The Mardešić Conjecture for countably compact spaces

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Space-filling curves

A famous theorem of G. Peano in 1890 says that there exists a continuous surjection from $[0, 1]$ onto $[0, 1] \times [0, 1]$.

It was a groundbreaking result, which challenged the notion of dimensions.

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For which linearly order topological spaces (LOTS) L, does there exist a continuous surjection from L onto $L \times L$?

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For every nondegenerate connected compact Suslin line S, there is no continuous surjection from S onto $S \times S$.

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In 1964, L. B. Treybig proved the following result.

Theorem (L. B. Treybig)

If X and Y are infinite Hausdorff spaces and $X \times Y$ is a continuous image of a compact LOTS, then both X and Y are metrizable.

Since X and Y are compact and metrizable, both of them are separable.

Corollary

If L is a compact LOTS that is not metrizable, then there exists no continuous surjection from L to $L \times L$.

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The Mardešić Conjecture

S. Mardešić proposed the following conjecture in 1970, which was proved by G. Martínez-Cervantes and G. Plebanek in 2019.

Theorem (G. Martínez-Cervantes and G. Plebanek)

Let d and s be positive integers. Let K_i be a compact LOTS for each $i < d$, X a compact subspace of $\prod_{i<d} K_i$, Z_j an infinite Hausdorff space for each $j < d + s$. If there exists a continuous surjection from X onto $\prod_{j<d+s} Z_j$, then there exist at least $s + 1$-many metrizable factors Z_j.

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What does it mean?

If $d = s = 1$, then it coincides with Treybig's Theorem.
By using Peano's Theorem, it is easy to see that for all positive integers d and s, there exists a continuous surjection from $[0,1]^d$ onto $[0,1]^{d+s}$. The Mardešić Conjecture implies even this seemingly weaker phenomenon does not happen to a nonmetrizable compact LOTS.

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Countably compact version

Recall that Čertanov's Theorem is the countably compact version of Treybig's Theorem. As the Mardešić Conjecture is a generalization of Treybig's Theorem, we may wonder if we can prove the countably compact version of the Mardešić Conjecture. We solved this problem positively, namely proved the following theorem.

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Let d and s be positive integers. Let K_i be a compact LOTS for each $i < d$, X a countably compact subspace of $\prod_{i<d} K_i$, Z_j an infinite Hausdorff space for each $j < d + s$. If there exists a continuous surjection f from X onto $\prod_{j<d+s} Z_j$, then there exist at least $s + 1$-many compact and metrizable factors Z_j.

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S. Purish proved that Stone-Čech compactification of any countably compact GO-space is a LOTS. By using this result, we can obtain the following corollary.

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Main Lemma

The following lemma is the biggest piece of the proof of the main theorem.

Lemma

Let d, K_i, X, Z_j, and f be as in the assumption of the main theorem. Suppose that Z_d is not separable. Then, there exists a compact LOTS $\tilde{K}_i$ for each $i < d-1$, a countably compact subspace $\tilde{X}$ of $\prod_{i<d-1} \tilde{K}_i$, and a continuous surjection $\tilde{f} : \tilde{X} \to \prod_{j<d} Z_j$.

Namely, if one of Z_j's is nonseparable, then there exist $\tilde{K}_i$, $\tilde{X}$, and $\tilde{f}$ that satisfies the assumption of the main theorem for $d-1$. This lemma is proved by heavy use of countable elementary submodels.

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I(K, M, p)

We shall sketch the proof of the main lemma.
In general, we shall define the following notations.

Definition

Let K be a compact LOTS and M a countable set. Then, for all $p \in K$, define

$$\eta(K, M, p) = \sup \{ u \in \mathrm{cl}(K \cap M) \mid u \leq p \}$$
$$\zeta(K, M, p) = \inf \{ u \in \mathrm{cl}(K \cap M) \mid u \geq p \}$$
$$I(K, M, p) = [\eta(L, M, p), \zeta(L, M, p)]$$

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Set up

Let d, K_i, X, Z_j, and f be as in the assumption of the main lemma.

- Let f_d be the d-th component function of f.
- Let $g : X \to \prod_{j<d} Z_j$ be defined by $g(x) = f(x) \restriction d$. (Technically, g must be a slight extension of this, but never mind.)
- Let M be a good coutable elementary submodel of $H(\theta)$ for a sufficiently large regular cardinal θ.

