THE MARDEŠIĆ CONJECTURE FOR COUNTABLY COMPACT SPACES

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Abstract. We shall show that if d and s are positive integers, K_i is a compact linearly ordered topological space for each $i < d$, X is a countably compact subspace of $\prod_{i<d} K_i$, Z_j is an infinite Hausdorff space for each $j < d + s$, and $f : X \to \prod_{j<d+s} Z_j$ is a continuous surjection, then there exist at least two indexes $j < d + s$ such that Z_j is compact and metrizable. This improves the Mardešić Conjecture, which was proved by G. Martínez-Cervantes and G. Plebanek in [8].

1. Introduction

G. Peano proved a famous theorem that there exists a continuous surjection from $[0, 1]$ onto $[0, 1] \times [0, 1]$. It is natural to ask if an analogous theorem can be proved for other compact LOTS. Some results were obtained by D. Kurepa in [5] and S. Mardešić and P. Papić in [7]. Then, L. B. Treybig proved the following amazing theorem.

Theorem 1.1 (L. B. Treybig [10]). *If X and Y are infinite Hausdorff spaces and $X \times Y$ is a continuous image of a compact LOTS, then both X and Y are metrizable.*

For example, if K is a compact nonmetrizable LOTS, then there exists no continuous surjection from K onto $K \times K$.

The following generalization of Theorem 1.1 was conjectured by S. Mardešić in [6] and proved by G. Martínez-Cervantes and G. Plebanek in [8].

Theorem 1.2 (The Mardešić Conjecture). *Let d and s be positive integers. If K_i is a compact LOTS for each $i < d$, Z_j is an infinite Hausdorff space for each $j < d + s$, and there exists a continuous surjection from $\prod_{i<d} K_i$ onto $\prod_{j<d+s} Z_j$, then there exist at least $s + 1$ indexes $j < d + s$ such that Z_j is metrizable.*

Notice that when $d = s = 1$, this is exactly Theorem 1.1. The proof in [8] was done by using a new dimension called *free dimension* for

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compact spaces. First, they observed that without loss of generality, we may assume $s = 1$. Then, they proved that when K_i and Z_j are as in the assumption of Theorem 1.2,

- The free dimension of $\prod_{i<d} K_i$ is at most d
- The free dimension does not increase by taking closed subspaces or continuous images, and
- If Z_j is not metrizable for all $j < d$, then the free dimension of $\prod_{j<d+1} Z_j$ is at least $d + 1$.

Not only did their argument solved a problem that were open for decades, but it also presented a clear and intuitive explanation why this theorem holds. In addition, we can prove the following stronger theorem.

Theorem 1.3. *Let d and s be positive integers. If K_i is a compact LOTS for each $i < d$, X is a closed subspace of $\prod_{i<d} K_i$, Z_j is an infinite Hausdorff spaces for each $j < d + s$, and there exists a continuous surjection from X onto $\prod_{j<d+s} Z_j$, then there exist at least two indexes $j < d + s$ such that Z_j is metrizable.*

Meanwhile, G. I. Čertanov proved the following theorem, which is a countably compact version of Theorem 1.1.

Theorem 1.4 (G. I. Čertanov in [11])**.** *If X and Y are infinite Hausdorff spaces and $X \times Y$ is a continuous image of a countably compact GO-space, then X and Y are compact and metrizable.*

The main purpose of this paper is to show that Theorem 1.3 holds even if we assume X is countably compact instead of X being closed. Our proof is significantly different from the one of G. Martínez-Cervantez and G. Plebanek and uses countable elementary substructures. This is another demonstration of the effectiveness of the applications of countable elementary substructures to analyze nonseparable structures, which was used by the author in [4].

The structure of this paper is as follows. In Section 2, we shall define some notations that were introduced by the author in [3] and [4]. They play crucial roles in Section 5.In Section 3 and 4, we shall prove lemmas that are used later. In Section 5, we shall show that under the assumption of the main theorem, if at least one of Z_j is not separable, then we may remove it while decrease the dimension of the domain by one. In Section 6, we shall prove a slight strengthening of S. Mardešić's theorem in [6]. Note that the original argument by Mardešić will work, but we shall give a more direct proof. In Section 7, we shall combine the results from the previous sections to prove the main theorem. We conclude the paper by presenting some open problems in 8.

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2. Definitions

We shall define some notions about LOTS.

Definition 2.1. Let K be a LOTS and $A \subseteq K$. We say that a subset B of A is a *cofinal subset of* A if and only if for every $p \in A$, there exists $q \in B$ such that $p \leq q$. Note that we consider that $\varnothing$ is a confinal subset of $\varnothing$ and if $p = \max A$, then $\{ p \}$ is a cofinal subset of A. Similarly, we say that a subset B of A is *coinitial subset of* A if and only if for every $p \in A$, there exists $q \in B$ such that $q \leq p$.

For every $p \in K$, define $(\leftarrow, p) = \{ q \in K \mid q < p \}$ and $(p, \rightarrow) = \{ q \in K \mid q > p \}$. For each $p \in K$, if p is not the least element, then let $\mathrm{cf}(p)$ be the least cardinality of cofinal subsets of $(\leftarrow, p)$. If p is not the greatest element, then let $\mathrm{ci}(p)$ be the least cardinality of coinitial subsets of $(p, \rightarrow)$.

The following notions are defined by the author in [3] and [4] to use countable elementary submodels to analyze nonseparable connected LOTS.

Definition 2.2. Let K be a GO space and M a countable set. Then, for all $p \in K$, define

$$\begin{aligned}
\eta(K, M, p) &= \sup \{ u \in \mathrm{cl}(K \cap M) \mid u \leq p \} \\
\zeta(K, M, p) &= \inf \{ u \in \mathrm{cl}(K \cap M) \mid u \geq p \} \\
I(K, M, p) &= [\eta(L, M, p), \zeta(L, M, p)]
\end{aligned}$$

if they exist.

Let $\mathcal{I}(K, M)$ be the set of all sets of the form $I(K, M, x)$ where $x \in K$.

Note that if $p \in \mathrm{cl}(K \cap M)$, then $\eta(K, M, p) = \zeta(K, M, p) = p$. This is true even when p is not a limit point of $K \cap M$ from below. If $p \notin \mathrm{cl}(K \cap M)$ and $\eta(K, M, p)$ exists, then it is easy to see that $\eta(K, M, p) = \sup \{ u \in K \cap M \mid u < p \}$.

Recall that if K is a compact LOTS, then $\min K$ and $\max K$ exist and for every subset A of K, $\inf A$ and $\sup A$ exist. So, for every countable set M and $p \in K$, $\eta(K, M, p)$, $\zeta(K, M, p)$, and $I(K, M, p)$ exist.

Definition 2.3. Let $S_0, \ldots, S_{n-1}$ be any set. Let $\theta_{S_0,\ldots,S_{n-1}}$ be the least regular cardinal θ such that $\mathcal{P}(S_0 \cup S_1 \cup \cdots \cup S_{n-1}) \in H(\theta)$.

Lemma 2.4. *Let K be a compact LOTS and M a countable elementary submodel of $H(\theta_K)$ with $K \in M$. Let $p_1, p_2, p_3, p_4 \in K$ be such that $p_1 < p_2 < p_3 < p_4$ and $I(K, M, p_1) \neq I(K, M, p_2) \neq I(K, M, p_3) \neq I(K, M, p_4)$. Then, there exists $x \in K \cap M$ such that $p_1 < x < p_4$.*

Proof.

Case 1. $p_2 \in \mathrm{cl}(K \cap M)$

Since (p_1, p_3) is an open neighborhood of p_2, we have $(p_1, p_3) \cap (K \cap M) \neq \varnothing$. Let x be an element of $(p_1, p_3) \cap M$.

Case 2. $p_2 \notin \mathrm{cl}(K \cap M)$

Then, $p_2 < \zeta(K, M, p_2)$. Since $I(K, M, p_2) \neq I(K, M, p_3)$, we have $\zeta(K, M, p_2) \leq p_3 < p_4$. Since $\zeta(K, M, p_2) \in \mathrm{cl}(K \cap M)$ and $p_2 < \zeta(K, M, p_2) < p_4$, there exists $x \in K \cap M$ with $p_2 < x < p_4$. □

3. Extenting LOTS

This section is devoted to the proof of the following lemma.

Lemma 3.1. *Let d be a positive integer and K_i a compact LOTS for each $i < d$. Then, there exist a compact LOTS K'_i for each $i < d$ and an injective function $\sigma : \prod_{i<d} K_i \to \prod_{i<d} K'_i$ such that for every continuous surjective function f from a countably compact subspace X of $\prod_{i<d} K_i$ onto an infinite Hausdorff space Z, if*

- *Y is a subset of $\prod_{i<d} K'_i$ defined by $y \in Y$ if and only if for some $z \in Z$, $y \in \mathrm{Cl}(\sigma^{\to} f^{\leftarrow} \{ z \})$, and*
- *g is a function from Y to Z defined by $g(y) = z$ if and only if $y \in \mathrm{Cl}(\sigma^{\to} f^{\leftarrow} \{ z \})$*

then Y is countably compact, g is well-defined, and for every $z \in Z$, $\mathrm{Cl}(g^{\leftarrow} \{ z \}) \subseteq Y$.

Proof. Let d be a positive integer and K_i a compact LOTS for each $i < d$. For each $i < d$, we shall define a LOTS K'_i to be the set of all $\langle p, m \rangle$ such that

(i) $p \in K_i$,
(ii) $m \in \{ -1, 0, 1 \}$,
(iii) if $m = -1$, then either $p = \min K_i$ or $\mathrm{cf}(p) > \aleph_0$, and
(iv) if $m = 1$, then either $p = \max K_i$ or $\mathrm{ci}(p) > \aleph_0$.

It is easy to show that for every $i < d$, K'_i is compact.

For each $i < d$, define $\sigma_i : K_i \to K'_i$ by $\sigma_i(p) = \langle p, 0\rangle$. Let $\sigma : \prod_{i<d} K_i \to \prod_{i<d} K'_i$ by $\sigma(x)(i) = \sigma_i(x(i)) = \langle x(i), 0\rangle$. We shall show that these work.

For each $i < d$, define $\tau_i : K'_i \to K_i$ by $\tau_i(\langle p, m\rangle) = p$. Notice that τ_i is continuous. Let $\tau : \prod_{i<d} K'_i \to K_i$ be defined by $\tau(y)(i) = \tau_i(y(i))$. For each $i < d$, define $\mu_i : K'_i \to \{ -1, 0, 1 \}$ by $\mu_i(\langle p, m\rangle) = m$.

To show that K'_i for $i < d$ and σ work, let X be a countably compact subspace X of $\prod_{i<d} K_i$, Z an infinite Hausdorff space, and $f : X \to Z$ a continuous surjection.

Define Y as in the statement of this lemma. First, we shall show that Y is countably compact. To do so, we shall first prove the following claim.

