Finte Products of Connected Nowhere Real Linearly Ordered Spaces

Why is it better to live in an Euledean space

Tetsuya Ishiu

Department of Mathematics
Miami University

February 22, 2018

Outline

Example: $\mathbb{R}^2$

Consider $\mathbb{R}^2$.
The function that swap the coordinates is a homemorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$.

$$f(x, y) = \langle y, x \rangle$$

If $g_1 : \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R} \to \mathbb{R}$ are homeomorphisms, from $\mathbb{R}$ to $\mathbb{R}$, then

$$g(x, y) = \langle g_1(x), g_2(y) \rangle$$

is a homemorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$. For example, $g(x, y) = \langle 2x + 1, y^3 + 1 \rangle$ is a homeomorphism.

Example: $\mathbb{R}^2$

Consider $\mathbb{R}^2$.
The function that swap the coordinates is a homemorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$.

$$f(x, y) = \langle y, x \rangle$$

If $g_1 : \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R} \to \mathbb{R}$ are homeomorphisms, from $\mathbb{R}$ to $\mathbb{R}$, then

$$g(x, y) = \langle g_1(x), g_2(y) \rangle$$

is a homemorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$. For example, $g(x, y) = \langle 2x + 1, y^3 + 1 \rangle$ is a homeomorphism.

Example: $\mathbb{R}^2$

Consider $\mathbb{R}^2$.
The function that swap the coordinates is a homemorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$.

$$f(x, y) = \langle y, x \rangle$$

If $g_1 : \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R} \to \mathbb{R}$ are homeomorphisms, from $\mathbb{R}$ to $\mathbb{R}$, then

$$g(x, y) = \langle g_1(x), g_2(y) \rangle$$

is a homemorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$. For example, $g(x, y) = \langle 2x + 1, y^3 + 1 \rangle$ is a homeomorphism.

Example: $\mathbb{R}^2$

Consider $\mathbb{R}^2$.
The function that swap the coordinates is a homemorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$.

$$f(x, y) = \langle y, x \rangle$$

If $g_1 : \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R} \to \mathbb{R}$ are homeomorphisms, from $\mathbb{R}$ to $\mathbb{R}$, then

$$g(x, y) = \langle g_1(x), g_2(y) \rangle$$

is a homemorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$. For example, $g(x, y) = \langle 2x + 1, y^3 + 1 \rangle$ is a homeomorphism.

Example :$\mathbb{R}^2$

These two types of homeomorphisms and their compositions form a class of coordinate-wise homeomorphisms.

Namely, a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is coordinate-wise if and only if there are functions $g_1 : \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R} \to \mathbb{R}$ such that either

1. for every $\langle x, y\rangle \in \mathbb{R}^2$, $f(x, y) = \langle g_1(x), g_2(y)\rangle$ or
2. for every $\langle x, y\rangle \in \mathbb{R}^2$, $f(x, y) = \langle g_1(y), g_2(x)\rangle$.

Example :$\mathbb{R}^2$

These two types of homeomorphisms and their compositions form a class of coordinate-wise homeomorphisms.
Namely, a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is coordinate-wise if and only if there are functions $g_1 : \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R} \to \mathbb{R}$ such that either

1. for every $\langle x, y\rangle \in \mathbb{R}^2$, $f(x, y) = \langle g_1(x), g_2(y)\rangle$ or
2. for every $\langle x, y\rangle \in \mathbb{R}^2$, $f(x, y) = \langle g_1(y), g_2(x)\rangle$.

Example: $\mathbb{R}^2$

It is very easy to construct homeomophisms from $\mathbb{R}^2$ to $\mathbb{R}^2$ that are NOT coordinate-wise.

$$f(x, y) = \langle x + y, x - y\rangle \qquad g(x, y) = \langle x, e^x y\rangle$$

Is it generally true for linearly ordered topological spaces?

Example: $\mathbb{R}^2$

It is very easy to construct homeomophisms from $\mathbb{R}^2$ to $\mathbb{R}^2$ that are NOT coordinate-wise.

$$f(x, y) = \langle x + y, x - y \rangle \qquad g(x, y) = \langle x, e^x y \rangle$$

Is it generally true for linearly ordered topological spaces?

Outline

Definition (Linarly ordered topological space)

Definition

Let L be a linearly ordered set without minimum or maximum elements.

Then, the set of all open intervals $(a, b) = \{x \in L : a < x < b\}$ is a basis for a topology. The topology generated by this basis is called the order topology. A space whose topology is an order topology is called a linearly ordered topological space(LOTS).

For example, $\mathbb{R}$ is a LOTS with ordinary order.

For simplicity, we assume that all linearly ordered sets have no minimum or maximum elements.

Definition (Linarly ordered topological space)

Definition

Let L be a linearly ordered set without minimum or maximum elements.
Then, the set of all open intervals $(a, b) = \{x \in L : a < x < b\}$ is a basis for a topology. The topology generated by this basis is called the order topology. A space whose topology is an order topology is called a linearly ordered topological space(LOTS).

For example, $\mathbb{R}$ is a LOTS with ordinary order.

For simplicity, we assume that all linearly ordered sets have no minimum or maximum elements.

Definition (Linarly ordered topological space)

Definition

Let L be a linearly ordered set without minimum or maximum elements.
Then, the set of all open intervals $(a, b) = \{x \in L : a < x < b\}$ is a basis for a topology. The topology generated by this basis is called the order topology. A space whose topology is an order topology is called a linearly ordered topological space(LOTS).

For example, $\mathbb{R}$ is a LOTS with ordinary order.

For simplicity, we assume that all linearly ordered sets have no minimum or maximum elements.

Definition (Linarly ordered topological space)

Definition

Let L be a linearly ordered set without minimum or maximum elements.
Then, the set of all open intervals $(a, b) = \{x \in L : a < x < b\}$ is a basis for a topology. The topology generated by this basis is called the order topology. A space whose topology is an order topology is called a linearly ordered topological space(LOTS).

For example, $\mathbb{R}$ is a LOTS with ordinary order.

For simplicity, we assume that all linearly ordered sets have no minimum or maximum elements.

Definition (Linarly ordered topological space)

Definition

Let L be a linearly ordered set without minimum or maximum elements.
Then, the set of all open intervals $(a, b) = \{x \in L : a < x < b\}$ is a basis for a topology. The topology generated by this basis is called the order topology. A space whose topology is an order topology is called a linearly ordered topological space(LOTS).

For example, $\mathbb{R}$ is a LOTS with ordinary order.

For simplicity, we assume that all linearly ordered sets have no minimum or maximum elements.

Separability

Definition

Let L be a linearly ordered set. $D \subseteq L$ is dense in L if and only if D is dense in the order topology of L, namely for every $x, y \in L$ with $(x, y) \neq \varnothing$, there exists $d \in D$ such that $x < d < y$.

