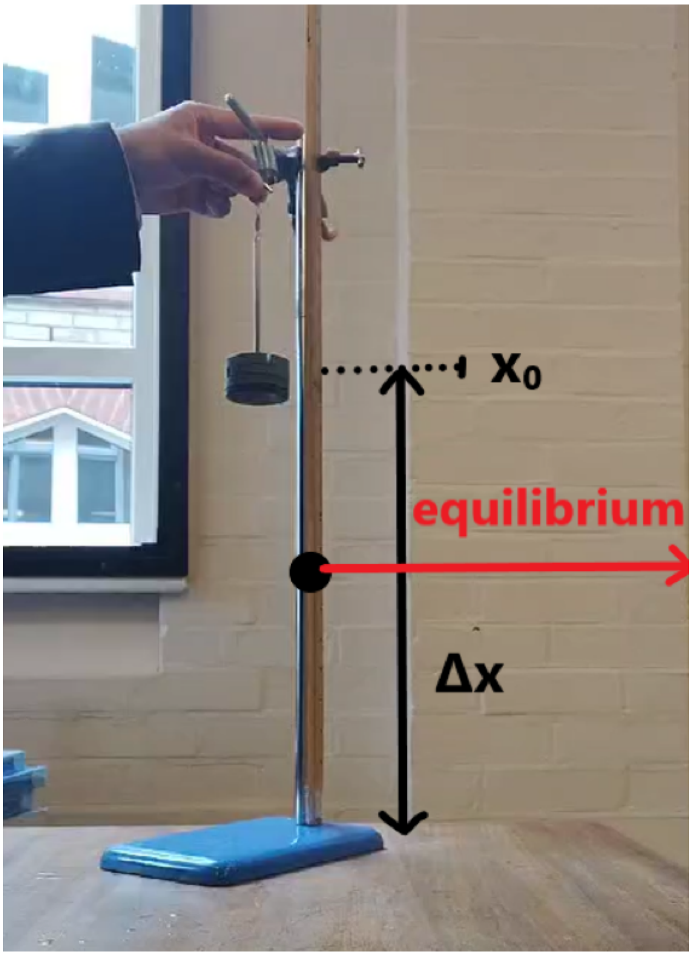

Figure 3: Visual representation of equilibrium.
Photo taken by candidate.

As shown in Figure 3, the equilibrium (ε) lies at the midpoint between 0m and 0.34m. Therefore ε can be considered to be 0.17m to (2 s.f).

When $t = \frac{1}{2\omega}$,

$$\mathcal{X}(t) = x_0 \cos\left(\frac{1}{2\omega}\pi\omega\right)$$

Omega cancels out.

$$\mathcal{X}(t) = x_0 \cos\left(\frac{1}{2}\pi\right) = 0$$

The function reads $\mathcal{X} = 0$. Meaning that the equilibrium position is currently at 0. In order to counteract this, 0.17m must be added to the function so that it will translate up by 1 amplitude length. Hence, repositioning it's equilibrium at 0.17m.

The other issue is that the amplitude x_0 is still set to 0.34m, so to counteract this, x_0 must be halved. Hence, the modified equation can be represented as follows.

$$\mathcal{X}(t) = \frac{1}{2}x_0 \cos(\omega t) + \varepsilon \tag{3.16}$$

The values for x_0, ω and ε are shown below.

$$\begin{cases} x_0 = 0.34, \\ \omega = 8.52, \\ \varepsilon = 0.17 \end{cases}$$

Substituting these values into equation (3.16) gives the following equation.

$$\mathcal{X}(t) = \frac{1}{2} \times 0.34 \cos(8.52t) + 0.17 \tag{3.17}$$

Simplifying fully gives the final result.

$$\mathcal{X}(t) = 0.17 \cos(8.52t) + 0.17 \tag{3.18}$$

3.3 Comparing Graphs

Given both the graphs (Figure 4 and Figure 5), equation (3.18) can now be compared against the raw data to see how accurate it is at predicting the motion of the oscillating mass.