Claim 1. Let $\langle y_n | n < \omega\rangle$ be a sequence in Y and $\langle z_n | n < \omega\rangle$ a sequence in Z such that

- for each $i < d$, $\langle y_n(i) | n < \omega\rangle$ is either constant or strictly monotone, and
- for each $n < \omega$, $y_n \in \mathrm{Cl}(\sigma^{\to} f^{\leftarrow} \{ z_n \})$.

Let y_ω be the limit point of $\langle y_n | n < \omega\rangle$. Let $a, b \in \prod_{i<d} K'_i$ such that $a < y_\omega < b$. Then,

(i) there exists $z \in Z$ and an infinite subset H of ω such that $\langle z_n | n \in H\rangle$ is convergent to z, and
(ii) $\sigma^{\to} f^{\leftarrow} \{ z \} \cap (a, b) \neq \varnothing$.

$\vdash$ Let

$$A_0 = \{ i < d \mid \langle y_n(i) | n < \omega\rangle \text{ is constant} \}$$
$$A_{-1} = \{ i < d \mid \langle y_n(i) | n < \omega\rangle \text{ is strictly increasing} \}$$
$$A_1 = \{ i < d \mid \langle y_n(i) | n < \omega\rangle \text{ is strictly decreasing} \}$$

Let $n_0 < \omega$ be so that $n_0 > 0$, and

- if $i \in A_{-1}$, then $y_{n_0}(i) > a(i)$, and
- if $i \in A_1$, then $y_{n_0}(i) < b(i)$

For each $n \in [n_0, \omega)$, we shall define $a_n, b_n, y'_n \in \prod_{i<d} K'_i$ such that (a_n, b_n) is an open neighborhood of y_n and $y'_n \in (a_n, b_n)$. For every $i < d$, if $i \in A_{-1}$, then let $a_n(i) = y_{n-1}(i)$ and $b_n(i) = y_{n+1}(i)$. If $i \in A_1$, let $a_n(i) = y_{n+1}(i)$ and $b_n(i) = y_{n-1}(i)$.

By induction on $n \in [n_0, \omega)$, we shall define $a_n(i)$ and $b_n(i)$ for $i \in A_0$. Let $a_0(i) = a(i)$ and $b_0(i) = b(i)$.

Suppose that $a_n(i)$ and $b_n(i)$ for all $i \in A_0$ are defined. So, a_n and b_n are defined. Since (a_n, b_n) is an open neighborhood of y_n and $y_n \in \mathrm{Cl}(\sigma^{\to} f^{\leftarrow} \{ z_n \})$, there exists $y'_n \in \sigma^{\to} f^{\leftarrow} \{ z_n \} \cap (a_n, b_n)$. Then, there exists $x'_n \in f^{\leftarrow} \{ z_n \}$ such that $\sigma(x'_n) = y'_n$.

Now, we shall define $a_{n+1}(i)$ and $b_{n+1}(i)$ for every $i \in A_0$. If $y'_n(i) < y_\omega(i)$, then let $a_{n+1}(i) = y'_n(i)$ and $b_{n+1}(i) = b_n(i)$. If $y'_n(i) > y_\omega(i)$, then let $a_{n+1}(i) = a_n(i)$ and $b_{n+1}(i) = y'_n(i)$. If $y'_n(i) = y_\omega(i)$, then let $a_{n+1} = a_n(i)$ and $b_{n+1}(i) = b_n(i)$. In all cases, we have $a_{n+1}(i) < y_\omega(i) < b_{n+1}(i)$. This completes the definition of a_n, b_n, and y'_n for all $n \in [n_0, \omega)$.

For every $i \in A_{-1}$, for every $n \in [n_0, \omega)$,

$$y_{n-1}(i) = a_n(i) < y'_n(i) < b_n(i) = y_{n+1} < y_\omega(i)$$

and $\langle y_n(i) | n < \omega \rangle$ converges to $y_\omega(i)$. So, $\langle y'_n(i) | n < \omega \rangle$ also converges to $y_\omega(i)$. Similarly, for every $i \in A_1$, $\langle y'_n(i) | n < \omega \rangle$ converges to $y_\omega(i)$.

For each $i \in A_0$, define

$$H_{i,0} = \{ n \in [n_0, \omega) \mid y'_n(i) = y_\omega(i) \}$$
$$H_{i,-1} = \{ n \in [n_0, \omega) \mid y'_n(i) < y_\omega(i) \}$$
$$H_{i,1} = \{ n \in [n_0, \omega) \mid y'_n(i) > y_\omega(i) \}$$

Then, trivially, $H_{i,0} \cup H_{i,-1} \cup H_{i,1} = [n_0, \omega)$. So, at least one of them is infinite. If $H_{i,0}$ is infinite, then $\langle y'_n(i) | n \in H_{i,0} \rangle$ is constantly $y_\omega(i)$ and hence converges to $y_\omega(i)$. If $H_{i,-1}$ is infinite, then for all $n_1, n_2 \in H_{i,-1}$, if $n_1 < n_2$, then $y'_{n_1}(i) = a_{n_1+1}(i) \leq a_{n_2}(i) < y'_{n_2}(i)$. So, $\langle y_n(i) | n \in H_{i,-1} \rangle$ is strictly increasing. Similarly, if $H_{i,1}$ is infinite, then $\langle y_n(i) | n \in H_{i,1} \rangle$ is strictly decreasing.

Then, there exists a function $h : d \to \{ -1, 0, 1 \}$ such that $\bigcap_{i<d} H_{i,h(i)}$ is infinite. Let $H = \bigcap_{i<d} H_{i,h(i)}$. Then, for every $i < d$, $\langle y'_n(i) | n \in H \rangle$ is either constant or strictly monotone. Thus, $\langle x'_n(i) | n \in H \rangle$ is also constant or strictly monotone. Since X is countably compact and for every $n \in H$, $x'_n \in X$, there exists a limit point $x'_\omega \in X$ of $\{ x'_n \mid n \in H \}$. Since for every $i < d$, $\langle x'_n(i) | n \in H \rangle$ is either constant or strictly monotone, $\langle x'_n | n \in H \rangle$ converges to x'_ω. Since f is continuous, $\langle f(x'_n) | n \in H \rangle$ converges to $f(x'_\omega)$. Since $f(x'_n) = z_n$, it implies that $\langle z_n | n \in H \rangle$ converges to $f(x'_\omega)$. Let $z = f(x'_\omega)$.

Let $y'_\omega = \sigma(x'_\omega)$. For every $i < d$, if $h(i) = 0$, then $y'_n(i) = y_\omega(i)$ for all $n < \omega$. Thus, since $\sigma(x'_n) = y'_n$ for all $n < \omega$, we have $y'_\omega(i) = y_\omega(i)$ and hence $a(i) < y'_\omega(i) < b(i)$. If $h(i) = -1$, then $a(i) < y'_{n_0}(i) = \sigma_i(x'_{n_0}(i)) < \sigma_i(x'_\omega(i)) = y'_\omega$. Also, for every $n \in H$, since $n \in H_{i,-1}$, $y'_n(i) < y_\omega(i)$ and hence $y'_\omega \leq y_\omega < b(i)$. Thus, $a(i) < y'_\omega(i) < b(i)$. Similarly, if $h(i) = 1$, then $a(i) < y'_\omega(i) < b(i)$. Therefore, $y'_\omega \in (a, b)$. $\dashv$ (Claim 1)

Claim 2. Y is countably compact.

$\vdash$ Let $\{ y_n \mid n < \omega \}$ be a countable subset of Y. We shall show that $\{ y_n \mid n < \omega \}$ has a limit point in Y. Without loss of generality, we may assume that for every $i < d$, $\langle y_n(i) | n < \omega \rangle$ is constant or strictly monotone. Thus, $\langle y_n | n < \omega \rangle$ converges to some $y_\omega \in \prod_{i<d} K'_i$.

For each $n < \omega$, since $y_n \in Y$, there exists $z_n \in Z$ such that $y_n \in \mathrm{Cl}(\sigma^{\rightarrow} f^{\leftarrow} \{ z_n \})$. By the previous claim, there exists an infinite subset H of ω such that $\langle z_n | n \in H \rangle$ converges to some $z \in Z$.

Now, we shall show that $y_\omega \in \mathrm{Cl}(\sigma^{\rightarrow} f^{\leftarrow} \{ z \})$. Let $a, b \in \prod_{i<d} K'_i$ such that $a < y_\omega < b$. Then by the preivous claim applied to $\langle y_n | n \in H \rangle$, there exists an infinite subset H' of H such that $\langle z_n | n \in H' \rangle$ converges of z and $\sigma^{\rightarrow} f^{\leftarrow} \{ z \} \cap (a, b) \neq \varnothing$. $\dashv$ (Claim 2)

Claim 3. For every $y \in Y$, there exists a unique $z \in Z$ such that $y \in \mathrm{Cl}(\sigma^{\rightarrow} f^{\leftarrow} \{ z \})$.

$\vdash$ The existence is by the definition of Y. To show the uniqueness, let $z_1, z_2 \in Z$ be so that $y \in \mathrm{Cl}(\sigma^{\rightarrow} f^{\leftarrow} \{ z_1 \}) \cap \mathrm{Cl}(\sigma^{\rightarrow} f^{\leftarrow} \{ z_2 \})$. Let $x = \tau(y)$. For each $m \in \{ -1, 0, 1 \}$, let

$$A_m = \{ i < d \mid \mu_i(y(i)) = m \}$$

Let

$$B_{-1} = \{ i \in A_0 \mid \mathrm{cf}(y(i)) \leq \omega \}$$
$$B_1 = \{ i \in A_0 \mid \mathrm{ci}(y(i)) \leq \omega \}$$

We shall define $a_n, b_n \in \prod_{i<d} K'_i$ for all $n < \omega$ with $a_n < y(i) < b_n$ for all $n < \omega$. If $i \in B_{-1}$, let $\langle a_n(i) | n < \omega \rangle$ be a nondecreasing cofinal sequence in $(\leftarrow, y(i))$. Note that $x(i)$ has the immediate successor p, then $\langle p, 0 \rangle$ is the immediate successor of $y(i)$ and for all sufficiently large $n < \omega$, $a_n(i) = \langle p, 0 \rangle$. If $i \in A_0 \setminus B_{-1}$, then $\langle x(i), -1 \rangle \in K'_i$ and $\langle x(i), -1 \rangle$ is the immediate successor of $y(i) = \langle x(i), 0 \rangle$. Let $a_n(i) = \langle x(i), -1 \rangle$ for all $n < \omega$. If $i \in B_1$, let $\langle b_n(i) | n < \omega \rangle$ be a nonincreasing coinitial sequence in $(y(i), \rightarrow)$. If $i \in A_0 \setminus B_1$, then let $b_n(i) = \langle x(i), 1 \rangle$ for all $n < \omega$.

If $i \in A_{-1}$, then $y(i) = \langle x(i), -1 \rangle < \langle x(i), 0 \rangle$. Let $b_n(i) = \langle x(i), 0 \rangle$ for all $n < \omega$. If $i \in A_1$, then $y(i) = \langle x(i), 1 \rangle > \langle x(i), 0 \rangle$. Let $a_n(i) = \langle x(i), 0 \rangle$.