L is separable if and only if L has a countable dense subset.

Fact

- *$\mathbb{R}$ is separable since $\mathbb{Q}$ is a countable dense subset.*
- *A linearly ordered set L embeds into $\mathbb{R}$ if and only if L is separable and has only countably many jumps.*

Separability

Definition

Let L be a linearly ordered set. $D \subseteq L$ is dense in L if and only if D is dense in the order topology of L, namely for every $x, y \in L$ with $(x, y) \neq \varnothing$, there exists $d \in D$ such that $x < d < y$.

L is separable if and only if L has a countable dense subset.

Fact

- $\mathbb{R}$ *is separable since* $\mathbb{Q}$ *is a countable dense subset.*
- *A linearly ordered set L embeds into* $\mathbb{R}$ *if and only if L is separable and has only countably many jumps.*

Separability

Definition

Let L be a linearly ordered set. $D \subseteq L$ is dense in L if and only if D is dense in the order topology of L, namely for every $x, y \in L$ with $(x, y) \neq \varnothing$, there exists $d \in D$ such that $x < d < y$.
L is separable if and only if L has a countable dense subset.

Fact

- *$\mathbb{R}$ is separable since $\mathbb{Q}$ is a countable dense subset.*
- *A linearly ordered set L embeds into $\mathbb{R}$ if and only if L is separable and has only countably many jumps.*

Separability

Definition

Let L be a linearly ordered set. $D \subseteq L$ is dense in L if and only if D is dense in the order topology of L, namely for every $x, y \in L$ with $(x, y) \neq \varnothing$, there exists $d \in D$ such that $x < d < y$.
L is separable if and only if L has a countable dense subset.

Fact

- *$\mathbb{R}$ is separable since $\mathbb{Q}$ is a countable dense subset.*
- *A linearly ordered set L embeds into $\mathbb{R}$ if and only if L is separable and has only countably many jumps.*

Characterization of $\mathbb{R}$

Theorem

Every complete separable linearly ordered space L that is dense in itself without maximum or minimum elements is order-isomorphic to $\mathbb{R}$. Here,

- *L is complete if and only if every subset of L bounded above has a least upper bound.*
- *L is dense in itself if and only if for every $x, y \in L$ with $x < y$, there exists $z \in L$ such that $x < z < y$.*

Characterization of $\mathbb{R}$

Theorem

Every complete separable linearly ordered space L that is dense in itself without maximum or minimum elements is order-isomorphic to $\mathbb{R}$. Here,

- *L is complete if and only if every subset of L bounded above has a least upper bound.*
- *L is dense in itself if and only if for every $x, y \in L$ with $x < y$, there exists $z \in L$ such that $x < z < y$.*

Characterization of $\mathbb{R}$

Theorem

Every complete separable linearly ordered space L that is dense in itself without maximum or minimum elements is order-isomorphic to $\mathbb{R}$. Here,

- *L is complete if and only if every subset of L bounded above has a least upper bound.*
- *L is dense in itself if and only if for every $x, y \in L$ with $x < y$, there exists $z \in L$ such that $x < z < y$.*

Topological and order-theoretic properties

Theorem

Let L be a linearly ordered set, equipped with the order topology.

- *L is Hausdorff.*
- *L is connected if and only if L is complete and dense in itself.*
- *If L is connected, every bounded closed subset of L is compact.*

Topological and order-theoretic properties

Theorem

Let L be a linearly ordered set, equipped with the order topology.

- *L is Hausdorff.*
- *L is connected if and only if L is complete and dense in itself.*
- *If L is connected, every bounded closed subset of L is compact.*

Topological and order-theoretic properties

Theorem

Let L be a linearly ordered set, equipped with the order topology.

- *L is Hausdorff.*
- *L is connected if and only if L is complete and dense in itself.*
- *If L is connected, every bounded closed subset of L is compact.*

Outline

Theorem of Eda and Kamijo

Theorem (K. Eda and R. Kamijo)

Let L be a connected linearly ordered space such that for every non-empty open interval I, there exists an uncountable monotone subset of I. Then every homeomorphism $f : L^n \to L^n$ is coordinate-wise for every natural number n.

Example

Recall that ω_1 is the uncountable well-ordered set whose all proper initial segments are countable.
Let L be the set of all functions s from ω_1 into $\{0, 1\}$ that is not eventually 1, i.e. for unboundedly many $\alpha \in \omega_1$, $s(\alpha) = 0$. Let L be ordered lexicographically. Namely, $s < t$ if and only if $s(\alpha) < t(\alpha)$ when α is the least element of ω_1 such that $s(\alpha) \neq t(\alpha)$.
Then, L satisfies the assumption of the previous theorem.
So, for every natural number n, every homeomorphism from L^n to L^n is coordinate-wise.

Example

Example

Recall that ω_1 is the uncountable well-ordered set whose all proper initial segments are countable.
Let L be the set of all functions s from ω_1 into $\{0, 1\}$ that is not eventually 1, i.e. for unboundedly many $\alpha \in \omega_1$, $s(\alpha) = 0$. Let L be ordered lexicographically. Namely, $s < t$ if and only if $s(\alpha) < t(\alpha)$ when α is the least element of ω_1 such that $s(\alpha) \neq t(\alpha)$.
Then, L satisfies the assumption of the previous theorem. So, for every natural number n, every homeomorphism from L^n to L^n is coordinate-wise.

Example

Example

Recall that ω_1 is the uncountable well-ordered set whose all proper initial segments are countable.
Let L be the set of all functions s from ω_1 into $\{0, 1\}$ that is not eventually 1, i.e. for unboundedly many $\alpha \in \omega_1$, $s(\alpha) = 0$. Let L be ordered lexicographically. Namely, $s < t$ if and only if $s(\alpha) < t(\alpha)$ when α is the least element of ω_1 such that $s(\alpha) \neq t(\alpha)$.
Then, L satisfies the assumption of the previous theorem. So, for every natural number n, every homeomorphism from L^n to L^n is coordinate-wise.

Example

Example

Recall that ω_1 is the uncountable well-ordered set whose all proper initial segments are countable.
Let L be the set of all functions s from ω_1 into $\{0, 1\}$ that is not eventually 1, i.e. for unboundedly many $\alpha \in \omega_1$, $s(\alpha) = 0$. Let L be ordered lexicographically. Namely, $s < t$ if and only if $s(\alpha) < t(\alpha)$ when α is the least element of ω_1 such that $s(\alpha) \neq t(\alpha)$.
Then, L satisfies the assumption of the previous theorem. So, for every natural number n, every homeomorphism from L^n to L^n is coordinate-wise.