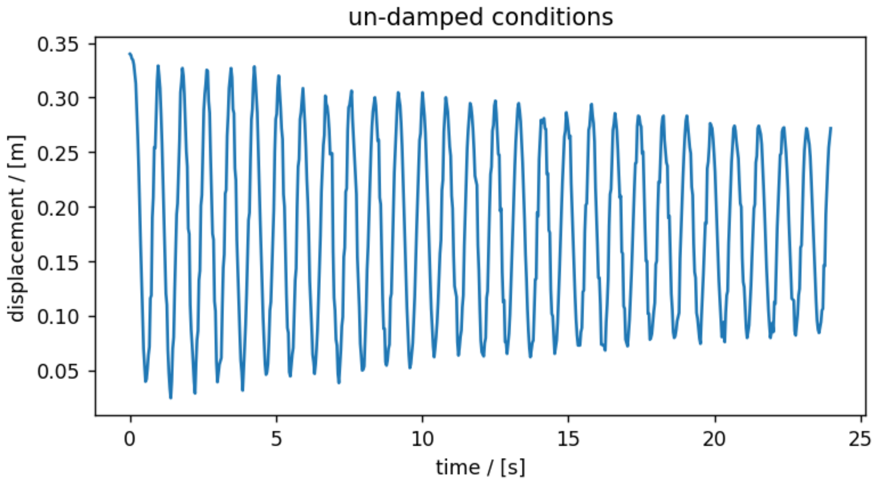

Figure 4: Raw Data. Graph generated by candidate using python version 3.12.8

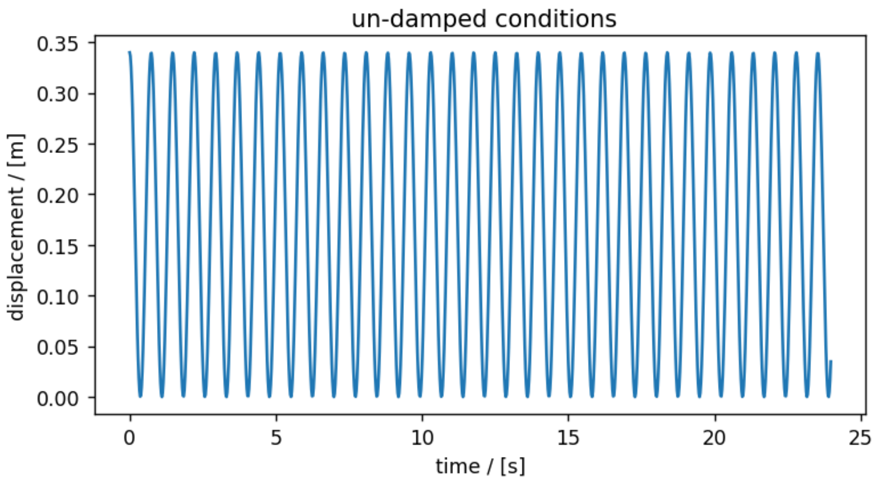

Figure 5: Equation (3.18). Graph generated by candidate using python version 3.12.8

The percentage error[11] of the area underneath each graph can be used to measure how accurate equation (3.18) is against the raw data.

$$\text{Percentage error (P.E)} = \frac{\text{Estimated value - Actual value}}{\text{Actual value}} \times 100$$

The integral over equation (3.18) can be evaluated easily which would give the area (A) under the graph between the limits : $\alpha \leq t \leq \beta$. where $\alpha = 0$ and $\beta = 23.97446$

$$
\begin{aligned}
A &= \int_{\alpha}^{\beta} \left(\frac{1}{2}x_0\cos(\omega t) + \varepsilon\right)\, dt \\
&= \int_{\alpha}^{\beta} \frac{1}{2}x_0\cos(\omega t)\, dt + \int_{\alpha}^{\beta} \varepsilon\, dt \\
&= \left[\frac{x_0}{2\omega}\sin(\omega t)\right]_{\alpha}^{\beta} + [\varepsilon t]_{\alpha}^{\beta} \\
&= \left[\frac{0.17}{2\times 8.52}\sin(8.52t)\right]_{0}^{23.97446} + \varepsilon\, [t]_{0}^{23.97446} \\
&= 0.0103 \times [\sin(204.26) - \sin(0)] + 0.17 \times 23.97446 - 0 \\
&= -0.00818803 + 4.0756582 \\
&= 4.067459169 \\
&= 4.07 \text{ to } (3 \text{ s.f})
\end{aligned} \tag{3.19}
$$

This result was verified with a graphing calculator[12] the value for the integral that it got was 4.07 to (3 s.f). This result was only slightly different to the result obtained in equation (3.19) because ω was rounded to (3 s.f). However this did not severely impact the result.

A computer algorithm employing Simpson's rule[13] (shown in Appendix) was used to approximate the area under the curve of raw data points, as it is impractical to attempt integrating anything without an equation. The result it obtained by the algorithm was 3.23 to (3 s.f).

Now substituting these values into the percentage error equation

$$P.E = \frac{|4.067459169 - 3.2333339397775305|}{3.2333339397775305} \times 100$$

$$= 25.79768266 \tag{3.20}$$

$$= 25.8\% \text{ to } (3 \text{ s.f})$$

Gives a percentage error of 25.79768266 or 25.8% to (3 s.f).

4 Damped SHM Oscillator

4.1 Experimental Method

The experimental method for the damped harmonic oscillator was not very different to the undamped oscillator. The same mass was used (0.3kg), however this time the mass was entirely submerged underwater in a large tube.

Method :

1. Set up a clamp stand on a flat surface.
2. Hold a ruler against the neck of the clamp stand.
3. Pin the ruler onto the clamp stand by securing a boss onto the top of it.
4. Hang a spring onto the long metal end of the boss
5. Prepare a mass hook and hang 300g of mass onto it.
6. Secure the mass hook onto the spring.
7. Set up camera so that the frame closely captures the experiment.
8. Start recording.

9. Grab the neck of the mass hook and palace it so that the mass is held 0.213m above the base of the clamp stand.

10. Fill a large tube with enough water to submerge the mass.

11. Hold the mass against the 0.213m mark making sure you are inspecting it at eye level without dropping.

12. Start recording with the Camera.

13. Drop the mass and let it oscillate until it stops or is close to stopping.

14. Cut out irrelevant parts of the video using video editing software.

15. Run the video through Tracker Online and generate data points.

16. Extract the data from Tracker Online

17. Clean the data using python's Pandas library.

18. Plot the data on a graph using python's Matplotlib library.

The setup for the experiment is shown in Figure 6.

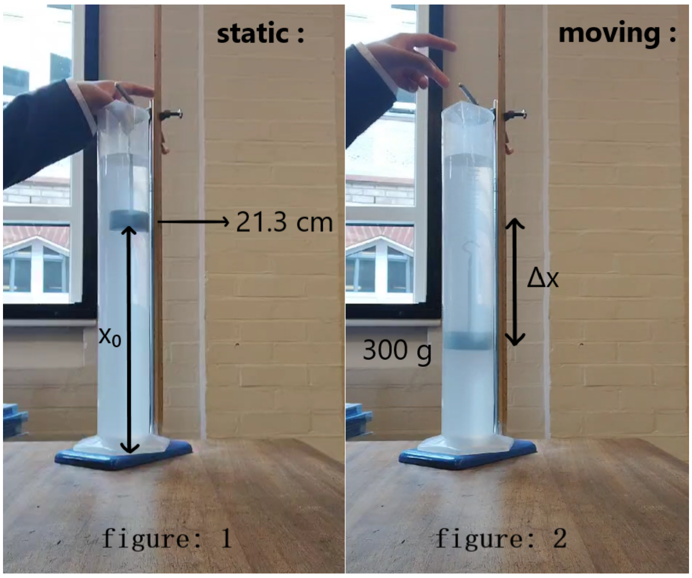

Figure 6: Damped SHM system setup.
Photo taken by candidate.