Now, to complete the definition of a_n and b_n, we need to specify $a_n(i)$ when $i \in A_{-1}$ and $b_n(i)$ when $i \in A_1$. While defining them, we also define a sequence $\langle x_{n,1}, x_{n,2} | n < \omega \rangle$ in X. For each $i \in A_{-1}$, let $a_0(i)$ be an arbitrary element $<y(i)$. For each $i \in A_1$, let $b_0(i)$ be an arbitrary element $>y(i)$.

Suppose that a_n, b_n are defined so that $a_n < y < b_n$. Since $y \in \mathrm{Cl}(\sigma^{\to} f^{\leftarrow}\{z_1\}) \cap \mathrm{Cl}(\sigma^{\to} f^{\leftarrow}\{z_2\})$, there exists $x_{n,1} \in f^{\leftarrow}\{z_1\}$ and $x_{n,2} \in f^{\leftarrow}\{z_2\}$ such that $\sigma(x_{n,1}), \sigma(x_{n,2}) \in (a_n, b_n)$.

We shall define $a_{n+1}(i)$ for all $i \in A_{-1}$ and $b_{n+1}(i)$ for all $i \in A_1$. Let $i \in A_{-1}$. Then, we have $b_n(i) = \langle x(i), 0\rangle$ and $y(i) = \langle x(i), -1\rangle$. So, $(a_n(i), b_n(i)) = (a_n(i), y(i)]$. Since $\mu_i(\sigma(x_{n,1})(i)) = \mu_i(\sigma(x_{n,2})(i)) = 0$ and $\mu_i(y(i)) = -1$, we have $\sigma(x_{n,1})(i) \neq y(i)$ and $\sigma(x_{n,2})(i) \neq y(i)$. Since $\sigma(x_{n,1})(i), \sigma(x_{n,2})(i) \in (a_n(i), b_n(i)) = (a_n(i), y(i)]$, we have $\sigma(x_{n,1})(i) < y(i)$ and $\sigma(x_{n,2})(i) < y(i)$. So,

$$\begin{aligned} a_n(i) &< \min\{\sigma(x_{n,1})(i), \sigma(x_{n,2})(i)\} \\ &\leq \max\{\sigma(x_{n,1})(i), \sigma(x_{n,2})(i)\} < y(i) \end{aligned}$$

Let $a_{n+1}(i) = \max\{\sigma(x_{n,1})(i), \sigma(x_{n,2})(i)\}$. Similarly, for each $i \in A_1$, let $b_{n+1}(i) = \min\{\sigma(x_{n,1})(i), \sigma(x_{n,2})(i)\}$. We can prove that $y(i) < b_{n+1} = \min\{\sigma(x_{n,1})(i), \sigma(x_{n,2})(i)\} \leq \max\{\sigma(x_{n,1})(i), \sigma(x_{n,2})(i)\} < b_n(i)$.

Subclaim 3.1. For every $i \in A_0$, both $\langle \sigma(x_{n,1}(i)) | n < \omega\rangle$ and $\langle \sigma(x_{n,1}(i)) | n < \omega\rangle$ converge to $y(i)$.

$\vdash$ If $i \in A_0 \setminus B_{-1}$, then for all $n < \omega$, we have $a_n(i) = \langle x(i), -1\rangle$, which is the immediate predecessor of $y(i)$. If $i \in B_{-1}$, then $\langle a_n(i) | n < \omega\rangle$ converges to $y(i)$. If $i \in A_0 \setminus B_1$, then for all $n < \omega$, we have $b_n(i) = \langle x(i), 1\rangle$, which is the immediate successor of $y(i)$. If $i \in B_1$, then $\langle b_n(i) | n < \omega\rangle$ converges to $y(i)$. Since for every $n < \omega$, $a_n(i) < \min\{\sigma(x_{n,1})(i), \sigma(x_{n,2})(i)\} \leq \max\{\sigma(x_{n,1})(i), \sigma(x_{n,2})(i)\} < b_n(i)$, both $\langle \sigma(x_{n,1}(i)) | n < \omega\rangle$ and $\langle \sigma(x_{n,1}(i)) | n < \omega\rangle$ converge to $y(i)$. $\dashv$ (Subclaim 3.1)

Subclaim 3.2. For every $i \in A_{-1}$, both $\langle \sigma(x_{n,1}(i)) | n < \omega\rangle$ and $\langle \sigma(x_{n,1}(i)) | n < \omega\rangle$ converge to the same point, denoted by $x'(i)$.

$\vdash$ Recall that for all $n < \omega$,

$$a_n(i) < \min\{\sigma(x_{n,1})(i), \sigma(x_{n,2})(i)\} \leq \max\{\sigma(x_{n,1})(i), \sigma(x_{n,2})(i)\} \leq a_{n+1}(i)$$

So, both $\langle \sigma(x_{n,1}(i)) | n < \omega\rangle$ and $\langle \sigma(x_{n,1}(i)) | n < \omega\rangle$ converge to the same point. $\dashv$ (Subclaim 3.2)

Similarly, we can prove that for every $i \in A_{-1}$, both $\langle \sigma(x_{n,1}(i)) | n < \omega\rangle$ and $\langle \sigma(x_{n,1}(i)) | n < \omega\rangle$ converge to the same point, denoted by $x'(i)$. For each $i \in A_0$, let $x'(i) = y(i)$. Then, x' is the limit point of both $\{\sigma(x_{n,1}) \mid n < \omega\}$ and $\{\sigma(x_{n,2}) \mid n < \omega\}$. Recall that for every $n < \omega$, $\sigma(x_{n,1}) \in f^{\leftarrow}\{z_1\}$ and $\sigma(x_{n,2}) \in f^{\leftarrow}\{z_1\}$. So, $f(x') = z_1 = z_2$.

$\dashv$ (Claim 3)

By the previous claim, the function g is well-defined.

Claim 4. For every $z \in Z$, $\mathrm{Cl}(g^{\leftarrow}\{z\}) \subseteq Y$.

$\vdash$ Let $z \in Z$ and $y_0 \in \mathrm{Cl}(g^{\leftarrow}\{z\})$. To show that $y_0 \in Y$, let U be an open neighborhood of y_0. Since $y_0 \in \mathrm{Cl}(g^{\leftarrow}\{z\})$, there exists $y_1 \in U \cap g^{\leftarrow}\{z\}$, i.e. $g(y_1) = z$. By definition, it implies $y_1 \in \mathrm{Cl}(\sigma^{\rightarrow} f^{\leftarrow}\{z\})$. Thus, there exists $y_2 \in U \cap \sigma^{\rightarrow} f^{\leftarrow}\{z\}$. Thus, $U \cap \sigma^{\rightarrow} f^{\leftarrow}\{z\} \neq \varnothing$. Hence, $y_0 \in \mathrm{Cl}(\sigma^{\rightarrow} f^{\leftarrow}\{z\})$ and so $y_0 \in Y$. $\dashv$ (Claim 4)

□

We shall prove the following lemma about the objects that satisfy the conclusion of the previous lemma.

Lemma 3.2. *Let K_i be a compact LOTS for each $i < d$, X a countably compact subspace of $\prod_{i<d} K_i$, Z an infinite Hausdorff space, and $f : X \to Z$ such that for every $z \in Z$, $\mathrm{Cl}(f^{\leftarrow}\{z\}) \subseteq X$. Then,*

(i) for every $z \in Z$, $f^{\leftarrow}\{z\}$ is closed, and
(ii) for every $z \in Z$, $i_0 < d$, and $r \in K_{i_0}$, if $z \notin f^{\rightarrow}\tau_{i_0}{}^{\leftarrow}\{r\}$, then there exists an open neighborhood U of r such that $z \notin f^{\rightarrow}\pi_{i_0}{}^{\leftarrow}U$.

Proof. For (i), let $z \in Z$. It suffices to prove $f^{\leftarrow}\{z\} = \mathrm{Cl}(f^{\leftarrow}\{z\})$. Trivially, $f^{\leftarrow}\{z\} \subseteq \mathrm{Cl}(f^{\leftarrow}\{z\})$. To show $\mathrm{Cl}(f^{\leftarrow}\{z\}) \subseteq f^{\leftarrow}\{z\}$, let $x \in \mathrm{Cl}(f^{\leftarrow}\{z\})$. By the assumption, we have $\mathrm{Cl}(f^{\leftarrow}\{z\}) \subseteq X$ and hence $x \in X$. It is easy to see that $f(x) = z$ and hence $x \in f^{\leftarrow}\{z\}$.

For (ii), let $z \in Z$, $i_0 < d$, and $r \in K_{i_0}$ be so that $z \notin f^{\rightarrow}\pi_{i_0}{}^{\leftarrow}\{r\}$. Then, for each $x \in \pi_{i_0}{}^{\leftarrow}\{r\}$, $f(x) \neq z$. Since f is continuous, there exists an open neighborhood W_x of x such that $W_x \cap f^{\leftarrow}\{z\} = \varnothing$. Since $\prod_{i<d} K_i$ is compact and $\pi_{i_0}{}^{\leftarrow}\{r\}$ is closed, there exist finitely many elements $x_0, x_1, \ldots, x_{k-1} \in \pi_{i_0}{}^{\leftarrow}\{r\}$ such that $\pi_{i_0}{}^{\leftarrow}\{r\} \subseteq W_{x_0} \cup W_{x_1} \cup \cdots \cup W_{x_{k-1}}$. Now it is easy to find an open neighborhood U of r such that $\pi_{i_0}{}^{\leftarrow}U \subseteq W_{x_0} \cup W_{x_1} \cup \cdots \cup W_{x_{k-1}}$. By the definition of W_x, we have $\pi_{i_0}{}^{\leftarrow}U \cap f^{\leftarrow}\{z\} = \varnothing$, so $z \notin f^{\rightarrow}\pi_{i_0}{}^{\leftarrow}U$ □(Lemma 3.2)

4. Finite union

We shall show the following lemma, which plays an important role in the next section. A similar lemma was proved by S. Mardešić in [6].

Lemma 4.1. *Let $n, d < \omega$. Suppose*

- *Z is a Hausdorff space, and*
- *for each $m < n$,*
  - *$K_{m,i}$ is a compact LOTS for each $i < d$,*
  - *X_m is a countably compact subspace of $\prod_{i<d} K_{m,i}$, and*
  - *$f_m : X_m \to Z$ is a continuous function, and*
- *$\bigcup_{m<n} \operatorname{ran}(f_m) = Z$.*

Then, there exist a compact LOTS K_i for each $i < d$, a countably compact subspace X of $\prod_{i<d} K_i$, and a continuous surjection $f : X \to Z$.

Proof. For each $i < d$, let $K_i = \bigcup_{m<n}(\{ m \} \times K_{m,i})$. We define the order on K_i by lexicographic ordering. For each $m < n$, define $\varphi_m : \prod_{i<d} K_{m,i} \to \prod_{i<d} K_i$ by $\varphi_m(\bar{x})(i) = \langle m, \bar{x}(i) \rangle$. Then, φ_m is a continuous injection.

Let $X = \bigcup_{m<n} \varphi_m{}^{\to} X_n$. For every $m < n$, since X_m is countably compact, so is $\varphi_m{}^{\to} X_m$. Thus, X is countably compact.