Outline

Question of Eda

Definition

A linearly ordered space L is *nowhere real* if and only if no nonempty open interval is separable.

(probably "*nowhere separable*" is better, but let me keep using it).

Eda asked the following question.

Question

Can we replace "every non-empty open interval contains an uncountable monotone sequence" by "nowhere real"?

Note that connectedness is necessary because we may just rearrange the connected components.

Question of Eda

Definition

A linearly ordered space L is *nowhere real* if and only if no nonempty open interval is separable.

(probably "*nowhere separable*" is better, but let me keep using it).
Eda asked the following question.

Question

Can we replace "every non-empty open interval contains an uncountable monotone sequence" by "nowhere real"?

Note that connectedness is necessary because we may just rearrange the connected components.

Question of Eda

Definition

A linearly ordered space L is *nowhere real* if and only if no nonempty open interval is separable.

(probably "*nowhere separable*" is better, but let me keep using it).
Eda asked the following question.

Question

Can we replace "every non-empty open interval contains an uncountable monotone sequence" by "nowhere real"?

Note that connectedness is necessary because we may just rearrange the connected components.

Aronszajn lines

We know that it is not vacuous question.

(In ZFC), there also exists a linearly ordered set that do not contain

- any uncountable separable suborder, or
- any uncountable monotone sequence.

Such a linearly ordered set is called an Aronszajn line. It is easy to construct an Aronszajn line that is dense in itself.

Aronszajn lines

We know that it is not vacuous question.
(In ZFC), there also exists a linearly ordered set that do not contain

- any uncountable separable suborder, or
- any uncountable monotone sequence.

Such a linearly ordered set is called an Aronszajn line. It is easy to construct an Aronszajn line that is dense in itself.

Aronszajn lines

We know that it is not vacuous question.
(In ZFC), there also exists a linearly ordered set that do not contain

- any uncountable separable suborder, or
- any uncountable monotone sequence.

Such a linearly ordered set is called an Aronszajn line. It is easy to construct an Aronszajn line that is dense in itself.

Aronszajn lines

We know that it is not vacuous question.
(In ZFC), there also exists a linearly ordered set that do not contain

- any uncountable separable suborder, or
- any uncountable monotone sequence.

Such a linearly ordered set is called an Aronszajn line. It is easy to construct an Aronszajn line that is dense in itself.

Let L be an Aronszajn line that is dense in itself. Consider the cut-completion $\hat{L}$ of L, namely the set $\hat{L}$ of all Dedekind cuts of L.

Then, $\hat{L}$ is a connected nowhere real linearly ordered space that has no uncountable monotone sequence. ($\hat{L}$ has both minimum and maximum elements, but it is easy to remove them).

So, Eda's question is not vacuous.

Cut-completion

Let L be an Aronszajn line that is dense in itself. Consider the cut-completion $\hat{L}$ of L, namely the set $\hat{L}$ of all Dedekind cuts of L.

Then, $\hat{L}$ is a connected nowhere real linearly ordered space that has no uncountable monotone sequence. ($\hat{L}$ has both minimum and maximum elements, but it is easy to remove them).

So, Eda's question is not vacuous.

Cut-completion

Let L be an Aronszajn line that is dense in itself. Consider the cut-completion $\hat{L}$ of L, namely the set $\hat{L}$ of all Dedekind cuts of L.

Then, $\hat{L}$ is a connected nowhere real linearly ordered space that has no uncountable monotone sequence. ($\hat{L}$ has both minimum and maximum elements, but it is easy to remove them).

So, Eda's question is not vacuous.

Cut-completion

Let L be an Aronszajn line that is dense in itself. Consider the cut-completion $\hat{L}$ of L, namely the set $\hat{L}$ of all Dedekind cuts of L.
Then, $\hat{L}$ is a connected nowhere real linearly ordered space that has no uncountable monotone sequence. ($\hat{L}$ has both minimum and maximum elements, but it is easy to remove them).
So, Eda's question is not vacuous.

Answer: Yes, Theorem of Eda and Kamijo can be extended to every connected nowhere real linearly ordered spaces.

And even more can be said.

Theorem

Let $K_0, K_1, \ldots, K_{n-1}, L_0, L_1, \ldots, L_{n-1}$ be connected nowhere real linearly ordered spaces. Let $f : \prod_{i<n} K_i \to \prod_{j<n} L_j$ be a continuous injective function. Then, f is coordinate-wise.

Main Theorem

Answer: Yes, Theorem of Eda and Kamijo can be extended to every connected nowhere real linearly ordered spaces. And even more can be said.

Theorem

Let $K_0, K_1, \ldots, K_{n-1}, L_0, L_1, \ldots, L_{n-1}$ be connected nowhere real linearly ordered spaces. Let $f : \prod_{i<n} K_i \to \prod_{j<n} L_j$ be a continuous injective function. Then, f is coordinate-wise.

Main Theorem

Answer: Yes, Theorem of Eda and Kamijo can be extended to every connected nowhere real linearly ordered spaces. And even more can be said.

Theorem

Let $K_0, K_1, \ldots, K_{n-1}, L_0, L_1, \ldots, L_{n-1}$ be connected nowhere real linearly ordered spaces. Let $f : \prod_{i<n} K_i \to \prod_{j<n} L_j$ be a continuous injective function. Then, f is coordinate-wise.

Main Theorem

Answer: Yes, Theorem of Eda and Kamijo can be extended to every connected nowhere real linearly ordered spaces. And even more can be said.

Theorem

Let $K_0, K_1, \ldots, K_{n-1}, L_0, L_1, \ldots, L_{n-1}$ be connected nowhere real linearly ordered spaces. Let $f : \prod_{i<n} K_i \to \prod_{j<n} L_j$ be a continuous injective function. Then, f is coordinate-wise.

Advantages of the Main Theorem

Main Theorem improves the Theorem of Eda and Kamijo in the following sense.

1. The domain and codomain do not have to be the same.
2. Neither the domain nor codomain has to be the power of a single linearly ordered space, but the product of different linearly ordered spaces.
3. The function does not have to be a homeomorphism, but a continuous injection.

Advantages of the Main Theorem

Main Theorem improves the Theorem of Eda and Kamijo in the following sense.

1. The domain and codomain do not have to be the same.
2. Neither the domain nor codomain has to be the power of a single linearly ordered space, but the product of different linearly ordered spaces.
3. The function does not have to be a homeomorphism, but a continuous injection.

Advantages of the Main Theorem

Main Theorem improves the Theorem of Eda and Kamijo in the following sense.

1. The domain and codomain do not have to be the same.
2. Neither the domain nor codomain has to be the power of a single linearly ordered space, but the product of different linearly ordered spaces.
3. The function does not have to be a homeomorphism, but a continuous injection.