It can be assumed that the initial conditions of the damped SHM system remain the same as the undamped SHM system.

4.2 Solution to Damped SHM Oscillator

In undamped conditions (Figure 4), the oscillator was critically under-damped. This time, the oscillations happen in water. The mass is still being hooked onto a spring, so Hooke's Law still applies. However, there is a buoyant force also acting against the mass (bv). Since the buoyant force is opposing the weight of the object, the direction of this force will be $-bv$. So the three forces involved in this system are $m\ddot{x}$ (total force), $-bv$ (drag force)[14] and $-kx$ (elastic force). Equating these forces gives the following differential equation.

$$m\ddot{x} = -b\dot{x} - kx \tag{4.1}$$

Since this differential equation is also second order, a similar approach to what was used when finding the solution to the undamped harmonic oscillator can be used. Assuming that the solution $\mathcal{X}(t)$ is an exponential, a substitution can be made to arrive at an auxiliary equation. Solving this and taking the real component will give the solution to the differential equation.

Before any substitutions are made, both sides are divided by m to re-substitute ω^2.

$$\begin{aligned} m\mathcal{X}''(t) &= -b\mathcal{X}'(t) - k\mathcal{X}(t) \\ \mathcal{X}''(t) &= -\frac{b}{m}\mathcal{X}'(t) - \frac{k}{m}\mathcal{X}(t) \\ \mathcal{X}''(t) &= -\frac{b}{m}\mathcal{X}'(t) - \omega^2\mathcal{X}(t) \end{aligned} \tag{4.2}$$

Here, a new substitution can be introduced.

$$\text{let } \frac{b}{m} = \zeta \tag{4.3}$$

Substituting this value back into equation (4.2) gives

$$\begin{aligned} \mathcal{X}''(t) &= -\zeta\mathcal{X}'(t) - \omega^2\mathcal{X}(t) \\ 0 &= \mathcal{X}''(t) + \zeta\mathcal{X}'(t) + \omega^2\mathcal{X}(t) \end{aligned} \tag{4.4}$$

The rest of the approach is straightforward. $\mathcal{X}(t)$ can be assumed to be some exponential solution $\psi e^{\lambda t}$ where $\psi, t \in \mathbb{R}$ and $\lambda \in \mathbb{C}$. Then differentiating twice to substitute for $\mathcal{X}'(t)$ and $\mathcal{X}''(t)$.

$$\begin{cases} \mathcal{X}(t) &= \psi e^{\lambda t}, \\ \mathcal{X}'(t) &= \psi\lambda e^{\lambda t}, \\ \mathcal{X}''(t) &= \psi\lambda^2 e^{\lambda t} \end{cases}$$

After substitution, the equation looks like this. Simplifying this equation further gives the auxiliary equation.

$$\begin{aligned} \psi\lambda^2 e^{\lambda t} + \zeta\psi\lambda e^{\lambda t} + \omega^2\psi e^{\lambda t} &= 0 \\ \psi e^{\lambda t}(\lambda^2 + \zeta\lambda + \omega^2) &= 0 \\ \lambda^2 + \zeta\lambda + \omega^2 &= 0 \end{aligned} \tag{4.5}$$

Just as done previously, the quadratic formula can be used to find the values for λ and λ^*.

$$\lambda,\ \lambda^* = \frac{-\zeta \pm \sqrt{\zeta^2 - 4\omega^2}}{2} \tag{4.6}$$

Simplifying fully gives the following.

$$\begin{aligned}\lambda,\ \lambda^* &= -\frac{1}{2}\zeta \pm \frac{1}{2}\sqrt{\zeta^2 - 4\omega^2} \\ &= -\frac{1}{2}\zeta \pm \sqrt{\frac{1}{4}\zeta^2 - \frac{4}{4}\omega^2} \\ &= -\frac{1}{2}\zeta \pm \sqrt{\frac{1}{4}\zeta^2 - \omega^2}\end{aligned} \tag{4.7}$$

Here, another substitution can be made for $-\frac{1}{2}\zeta$ since, there exists a term inside the square root which is exactly the same as $(-\frac{1}{2}\zeta)^2$.

$$\text{let } \gamma = -\frac{1}{2}\zeta \tag{4.8}$$

Now, γ can be substituted back into equation (4.7).

$$\lambda,\ \lambda^* = \gamma \pm \sqrt{\gamma^2 - \omega^2} \tag{4.9}$$

$(\sqrt{\gamma^2 - \omega^2})$ can directly be replaced with $(i\sqrt{\omega^2 - \gamma^2})$ by taking out a factor of (-1) from both terms inside the square root.

Therefore, the following substitution can be made.

$$\text{let } i\Omega := i\sqrt{\omega^2 - \gamma^2} \tag{4.10}$$

Now, $i\Omega$ can be substituted into equation (4.9) giving the following equation.

$$\lambda,\ \lambda^* = \gamma \pm i\Omega \tag{4.11}$$

Finally the solution can be obtained by solving the following equation.

$$\begin{aligned}
\mathcal{X}(t) &= Ae^{\lambda t} + Be^{\lambda^* t} \\
&= A[e^{t(\gamma+i\Omega)}] + B[e^{t(\gamma-i\Omega)}] \\
&= A[e^{\gamma t}\cdot e^{i\Omega t}] + B[e^{\gamma t}\cdot e^{-i\Omega t}] \\
&= e^{\gamma t}(A[\cos(\Omega t) + i\sin(\Omega t)] + B[\cos(\Omega t) - i\sin(\Omega t)]) \\
&= e^{\gamma t}(A\cos(\Omega t) + Ai\sin(\Omega t) + B\cos(\Omega t) - Bi\sin(\Omega t)) \\
&= e^{\gamma t}[A+B]\cos(\Omega t) + e^{\gamma t}[A-B]i\sin(\Omega t)
\end{aligned} \tag{4.12}$$

Letting $[A+B] = x_0$ and taking the real part of the equation gives the solution.

