There is no space-filling curve for a countably compact connected nowhere separable linearly ordered topological space

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Outline

1 Introduction
- What are the reals?
- Nowhere separable LOTS

2 Coordinate-wise theorem
- Statement
- Ideas
- Isn't it nice?

3 Space-filling curves
- Space-filling curves

4 Linearly Ordered Semigroups
- Connected Linearly Ordered Topological Spaces
- Linearly ordered semigroups

5 Open Problems
- Open Problems

The characterization of $\mathbb{R}$

The real line, $\mathbb{R}$, is the unique complete self-dense separable linearly ordered sets without endpoints.

By the way, when I say the reals in this talk, I do not mean 2^ω or ω^ω, but the genuine real line, which is complete.

A linearly ordered set can be topologized by the order topology. Such a topological space is called a *linearly ordered topological space (LOTS)*.

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Connectedness

Lemma

Let L be a LOTS. Then the following are equivalent.

1. *L is connected as a topological space (i.e. there are no two open sets U and W such that $U \cap W = \varnothing$ and $U \cup W = L$).*
2. *L is complete and self-dense.*

So, $\mathbb{R}$ is characterized as a

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'Without endpoints' is a superficial condition.

'Connectedness' seems important to state continuous phenomenon (I know there is an objection, but hold on).

Being 'linearly ordered' seems necessary to compare the quantities.

How about 'separability'? What is wrong to abandon it?

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Nowhere separable spaces

Definition

A topological space X is *nowhere separable* if and only if no nonempty open set is separable.

Then, a connected LOTS is nowhere separable if and only if there is no nonempty open interval that is homeomorphic to $\mathbb{R}$.

Question

How similar to $\mathbb{R}$ can a connected nowhere separable LOTS be?

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How similar to $\mathbb{R}$ can a connected nowhere separable LOTS be?

Examples

Example

Let κ be an uncountable regular cardinal. Define

$$L = \{ f \in 2^\kappa \mid f \text{ is not eventually } 1 \}$$

ordered by lexicographical ordering.

Then, L is a connected nowhere separable LOTS.

This set has a dense set of points of cofinality κ.

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Example (cont.)

Example

Let L be a self-dense Aronszajn line and $\hat{L}$ the Dedekind completion of L.
Then, $\hat{L}$ is a connected nowhere separable LOTS.

Such a linearly ordered set is called an *Aronszajn continuum* in Todorcevic's article in the Handbook of Set Theoretic Topology, but there seems not much research afterwards (correct me if I am wrong).

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The differences between $\mathbb{R}$ and connected nowhere separable LOTS.

In fact, there are qualitative differences between $\mathbb{R}$ and connected nowhere separable LOTS.

We shall mention two such differences.

- Every continuous injection from the product into the product is coordinate-wise.
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Coordinate-wise Theorem

Theorem (Ishiu)

Let n be a non-zero natural number, $K_0, \ldots, K_{n-1}$, $L_0, \ldots, L_{n-1}$ be connected nowhere separable LOTS, and $f : \prod_{i<n} K_i \to \prod_{i<n} L_i$ a continuous injection. Then, f is coordinate-wise, namely there exist a bijection $h : n \to n$ and a function $p_i : K_i \to L_{h(i)}$ for each $i < n$ such that for all $x \in \prod_{i<n} K_i$ and $i < n$, $f(x)(h(i)) = p_i(x(i))$

This theorem extends the result of Eda and Kamijo about LOTS that has densely many points of uncountable cofinality or coinitiality.

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Two-dimensional version

The previous theorem is not that intuitive, so let me state the two-dimensional version.

Theorem (Ishiu)

Let K_0, K_1, L_0, L_1 be connected nowhere separable LOTS. Then every continuous injection $f : K_0 \times K_1 \to L_0 \times L_1$ is coordinate-wise, namely either

1. *there exist functions $g_0 : K_0 \to L_0$ and $g_1 : K_1 \to L_1$ such that $f(x, y) = \langle g_0(x), g_1(y) \rangle$, or*
2. *there exist functions $g_0 : K_0 \to L_1$ and $g_1 : K_1 \to L_0$ such that $f(x, y) = \langle g_1(y), g_0(x) \rangle$.*

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Countable elementary submodels

Suppose that N is a countable elementary submodel of $H(\theta)$ for some sufficiently large regular cardinal θ) such that everything in the context belongs to N.

Since L is nowhere separable, for every nonempty open subset U of L,

$$U \setminus \mathrm{Cl}(L \cap N) \neq \varnothing$$

This is a huge difference between separable and nowhere separable topological spaces.