Advantages of the Main Theorem

Main Theorem improves the Theorem of Eda and Kamijo in the following sense.

1. The domain and codomain do not have to be the same.
2. Neither the domain nor codomain has to be the power of a single linearly ordered space, but the product of different linearly ordered spaces.
3. The function does not have to be a homeomorphism, but a continuous injection.

Definition (coordinate-wise functions)

First, let me formally define coordinate-wise functions.

Definition

Let $\prod_{i<n} K_i$ and $\prod_{j<n} L_j$ be the products of sets. We say that a function $f : \prod_{i<n} K_i \to \prod_{j<n} L_j$ is coordinate-wise if and only if for every $j < n$, there exists $i < n$ such that the value of the j-th coordinate of $f(x)$ depends only on the i-th coordinate of x.

i.e. there exist a function $h : \{ j : j < n \} \to \{ i : i < n \}$, and $g_j : K_{h(j)} \to L_j$ for each $j < n$ such that for every $x \in \prod_{i<n} K_i$, the j-th coordinate of $f(x)$ equals to $g_j(x(h(j)))$.

$$f(x)_j = g_j(x_{h(j)})$$

Definition (coordinate-wise functions)

First, let me formally define coordinate-wise functions.

Definition

Let $\prod_{i<n} K_i$ and $\prod_{j<n} L_j$ be the products of sets. We say that a function $f : \prod_{i<n} K_i \to \prod_{j<n} L_j$ is coordinate-wise if and only if for every $j < n$, there exists $i < n$ such that the value of the j-th coordinate of $f(x)$ depends only on the i-th coordinate of x.

i.e. there exist a function $h : \{ j : j < n \} \to \{ i : i < n \}$, and $g_j : K_{h(j)} \to L_j$ for each $j < n$ such that for every $x \in \prod_{i<n} K_i$, the j-th coordinate of $f(x)$ equals to $g_j(x(h(j)))$.

$$f(x)_j = g_j(x_{h(j)})$$

Definition (coordinate-wise functions)

First, let me formally define coordinate-wise functions.

Definition

Let $\prod_{i<n} K_i$ and $\prod_{j<n} L_j$ be the products of sets. We say that a function $f : \prod_{i<n} K_i \to \prod_{j<n} L_j$ is coordinate-wise if and only if for every $j < n$, there exists $i < n$ such that the value of the j-th coordinate of $f(x)$ depends only on the i-th coordinate of x.

i.e. there exist a function $h : \{ j : j < n \} \to \{ i : i < n \}$, and $g_j : K_{h(j)} \to L_j$ for each $j < n$ such that for every $x \in \prod_{i<n} K_i$, the j-th coordinate of $f(x)$ equals to $g_j(x(h(j)))$.

$$f(x)_j = g_j(x_{h(j)})$$

Definition (coordinate-wise functions)

First, let me formally define coordinate-wise functions.

Definition

Let $\prod_{i<n} K_i$ and $\prod_{j<n} L_j$ be the products of sets. We say that a function $f : \prod_{i<n} K_i \to \prod_{j<n} L_j$ is coordinate-wise if and only if for every $j < n$, there exists $i < n$ such that the value of the j-th coordinate of $f(x)$ depends only on the i-th coordinate of x.
i.e. there exist a function $h : \{ j : j < n \} \to \{ i : i < n \}$, and $g_j : K_{h(j)} \to L_j$ for each $j < n$ such that for every $x \in \prod_{i<n} K_i$, the j-th coordinate of $f(x)$ equals to $g_j(x(h(j)))$.

$$f(x)_j = g_j(x_{h(j)})$$

Elementary submodels

We shall outline the proof in case $n = 2$. The case $n > 2$ is done by the same idea with a little more tricks and notational difficulties.
Fortunately, it has nothing to do with forcing, large cardinals, relative consitency, and so on.
However, we need "elementary submodels".

Elementary submodels

We shall outline the proof in case $n = 2$.The case $n > 2$ is done by the same idea with a little more tricks and notational difficulties.

Fortunately, it has nothing to do with forcing, large cardinals, relative consitency, and so on.

However, we need “elementary submodels”.

Elementary submodels

We shall outline the proof in case $n = 2$.The case $n > 2$ is done by the same idea with a little more tricks and notational difficulties.
Fortunately, it has nothing to do with forcing, large cardinals, relative consitency, and so on.
However, we need "elementary submodels".

Elementary submodels

We shall outline the proof in case $n = 2$.The case $n > 2$ is done by the same idea with a little more tricks and notational difficulties.
Fortunately, it has nothing to do with forcing, large cardinals, relative consitency, and so on.
However, we need “elementary submodels”.

Sufficiently large part of the universe

In ZFC, for some fixed objects (such as sequences of linearly ordered spaces), we can show that there is a very large set $H(\theta)$ such that

1. almost all axioms of ZFC are satisfied (except the Power Set Axiom), and
2. everything related to those fixed objects can be done inside $H(\theta)$ correctly.

(Technical remark) By Gödel's Incompleteness Theorem, there may not exist a set that satisfies ZFC.

Sufficiently large part of the universe

In ZFC, for some fixed objects (such as sequences of linearly ordered spaces), we can show that there is a very large set $H(\theta)$ such that

1. almost all axioms of ZFC are satisfied (except the Power Set Axiom), and
2. everything related to those fixed objects can be done inside $H(\theta)$ correctly.

(Technical remark) By Gödel's Incompleteness Theorem, there may not exist a set that satisfies ZFC.

Sufficiently large part of the universe

In ZFC, for some fixed objects (such as sequences of linearly ordered spaces), we can show that there is a very large set $H(\theta)$ such that

1. almost all axioms of ZFC are satisfied (except the Power Set Axiom), and
2. everything related to those fixed objects can be done inside $H(\theta)$ correctly.

(Technical remark) By Gödel's Incompleteness Theorem, there may not exist a set that satisfies ZFC.

Sufficiently large part of the universe

In ZFC, for some fixed objects (such as sequences of linearly ordered spaces), we can show that there is a very large set $H(\theta)$ such that

1. almost all axioms of ZFC are satisfied (except the Power Set Axiom), and
2. everything related to those fixed objects can be done inside $H(\theta)$ correctly.

(Technical remark) By Gödel's Incompleteness Theorem, there may not exist a set that satisfies ZFC.

Downward Löwenheim-Skolem Theorem

By Löwenheim-Skolem Theorem, there exists a countable elementary submodel M of such an $H(\theta)$.

Namely, for every first-order formula φ and $a_1, a_2, \ldots, a_n \in M$, $H(\theta)$ satisfies $\varphi(a_1, a_2, \ldots, a_n)$ if and only if M satisfies $\varphi(a_1, a_2, \ldots, a_n)$.