$$\mathcal{X}(t) = e^{\gamma t}x_0\cos(\Omega t) \tag{4.13}$$

Once again modifying the equation to adjust its equilibrium position.

$$\mathcal{X}(t) = \frac{1}{2}x_0 e^{-\frac{b}{2m}t}\cos\left(t\cdot\sqrt{\omega^2 - \frac{b^2}{4m^2}}\right) + \varepsilon \tag{4.14}$$

As previously stated, the initial conditions of the system remain the same except from the equilibrium which changes to 0.213m (as mentioned in the experimental method). ω has not changed because there is no variation to k or m therefore, these values were reused from the previous experiment. Finally substituting in the values for x_0, m, ω and ε gives the following equation.

$$\mathcal{X}(t) = 0.17e^{-\frac{b}{0.6}t}\cos\left(t\cdot\sqrt{73.01 - \frac{b^2}{0.36}}\right) + 0.213 \tag{4.15}$$

4.3 Finding the Dampening Coefficient

When it came to comparing the graphs of equation (4.15) and the raw data, there was a problem. The solution to the damped harmonic oscillator has an unknown value (b) where $b \in \mathbb{R}$. Equation (4.15) takes the form of a cosine wave being damped by an exponential curve (Figure 7).

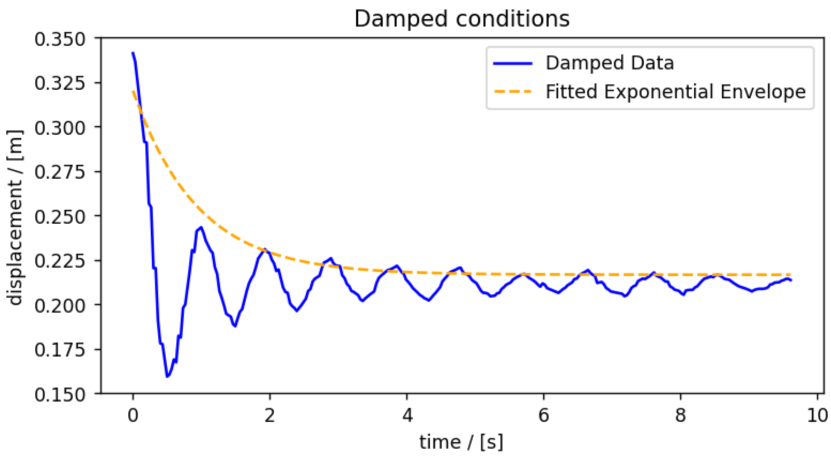

Figure 7: Exponential envelope example.
Graph generated by candidate using python version 3.12.8

The amplitudes of the cosine wave decrease exponentially as time progresses shown by the orange dotted line in Figure 7. From this, it is possible that there must be some exponential function of time that envelopes[15] the cosine wave which keeps the amplitudes decreasing overtime. A similar kind of exponential appears at the start of equation (8.16).

Consider the equation of the envelope of the curve derived in equation (8.16) to be the following.

$$E(t) = \frac{1}{2}x_0 e^{-\frac{b}{2m}t} \tag{4.16}$$

Observing the trend shown in Figure 7, the amplitudes of the graphs decrease at a similar rate to the exponential envelope. From this, it can be deduced that the peaks of the wave align with equation (4.16) which means the value of b can be estimated.

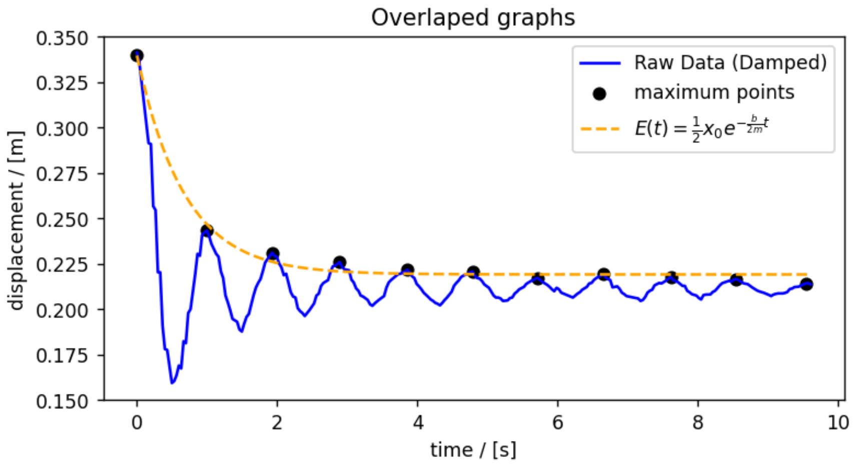

Figure 8: Fitted Envelope.
Graph generated by candidate using python version 3.12.8

Now, knowing the equation of this exponential in terms of variables and given a set of time coordinates e.g. $T = \{t_0,\ t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ \dots\ ,\ t_n\}$ and a set of displacement coordinates e.g. $X = \{x_0,\ x_1,\ x_2,\ x_3,\ x_4,\ ,x_5,\ \dots\ ,\ x_n\}$, it should be possible to substitute each time and displacement value into the equation and generate a value for b. Once this is done for all the values of time and displacement, each value of b can be averaged.

Firstly $E(t_j)$ can be represented as x_j (where $0 \leq j \leq n$) since it is known that $E(t)$ returns a displacement value as an output for any given time value.

$$E(t_j) = x_j, \qquad \text{where } j \in \mathbb{Z}^+$$

Next, equation (4.16) can be rewritten in terms of x_j and t_j.

$$x_j = \frac{1}{2}x_0 e^{-\frac{b}{2m}t_j}$$

Dividing by x_0 and multiply by 2 on both sides.

$$\frac{2x_j}{x_0} = e^{-\frac{b}{2m}t_j}$$

Taking a natural log on both sides.

$$\ln\left(\frac{2x_j}{x_0}\right) = -\frac{b}{2m}t_j$$

Rearranging for b,

$$-\frac{2m}{t_j} \cdot \ln\left(\frac{2x_j}{x_0}\right) = b$$

Gives the value of b as a function of t and x. Iteratively substituting every value of t and x will generate several values of b. The average of all these b values can be taken to find a value of b that works for all values of t and x. This will give an approximate value ($\bar{b}$).