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Some notations

Let

$$J(L, N) = \{\, x \in L \setminus \mathrm{Cl}(L \cap N) \mid \exists x_0, x_1 \in L \cap N(x_0 < x < x_1)\,\}$$

For each $\hat{x} \in J(L, N)$, define

$$\begin{aligned} \eta(\hat{x}) &= \eta(L, N, \hat{x}) = \sup \{\, x \in L \cap N \mid x < \hat{x} \,\} \\ \zeta(\hat{x}) &= \zeta(L, N, \hat{x}) = \inf \{\, x \in L \cap N \mid x > \hat{x} \,\} \\ I(\hat{x}) &= I(L, N, \hat{x}) = [\eta(L, N, \hat{x}), \zeta(L, N, \hat{x})] \end{aligned}$$

i.e. $I(\hat{x})$ is the largest interval which has $\hat{x}$ as an element and whose interior is disjoint from N.

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Properties

Let $g : K \to L$ be a continuous function (with $g \in N$) and $\hat{x} \in J(K, N)$.

Then, g has very nice properties on $I(\hat{x})$.

- $g \upharpoonright I(\hat{x})$ has a maximum and a minimum at the endpoints.
- If $g(\hat{x}) \in N$, then $g \upharpoonright I(\hat{x})$ is constant.
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Cell-to-cell

By using the facts in the previous slide, we can prove:

Lemma

Let $\hat{x}_0 \in J(K_0, N)$ and $\hat{x}_1 \in J(K_1, N)$. Define $\langle \hat{y}_0, \hat{y}_1 \rangle = f(\hat{x}_0, \hat{x}_1)$. Then,

$$f^{\rightarrow}(I(\hat{x}_0) \times I(\hat{x}_1)) = I(\hat{y}_0) \times I(\hat{y}_1)$$

(To be honest, it requires many technical arguments to get here from the previous slide...)

By using the previous lemma, we can prove that f is coordinate-wise.

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Power of countable elementary submodels

I really like the outline of this proof (as much as I hate annoying details in it).

This has nothing to do with consistency, inner models, large cardinals, and other ordinary set theory things, but it demonstrates how strong elementary submodel arguments are.

Can't we prove any other theorems this way?

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Space-filling curves on $\mathbb{R}^2$

It is well-known that there exists a continuous surjection from $[0,1]$ onto $[0,1] \times [0,1]$. Such a curve is called a *space-filling curve*.

Because G. Peano discovered the first example, space-filling curves are sometimes called *Peano curves* (at least according to Wikipedia). But the *Peano curve* more often refers to the particular example that G. Peano discovered.

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What about connected nowhere separable LOTS?

It is natural to ask if there is a space-filling curve onto the product of connected nowhere separable LOTS?

Question

Let K be a connected nowhere separable LOTS. Is there a continuous surjection from K onto $K \times K$?
More generally: Let K, L_0, L_1 be connected nowhere separable LOTS. Is there a continuous surjection from K onto $L_0 \times L_1$?

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$[0, 1]$ onto $L_0 \times L_1$?

By the way, we can show that there is no space-filling curve onto $L_0 \times L_1$ in the original sense for connected nowhere separable LOTS L_0, L_1. Namely,

Proposition

Let L_0, L_1 be connected nowhere separable LOTS. Then, there is no continuous surjection from $[0, 1]$ onto $L_0 \times L_1$.

This is an easy corollary of the fact that every continuous function from an interval of $\mathbb{R}$ into a nowhere separable LOTS is constant.

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K onto $L_0 \times L_1$?

So, it is more reasonable to ask if there is a continuous surjection from K onto $L_0 \times L_1$.

As the title of this talk suggests, the answer is NO if K is countably compact.

Theorem (Ishiu)

Let K, L_0, L_1 be connected nowhere separable LOTS such that K is countably compact. Then, there exists no continuous surjection from K onto $L_0 \times L_1$.

This can be proved by the lemmas for the 'Coordinate-wise' theorem.

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Proof

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Suppose that there exists a continuous surjection $f : K \to L_0 \times L_1$. Let g_0, g_1 be its component functions, i.e. $f(t) = \langle g_0(t), g_1(t) \rangle$ for all $t \in K$.

Let N be a countable elementary submodel of $H(\theta)$ for some sufficiently large regular cardinal θ with $K, L_0, L_1, f \in N$

Let $\langle x_i | i < \omega \rangle$ be a sequence of distinct elements of $L_0 \cap N$ and $y \in J(L_1, N)$. Since f is surjective, for each $i < \omega$, there exist $t_i \in K$ such that $f(t_i) = \langle x_i, y \rangle$. □

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Proof (2)

Proof (Cont.)

Let $i < \omega$.

Since $g_1(t_i) = y \in J(L_1, N)$, we have $t_i \in J(K, N)$ unless $t_i < \inf(K \cap N)$ or $t_i > (K \cap N)$.

Since $g_0(t_i) = x_i \in N$, $g_0 \restriction I(t_i)$ is constant.

Since $g_1(t_i) = y \in J(L_1, N)$, $g_1{}^{\to} I(t_i) = I(y)$.