More practically, it means that M has all "definable" objects as elements, and is closed under every function in M.

Downward Löwenheim-Skolem Theorem

By Löwenheim-Skolem Theorem, there exists a countable elementary submodel M of such an $H(\theta)$.
Namely, for every first-order formula φ and $a_1, a_2, \ldots, a_n \in M$, $H(\theta)$ satisfies $\varphi(a_1, a_2, \ldots, a_n)$ if and only if M satisfies $\varphi(a_1, a_2, \ldots, a_n)$.
More practically, it means that M has all "definable" objects as elements, and is closed under every function in M.

Downward Löwenheim-Skolem Theorem

By Löwenheim-Skolem Theorem, there exists a countable elementary submodel M of such an $H(\theta)$.
Namely, for every first-order formula φ and $a_1, a_2, \ldots, a_n \in M$, $H(\theta)$ satisfies $\varphi(a_1, a_2, \ldots, a_n)$ if and only if M satisfies $\varphi(a_1, a_2, \ldots, a_n)$.
More practically, it means that M has all "definable" objects as elements, and is closed under every function in M.

Example

Let M be a countable elementary submodel of a sufficiently large $H(\theta)$.

Then,

1. $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and all definable objects are in M.
2. All definable fuctions are in M.
3. Since $\mathbb{R}$ is complete in $H(\theta)$, $\mathbb{R}$ is complete in M. Namely, if A is a subset of $\mathbb{R}$ that is bounded above and $A \in M$, then A has the least upper bound in M.
4. The supremum of $\omega_1 \cap M$ is NOT in M. It is OK since $\omega_1 \cap M$ is not an element of M.

Example

Let M be a countable elementary submodel of a sufficiently large $H(\theta)$.
Then,

1. $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and all definable objects are in M.
2. All definable fuctions are in M.
3. Since $\mathbb{R}$ is complete in $H(\theta)$, $\mathbb{R}$ is complete in M. Namely, if A is a subset of $\mathbb{R}$ that is bounded above and $A \in M$, then A has the least upper bound in M.
4. The supremum of $\omega_1 \cap M$ is NOT in M. It is OK since $\omega_1 \cap M$ is not an element of M.

Example

Let M be a countable elementary submodel of a sufficiently large $H(\theta)$.
Then,

1. $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and all definable objects are in M.
2. All definable fuctions are in M.
3. Since $\mathbb{R}$ is complete in $H(\theta)$, $\mathbb{R}$ is complete in M. Namely, if A is a subset of $\mathbb{R}$ that is bounded above and $A \in M$, then A has the least upper bound in M.
4. The supremum of $\omega_1 \cap M$ is NOT in M. It is OK since $\omega_1 \cap M$ is not an element of M.

Example

Let M be a countable elementary submodel of a sufficiently large $H(\theta)$.
Then,

1. $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and all definable objects are in M.
2. All definable fuctions are in M.
3. Since $\mathbb{R}$ is complete in $H(\theta)$, $\mathbb{R}$ is complete in M.
   Namely, if A is a subset of $\mathbb{R}$ that is bounded above and $A \in M$, then A has the least upper bound in M.
4. The supremum of $\omega_1 \cap M$ is NOT in M. It is OK since $\omega_1 \cap M$ is not an element of M.

Example

Let M be a countable elementary submodel of a sufficiently large $H(\theta)$.
Then,

1. $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and all definable objects are in M.
2. All definable fuctions are in M.
3. Since $\mathbb{R}$ is complete in $H(\theta)$, $\mathbb{R}$ is complete in M. Namely, if A is a subset of $\mathbb{R}$ that is bounded above and $A \in M$, then A has the least upper bound in M.
4. The supremum of $\omega_1 \cap M$ is NOT in M. It is OK since $\omega_1 \cap M$ is not an element of M.

Example

Let M be a countable elementary submodel of a sufficiently large $H(\theta)$.
Then,

1. $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and all definable objects are in M.
2. All definable fuctions are in M.
3. Since $\mathbb{R}$ is complete in $H(\theta)$, $\mathbb{R}$ is complete in M. Namely, if A is a subset of $\mathbb{R}$ that is bounded above and $A \in M$, then A has the least upper bound in M.
4. The supremum of $\omega_1 \cap M$ is NOT in M. It is OK since $\omega_1 \cap M$ is not an element of M.

Outline

Characterization of connected nowhere real LOTS

Fix a connected nowhere real linearly ordered space K and a sufficiently good countable elementary submodel M with $K \in M$.

Let $a < b$ both in K. Since K is nowhere real, $(a, b) \cap M$ is not dense in (a, b). So, $\mathrm{Cl}((a, b) \cap M)$ does not contain (a, b).

We will play with this gap. By the way, this gap does not exists in case of $\mathbb{R}$ and it explains why a similar argument does not work for $\mathbb{R}$.

Characterization of connected nowhere real LOTS

Fix a connected nowhere real linearly ordered space K and a sufficiently good countable elementary submodel M with $K \in M$.

Let $a < b$ both in K. Since K is nowhere real, $(a, b) \cap M$ is not dense in (a, b). So, $\mathrm{Cl}((a, b) \cap M)$ does not contain (a, b).

We will play with this gap. By the way, this gap does not exists in case of $\mathbb{R}$ and it explains why a similar argument does not work for $\mathbb{R}$.

Characterization of connected nowhere real LOTS

Fix a connected nowhere real linearly ordered space K and a sufficiently good countable elementary submodel M with $K \in M$.

Let $a < b$ both in K. Since K is nowhere real, $(a, b) \cap M$ is not dense in (a, b). So, $\mathrm{Cl}((a, b) \cap M)$ does not contain (a, b).

We will play with this gap. By the way, this gap does not exists in case of $\mathbb{R}$ and it explains why a similar argument does not work for $\mathbb{R}$.

Characterization of connected nowhere real LOTS

Fix a connected nowhere real linearly ordered space K and a sufficiently good countable elementary submodel M with $K \in M$.
Let $a < b$ both in K. Since K is nowhere real, $(a, b) \cap M$ is not dense in (a, b). So, $\mathrm{Cl}((a, b) \cap M)$ does not contain (a, b).
We will play with this gap. By the way, this gap does not exists in case of $\mathbb{R}$ and it explains why a similar argument does not work for $\mathbb{R}$.

Notations

Definition

Let $J(K, M)$ be the set of all $x \in K$ such that $x \notin \mathrm{Cl}(K \cap M)$, and there exist $a, b \in K \cap M$ such that $a \leq x \leq b$.

By the previous slide, for every $a, b \in K$ with $a < b$, there exists $x \in J(K, M)$ such that $a < x < b$.