Define $f : X \to Z$ by $f(x) = z$ if and only if there exist $m < n$ and $\bar{x} \in X_m$ such that $x = \varphi_n(\bar{x})$ and $f_m(\bar{x}) = z$. We shall show that f is well-defined. Suppose that $x = \varphi_m(\bar{x}) = \varphi_{m'}(\bar{x}')$. Then, for every $i < d$, $x(i) = \varphi_m(\bar{x})(i) = \langle m, \bar{x}(i) \rangle$ and $x(i) = \varphi_{m'}(\bar{x}')(i) = \langle m', \bar{x}'(i) \rangle$. So, $m = m'$ and $\bar{x} = \bar{x}'$. It is easy to see that f is a continuous surjection. $\square$(Lemma 4.1)

5. Toward separability

This section is devoted to the proof of the following lemma.

Lemma 5.1. *Let d be an integer with $d \geq 2$. Let K_i be a compact LOTS for each $i < d$ and X a countably compact subspace of $\prod_{i<d} K_i$. Let Z_j be an infinite Hausdorff space for each $j \leq d$ such that Z_d is not separable. Suppose that there exists a continuous surjection $f : X \to \prod_{j \leq d} Z_d$. Then, there exist a compact LOTS $\tilde{K}_i$ for each $i < d-1$, a countably compact subspace $\tilde{Y}$ of $\prod_{i<d-1} \tilde{K}_i$, and a continuous surjection $\tilde{f} : \tilde{Y} \to \prod_{j<d} Z_d$.*

Proof. Suppose not. Let d be the least such that this lemma does not hold. Let K_i for each $i < d$, X, Z_j for each $j \leq d$, and f be as in the assumption.

Let K'_i for $i < d$ and $\sigma : \prod_{i<d} K_i \to \prod_{i<d} K'_i$ be as in the conclusion of Lemma 3.1. Let $X' \subseteq \prod_{i<d} K'_i$ be defined by $x' \in X'$ if and only if for some $z \in \prod_{j \leq d} Z_j$, $x' \in \operatorname{Cl}(\sigma^{\to} f^{\leftarrow} \{ z \})$ and let $f' : X' \to \prod_{j \leq d} Z_j$ be defined by $f'(x') = z$ if and only if $x' \in \operatorname{Cl}(\sigma^{\to} f^{\leftarrow} \{ z \})$. Then, by

the definition of K_i' and σ, X' is countably compact, f' is well-defined, and for all $z \in \prod_{j\le d} Z_j$, $\mathrm{Cl}((f')^{\leftarrow}\{z\}) \subseteq X'$. Let $g : X \to \prod_{j<d} Z_j$ be defined by $g(x)(i) = f(x)(i)$, namely $g(x)$ can be obtained by removing the last coordinate of $f(x)$. Let $Y' \subseteq \prod_{i<d} K_i'$ be defined by $x' \in Y'$ if and only if for some $\bar{z} \in \prod_{j<d} Z_j$, $x' \in \mathrm{Cl}(\sigma^{\rightarrow} g^{\leftarrow}\{\bar{z}\})$ and let $g' : Y' \to \prod_{j<d} Z_j$ be defined by $g'(x') = \bar{z}$ if and only if $x' \in \mathrm{Cl}(\sigma^{\rightarrow} g^{\leftarrow}\{\bar{z}\})$. By the definition of K_i' and σ again, Y' is countably compact, g' is well-defined, and for all $\bar{z} \in \prod_{j<d} Z_j$, $\mathrm{Cl}((g')^{\leftarrow}\{\bar{z}\}) \subseteq Y'$.

By renaming K_i', X', f', Y', and g' as K_i, X, f, Y, and g if necessarily, in addition to the original assumptions, we may assume

- for every $z \in \prod_{j\le d} Z_j$, $\mathrm{Cl}(f^{\leftarrow}\{z\}) \subseteq X$,
- for every $\bar{z} \in \prod_{j<d} Z_j$, $\mathrm{Cl}(g^{\leftarrow}\{\bar{z}\}) \subseteq Y$,
- $X \subseteq Y$, and
- for every $x \in X$, $g(x) = f(x) \restriction d$.

Let $\theta = \theta_{\langle K_i | i<d\rangle, \langle Z_j | j\le d\rangle}$and M a countable elementary submodel of $H(\theta)$ such that $\langle K_i | i < d\rangle, \langle Z_j | j \le d\rangle, f, g \in M$. For each $i \le d$, let π_i be the projection onto the i-th coordinate. Define $f_i = \pi_i \circ f$.

Claim 1. Let $r \in Z_d \setminus \mathrm{Cl}(Z_d \cap M)$. Then, there exists a finite subset A of X such that for all $x \in X$, if $f_d(x) = r$, then there exists $y \in A$ and $i < d$ such that $x(i) \in I(K_i, M, y(i))$.

$\vdash$ Suppose that the conclusion is false, i.e. for all finite subsets A of X, there exists $x \in X$ such that $f_d(x) = r$ and for all $y \in A$ and $i < d$, we have $x(i) \notin I(K_i, M, y(i))$.

We shall construct a sequence $\langle x_n | n < \omega\rangle$ in X as follows. Let x_0 be an arbitrary element of X such that $f_d(x) = r$. Suppose that $\langle x_m | m \le n\rangle$ has been defined. By assumption, there exists $x_{n+1} \in X$ such that $f_d(x_{n+1}) = r$ and for all $m < n$ and $i < d$, we have $x_{n+1}(i) \notin I(K_i, M, x_m(i))$.

It is easy to find an infinite subset H of ω such that for all $i < d$, $\langle x_n(i) | n < \omega\rangle$ is strictly monotone. Since X is countably compact, there exists a limit point x_ω of $\{ x_n \mid n \in H \}$. Since for all $i < d$, $\langle x_n(i) | n < \omega\rangle$ is strictly monotone, $\langle x_n | n \in H\rangle$ converges to x_ω. Since x_ω is a limit point of $\langle x_n | n \in H\rangle$ and for all $n \in H$, $f_d(x_n) = r$, we have $f_d(x_\omega) = r$.

Now, we shall show $f_d(x_\omega) \in \mathrm{Cl}(Z \cap M)$. It will derive a contradiction since $r \notin \mathrm{Cl}(Z \cap M)$ and we are done. It suffices to show that x_ω is a limit point of $X \cap M$. Let $a, b \in \prod_{i<d}(K_i \cap M)$ be so that $a < x_\omega < b$. Then, there exists $n_0 < \omega$ such that $x_{n_0}(i) \in (a, b)$. Note that it implies for all $n \in [n_0, \omega)$, $x_n(i) \in (a, b)$.

We shall define $a', b' \in \prod_{i<d} K_i \cap M$ as follows. Let $i < d$. If $\langle x_n(i) | n < \omega \rangle$ is strictly increasing, then by applying Lemma 2.4 to $x_{n_0}(i), x_{n_0+1}(i), x_{n_0+2}(i), x_{n_0+3}(i)$ and $x_{n_0+4}(i), x_{n_0+5}(i), x_{n_0+6}(i), x_{n_0+7}(i)$, we can find $a'(i), b'(i) \in K_i \cap M$ such that $x_{n_0}(i) < a'(i) < x_{n_0+3}(i)$ and $x_{n_0+4}(i) < b'(i) < x_{n_0+7}(i)$. If $\langle x_n(i) | n < \omega \rangle$ is strictly decreasing, similarly we can find $a'(i), b'(i)$ such that $x_{n_0+7}(i) < a'(i) < x_{n_0+4}(i) < x_{n_0+3}(i) < b'(i) < x_{n_0}(i)$.

Then, $x_{n_0+3} \in (a', b')$ and $(a', b') \subseteq (a, b)$. Note $x_{n_0+3} \in (a', b')$ and hence $(a', b') \cap X \neq \varnothing$. Since $(a', b') \in M$, there exists $x' \in X \cap (a', b') \cap M$. Since $(a', b') \subseteq (a, b)$, we have $x' \in X \cap M \cap (a, b)$. So, x_ω is a limit point of $X \cap M$. $\dashv$ (Claim 1)

Claim 2. Let $r \in Z_d \setminus \mathrm{Cl}(Z_d \cap M)$. Then, there exist finite sequences $\langle i_n | n < \hat{n} \rangle$ and $\langle p_n | n < \hat{n} \rangle$ such that for all $n < \hat{n}$, $i_n < d$ and $p_n \in K_{i_n}$ such that for all $x \in X$, if $f_d(x) = r$, then there exists $n < \hat{n}$ such that $x(i_n) \in I(K_{i_n}, M, p_n)$.

$\vdash$ Let A be as in the conclusion of Claim 1. Enumerate $d \times A$ as $\langle \langle i_n, x_n \rangle | n < \hat{n} \rangle$. Let $p_n = x_n(i_n)$. It is easy to verify that $\langle i_n | n < \hat{n} \rangle$ and $\langle p_n | n < \hat{n} \rangle$ work. $\dashv$ (Claim 2)

Claim 3. There exist finite sequences $\langle i_n | n < \hat{n} \rangle$ and $\langle p_n | n < \hat{n} \rangle$ such that for all $n < \hat{n}$, $i_n < d$ and $p_n \in K_{i_n}$, and for all $\bar{z} \in \bar{Z}$, there exist $x \in Y$ and $n < \hat{n}$ such that $g(x) = \bar{z}$ and $x(i_n) \in I(K_{i_n}, M, p_n)$.

$\vdash$ Since Z_d is not separable, there exists $r \in Z_d \setminus \mathrm{Cl}(Z_d \cap M)$. Let $\langle i_n | n < \hat{n} \rangle$ and $\langle p_n | n < \hat{n} \rangle$ be as in the conclusion of Claim 2. We shall show that for all $\bar{z} \in \bar{Z}$, there exists $x \in X$ and $n < \hat{n}$ such that $g(x) = \bar{z}$ and $x(i_n) \in I(K_{i_n}, M, p_n)$.

Let $\bar{z} \in \bar{Z}$. Define $z \in \prod_{j \leq d} Z_j$ by $z \restriction d = \bar{z}$ and $z(d) = r$. Since f is surjective, there exists $x \in X$ such that $f(x) = z$. Since $X \subseteq Y$, we have $x \in Y$ and $g(x) = f(x) \restriction d = \bar{z}$. Moreover, by the definition of $\langle i_n | n < \hat{n} \rangle$ and $\langle p_n | n < \hat{n} \rangle$, there exists $n < \hat{n}$ such that $x(i_n) \in I(K_{i_n}, M, p_n)$. $\dashv$ (Claim 3)

The advantage of Claim 3 is that we can describe it without r, which does not belong to M. Let $\langle i_n | n < \hat{n} \rangle$ and $\langle p_n | n < \hat{n} \rangle$ be the witness of the previous claim with the least $\hat{n}$.

Let E be the minimal subset of d such that for all $\bar{z} \in \prod_{j<d} Z_j$, there exist $y \in Y$ and $n < \hat{n}$ such that $g(y) = \bar{z}$ and for all $n < \hat{n}$, if $n \in E$, then $y(i_n) \in I(K_{i_n}, M, p_n)$ and if $n \notin E$, then $y(i_n) \in C(K_{i_n}, M, p_n)$.