Hence, the average dampening coefficient formula can be written as follows.

$$\bar{b} = \frac{\sum_{j=0}^{n}\left[-\frac{2m}{t_j} \cdot \ln\left(\frac{2x_j}{x_0}\right)\right]}{n} \tag{4.17}$$

Simplifying,

$$\bar{b} = \frac{\sum_{j=0}^{n}\left[-\frac{2m}{t_j} \cdot \ln\left(\frac{x_j}{x_0}\right)\right]}{n}$$

Substituting $m = 0.6$,

$$\bar{b} = \frac{\sum_{j=0}^{n}\left[-\frac{0.6}{t_j} \cdot \ln\left(\frac{2x_j}{0.17}\right)\right]}{n}$$

Then bringing -0.6 into the logarithm as a power.

$$\bar{b} = \frac{\sum_{j=0}^{n} \left[\frac{1}{t_j} \cdot \ln\left(\frac{2^{-0.6} x_i^{-0.6}}{0.17^{-0.6}} \right) \right]}{n}$$

Taking the reciprocal of the fraction inside the logarithm will make the powers turn positive.

$$\bar{b} = \frac{\sum_{j=0}^{n} \left[\frac{1}{t_j} \cdot \ln\left(\frac{0.17^{0.6}}{2^{0.6} x_i^{0.6}} \right) \right]}{n}$$

Performing the division inside the logarithm.

$$\bar{b} = \frac{\sum_{j=0}^{n} \left[\frac{1}{t_j} \cdot \ln\left(\frac{0.227851}{x_j^{0.6}} \right) \right]}{n}$$

Using logarithm rules, this can be rearranged in to the form of $\ln(A) - \ln(B)$

$$\bar{b} = \frac{\sum_{j=0}^{n} \left[\frac{\ln(0.227851) - \ln(x_j^{0.6})}{t_j} \right]}{n}$$

After some simplification, it can also be written as follows.

$$\bar{b} = \frac{\sum_{j=0}^{n} \left[\frac{-1.48 - 0.6 \ln(x_j)}{t_j} \right]}{n} \tag{4.18}$$

The following data points represent the time and displacement points at each peak on the wave respectively (discarding $t = 0$).

t_0	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9
0.340	0.998	1.929	2.893	3.857	4.788	5.719	6.650	7.615	8.546

Table 1: values for t at every peak of the wave.

x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
0.243	0.231	0.226	0.222	0.221	0.217	0.219	0.218	0.217	0.214

Table 2: values for x at every peak of the wave.

Counting the number of entries in each table, shows that there are 10 values for both displacement and time, therefore the value of n from equation (4.18) is set to 10.

$$\bar{b} \approx \frac{\sum_{j=0}^{10}\left[\frac{-1.48-0.6\ln(x_j)}{t_j}\right]}{10} \tag{4.19}$$

Now all that remains is to substitute each t value with its corresponding x value into the equation.

$$\bar{b} \approx \frac{\left(\frac{-1.48-0.6\ln(0.243)}{0.340}\right)+\left(\frac{-1.48-0.6\ln(0.231)}{0.998}\right)+\left(\frac{-1.48-0.6\ln(0.226)}{1.929}\right)+\ \ldots}{10}$$

After fully simplifying

$$\bar{b} = -0.3880445689293691$$

Which can be rounded off to

$$\bar{b} = -0.388 \text{ to } (3 \text{ s.f})$$

Taking the absolute value of $\bar{b}$ gives 0.388 which can now be substituted into equation (4.15)

4.4 Comparing Graphs

Now, substituting $|\bar{b}|$ back into equation (4.15).

$$
\begin{aligned}
\mathcal{X}(t) &= 0.17e^{-\frac{0.388}{0.6}t}\cos\left(t\cdot\sqrt{73.01-\frac{0.388^2}{0.36}}\right)+0.213 \\
&= 0.17e^{-0.647t}\cos\left(t\cdot\sqrt{73.01-0.4182}\right)+0.213 \\
&= 0.17e^{-0.647t}\cos(8.52t)+0.213
\end{aligned}
\tag{4.20}
$$

This gives a simplified equation which can now be graphed and compared.

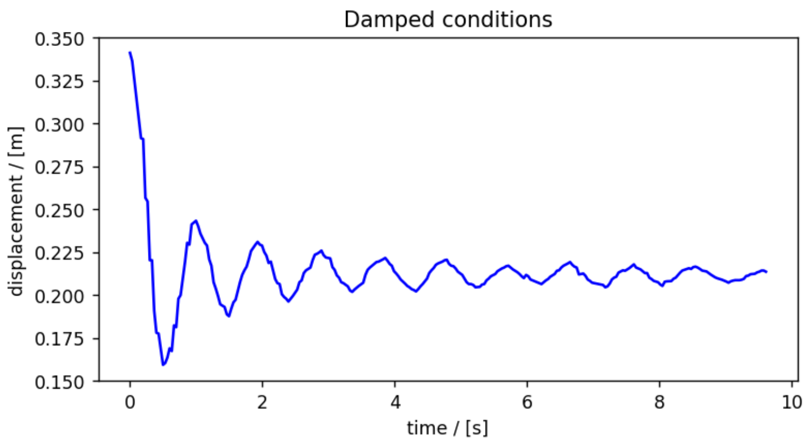

Figure 9: Raw data. Graph generated by candidate using python version 3.12.8

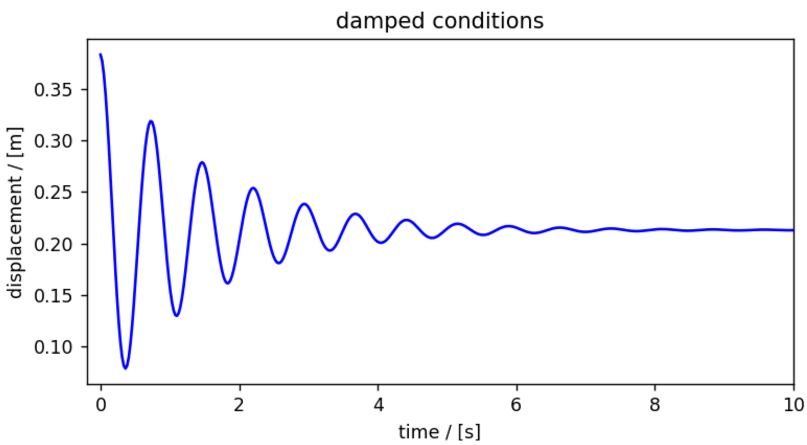

Figure 10: Equation (4.20). Graph generated by candidate using python version 3.12.8