Notations

Definition

Let $J(K, M)$ be the set of all $x \in K$ such that $x \notin \mathrm{Cl}(K \cap M)$, and there exist $a, b \in K \cap M$ such that $a \leq x \leq b$.

By the previous slide, for every $a, b \in K$ with $a < b$, there exists $x \in J(K, M)$ such that $a < x < b$.

Notations

Definition

Let $J(K, M)$ be the set of all $x \in K$ such that $x \notin \mathrm{Cl}(K \cap M)$, and there exist $a, b \in K \cap M$ such that $a \leq x \leq b$.

By the previous slide, for every $a, b \in K$ with $a < b$, there exists $x \in J(K, M)$ such that $a < x < b$.

Notations(cont)

Definition

$$\begin{aligned}
\eta(K, M, x) &= \sup \{ y \in \mathrm{Cl}(K \cap M) : y \leq x \} \\
\zeta(K, M, x) &= \inf \{ y \in \mathrm{Cl}(K \cap M) : y \geq x \} \\
I(K, M, x) &= [\eta(K, M, x), \zeta(K, M, x)] \\
C(K, M, x) &= \{ \eta(K, M, x), \zeta(K, M, x) \}
\end{aligned}$$

if they exist.

Note that if $x \in J(K, M)$, then all of them exist.
I is for the "interval", and C is for the "corner".
When K or M are clear from the context, we omit them.

Notations(cont)

Definition

$$
\begin{aligned}
\eta(K, M, x) &= \sup \{ y \in \mathrm{Cl}(K \cap M) : y \leq x \} \\
\zeta(K, M, x) &= \inf \{ y \in \mathrm{Cl}(K \cap M) : y \geq x \} \\
I(K, M, x) &= [\eta(K, M, x), \zeta(K, M, x)] \\
C(K, M, x) &= \{ \eta(K, M, x), \zeta(K, M, x) \}
\end{aligned}
$$

if they exist.

Note that if $x \in J(K, M)$, then all of them exist.

I is for the “interval”, and C is for the “corner”.

When K or M are clear from the context, we omit them.

Notations(cont)

Definition

$$\begin{aligned}
\eta(K, M, x) &= \sup\{\, y \in \mathrm{Cl}(K \cap M) : y \leq x \,\} \\
\zeta(K, M, x) &= \inf\{\, y \in \mathrm{Cl}(K \cap M) : y \geq x \,\} \\
I(K, M, x) &= [\eta(K, M, x), \zeta(K, M, x)] \\
C(K, M, x) &= \{\, \eta(K, M, x), \zeta(K, M, x) \,\}
\end{aligned}$$

if they exist.

Note that if $x \in J(K, M)$, then all of them exist.
I is for the “interval”, and C is for the “corner”.
When K or M are clear from the context, we omit them.

Notations(cont)

Definition

$$
\begin{aligned}
\eta(K, M, x) &= \sup \{ y \in \mathrm{Cl}(K \cap M) : y \leq x \} \\
\zeta(K, M, x) &= \inf \{ y \in \mathrm{Cl}(K \cap M) : y \geq x \} \\
I(K, M, x) &= [\eta(K, M, x), \zeta(K, M, x)] \\
C(K, M, x) &= \{ \eta(K, M, x), \zeta(K, M, x) \}
\end{aligned}
$$

if they exist.

Note that if $x \in J(K, M)$, then all of them exist.
I is for the "interval", and C is for the "corner".
When K or M are clear from the context, we omit them.

A little lemma

Lemma

Let $\hat{x} \in J(K, M)$. Then, either $\eta(\hat{x}) \notin M$ or $\zeta(\hat{x}) \notin M$.

Proof.

Suppose that both $\eta(\hat{x}) \in M$ and $\zeta(\hat{x}) \in M$. Since K is dense in itself and M is an elementary submodel, there exists $x \in M$ such that $\eta(\hat{x}) < x < \zeta(\hat{x})$. If $x < \hat{x}$, then by the definition $\eta(\hat{x}) \geq x$. This is a contradiction. Similarly if $x > \hat{x}$. $\square$

Outline

Max and min at the corners

Now consider two connected nowhere real linearly ordered spaces K and L, a continuous function $g : K \to L$, and a suffienctly good countable elementsry suubmodel M with $K, L, g \in M$.

Theorem

Let $\hat{x} \in J(K, M)$. Then, $g \restriction I(\hat{x})$ has maximum and minimum at either $\eta(\hat{x})$ or $\zeta(\hat{x})$. In particular, if $g(\eta(\hat{x})) = g(\zeta(\hat{x}))$, then g is constant on $I(\hat{x})$.

Max and min at the corners

Now consider two connected nowhere real linearly ordered spaces K and L, a continuous function $g : K \to L$, and a suffienctly good countable elementsry suubmodel M with $K, L, g \in M$.

Theorem

Let $\hat{x} \in J(K, M)$. Then, $g \upharpoonright I(\hat{x})$ has maximum and minimum at either $\eta(\hat{x})$ or $\zeta(\hat{x})$. In particular, if $g(\eta(\hat{x})) = g(\zeta(\hat{x}))$, then g is constant on $I(\hat{x})$.

Max and min at the corners

Now consider two connected nowhere real linearly ordered spaces K and L, a continuous function $g : K \to L$, and a suffienctly good countable elementsry suubmodel M with $K, L, g \in M$.

Theorem

Let $\hat{x} \in J(K, M)$. Then, $g \restriction I(\hat{x})$ has maximum and minimum at either $\eta(\hat{x})$ or $\zeta(\hat{x})$. In particular, if $g(\eta(\hat{x})) = g(\zeta(\hat{x}))$, then g is constant on $I(\hat{x})$.

No elements of M in $g^{\rightarrow} I(\hat{x})$

Theorem

Let $\hat{x} \in J(K, M)$ with $g(\hat{x}) \in M$. Then, g is constant on $I(\hat{x})$.

Theorem

Let $\hat{x} \in J(K, M)$ with $g(\hat{x}) \in J(L, M)$. Then,

$$\{ g(\eta(\hat{x})), g(\zeta(\hat{x})) \} = \{ \eta(g(\hat{x})), \zeta(g(\hat{x})) \}$$

So, the behavior of g is very restricted by M.

No elements of M in $g^{\rightarrow} I(\hat{x})$

Theorem

Let $\hat{x} \in J(K, M)$ with $g(\hat{x}) \in M$. Then, g is constant on $I(\hat{x})$.

Theorem

Let $\hat{x} \in J(K, M)$ with $g(\hat{x}) \in J(L, M)$. Then,

$$\{ g(\eta(\hat{x})), g(\zeta(\hat{x})) \} = \{ \eta(g(\hat{x})), \zeta(g(\hat{x})) \}$$

So, the behavior of g is very restricted by M.