Claim 4. $E \neq \varnothing$.

$\vdash$ Suppose $E = \varnothing$. Then, $g^{\rightarrow} \bigcup_{n<\hat{n}} \pi_{i_n}{}^{\leftarrow} C(K_{i_n}, M, p_n) = \prod_{j<d} Z_j$.

For each $n < \hat{n}$, let $Y_{2n} = \{ y \restriction d \setminus \{ i_n \} \mid y \in Y \wedge y(i_n) = \eta(K_{i_n}, M, p_n) \}$ and $Y_{2n+1} = \{ y \restriction d \setminus \{ i_n \} \mid y \in Y \wedge y(i_n) = \zeta(K_{i_n}, M, p_n) \}$. Define a function $g_{2n} : Y_{2n} \to \prod_{j<d} Z_j$ by $g_{2n}(\bar{y}) = z$ if and only if there exists $y \in Y$ such that $y(i_n) = \eta(K_{i_n}, M, p_n)$, $y \restriction d \setminus \{ i_n \} = \bar{y}$, and $g(y) = z$. Similarly, define a function $g_{2n+1} : Y_{2n+1} \to \prod_{j<d} Z_j$ if and only if there exists $y \in Y$ such that $y(i_n) = \zeta(K_{i_n}, M, p_n)$, $y \restriction d \setminus \{ i_n \} = \bar{y}$, and $g(y) = z$. Since Y is countably compact, both Y_{2n} and Y_{2n+1} are countably compact subspaces in $\prod_{i \in d \setminus \{ i_n \}} K_i$. Clearly, $g^{\rightarrow} \pi_{i_n}{}^{\leftarrow} C(K_{i_n}, M, p_n) = \operatorname{ran}(g_{2n}) \cup \operatorname{ran}(g_{2n+1})$.

Then, we have $\bigcup_{m<2\hat{n}} \operatorname{ran} g_m = g^{\rightarrow} \bigcup_{n<\hat{n}} \pi_{i_n}{}^{\leftarrow} C(K_{i_n}, M, p_n) = \prod_{j<d} Z_j$. By Lemma 4.1, there exist a compact LOTS $\tilde{K}_i$ for each $i < d-1$, a countable compact subspace $\tilde{Y}$ of $\prod_{i<d-1} K'_i$, and a continuous surjection $\tilde{f} : \tilde{Y} \to \prod_{j<d} Z_j$. This is a contradiction to the minimality of d. $\dashv$ (Claim 4)

Then, by the minimality of $\hat{n}$ and E, for each $m \in E$, there exists $\bar{z}_m \in \prod_{j<d} Z_j$ such that for all $y \in Y$ and $n < \hat{n}$, if $g(y) = \bar{z}_m$ and $y(i_n) \in I(K_{i_n}, M, p_n)$, then $n = m$ and $y(i_n) \notin C(K_{i_n}, M, p_n)$. In particular, for every $m, n < \hat{n}$, $\bar{z}_m \notin g^{\rightarrow} \pi_{i_n}{}^{\leftarrow} C(K_{i_n}, M, p_n)$. By Lemma 3.2, for every $n < \hat{n}$, we can find $a_n, b_n \in K_{i_n} \cap M$ such that for every $m < \hat{n}$, $\bar{z}_m \notin g^{\rightarrow} \pi_{i_n}{}^{\leftarrow}([a_n, \eta(K_{i_n}, M, p_n)] \cup [\zeta(K_{i_n}, M, p_n), b_n])$. Without loss of generality, we may assume that if $\eta(K_{i_n}, M, p_n) \in M$, then $a_n = \eta(K_{i_n}, M, p_n)$ and if $\zeta(K_{i_n}, M, p_n) \in M$, then $b_n = \zeta(K_{i_n}, M, p_n)$.

Let Q be the set of all functions q with $\operatorname{dom}(q) = \hat{n}$ such that for all $n < \hat{n}$, $q(n)$ is a subset of K_{i_n} with $q(n) \subseteq [a_n, b_n]$, and if $n \in E$, then $q(n)$ is a closed interval and if $n \notin E$, then $|q(n)| = 2$, and $g^{\rightarrow} \left(\bigcup_{n<\hat{n}} \pi_{i_n}{}^{\leftarrow} q(n) \right) = \prod_{j<d} Z_j$. The order on Q is defined by $q_1 \leq q_2$ if and only if for every $n \in E$, $q_1(n) \subseteq q_2(n)$. Notice that if we define a function $\bar{q}$ with $\operatorname{dom}(\bar{q}) = \hat{n}$ by for every $n \in E$, $\bar{q}(n) = I(K_{i_n}, M, p_n)$ and for every $n \notin E$, $\bar{q}(n) = C(K_{i_n}, M, p_n)$, then $\bar{q} \in Q$. Also notice that $Q \in M$.

Since $Q \in M$, there exists a maximal decreasing sequence $\langle q_\gamma | \gamma < \alpha \rangle \in M$.

Case 1. α is a successor ordinal.

Let δ be an ordinal with $\alpha = \delta + 1$. By the definition of Q, we have $g^{\rightarrow} \bigcup_{n<\hat{n}} \pi_{i_n}{}^{\leftarrow} q_\delta(n) = \prod_{j<d} Z_j$. We shall show that $\bar{q} < q_\delta$, which contradicts the assumption that $\langle q_\gamma | \gamma < \alpha \rangle$ is maximal.

It suffices to show that for every $m \in E$, $\bar{q}(m) = I(K_{i_m}, M, p_m) \subsetneq q(m)$. Let $m \in E$. There exists $y \in \bigcup_{n<\hat{n}} \pi_{i_n}{}^{\leftarrow} q_\delta(n)$ such that $g(y) =$

$\bar{z}_m$. So, there exists $n < \hat{n}$ such that $y \in \pi_{i_n}{}^{\leftarrow} q_\delta(n)$, i.e. $y(i_n) \in q_\delta(n)$. Since $q_\delta \in Q$, we have $q_\delta(n) \subseteq [a_n, b_n]$. By the definition of a_n, b_n, we have $\bar{z}_m \notin g^{\rightarrow}\pi_{i_n}{}^{\leftarrow}([a_n, \eta(K_{i_n}, M, p_n)] \cup [\zeta(K_{i_n}, M, p_n), b_n])$. Since $g(y) = \bar{z}_m$, we have $y \notin \pi_{i_n}{}^{\leftarrow}([a_n, \eta(K_{i_n}, M, p_n)] \cup [\zeta(K_{i_n}, M, p_n), b_n])$, i.e. $y(i_n) \notin ([a_n, \eta(K_{i_n}, M, p_n)] \cup [\zeta(K_{i_n}, M, p_n), b_n])$. Thus, we get $\eta(K_{i_n}, M, p_n) < y(i_n) < \zeta(K_{i_n}, M, p_n)$. By the definition of $\bar{z}_m$, we have $m = n$. Since $y(i_m) \in q_\delta(m) \in M$ and $q_\delta(m)$ is a closed interval, we have $\min q_\delta(n) \leq \eta(K_{i_n}, M, p_n)$ and $\zeta(K_{i_n}, M, p_n) \leq \max q_\delta(n)$. So, $I(K_{i_m}, M, p_m) = [\eta(K_{i_m}, M, p_m), \zeta(K_{i_m}, M, p_m)] \subseteq q_\delta(n)$. Since $q_\delta \in M$ but $I(K_{i_m}, M, p_m) \notin M$, we have $I(K_{i_m}, M, p_m) \subsetneq q_\delta(n)$. Thus, we are done.

Case 2. α is a limit ordinal.

By induction on n, we shall define a decreasing sequence $\langle D_n | n \leq \hat{n} \rangle$ of unbounded subsets of α lying in M such that there exists $\delta < \alpha$ such that for every $n \in \hat{n} \setminus E$, $m \in E$, and $\gamma \in D_{n+1} \setminus \delta$, we have $\bar{z}_m \notin g^{\rightarrow}\pi_{i_n}{}^{\leftarrow} q_\gamma(n)$. Let $D_0 = \alpha$.

Suppose that D_n is defined. If $n \in E$, then let $D_{n+1} = D_n$. Suppose $n \notin E$. Let S be the set of all $r \in K_{i_n}$ such that for all open neighborhood U of r, $\{ \gamma \in D_n \mid \min q_\gamma(n) \in U \}$ is unbounded in α. Since K_{i_n} is compact, S is nonempty. Note that $S \in M$.

We shall find an unbounded subset D'_n of D_n such that for every $\gamma \in D'_n$ and every $m \in E$, $\bar{z}_m \notin g^{\rightarrow}\pi_{i_n}{}^{\leftarrow} \{ \min q_\gamma(n) \}$.

Subcase 2.1. $(S \cap M) \setminus C(K_{i_n}, M, p_n) \neq \varnothing$.

Let $r_n \in (S \cap M) \setminus C(K_{i_n}, M, p_n)$. Since r_n is a limit point of $\{ \min q_\gamma(n) \mid \gamma \in D_n \}$ and for every $\gamma \in D_n$, $q_\gamma(n) \subseteq [a_n, b_n]$, we have $r_n \in [a_n, b_n]$. Since $r_n \in M \setminus C(K_{i_n}, M, p_n)$, we have $r_n \in [a_n, \eta(K_{i_n}, M, p_n)) \cup (\zeta(K_{i_n}, M, p_n), b_n]$.

Claim 5. There exists an open neighborhood $U \in M$ of r_n such that $U \cap I(K_{i_n}, M, p_n) = \varnothing$.

$\vdash$ If $r_n \in [a_n, \eta(K_{i_n}, M, p_n))$, then $a_n < \eta(K_{i_n}, M, p_n)$. Then, by the definition of a_n, $\eta(K_{i_n}, M, p_n) \notin M$ and hence $\eta(K_{i_n}, M, p_n)$ is a limit point of $K_{i_n} \cap M$ from below. So, there exists $b'_n \in K_{i_n} \cap M$ such that $a_n < b'_n < \eta(K_{i_n}, M, p_n)$. Let $U = (\leftarrow, b'_n)$. Since $b'_n \in M$, we have $U \in M$.

If $r_n \in (\zeta(K_{i_n}, M, p_n), b_n]$, then $\zeta(K_{i_n}, M, p_n) < b_n$. By the definition of b_n, we have $\zeta(K_{i_n}, M, p_n) \notin M$ and hence $\zeta(K_{i_n}, M, p_n)$ is a limit point of $K_{i_n} \cap M$. So, there exists $a'_n \in K_{i_n} \cap M$ such that $\zeta(K_{i_n}, M, p_n) < a'_n < b_n$. Let $U = (a'_n, \rightarrow)$. Since $a'_n \in M$, we have $U \in M$. $\dashv$ (Subclaim 5.0)

Let $D'_n = \{ \gamma \in D_n \mid \min q_\gamma(n) \in U \}$. By the definition of S, since $r_n \in S$, D'_n is unbounded in α. Clearly, $D'_n \in M$. Note that for every $\gamma \in D'_n$, since $\min q_\gamma(n) \in U \cap [a_n, b_n]$, we have $\min q_\gamma(n) \in [a_n, \eta(K_{i_n}, M, p_n)) \cup (\zeta(K_{i_n}, M, p_n), b_n]$ and so $\bar{z}_m \notin g^{\rightarrow}{\pi_{i_n}}^{\leftarrow}\{ \min q_\gamma(n) \}$.