Observing the graphs, it is noticeable that the exponential property of the curve is much more evident in both cases.

The accuracy of equation (4.20) can be judged by integrating to find the area underneath the curve and the raw data and calculating a percentage error. To make the integration easier, a few parts of equation (4.20) can be replaced with new constants and then the original values can be re-substituted back into the solution at the end and simplified. This is to avoid writing out long decimals.

Firstly, assigning new variables to the constants in equation (10.1)

$$\begin{cases} \mu &= 0.34, \\ \gamma &= -0.647, \\ \Omega &= 8.52, \\ \varepsilon &= 0.213 \end{cases} \qquad \textit{(all rounded to 3 s.f)}$$

Now beginning to integrate. (Consider the area under the graph to be A).

$$A = \int_{\alpha}^{\beta} \left(\frac{1}{2}\mu e^{\gamma t}\cos(\Omega t) + \varepsilon\right)\, dt, \qquad \textit{where } \alpha, \beta \in \mathbb{R} \tag{4.21}$$

The values for α and β are 0.0 and 9.61 respectively. However it is sensible for now to keep them as letters (in consideration of the limited amount of space on the page).

The integral can be split into parts to integrate the longer expression first (A_1) and then the smaller expression (A_2). Then taking a sum of both A_1 and A_2 gives the total area under the curve.

$$A_1 = \int_{\alpha}^{\beta} \frac{1}{2}\mu e^{\gamma t}\cos(\Omega t)\, dt, \qquad A_2 = \int_{\alpha}^{\beta} \varepsilon\, dt$$

Calculating for A_1, the constants can be factored out of the integral.

$$A_1 = \frac{1}{2}\mu \int_{\alpha}^{\beta} e^{\gamma t} \cos(\Omega t)\ dt$$

This expression can now be evaluated through integration by parts. By taking the following terms as u, dv, v and du.

$$\begin{cases} u &= e^{\gamma t}, \\ dv &= \cos(\Omega t)\ dt, \\ v &= \frac{1}{\Omega}\sin \Omega t, \\ du &= \gamma e^{\gamma t}\ dt \end{cases}$$

Substituting into the integration by parts equation.

$$A_1 = \frac{1}{2}\mu \left(\left[\frac{1}{\Omega}\sin(\Omega t) \cdot e^{\gamma t}\right]_{\alpha}^{\beta} - \int_{\alpha}^{\beta} \frac{1}{\Omega}\sin(\Omega t) \cdot \gamma e^{\gamma t}\ dt \right)$$

Factoring out constants gives the following expression.

$$A_1 = \frac{1}{2}\mu \left[\frac{e^{\gamma t}}{\Omega}\sin(\Omega t)\right]_{\alpha}^{\beta} - \frac{\gamma\mu}{2\Omega}\int_{\alpha}^{\beta} e^{\gamma t} \cdot \sin(\Omega t)\ dt \tag{4.22}$$

Now, applying integration by parts once again, but this time taking dv as $\sin(\Omega t)\ dt$, then v becomes $-\frac{1}{\Omega}\cos(\Omega t)$. Hence separating out the integral above from the rest of the expression gives the following expression.

$$\int_{\alpha}^{\beta} e^{\gamma t} \cdot \sin(\Omega t)\ dt = \left[-\frac{e^{\gamma t}}{\Omega}\cos(\Omega t)\right]_{\alpha}^{\beta} - \int_{\alpha}^{\beta} -\frac{1}{\Omega}\cos(\Omega t) \cdot \gamma e^{\gamma t} dt$$

Which simplifies to

$$\int_{\alpha}^{\beta} e^{\gamma t} \cdot \sin(\Omega t)\ dt = \left[-\frac{e^{\gamma t}}{\Omega}\cos(\Omega t)\right]_{\alpha}^{\beta} + \frac{\gamma}{\Omega}\int_{\alpha}^{\beta} \cos(\Omega t) \cdot e^{\gamma t} dt$$

Observing the integral at the end of the expression, it resembles the original integral. Hence, it can be substituted out for A_1.

$$\int_{\alpha}^{\beta} e^{\gamma t} \cdot \sin(\Omega t)\ dt = \left[-\frac{e^{\gamma t}}{\Omega}\cos(\Omega t)\right]_{\alpha}^{\beta} + \frac{\gamma}{\Omega} \cdot A_1$$

Substituting into equation (4.22)

$$A_1 = \frac{1}{2}\mu\left[\frac{e^{\gamma t}}{\Omega}\sin(\Omega t)\right]_{\alpha}^{\beta} - \frac{\gamma\mu}{2\Omega}\left(\left[-\frac{e^{\gamma t}}{\Omega}\cos(\Omega t)\right]_{\alpha}^{\beta} + \frac{\gamma}{\Omega} \cdot A_1\right)$$

Now expanding,

$$A_1 = \frac{1}{2}\mu\left[\frac{e^{\gamma t}}{\Omega}\sin(\Omega t)\right]_{\alpha}^{\beta} - \frac{\gamma\mu}{2\Omega}\left[-\frac{e^{\gamma t}}{\Omega}\cos(\Omega t)\right]_{\alpha}^{\beta} - \frac{\gamma^2\mu}{2\Omega^2} \cdot A_1$$