No elements of M in $g^{\rightarrow} I(\hat{x})$

Theorem

Let $\hat{x} \in J(K, M)$ with $g(\hat{x}) \in M$. Then, g is constant on $I(\hat{x})$.

Theorem

Let $\hat{x} \in J(K, M)$ with $g(\hat{x}) \in J(L, M)$. Then,

$$\{ g(\eta(\hat{x})), g(\zeta(\hat{x})) \} = \{ \eta(g(\hat{x})), \zeta(g(\hat{x})) \}$$

So, the behavior of g is very restricted by M.

Outline

1. Question of Eda
   - $\mathbb{R}^2$
   - Linearly ordered topological spaces
   - Theorem of Eda and Kamijo
   - Question
2. A little flavor of logic
3. Proof
   - K
   - $g : K \to L$
   - $f : K_0 \times K_1 \to L$
   - $f : K_0 \times K_1 \to L_0 \times L_1$
4. Concluding remarks
   - Corollaries
   - Open Problems

Product

Now suppose

- K_0, K_1, L are connected nowhere real linearly ordered spaces,
- $f : K_0 \times K_1 \to L$ is a continuous function, and
- M is a sufficiently good countable elementary submodl with $K_0, K_1, L, f \in M$.

Notation

Definition

$$I\left(K_0 \times K_1, M, \langle x, y\rangle\right) = I(\hat{x}) \times I(\hat{y})$$
$$C\left(K_0 \times K_1, M, \langle x, y\rangle\right) = C(\hat{x}) \times C(\hat{y})$$

When $\hat{y} \in M$, ...

Lemma

Let $\hat{x} \in J(K_0, M)$ and $\hat{y} \in M$. Then,

1. *$f \restriction (I(\hat{x}) \times \{ \hat{y} \})$ has maximum and minimum at either $\langle \eta(\hat{x}), \hat{y} \rangle$ or $\langle \zeta(\hat{x}), \hat{y} \rangle$,*
2. *if $f(\hat{x}, \hat{y}) \in M$, then $f \restriction (I(\hat{x}) \times \{ \hat{y} \})$ is constant, and*
3. *if $f(\hat{x}, \hat{y}) \in J(L, M)$, then*

$$f^{\rightarrow}(I(\hat{x}) \times \{ \hat{y} \}) = I(f(\hat{x}, \hat{y}))$$
$$\{ f(\eta(\hat{x}), \hat{y}), f(\zeta(\hat{x}), \hat{y}) \} = \{ \eta(f(\hat{x}, \hat{y})), \zeta(f(\hat{x}, \hat{y})) \}$$

Proof.

Note that $f \restriction (K_0 \times \{ \hat{y} \})$ is in M. Apply the result in the previous section. □

When $\hat{y} \in M$, ...

Lemma

Let $\hat{x} \in J(K_0, M)$ and $\hat{y} \in M$. Then,

1. *$f \restriction (I(\hat{x}) \times \{ \hat{y} \})$ has maximum and minimum at either $\langle \eta(\hat{x}), \hat{y} \rangle$ or $\langle \zeta(\hat{x}), \hat{y} \rangle$,*
2. *if $f(\hat{x}, \hat{y}) \in M$, then $f \restriction (I(\hat{x}) \times \{ \hat{y} \})$ is constant, and*
3. *if $f(\hat{x}, \hat{y}) \in J(L, M)$, then*

$$f^{\to}(I(\hat{x}) \times \{ \hat{y} \}) = I(f(\hat{x}, \hat{y}))$$

$$\{ f(\eta(\hat{x}), \hat{y}), f(\zeta(\hat{x}), \hat{y}) \} = \{ \eta(f(\hat{x}, \hat{y})), \zeta(f(\hat{x}, \hat{y})) \}$$

Proof.

Note that $f \restriction (K_0 \times \{ \hat{y} \})$ is in M. Apply the result in the previous section. □

When $\hat{y}$ is a limit point, ...

Lemma

Let $\hat{x} \in J(K_0, M)$ and $\hat{y}$ is a limit point of $K_1 \cap M$. Suppose $f(\hat{x}, \hat{y}) \in J(L, M)$. Then,

$$f^{\to}(I(\hat{x}) \times \{\hat{y}\}) = I(f(\hat{x}, \hat{y}))$$

$$\{f(\eta(\hat{x}), \hat{y}), f(\zeta(\hat{x}), \hat{y})\} = \{\eta(f(\hat{x}, \hat{y})), \zeta(f(\hat{x}, \hat{y}))\}$$

Proof.

If $y \in K_1 \cap M$ and $f(\hat{x}, y) \in I(f(\hat{X}, \hat{y}))$, then

$$\begin{aligned}\{f(\eta(\hat{x}), y), f(\zeta(\hat{x}), y)\} &= \{\eta(f(\hat{x}, y)), \zeta(f(\hat{x}, y))\}\\ &= \{\eta(f(\hat{x}, \hat{y})), \zeta(f(\hat{x}, \hat{y}))\}\end{aligned}$$

Note that $\hat{y}$ is a limit point of such y's. □

When $\hat{y}$ is a limit point, ...

Lemma

Let $\hat{x} \in J(K_0, M)$ and $\hat{y}$ is a limit point of $K_1 \cap M$. Suppose $f(\hat{x}, \hat{y}) \in J(L, M)$. Then,

$$f^{\to}(I(\hat{x}) \times \{\hat{y}\}) = I(f(\hat{x}, \hat{y}))$$
$$\{ f(\eta(\hat{x}), \hat{y}), f(\zeta(\hat{x}), \hat{y}) \} = \{ \eta(f(\hat{x}, \hat{y})), \zeta(f(\hat{x}, \hat{y})) \}$$

Proof.

If $y \in K_1 \cap M$ and $f(\hat{x}, y) \in I(f(\hat{X}, \hat{y}))$, then

$$\begin{aligned} \{ f(\eta(\hat{x}), y), f(\zeta(\hat{x}), y) \} &= \{ \eta(f(\hat{x}, y)), \zeta(f(\hat{x}, y)) \} \\ &= \{ \eta(f(\hat{x}, \hat{y})), \zeta(f(\hat{x}, \hat{y})) \} \end{aligned}$$

Note that $\hat{y}$ is a limit point of such y's. □

When $\hat{y}$ is a limit point, ...

Lemma

Let $\hat{x} \in J(K_0, M)$ and $\hat{y}$ is a limit point of $K_1 \cap M$. Suppose $f(\hat{x}, \hat{y}) \in J(L, M)$. Then,

$$f^{\to}(I(\hat{x}) \times \{\hat{y}\}) = I(f(\hat{x}, \hat{y}))$$
$$\{f(\eta(\hat{x}), \hat{y}), f(\zeta(\hat{x}), \hat{y})\} = \{\eta(f(\hat{x}, \hat{y})), \zeta(f(\hat{x}, \hat{y}))\}$$

Proof.