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Let $i < \omega$.
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Since K is countably compact, there is a limit point t of $\{ t_i \mid i < \omega \}$. Clearly, t is also a limit point of $\langle t'_i | i < \omega \rangle$. Then,

$$g_1(t) = \lim_{i \to \infty} g_1(t_i) = y$$

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We can prove the following easy corollary:

Corollary

Let K be a compact connected LOTS. Then, there exists a continuous surjection from K onto $K \times K$ if and only if K is separable, i.e. K is homeomorphic to $[0, 1]$.

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A corollary of the Coordinate-wise theorem

As a corollary of the Coordinate-wise theorem, we can prove the following:

Definition

Let S be a semigroup. We say that S is *cancellative* if and only if for all $a, b, c \in S$, $ac = bc$ implies $a = b$ and $ca = cb$ implies $a = b$.

Corollary

Let S be a cancellative topological semigroup that is a connected LOTS as a topological space. Then, S is separable.

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Let S be a cancellative topological semigroup that is a connected LOTS. Define $f : S \times S \to S \times S$ by

$$f(a,b) = \langle a, ab \rangle$$

Then, f is not coordinate-wise. Thus, S is not nowhere separable. From this, we can prove that S is separable. □

But a stronger result was known long before this result.

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The following theorem was first proved by J. Aczél in 1949, and the proof was simplified by R. Craigen and Z. Páles in 1989.

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A topological semigroup that is a LOTS as a topological space is called a *linearly ordered topological semigroup*.

Theorem (J. Aczel(1949))

Let S be a connected linearly ordered topological semigroup. Then, S is order- and semigroup-isomorphic to a subsemigroup of $(\mathbb{R}, +)$.

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Linearly ordered semigroups

In fact, semigroups on linearly ordered sets have been studied so much, but often they are not linearly ordered topological semigroups.

Definition

Let S be a set with a semigroup operation $\cdot$ and a linear ordering $\leq$. We say that S is a *linearly ordered semigroup* if and only if for all $a, b, c \in S$, $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$.

These two notions are independent from each other. Namely, there exists a linearly ordered semigroup that is not linearly ordered topological semigroup and vice versa.

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Positively ordered

There are several results that says that S is order- and semigroup-isomorhpic to a subsemigroup of $(\mathbb{R}, +, \leq)$. By the way, we get the commutativity as a conclusion.

Definition

Let S be a linearly ordered semigroup. We say that S is *positively ordered* iff $ab \geq a$ and $ab \geq b$ for all $a, b \in S$.

Example

$([0, \infty), +, \leq)$ is positively ordered, but $(\mathbb{R}, +, \leq)$ is not positively ordered.

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Archimedean and anomalous pairs

In this talk, when we say S is positively ordered, we shall implicitly assume S is linearly ordered.

Definition

Let S be a linearly ordered semigroup that is positively ordered.

- S is *archimedean* iff for every $a, b \in S$, whenever a is not an identity, there exists $n \in \mathbb{N}$ such that $a^n \geq b$.
- $a, b \in S$ form an *anomalous pair* iff $a \neq b$, and for all positive natural number n, $a^n < b^{n+1}$ and $b^n < a^{n+1}$.

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What does a positively ordered semigroup have anomalous pairs?

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Example of anomalous pairs

Example

Define a linearly ordered semigroup $S = (0, \infty) \times [0, \infty)$ by $\langle a, x\rangle \cdot \langle b, y\rangle = \langle a + b, x + y\rangle$ and letting S be ordered by the lexicographical ordering.

It is easy to see that S is a linearly ordered semigroup that is positively ordered and archimedean.

For every $a \in (0, \infty)$ and $x \neq y \in [0, \infty)$, $\langle a, x\rangle$ and $\langle a, y\rangle$ form an anomalous pair.

Note that $[0, \infty)$ may be replaced by any linearly ordered semigroup.

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Fuchs' Theorem

The following is one of the theorems that give an equivalent condition for a positively ordered semigroup to be order- and semigroup-isomorphic to a subsemigroup of $([0,\infty),+)$.

Theorem (L. Fuchs)

Let S be a positively ordered semigroup. Then, S is order- and semigroup-isomorphic to a subsemigroup of $([0,\infty),+)$ *if and only if*

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What if it has anomalous pairs?

In case that S has an anomalous pair, I believe I proved the following.

Conjecture

Let S be a positively ordered archimedean semigroup with no maximal element. Define an equivalence relation $\sim$ on S by $a \sim b$ if and only if either $a = b$ or a and b form an anomalous pair.

If we naturally define $S/\sim$, then $S/\sim$ is order- and semigroup-isomorphic to a subsemigroup of $([0, \infty), +)$.

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Let S be a positively ordered semigroup. We may define S-metrizability and S-ultrametrizability of a topological space.

Proposition

Assume the previous conjecture holds. Suppose that X is S-metrizable for some positively ordered semigroup S. If S is archimedean, then X is metrizable.

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