Grouping together like-terms,

$$A_1 + A_1 \cdot \frac{\gamma^2\mu}{2\Omega^2} = \frac{1}{2}\mu\left[\frac{e^{\gamma t}}{\Omega}\sin(\Omega t)\right]_{\alpha}^{\beta} - \frac{\gamma\mu}{2\Omega}\left[-\frac{e^{\gamma t}}{\Omega}\cos(\Omega t)\right]_{\alpha}^{\beta}$$

Simplifying,

$$A_1 \cdot \left(1 + \frac{\gamma^2\mu}{2\Omega^2}\right) = \frac{1}{2}\mu\left[\frac{e^{\gamma t}}{\Omega}\sin(\Omega t)\right]_{\alpha}^{\beta} - \frac{\gamma\mu}{2\Omega}\left[-\frac{e^{\gamma t}}{\Omega}\cos(\Omega t)\right]_{\alpha}^{\beta}$$

Rearranging the equation so that the left hand side of the equation only contains A_1.

$$A_1 = \frac{\frac{1}{2}\mu\left[\frac{e^{\gamma t}}{\Omega}\sin(\Omega t)\right]_\alpha^\beta - \frac{\gamma\mu}{2\Omega}\left[-\frac{e^{\gamma t}}{\Omega}\cos(\Omega t)\right]_\alpha^\beta}{\left(1+\frac{\gamma^2\mu}{2\Omega^2}\right)} \tag{4.23}$$

Now, substituting the numerical values for μ, γ, Ω, α and β into equation (4.23).

$$A_1 = \frac{\frac{1}{2}\times 0.34\times\left[\frac{e^{-0.647t}}{8.52}\sin(8.52t)\right]_0^{9.61} - \frac{-0.647\times 0.34}{2\times 8.52}\times\left[-\frac{e^{-0.647t}}{8.52}\cos(8.52t)\right]_0^{9.61}}{\left(1+\frac{(-0.647)^2\times 0.34}{2\times 8.52^2}\right)} \tag{4.24}$$

To simplifying the constant term in the denominator.

$$A_1 = \frac{0.17\times\left[\frac{e^{-0.647t}}{8.52}\sin(8.52t)\right]_0^{9.61} + 0.013\times\left[-\frac{e^{-0.647t}}{8.52}\cos(8.52t)\right]_0^{9.61}}{1.000980344}$$

Now, evaluating the terms with the limits.

$$A_1 = \frac{0.17\times(2.31\times 10^{-4} - 0) + 0.0129\times(-3.31\times 10^-5 + 0.11737)}{1.000980344}$$

Simplifying fully.

$$A_1 = \frac{-3.30665\times 10^{-5} + 1.51479\times 10^{-3}}{1.000980344}$$

$$= \frac{1.55417\times 10^{-3}}{1.000980344}$$

$$= 1.55\times 10^{-3} \text{ to (3 s.f)}$$

Now that the value for A_1 is found, the value for A_2 must be found also.

This is a standard integral.

$$\int_{\alpha}^{\beta} \varepsilon \; dt$$

This integral evaluates to

$$A_2 = [\varepsilon \cdot t]_{\alpha}^{\beta}$$

Substituting values for ε, α and β as done previously gives

$$A_2 = [0.213 \times t]_0^{9.61}$$

Which evaluates to

$$A_2 \approx 2.04693$$

Hence, by taking the sum of A_1 and A_2, the total area underneath the graph is approximately 2.048 to (4 s.f).

$$A = A_1 + A_2$$

$$= 1.55 \times 10^{-3} + 2.04693$$

$$= 2.04848265$$

$$= 2.048 \text{ to } (4 \text{ s.f})$$

Once again using a computer algorithm, the area under the raw data curve approximates to 2.0499822151654996 which can be rounded off to 2.05 to (3 s.f). So finally, substituting in the values into the percentage error equation as shown previously gives the error between the raw data and equation (4.20).

$$P.E = \frac{|2.04848265 - 2.0499822151654996|}{2.0499822151654996} \times 100$$

$$= 0.07315015488$$

$$= 0.0732\% \text{ to } (3 \text{ s.f})$$

5 Conclusion

Through this investigation, it is conclusive that equation (4.20) is much more accurate at modeling the effect of damping on a simple harmonic oscillator in comparison to equation (3.18). Equation (4.20) has an error of 0.0732% to (3 s.f) and equation (3.18) has an error of 25.7% to (3 s.f). This is roughly a 351 times increase in accuracy meaning that equation (4.20) is very successful at describing the motion of a damped simple harmonic oscillator. In relation to the title question, equation (2.40) proves that there is an exponentially decreasing displacement from the equilibrium as time passes. Given the correct value for b the equation (2.40) is able to approximate the displacement of moving mass with a very high accuracy. With equation (2.40), other aspects of the system can be found such as it's average velocity and acceleration between any point on the graph with a high degree of accuracy. Additionally, it is confirmed that the approach to find the dampening coefficient and the formula derived in equation (4.19) was successful at providing a fitting exponential function for any given dataset. It was initially hypothesized that the dampening effect would be greater in water because of the drag force. Ultimately, this hypothesis proved to be correct due to equation (2.40) having a steeper exponential envelope gradient in comparison to equation (3.18).