Subcase 2.2. $(S \cap M) \setminus C(K_{i_n}, M, p_n) = \varnothing$.

So, we have $S \cap M \subseteq C(K_{i_n}, M, p_n)$. Since $C(K_{i_n}, M, p_n) \cap M$ has at most one element, $S \cap M$ has exactly one element, say r_n. By the elementarity of S, it implies $S = \{ r_n \}$. Note $C(K_{i_n}, M, p_n) \cap M = \{r_n\}$.

For every $m < \hat{n}$, by the definition of $\bar{z}_m$, we have $\bar{z}_m \notin g^{\rightarrow}{\pi_{i_n}}^{\leftarrow}C(K_{i_n}, M, p_n)$. So, $\bar{z} \notin g^{\rightarrow}{\pi_{i_n}}^{\leftarrow}\{ r_n \}$. By Lemma 3.2, there exists an open neighborhood U of r_n such that $\bar{z}_m \notin g^{\rightarrow}{\pi_{i_n}}^{\leftarrow}U$. Since $S = \{ r_n \}$, it is easy to see that there exists $\delta < \alpha$ such that for every $\gamma \in D_n \setminus \delta$, $\min q_\gamma(n) \in U$ and hence $\bar{z}_m \notin g^{\rightarrow}{\pi_{i_n}}^{\leftarrow}\{ \min q_\gamma(n) \}$. Let $D'_n = D_n \setminus \delta$. Note that δ may not be in M and so $D_n \setminus \delta$ is not necessarily in M, but we have $D'_n \in M$.

By a similar argument, we can define an unbounded subset D_{n+1} of D'_n such that there exists $\delta < \alpha$ such that for every $\gamma \in D_{n+1} \setminus \delta$ and $m \in E$, $\bar{z}_m \notin g^{\rightarrow}{\pi_{i_n}}^{\leftarrow}\{ \max q_\gamma(n) \}$. So, there exists $\delta < \alpha$ such that for every $\gamma \in D_{n+1} \setminus \delta$ and $m \in E$, $\bar{z}_m \notin g^{\rightarrow}{\pi_{i_n}}^{\leftarrow}q_\gamma(n)$.

Suppose that $D_{\hat{n}}$ is defined. Then, there exists $\delta < \alpha$ such that for every $\gamma \in D_{\hat{n}} \setminus \delta$, $n \in \hat{n} \setminus E$, and $m < \hat{n}$, we have $\bar{z}_m \notin g^{\rightarrow}{\pi_{i_n}}^{\leftarrow}q_\gamma(n)$.

Claim 6. For every $\gamma \in D_{\hat{n}} \setminus \delta$ and $m \in E$, $\bar{z}_m \in g^{\rightarrow}{\pi_{i_m}}^{\leftarrow}(q_\gamma(m) \cap (\eta(K_{i_m}, M, p_m), \zeta(K_{i_m}, M, p_m))$.

$\vdash$ Let $\gamma \in D_{\hat{n}} \setminus \delta$ and $m \in E$. By the definition of Q, $\bar{z}_m \in g^{\rightarrow} \bigcup_{n<\hat{n}} {\pi_{i_n}}^{\leftarrow}q_\gamma(n)$. But by the definition of $D_{\hat{n}}$, we have $\bar{z}_m \notin g^{\rightarrow} \bigcup_{n\in\hat{n}\setminus E} {\pi_{i_n}}^{\leftarrow}q_\gamma(n)$. So, $\bar{z}_m \in g^{\rightarrow}{\pi_{i_n}}^{\leftarrow} \bigcup_{n\in E} q_\gamma(n)$. Let $n \in E$ be so that $\bar{z}_m \in g^{\rightarrow}{\pi_{i_n}}^{\leftarrow}q_\gamma(n)$. By the definition of Q, $q_\gamma(n) \in [a_n, b_n]$. By the definition of a_n and b_n, we have $\bar{z}_m \notin g^{\rightarrow}{\pi_{i_n}}^{\leftarrow}([a_n, \eta(K_{i_n}, M, p_n)] \cup [\zeta(K_{i_n}, M, p_n), b_n])$. So, $\bar{z}_m \in g^{\rightarrow}{\pi_{i_n}}^{\leftarrow}(q_\gamma(n) \cap (\eta(K_{i_n}, M, p_n), \zeta(K_{i_n}, M, p_n)))$. By the definition of $\bar{z}_m$, it implies $n = m$. So, $\bar{z}_m \in g^{\rightarrow}(\eta(K_{i_m}, M, p_m), \zeta(K_{i_m}, M, p_m))$.
$\dashv$ (Claim 6)

Claim 7. For every $\gamma \in D_{\hat{n}} \setminus \delta$ and $m \in E$, $\min q_\gamma(m) < \zeta(K_{i_m}, M, p_m)$ and $\max q_\gamma(m) > \eta(K_{i_m}, M, p_m)$

$\vdash$ By the previous claim, we have $q_\gamma(m) \cap (\eta(K_{i_m}, M, p_m), \zeta(K_{i_m}, M, p_m)) \neq \varnothing$. Since $q_\gamma(m)$ is a closed interval, we have $\min q_\gamma(m) < \zeta(K_{i_m}, M, p_m)$ and $\max q_\gamma(m) > \eta(K_{i_m}, M, p_m)$. $\dashv$ (Claim 7)

For each $n \in E$, let $q(n) = \bigcap_{\gamma \in D_{\hat{n}}} q_\gamma(n)$. Note that $q \in E$.

Claim 8. For every $n \in E$, $a_n \leq \min q(n) \leq \zeta(K_{i_n}, M, p_n)$ and $\eta(K_{i_n}, M, p_n) \leq \max q(n) \leq b_n$.

$\vdash$ Let $n \in E$. By the definition of Q, for every $\gamma < \alpha$, $a_n \leq \min q_\gamma(n)$. So, $a_n \leq \min q(n)$. Similarly, $\max q(n) \leq b_n$.

Since each $\langle q_\gamma(n) | \gamma \in D_{\hat{n}} \rangle$ is a decreasing sequence of closed intervals in K_{i_n}, $q(n)$ is a nonempty closed interval. By Claim 7, for every $\gamma \in D_{\hat{n}} \setminus \delta$, $\min q_\gamma(n) < \zeta(K_{i_n}, M, p_n)$ and $\max q_\gamma(n) > \eta(K_{i_n}, M, p_n)$. So, we have $\min q(n) \leq \zeta(K_{i_n}, M, p_n)$ and $\max q(n) \geq \eta(K_{i_n}, M, p_n)$. $\dashv$ (Claim 8)

Claim 9. For every $n \in E$, $\bar{z}_n \in g^{\rightarrow} \pi_{i_n}{}^{\leftarrow} q(n)$

$\vdash$ Let $n \in E$. By Claim 6, for every $\gamma \in D_{\hat{n}} \setminus \delta$, $\bar{z}_m \in g^{\rightarrow} \pi_{i_n}{}^{\leftarrow} q_\gamma(n)$. So, $g^{\leftarrow} \{ \bar{z}_m \} \cap \pi_{i_n}{}^{\leftarrow} q_\gamma(n) \neq \varnothing$. Moreover, $g^{\leftarrow} \{ \bar{z}_m \} \cap \pi_{i_n}{}^{\leftarrow} q_\gamma(n)$ is closed.

So, $\bigcap_{\gamma \in D_{\hat{n}} \setminus \delta} (g^{\leftarrow} \{ \bar{z}_m \} \cap \pi_{i_n}{}^{\leftarrow} (q_\gamma(n)))$ is an intersection of a decreasing sequence of nonempty closed sets in a compact space $\prod_{i<d} K_i$. Thus, it is nonempty. Note that

$$\begin{aligned}
\bigcap_{\gamma \in D_{\hat{n}} \setminus \delta} (g^{\leftarrow} \{ \bar{z}_n \} \cap \pi_{i_n}{}^{\leftarrow} q_\gamma(n)) &= g^{\leftarrow} \{ \bar{z}_n \} \cap \bigcap_{\gamma \in D_{\hat{n}} \setminus \delta} \pi_{i_n}{}^{\leftarrow} q_\gamma(n) \\
&= g^{\leftarrow} \{ \bar{z}_n \} \cap \pi_{i_n}{}^{\leftarrow} \left(\bigcap_{\gamma \in D_{\hat{n}} \setminus \delta} q_\gamma(n) \right) \\
&= g^{\leftarrow} \{ \bar{z}_n \} \cap \pi_{i_n}{}^{\leftarrow} q(n)
\end{aligned}$$

So, $\bar{z}_n \in g^{\rightarrow} \pi_{i_n}{}^{\leftarrow} q(n)$. $\dashv$ (Claim 9)

Claim 10. There exists an $n \in E$ such that either $\min q(n) > \eta(K_{i_n}, M, p_n)$ or $\zeta(K_{i_n}, M, p_n) > \max q(n)$.

$\vdash$ Suppose not, i.e. for every $n \in E$, $\min q(n) \leq \eta(K_{i_n}, M, p_n)$ and $\zeta(K_{i_n}, M, p_n) \leq \max q(n)$. Then $\bar{q}(n) \subseteq q(n) \subseteq q_\gamma(n)$ for every $\gamma \in D_{\hat{n}}$. So, $\bar{q} < q_\gamma$ for every $\gamma < \alpha$, which contradicts the maximality of $\langle q_\gamma | \gamma < \alpha \rangle$. So, there exists an $n \in E$ such that either $\min q(n) > \eta(K_{i_n}, M, p_n)$ or $\zeta(K_{i_n}, M, p_n) > \max q(n)$. $\dashv$ (Claim 10)

Let $n \in E$ be as in the previous claim. First, suppose $\min q(n) > \eta(K_{i_n}, M, p_n)$. Since $\min q(n) \in M$, we have $\min q(n) \geq \zeta(K_{i_n}, M, p_n)$. Since we also have $\min q(n) \leq \zeta(K_{i_n}, M, p_n)$, we have $\min q(n) =$

$\zeta(K_{i_n}, M, p_n)$. Recall $\max q(n) \leq b_n$. So, $q(n) \subseteq [\zeta(K_{i_n}, M, p_n), b_n]$. By the definition of b_n, we have $\bar{z}_n \notin g^{\rightarrow}\pi_{i_n}{}^{\leftarrow}q(n)$. This is a contradiction to Claim 9. Similarly, we can derive a contradiction when $\max q(n) < \zeta(K_{i_n}, M, p_n)$. $\square$(Lemma 5.1)

By repeatedly apply the previous lemma, we can show the following.