If $y \in K_1 \cap M$ and $f(\hat{x}, y) \in I(f(\hat{X}, \hat{y}))$, then

$$\begin{aligned}\{f(\eta(\hat{x}), y), f(\zeta(\hat{x}), y)\} &= \{\eta(f(\hat{x}, y)), \zeta(f(\hat{x}, y))\} \\ &= \{\eta(f(\hat{x}, \hat{y})), \zeta(f(\hat{x}, \hat{y}))\}\end{aligned}$$

Note that $\hat{y}$ is a limit point of such y's. $\square$

Outline

Now suppose

- K_0, K_1, L_0, L_1 are connected nowhere real linearly ordered spaces,
- $f : K_0 \times K_1 \to L_0 \times L_1$ is a continuous injection, and
- M is a sufficiently good countable elementary submodl with $K_0, K_1, L_0, L_1, f \in M$.

$f : K_0 \times K_1 \to L_0 \times L_1$

Now suppose

- K_0, K_1, L_0, L_1 are connected nowhere real linearly ordered spaces,
- $f : K_0 \times K_1 \to L_0 \times L_1$ is a continuous injection, and
- M is a sufficiently good countable elementary submodl with $K_0, K_1, L_0, L_1, f \in M$.

$f : K_0 \times K_1 \to L_0 \times L_1$

Now suppose

- K_0, K_1, L_0, L_1 are connected nowhere real linearly ordered spaces,
- $f : K_0 \times K_1 \to L_0 \times L_1$ is a continuous injection, and
- M is a sufficiently good countable elementary submodl with $K_0, K_1, L_0, L_1, f \in M$.

Yet another analogous lemma

Lemma

Let $\hat{x} \in J(K_0, M)$ and $\hat{y} \in J(K_1, M)$. Then,

$$f^{\to}(I(\hat{x}, \hat{y})) = I(f(\hat{x}, \hat{y}))$$
$$f^{\to}(C(\hat{x}, \hat{y})) = C(f(\hat{x}, \hat{y}))$$

Here, we need the assumption that f is injective.

Yet another analogous lemma

Lemma

Let $\hat{x} \in J(K_0, M)$ and $\hat{y} \in J(K_1, M)$. Then,

$$f^{\rightarrow}(I(\hat{x}, \hat{y})) = I(f(\hat{x}, \hat{y}))$$
$$f^{\rightarrow}(C(\hat{x}, \hat{y})) = C(f(\hat{x}, \hat{y}))$$

Here, we need the assumption that f is injective.

$f \restriction (\eta(\hat{x}), K_1)$ is (almost) constant

Lemma

Let $\hat{x} \in J(K_0, M)$*. Then, there exists* $i \in \{0, 1\}$ *such that* $f \restriction (\eta(\hat{x}) \times [\inf(K_1 \cap M), \sup(K_1 \cap M)])$ *is constant.*

Wrapping all up

By changing M and a little more argument, we can show that f is coordinate-wise.

Theorem

Let $K_0, K_1, \ldots, K_{n-1}, L_0, L_1, \ldots, L_{n-1}$ be connected nowhere real linearly ordered spaces. Let $f : \prod_{i<n} K_i \to \prod_{j<n} L_j$ be a continuous injective function. Then, f is coordinate-wise.

Wrapping all up

By changing M and a little more argument, we can show that f is coordinate-wise.

Theorem

Let $K_0, K_1, \ldots, K_{n-1}, L_0, L_1, \ldots, L_{n-1}$ be connected nowhere real linearly ordered spaces. Let $f : \prod_{i<n} K_i \to \prod_{j<n} L_j$ be a continuous injective function. Then, f is coordinate-wise.

Outline

Corollary 1.

Corollary

Let K_0, K_1, L_0, L_1 be connected nowhere real linearly ordered spaces. Suppose that $K_0 \times K_1$ and $L_0 \times L_1$ are homeomorphic. Then, either

- *K_0 and L_0 are homeomorphic and K_1 and L_1 are homeomorphic, or*
- *K_0 and L_1 are homeomorphic and K_1 and L_0 are homeomorphic.*

Corollary 1.

Corollary

Let K_0, K_1, L_0, L_1 be connected nowhere real linearly ordered spaces. Suppose that $K_0 \times K_1$ and $L_0 \times L_1$ are homeomorphic. Then, either

- *K_0 and L_0 are homeomorphic and K_1 and L_1 are homeomorphic, or*
- *K_0 and L_1 are homeomorphic and K_1 and L_0 are homeomorphic.*

Corollary 1.

Corollary

Let K_0, K_1, L_0, L_1 be connected nowhere real linearly ordered spaces.Suppose that $K_0 \times K_1$ and $L_0 \times L_1$ are homeomorphic.Then, either

- *K_0 and L_0 are homeomorphic and K_1 and L_1 are homeomorphic, or*
- *K_0 and L_1 are homeomorphic and K_1 and L_0 are homeomorphic.*

Corollary 2.

Corollary

Let K be a connected nowhere real linearly ordered spaces. Then, there is no group operation that makes K a topological group.

Proof.

If K is a topological group, then $\langle x, y\rangle \mapsto \langle x, x \cdot y\rangle$ is a homeomorphism that is not coordinate-wise. This is a contradiction. □

Question

Is it known? More direct proof?

Corollary 2.

Corollary

Let K be a connected nowhere real linearly ordered spaces. Then, there is no group operation that makes K a topological group.

Proof.

If K is a topological group, then $\langle x, y\rangle \mapsto \langle x, x \cdot y\rangle$ is a homeomorphism that is not coordinate-wise. This is a contradiction. □

Question

Is it known? More direct proof?

Corollary 2.

Corollary

Let K be a connected nowhere real linearly ordered spaces. Then, there is no group operation that makes K a topological group.

Proof.

If K is a topological group, then $\langle x, y \rangle \mapsto \langle x, x \cdot y \rangle$ is a homeomorphism that is not coordinate-wise. This is a contradiction. $\square$

Question

Is it known? More direct proof?

Outline

Open problems

Problem

Can we extend this to more spaces? Maybe one-dimensional in some sense?

Problem

Can other properties of $\mathbb{R}$ be extended to other connected linearly ordered spaces?

Open problems

Problem

Can we extend this to more spaces? Maybe one-dimensional in some sense?

Problem

Can other properties of $\mathbb{R}$ be extended to other connected linearly ordered spaces?

Open problems

Problem

Can we extend this to more spaces? Maybe one-dimensional in some sense?

Problem

Can other properties of $\mathbb{R}$ be extended to other connected linearly ordered spaces?