6 Evaluation

Despite the successes of equation (2.40), there are a few drawbacks that it encountered. The most prevalent issue is that the graph does not properly align with the raw data. An example of this can be seen in Figure 11.

s

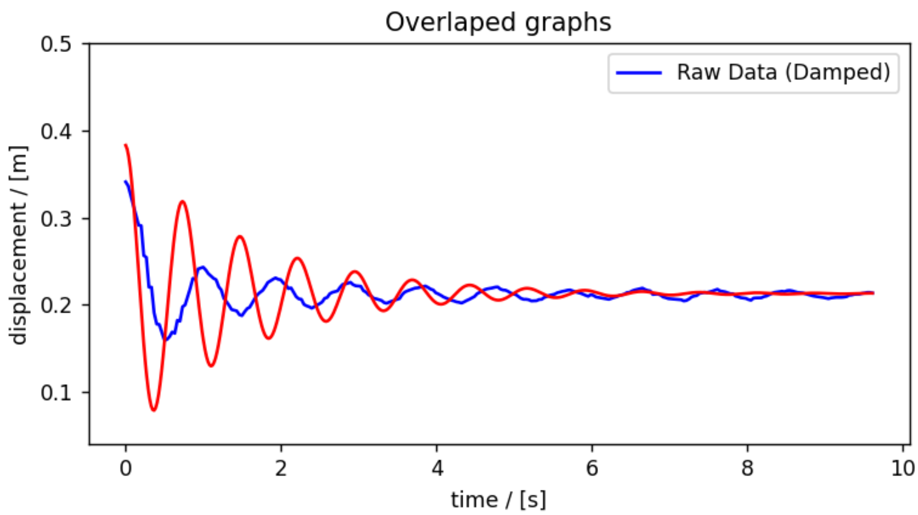

Figure 11: Overlap Comparison.
Graph generated by candidate using python version 3.12.8

Figure 11 shows the equation (2.40) and the raw data overlapping each other, Where the red graph represents equation (2.40) and the blue graph represents the raw data. It is visually apparent that equation (2.40) is inaccurate between points $0 \leq t \leq 4$, where the maximum/minimum point of the blue graph is much greater than that of the red graph. For example, at point $t = 1.49$s, the raw data shows the instantaneous displacement[14] value to be $\mathcal{X} = 0.189$m (to 3 s.f). Alternatively, equation (4.20) predicts $\mathcal{X} = 0.279$m (to 3 s.f). This large difference in results is due to various reasons. There is two reasons as to why this happens.

Firstly, there was an experimental error when conducting the experiment for the damped oscillator. Initially, the mass was meant to be hung at the same initial displacement $x_0 = 0.34m$ but because the setup was positioned next to an open window (which allowed light rays to refract through the large glass tube) the perceived position of the mass was different to its actual location leading to an inaccuracy in initial displacement measurement. This is a major component to why the model is inaccurate in few aspects such as e.g. predicting instantaneous values for displacement.

The second issue is that the mass was dropped from a medium of low density to a medium of high density. In the experimental method for the damped harmonic oscillator, the mass drops from air into water. Water has a higher density than air, meaning that there would be a greater drag force in water. Due to the mass dropping into the water in under a second, the rate of damping became very high (which explains the unusually steep dip in the raw data). This sudden dip in amplitude is not as noticeable in equation (2.40) as the calculation did not account for any changes in medium. Furthermore, water is liquid and the force of buoyancy[16] was not taken into account in the calculation. If these factors were to be taken into consideration, equation (4.20) would be more accurate.

In the end, equation (4.20) is quite successful at predicting average changes in displacement, velocity, acceleration. The function $\mathcal{X}(t)$ forms the basis of the model and then this function can be differentiated to find the velocity-time graph of the function. The change in velocity over 2 points and divided by the change in time between those 2 points gives the average velocity for that time period, and similarly done for acceleration.

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7 Appendix

```
#                         Script to generate equation (2.40) :


def X(t):
```

```
    b = 0.015                                   # dampening coefficient
    m = 0.3                                     # mass (kg)
    x_o = 0.17                                  # initial displacement (m)
    epsilon = 0.17                              # initial position X(o)
    omega = 8.544751264289932                   # angular frequency
    phi = 0.0                                   # phase difference
    coeff = -b/(2*m)                            # exponent constant
    exp = np.power(np.e, coeff*t)               # exponent
    para1 = omega**2                            # inside square root 1
    para2 = (b**2)/(4*(m**2))                   # inside square root 2
    cosine = np.cos(t*np.sqrt(para1 - para2)+phi) # cosine function
    func = (exp*x_o*cosine) + epsilon           # adjusting initial pos
    return func

#                       Script for Simpson's rule integration :

def simpsons_rule():
    I = sp.integrate.quad(lambda t: X(t),0, 023.97446)
    print(f"Area under X(t) is : {I}")
    # using simpson's rule to integrate under the raw data curve.

#                       Script to find spring constant :

import matplotlib.pyplot as plt
import numpy as np
dx1 = 34-29.5
dx2 = 34-25.3
dx3 = 34-20.1
dx4 = 34-16
dx5 = 34-12
dx6 = 34-7
#python -m springconstant.py
# y = Force
```

```
y = np.array([0,0.1*9.81,(0.2*9.81), (0.3*9.81), (0.4*9.81), (0.5*9.81),
    (0.6*9.81)])
# x = displacement
x = np.array([0,dx1,dx2,dx3,dx4,dx5,dx6])
for i in range(0, len(x)):
    x[i] = (x[i])/100
print(x)
print(y)
plt.xlabel("x [m]")
plt.ylabel("F [N]")
xerr = 0.002
yerr = 0.00001
slope, intercept = np.polyfit(x, y, 1)
mass = 0.3 #Kg
a, b = np.polyfit(x, y, 1)
plt.title("graph of force against displacement (Hooke's law)")
plt.errorbar(x, y, xerr, yerr, fmt="o", color="r")
plt.plot(np.unique(x), np.poly1d(np.polyfit(x, y, 1))(np.unique(x)), color = 'k')
plt.plot(x, a*x+b)
plt.scatter(x, y)
k = slope
m = 0.3
print(f"spring constant (k) = {slope}")
print(f"angular velocity (omega) = {np.sqrt(k/m)}")
plt.show()

#                   Script to find the dampening coefficient (b) :

    def dampeningCoefficient(self):
        # Ensure the inputs are numpy arrays for easy computation
        t_dash = np.array(self.t)
        x_dash = np.array(self.x)

        # Compute the sum in the equation
```

```
summation = np.sum((-1.48 - 0.6 * np.log(x_dash)) / t_dash)
# Divide the sum by 10
result = summation / 10
self.setB(result)
return self.b
```