Lemma 5.2. *Let d and e be integers with $2 \leq e \leq d$. Let K_i be a compact LOTS for each $i < d$ and X a countably compact subspace of $\prod_{i<d} K_i$. Let Z_j be an infinite Hausdorff space for each $j \leq d$ such that for every $j \leq d$, if $j \geq e$, then Z_j is not separable. Suppose that there exists a continuous surjection $f : X \to \prod_{j \leq d} Z_d$. Then, there exist compact LOTS $\tilde{K}_i$ for each $i < e-1$, a countably compact subspace $\tilde{Y}$ of $\prod_{i<e} \tilde{K}_i$, and a continuous surjection $\tilde{f} : \tilde{Y} \to \prod_{j<e} Z_j$.*

We can derive the following lemma.

Lemma 5.3. *Let d be a positive integer. Let K_i be a compact LOTS for each $i < d$, X a countably compact subspace of $\prod_{i<d} K_i$, and Z_j an infinite Hausdorff space for each $j \leq d$. Suppose that there exists a continuous surjection $f : X \to \prod_{j \leq d} Z_d$. Then, there exist at least two indexes $j \leq d$ such that Z_j is separable.*

Proof. Suppose not, i.e. there exists at most one $j \leq d$ such that Z_j is separable. Without loss of generality, we may assume that for every $j \leq d$, if $j > 0$, then Z_j is not separable. By applying Lemma 5.2 with $e = 2$, there exist a compact LOTS $\tilde{K}_0$, a countably compact subspace $\tilde{Y}$ of $\tilde{K}_0$, and a continous surjection $\tilde{f} : \tilde{Y} \to \prod_{j<2} Z_j$. But since $\tilde{Y}$ is a subspace of a LOTS $\tilde{K}_0$, $\tilde{Y}$ is a GO-space. So, by Theorem 1.4, there exists at least two indexes $j \leq d$ such that Z_j is compact and metrizable. In particular, there exists $j \leq d$ with $d \neq 0$ such that Z_j is compact and metrizalbe. So, Z_j is separable. This is a contradiction. $\square$(Lemma 5.3)

6. Strengthening of Mardešić's Theorem

We shall use the following lemma to complete the proof of the main theorem. It is a slight strengthening of a theorem proved by S. Mardešić in [6]. It can be proved by the same argument, but we shall give a little more direct proof for the completeness.

Lemma 6.1. *Let d and s be positive integers, K_i a compact LOTS for each $i < d$, X a countably compact subspace of $\prod_{i<d} K_i$, Z_j an infinite*

separable Hausdorff space for each $j < d+s$, and $f : X \to \prod_{j<d+s} Z_j$ a continuous surjection. Then,

(i) for every $j < d+s$, Z_j is compact, and
(ii) there exist at least $s+1$-many indexes $j < d+s$ such that Z_j is metrizable.

First, we shall show the following lemma, which says that when proving Lemma 6.1, we may focus on the case when K_i is separable for every $i < d$ and X is compact.

Lemma 6.2. *Let d be a positive integer, K_i a compact LOTS for each $i < d$, X a countably compact subspace of $\prod_{i<d} K_i$, Z a separable Hausdorff space, and $f : X \to Z$ a continuous surjection. Then, there exist a separable compact LOTS K'_i for each $i < d$, X' a compact subspace of $\prod_{i<d} K'_i$, and a continuous surjection $f' : X' \to Z$.*

Proof. Since Z is separable, there exists a countable dense subset E of Z. For each element $z \in E$, since f is surjective, there exists $x_z \in X$ such that $f(x_z) = z$. Let $D = \{ x_z \mid z \in E \}$. Since E is countable, D is also countable. For each $i < d$, let $D_i = {\pi_i}^{\to} D$. Clearly, each D_i is countable.

Claim 1. $\mathrm{cl}(D) \subseteq X$.

$\vdash$ Let $x \in \mathrm{cl}(D)$. If $x \in D$, then since $D \subseteq X$, clearly $x \in X$.

Suppose $x \notin D$. Then, x is a limit point of D. Since D is countable, it is easy to construct a sequence $\langle x_n | n < \omega \rangle$ such that $\langle x_n | n < \omega \rangle$ converges to x and for every $i < d$, $\langle x_n(i) | n < \omega \rangle$ is either constant or strictly monotone. Since X is countably compact and $x_n \in D \subseteq X$ for every $n < \omega$, we have $x \in X$. $\dashv$ (Claim 1)

For each $i < d$, let $K'_i = {\pi_i}^{\to} \mathrm{cl}(D)$. It is easy to see that K'_i is compact and D_i is a dense subset of K'_i. Let $X' = \mathrm{cl}(D)$. Clearly, $X' \subseteq \prod_{i<d} K'_i$. Since X' is a closed subset of a compact space $\prod_{i<d} K_i$, X' is compact. Define $f' = f \restriction X'$. We have $\mathrm{ran}\, f' = f^{\to} X' = f^{\to} \mathrm{cl}(D) = \mathrm{cl}\, f^{\to} D = \mathrm{cl}\, E = Z$. □(Lemma 6.2)

We shall define the following notation. Note that it is similar but slightly modified from the definition by S. Mardešić in [6].

Definition 6.3. Let K be a LOTS. For each $p \in K$, let $G(p)$ be the set of all $q \in K$ such that there exist at most only finitely many elements between p and q. Namely, $q \in G(p)$ if and only if $[p, q]$ is finite when $p \leq q$ and $[q, p]$ is finite when $q < p$.

It is trivial that for all $p, q \in G(p)$, $G(p)$ is a finite convex set and either $G(p) = G(q)$ or $G(p) \cap G(q) = \varnothing$. Also, if $\min G(p)$ is not the least point of K, then $\min G(p)$ is a limit point of K from below. Similarly, if $\max G(p)$ is not the greatest point of K, then $\max G(p)$ is a limit point of K from above.

Definition 6.4. Let d be a positive integer and K_i a compact LOTS for each $i < d$. For every $x \in \prod_{i<d} K_i$, let $G(x) = \prod_{i<d} G(x(i))$.

Lemma 6.5. *Let d be a positive integer and K_i a compact LOTS for each $i < d$. For each $i < d$, let D_i be a dense subset of K_i with $\min K_i, \max K_i \in D_i$. Let $x \in \prod_{i<d} K_i$ and U an open subset of $\prod_{i<d} K_i$ such that $G(x) \subseteq U$. Then there exist $a, b \in \prod_{i<d} D_i$ such that for every $i < d$, $a(i) \leq \min G(x(i))$ and $\max G(x(i)) \leq b(i)$, and $[a, b] \subseteq U$.*

Proof. For each element $y \in G(x)$, since $y \in G(x) \subseteq U$, there exist an open interval $U_{y,i}$ in K_i for each $i < d$ such that $y \in \prod_{i<d} U_{y,i} \subseteq U$.

For each $i < d$, we shall define $a(i) \in D_i$ as follows. If $\min G(x(i)) = \min K_i$, then let $a(i) = \min K_i$. By assumption, $a(i) \in D_i$. Suppose not, i.e. $\min G(x(i)) > \max K_i$. Then, for every $y \in G(x)$ with $y(i) = \min G(x(i))$, $y(i)$ is a limit point of D_i from below. So, there exists $a_y(i) \in D_i$ such that $a_y(i) < \min G(x(i))$ and $a_y(i) \in U_{y,i}$. Let

$$a(i) = \max \{\, a_y(i) \mid y \in G(x) \wedge y(i) = \min G(x(i)) \,\}$$

For each $i < d$, we shall define $b(i) \in D_i$ as follows. If $\max G(x(i)) = \max K_i$, then let $b(i) = \max K_i$. Suppose not, i.e. $\max G(x(i)) < \max K_i$. Then, for every $y \in G(x)$ with $y(i) = \max G(x(i))$, $y(i)$ is a limit point of D_i from above. So, there exists $b_y(i) \in D_i$ such that $\max G(x(i)) < b_y(i)$ and $b_y(i) \in U_{y,i}$. Let

$$b(i) = \min \{\, b_y(i) \mid y \in G(x) \wedge y(i) = \max G(x(i)) \,\}$$

By definition, clearly $a(i) \leq \min G(x(i))$ and $\max G(x(i)) \leq b(i)$. We shall show $[a, b] \subseteq U$. Let $x' \in [a, b]$. Define $y \in G(x)$ as follows. For every $i < d$,

$$y(i) = \begin{cases} x'(i) & \text{if } x'(i) \in G(x(i)) \\ \min G(x(i)) & \text{if } x'(i) < \min G(x(i)) \\ \max G(x(i)) & \text{if } x'(i) > \max G(x(i)) \end{cases}$$

Claim 1. $x' \in \prod_{i<d} U_{y,i}$.

$\vdash$ It suffices to show that for each $i < d$, $x'(i) \in U_{y,i}$. Let $i < d$. If $x'(i) \in G(x(i))$, then $y(i) = x'(i)$. So, clearly $x'(i) \in U_{y,i}$.

Suppose $x'(i) \notin G(x(i))$. Then, either $x'(i) < \min G(x(i))$ or $\max G(x(i)) < x'(i)$. Suppose $x'(i) < \min G(x(i))$. Then, $\min G(x(i)) > \min K_i$. So, $a_y(i)$ was defined and $a_y(i) \leq a(i) \leq x'(i) < \min G(x(i)) = y(i)$. Since $a_y(i), y(i) \in U_{y,i}$ and $U_{y,i}$ is convex, we have $x'(i) \in U_{y,i}$.

Suppose $x'(i) > \max G(x(i))$. Then, $\max G(x(i)) < \max K_i$. So, $b_y(i)$ was defined and $b_y(i) \geq b(i) \geq x'(i) > \max G(x(i)) = y(i)$. Since $b_y(i), y(i) \in U_{y,i}$ and $U_{y,i}$ is convex, we have $x'(i) \in U_{y,i}$. $\dashv$ (Claim 1)

By the previous claim, since $\prod_{i<d} U_{y,i} \subseteq U$, we have $x' \in U$. □(Lemma 6.5)

We shall also use the following theorem proved by A. Arhangel'skiĭ in [1]. This was proved as Theorem 3.1.19 in [2].

Definition 6.6. Let X be a topological space. A family $\mathcal{N}$ of subsets of X is called a *network for* X if and only if for every $x \in X$ and every neighborhood U of x, there exists an $N \in \mathcal{N}$ such that $x \in N \subseteq U$. The *network weight of* X is defined to be the least cardinality of networks of X.

Theorem 6.7 (A. Arhangel'skiĭ [1])**.** *For every compact space X, we have $nw(X) = w(X)$.*

Now, we are ready to prove Lemma 6.1.

Proof of Lemma 6.1. As G. Martínez-Cervantes and G. Plebanek [8] pointed out, it suffices to show the case when $s = 1$. So, we shall focus on the case $s = 1$. By Lemma 6.2, without loss of generality, we may assume that for every $i < d$, K_i is separable and X is compact. Then, $\prod_{j\leq d} Z_j$ is a continuous image of a compact space X and hence compact. Thus, for every $j \leq d$, Z_j is compact. Now, we shall show (ii). It suffices to prove the following statement for all positive integer d.