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The First Claim

First, we can prove the following claim.

Claim

Let $r \in Z_d \setminus \mathrm{Cl}(Z_d \cap M)$. Then, there exist finite sequences $\langle i_n | n < \hat{n} \rangle$ and $\langle p_n | n < \hat{n} \rangle$ such that for all $n < \hat{n}$, $i_n < d$ and $p_n \in K_{i_n}$, and

$$f_d^{\leftarrow}\{r\} \subseteq \bigcup_{n<\hat{n}} \pi_{i_n}^{\leftarrow} I(K_{i_n}, M, p_n)$$

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The Second Claim

We can remove the reference to r, which does not belong to M, by proving the following claim.

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There exist finite sequences $\langle i_n | n < \hat{n} \rangle$ and $\langle p_n | n < \hat{n} \rangle$ such that for all $n < \hat{n}$, $i_n < d$ and $p_n \in K_{i_n}$, and

$$\prod_{j<d} Z_j = g^{\rightarrow} \left(X \cap \bigcup_{n<\hat{n}} \pi_{i_n}{}^{\leftarrow} I(K_{i_n}, M, p_n) \right)$$

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The Third Claim

An argument with elementarity proves the following third claim.

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By 'combining' $\pi_{i_n}{}^{\leftarrow} \{ \eta(K_{i_n}, M, p_n), \zeta(K_{i_n}, M, p_n) \}$ for all $n < \hat{n}$, we can finish proving the main lemma.

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Toward separability

Let d, K_i, X, Z_j, and f be as in the assumption of the main theorem.

Let $e = |\{ j < d+1 \mid Z_j \text{ is separable} \}|$. Without loss of generality, we may assume that for all $j < d+1$, Z_j is separable if and only if $j < e$.

By repeatedly applying the main lemma, we can prove the following:

- $e \geq 2$, and
- there exists a compact LOTS $\tilde{K}_i$ for each $i < e-1$, a countably compact subspace $\tilde{X}$ of $\prod_{i<e-1} \tilde{K}_i$, and a continuous surjection $\tilde{f} : \tilde{X} \to \prod_{j<e} Z_j$.

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Note that $\prod_{j<e} Z_j$ is separable.

Toward separability

Let d, K_i, X, Z_j, and f be as in the assumption of the main theorem.
Let $e = |\{ j < d+1 \mid Z_j \text{ is separable} \}|$. Without loss of generality, we may assume that for all $j < d+1$, Z_j is separable if and only if $j < e$.
By repeatedly applying the main lemma, we can prove the following:

- $e \geq 2$, and
- there exists a compact LOTS $\tilde{K}_i$ for each $i < e-1$, a countably compact subspace $\tilde{X}$ of $\prod_{i<e-1} \tilde{K}_i$, and a continuous surjection $\tilde{f} : \tilde{X} \to \prod_{j<e} Z_j$.

Note that $\prod_{j<e} Z_j$ is separable.

Compactness

By using the fact that $\prod_{j<e} Z_j$ is separable, there exist a separable compact LOTS K_i' for each $i < e - 1$, X' a compact subspace of $\prod_{i<e-1} K_i'$, and a continuous surjection $f' : X' \to \prod_{j<e} Z_j$.

Now, by using the original Mardešić Conjecture, there exist at least two factors Z_j that is compact and metrizable.

Compactness

By using the fact that $\prod_{j<e} Z_j$ is separable, there exist a separable compact LOTS K'_i for each $i < e - 1$, X' a compact subspace of $\prod_{i<e-1} K'_i$, and a continuous surjection $f' : X' \to \prod_{j<e} Z_j$.
Now, by using the original Mardešić Conjecture, there exist at least two factors Z_j that is compact and metrizable.

Mardešić's Theorem

In fact, S. Mardešić proved that the Mardešić Conjecture holds when all Z_j's are separable. So, we can finish proving the main theorem without using the full Mardešić Conjecture.
Thus, this argument gives another proof of the (original) Mardešić Conjecture.

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Thus, this argument gives another proof of the (original) Mardešić Conjecture.

Open Questions

Question

Can we extended the main theorem to a wider class than LOTS?

Question

What can we say when there is a continuous surjection from $\prod_{i<d} K_i$ onto $\prod_{i<d} Z_i$?

We know that if all Z_i's are nonseparable, then Z_i is an image of some LOTS.