$(*)_d$ Let K_i a compact LOTS for each $i < d$, X a compact subspace of $\prod_{i<d} K_i$, Z_j an infinite separable Hausdorff space for each $j \leq d$, and $f : X \to \prod_{j\leq d} Z_j$ a continuous surjection. Then, there exist at least two indexes $j \leq d$ such that Z_j is metrizable.

By induction on d, we shall prove $(*)_d$ for every positive integer d. When $d = 1$, this follows from Theorem 1.4. Suppose $(*)_{d'}$ holds for all $d' < d$. Let K_i for each $i < d$, X, Z_j for each $j \leq d$, and f be a counterexample to $(*)_d$. As this is a counterexample, there exist at most one $j \leq d$ such that Z_j is metrizable. So, without loss of generality, we may assume that for every $j \leq d$ except $j = 0$, Z_j is not metrizable. We may also assume that for every $x \in X$ and $i < d$,

$x(i) \neq \min K_i$ and $x(i) \neq \max K_i$. For each $i < d$, let D_i be a countable dense subset of K_i with $\min K_i, \max K_i \in D_i$.

Define $g : X \to \prod_{j<d} Z_j$ by $g(x) = f(x) \restriction d$ and $h(x) = f(x)(d)$.

Note that $\prod_{j \leq d} Z_j = f^{\to} X$ and hence $\prod_{j \leq d} Z_j$ is compact. Thus, Z_j is compact for each $j \leq d$.

Claim 1. For every $z_d \in Z_d$ and for every finite subset Y of X, there exists $x \in X$ such that $h(x) = z_d$ and for every $y \in Y$ and $i < d$, $x(i) \notin G(y(i))$.

$\vdash$ Let $z_d \in Z_d$. Suppose that there exists a finite subset Y of X such that for every $x \in X$ with $h(x) = z_d$, there exist $y \in Y$ and $i < d$ such that $x(i) \in G(y(i))$. For each $i < d$, let $S_i = \bigcup_{y \in Y} G(y(i))$. Since Y is finite and for every $y \in Y$, $G(y(i))$ is finite, S_i is finite. For each $i < d$ and $p \in S_i$, let $P_{i,p} = \pi_i^{\leftarrow} \{ p \}$. It is easy to see that for every $x \in X$ with $h(x) = z_d$, there exist $i < d$ and $p \in S_i$ such that $x \in P_{i,p}$.

We shall show that for every $\bar{z} \in \prod_{j<d} Z_j$, there exist $i < d$ and $p \in S_i$ such that $\bar{z} \in g^{\to} P_{i,p}$. Define $z \in \prod_{j \leq d} Z_j$ by $z \restriction d = \bar{z}$ and $z(d) = z_d$. Since $f^{\to} X = \prod_{j \leq d} Z_j$, there exists $x \in X$ such that $f(x) = z$. Then, $h(x) = z_d$. By assumption, there exist $i < d$ and $p \in S_i$ such that $x \in P_{i,p}$. Note $g(x) = z \restriction d = \bar{z}$. So, $\bar{z} \in g^{\to} P_{i,p}$.

Thus, $g^{\to} \bigcup_{i<d} \bigcup_{p \in S_i} P_{i,p} = \prod_{j<d} Z_j$. Note that for every $i < d$ and $p \in S_i$, $P_{i,p}$ is homeomorphic to $\prod_{k<d, k \neq i} K_k$. By Lemma 4.1, there exist a compact LOTS K'_i for each $i < d-1$, a compact subspace X' of $\prod_{i<d-1} K'_i$, and a continuous surjection $f' : X' \to \prod_{j<d} Z_j$. By the inductive hypothesis, it implies that there exist at least two $j < d$ such that Z_j is metrizable. This is a contradiction. $\dashv$ (Claim 1)

Let $\mathcal{N}$ be the set of subsets N of Z_d such that there exist $a, b \in \prod_{i<d} D_i$ such that $a \leq b$ and $N = h^{\to}(X \cap [a, b])$. Clearly, $\mathcal{N}$ is countable.

We shall show that $\mathcal{N}$ is a network for Z_d. Let $z_d \in Z_d$ and U an open neighborhood of z_d. We shall show that there exists $N \in \mathcal{N}$ such that $z_d \in N \subseteq U$.

First, suppose that $z_d \in h^{\to} \prod_{i<d} D_i$. Let $x \in \prod_{i<d} D_i$ be so that $h(x) = z_d$. Then, $\{ z_d \} = \{ h(x) \} = h^{\to}(X \cap [x, x]) \in \mathcal{N}$. Trivially, $z_d \in \{ z_d \} \subseteq U$.

Suppose not i.e. $z_d \notin h^{\to} \prod_{i<d} D_i$.

Claim 2. There exists $x \in X$ such that $h(x) = z_d$ and $h^{\to}(X \cap G(x)) \subseteq U$.

$\vdash$ Suppose not, i.e. for every $x \in X$, if $h(x) = z_d$, then $h^{\to}(X \cap G(x)) \nsubseteq U$. By Claim 1, there exists a sequence $\langle x_n | n < \omega \rangle$ in X

such that for every $n < \omega$, $h(x) = z_d$, and for every $m < n$ and $i < d$, $x_n(i) \neq x_m(i)$. Without loss of generality, we may assume that for every $i < d$, $\langle x_n(i) | n < \omega\rangle$ is strictly monotone. By assumption, for every $n < \omega$, we have $h^{\rightarrow}(X \cap G(x_n)) \nsubseteq U$. So, there exists $y_n \in X \cap G(x_n)$ such that $h(y_n) \notin U$.

Since $\langle x_n(i) | n < \omega\rangle$ is strictly monotone for every $i < d$, $\langle x_n | n < \omega\rangle$ converges to some $x_\omega \in X$. Since $h(x_n) = z_d$ for every $n < \omega$, we have $h(x_\omega) = z_d$. In particular, $h(x_\omega) \in U$. Since $\langle G(x_n(i)) | n < \omega\rangle$ is a disjoint sequence of convex sets for every $i < d$, $\langle y_n | n < \omega\rangle$ also converges to x_ω. However, $h(y_n) \notin U$ for every $n < \omega$. This is a contradiction. $\dashv$ (Claim 2)

Let $x \in X$ be as in the previous claim. Since $h^{\rightarrow}(X \cap G(x)) \subseteq U$, we have $G(x) \subseteq (\prod_{i<d} K_i \setminus X) \cup h^{\leftarrow}U$. Since h is continuous, $h^{\leftarrow}U$ is open. So, $(\prod_{i<d} K_i \setminus X) \cup h^{\leftarrow}U$ is open. By Lemma 6.5, there exist $a, b \in \prod_{i<d} D_i$ such that for every $i < d$, $a(i) \leq \min G(x(i))$ and $\max G(x(i)) \leq b(i)$, and $[a, b] \subseteq (\prod_{i<d} K_i \setminus X) \cup h^{\leftarrow}U$. So, $h^{\rightarrow}(X \cap [a, b]) \subseteq U$. Therefore, $\mathcal{N}$ is a network for Z_d.

Thus, $nw(Z_d) = \aleph_0$. By Theorem 6.7, we have $w(Z_d) = \aleph_0$, i.e. Z_d is second countable. So, Z_d is metrizable. This is a contradiction to the assumption that Z_d is not metrizable. □

7. The proof of the main theorem

Now, we can prove the main theorem.

Theorem 7.1. *Let d and s be positive integers. Let K_i be a compact LOTS for each $i < d$, X a countably compact subspace of $\prod_{i<d} K_i$, Z_j an infinite Hausdorff space for each $j < d+s$, and $f : X \to \prod_{j<d} Z_j$ a continuous surjection. Then, there exist at least $s+1$ indexes $j < d+s$ such that Z_j is compact and metrizable.*

Proof. By using the remark of G. Martínez-Cervantes and G. Plebanek [8], we may assume $s = 1$.

Claim 1. There are at least two indexes $j \leq d$ such that Z_j is separable.

$\vdash$ Suppose not. Without loss of generality, we may assume that for every $j \leq d$, if $j \neq 0$, then Z_j is not separable. By Lemma 5.2, there exist a compact LOTS $\tilde{K}$, a countably compact subspace $\tilde{X}$ of $\tilde{K}$, and a continuous surjection $\tilde{f} : \tilde{X} \to Z_0 \times Z_1$. Note that $\tilde{X}$ is a countably compact GO-space. By Theorem 1.4, both Z_0 and Z_1 are compact and metrizable. So, Z_1 is separable. This is a contradiction. $\dashv$ (Claim 1)

Let $e = |\{ j \leq d \mid Z_j \text{ is separable} \}|$. By the previous claim, we have $e \geq 2$. Without loss of generality, we may assume that for every $j \leq d$, Z_j is separable if and only if $j < e$.

By Lemma 5.2, there exist a compact LOTS $\tilde{K}_i$ for each $i < d$, a countably compact subspace $\tilde{X}$ of $\prod_{i<d} \tilde{K}_i$, and a continuous surjection $\tilde{f} : \tilde{X} \to \prod_{j<e} Z_j$. By Lemma 6.1, there exist at least two indexes $j \leq d$ such that Z_j is compact and metrizable. $\square$(Theorem 7.1)

The following theorem is proved by S. Purisch in [9].

Theorem 7.2 (S. Purisch [9])**.** *For every Tychonoff space X, βX is homeomorphic to some LOTS if and only if X is pseudocompact and homeomorphic to some GO-space.*

Fact 7.3. Let X be a GO-space. Then, the following are equivalent.

(i) X is sequentially compact.
(ii) X is countably compact.
(iii) X is pseudocompact.

By using these, we can prove the following corollary.

Corollary 7.4. *Let d be a positive integer. Let K_i be a countably compact GO-space for each $i < d$, X a countably compact subspace of $\prod_{i<d} K_i$, Z_j an infinite Hausdorff space for each $j \leq d$, and $f : X \to \prod_{j\leq d} Z_j$ a continuous surjection. Then, there exist at least two indexes $j \leq d$ such that Z_j is metrizable.*

Proof. Let d, K_i, X, Z_j be as in the assumption. By Theorem 7.2 and Fact 7.3, for each $i < d$, βK_i is a compact LOTS. Then, X is a countably compact subspace of $\prod_{i<d} \beta K_i$. By Theorem 7.1, there exist at least two indexes $j \leq d$ such that Z_j is metrizable. $\square$(Corollary 7.4)

8. Open problems

The main theorem works only when the domain of the surjection is a countably compact subspace of the product of compact LOTS, which is very narrow. So, it is natural to ask the following question.

Question 1. Can we generalize the main theorem to a wider class than LOTS? What if we replace 'metrizable' by 'separable'?

The argument by G. Martínez-Cervantes and G. Plebanek in [8] define a useful invariant called 'free dimension', which beautifully explains why the Mardesic Conjecture holds. While the main theorem is stronger than their result, the proof fails to present such an invariant. So, we may ask the following question.

Question 2. Can we define a new type of dimension that can be used to prove the main theorem?

This article is another demonstration of the use of countable elementary substructures to analyze nonseparable spaces. See [3] and [4] for earlier applications of this method.

Question 3. Are there more problems about nonseparable spaces that can be solved by using countable elementary substructures?

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