THE FINITE PRODUCT OF CONNECTED LINEARLY ORDERED SETS CANNOT BE EMBEDDED INTO THE PRODUCT OF A SMALLER DIMENSION

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1. Introduction

L. E. J. Brouwer [1] proved the following theorem, called invariance of domain.

Theorem 1.1. *Every continuous injective function from an open subset of* $\mathbb{R}^n$ *into* $\mathbb{R}^n$ *is an open map.*

From this theorem, we can show the following corollary.

Corollary 1.2. *Let* $m < n$ *be positive integers. Then, there is no continuous injective function from an open subset of* $\mathbb{R}^n$ *into* $\mathbb{R}^m$*.*

The main purpose of this paper is to prove Theorem 5.1, which extends this result to the finite product of connected linearly ordered topological spaces(LOTS). Namely, we shall prove that for all positive integers $m < n$, if K_i is a connected LOTS with at least two points for all $i < n$ and L_j is a connected LOTS for all $j < m$, then there is no continuous injective function from an open subset of $\prod_{i<n} K_i$ into $\prod_{j<m} L_j$.

It is known that the finite product of nonseparable LOTS behaves differently from the Euclidean spaces. After For example, L. B. Treybig [9] proved the following theorem. Note that some partial results were obtained by D. Kurepa [5] and by S. Mardešić and P. Papić [7].

Theorem 1.3 (L. B. Treybig [9])**.** *If* X *and* Y *are infinite Hausdorff spaces and* $X \times Y$ *is a continuous image of a compact LOTS, then both* X *and* Y *are metrizable.*

Notice that in the conclusion of this theorem, X and Y are compact and metrizable and hence separable. A consequence of this theorem is that there is no analogue of a space-filling curve on the Euclidean space.

The following generalization of Theorem 1.3 was conjectured by S.Mardešić in [6] and proved by G. Martínez-Cervantes and G. Plebanek in [8].

Theorem 1.4. *Let* d *and* s *be positive integers. If* K_i *is a compact LOTS for each* $i < d$*,* Z_j *is an infinite Hausdorff space for each* $j < d + s$*, and there exists a continuous surjective function from* $\prod_{i<d} K_i$ *onto* $\prod_{<d+s} Z_j$*,*

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then there exists at least $(s+1)$-many indexes $j < d+s$ such that Z_j is metrizable.

The author showed an improvement of this theorem in which, for example, K_i is merely assumed to be countably compact instead of compact.

The following theorem, proved by the author [2], demonstrates another difference between the behaviors of the Euclidean spaces and the finite product of nonseparable LOTS. As usual in set theory, a natural number n is identified with the set of all natural numbers less than n, i.e. $n = \{0, 1, 2, \ldots, n-1\}$. In this paper, an n-tuple is identified with a function whose domain is n. Namely, $\langle x_0, x_1, \ldots, x_{n-1}\rangle$ is the function x with $\mathrm{dom}(x) = n$ such that $x(i) = x_i$ for each $i < n$.

Theorem 1.5 (T. Ishiu [2])**.** *Let n be a positive integer. Let K_i and L_i be connected nowhere separable LOTS for each $i < n$. If $f : \prod_{i<n} K_i \to \prod_{i<n} L_i$ is a continuous injective function, then f is coordinate-wise, namely, there exists a bijection $h : n \to n$ and a function $\tau_i : K_{h(i)} \to L_i$ for each $i < n$ such that for all $x \in \prod_{i<n} K_i$ and $i < n$,*

$$f(x)(i) = \tau_i(x(h(i)))$$

Here, a topological space is nowhere separable if and only if there is no separable nonempty open subspace.

So, many properties of the Euclidean spaces cannot be generalized to the finite product of (connected) LOTS. However, the author [4] discovered that Brouwer's fixed-point theorem and the Poincaré-Miranda theorem can be generalized as follows.

Theorem 1.6 (T. Ishiu [4])**.** *Let n be a positive integer. Let K_i be a compact connected linearly ordered topological space for each $i < n$. Then, every continuous function $f : \prod_{i<n} K_i \to \prod_{i<n} K_i$ has a fixed point, namely there exists an $x \in \prod_{i<n} K_i$ such that $f(x) = x$.*

Theorem 1.7 (T. Ishiu [4])**.** *Let n be a positive integer. Let K_i and L_i be compact connected LOTS for each $i < n$. Define $\vec{K} = \prod_{i<n} K_i$, $\vec{L} = \prod_{i<n} L_i$, $a = \min \vec{K}$, and $b = \max \vec{K}$. Let $f : \vec{K} \to \vec{L}$ be a continuous function. Let $z \in \vec{L}$ be so that for every $i < n$ and $x \in \vec{K}$,*

- *if $x(i) = a(i)$, then $f(x)(i) < z$, and*
- *if $x(i) = b(i)$, then $f(x)(i) > z$*

Then $z \in \mathrm{ran}(f)$.

Theorem 5.1 is another example of the theorems on the Euclidean spaces that can be generalized. Notice that when each K_i and L_i are nowhere separable, it is an easy corollary of Theorem 1.5 though it holds in a very different way from the Euclidean spaces. So, the main theorem is an amalgamation of quite different phenomena.

The proof of Theorem 5.1 uses the machinery developed in [4]. It is based on the same ideas as in [2] and [3], but it is modified so that it works better for connected LOTS that are not necessarily nowhere separable. In particular, we define equivalence relations $\sim_{K,M}$ and $\sim_{\vec{K},M}$, and the definition of $C(K, M, x)$ is different. By using this method, we can split a function from the finite product of connected LOTS to another finite product of connected LOTS into two parts, one that is essentially a function on Euclidean space and one that is similar to a coordinate-wise function.

We conjecture that the invariance of domain theorem can be extended to the finite product of connected LOTS. While the main theorem is weaker than this conjecture, we hope that it helps resolve this question.

2. Preliminary

We shall define the following notation to concisely describe a sufficiently large regular cardinal.

Definition 2.1. Let $S_0, \dots, S_{n-1}$ be any set. Let $\theta_{S_0,\dots,S_{n-1}}$ be the least regular cardinal θ such that $\mathcal{P}(S_0 \cup S_1 \cup \cdots \cup S_{n-1}) \in H(\theta)$.

The following properties can be easily shown.

Fact 2.2. Let K be a LOTS.

- (i) If K is connected, then K is self-dense and has the least upper-bound property.
- (ii) If K is separable and connected, then K is order-isomorphic to a convex subset of $\mathbb{R}$, i.e. one of $\{0\}$, $[0, 1]$, $[0, 1)$, $(0, 1]$, and $(0, 1)$.
- (iii) If K is connected and has both endpoints, then K is compact.
- (iv) If K is compact, then K has both endpoints.

The following lemma allows us to avoid notational complications.

Lemma 2.3. *Let K be a connected LOTS. Then, there exists a connected LOTS K' extending K such that K' has neither a minimum nor maximum element.*

Proof. Let K be a connected LOTS.

We shall first build a connected LOTS K' extending K such that K' has no minimum element. If K has no minimum element, then let $K' = K$. Suppose K has a minimum element. Let K' be the concatenation of $\mathbb{R}$ and K. Then, it is easy to observe that K' is a connected LOTS without a minimum element.

Then, we shall build a connected LOTS K'' extending K' such that K' has no minimum or maximum element. If K' has no maximum element, then let $K'' = K'$. Suppose that K' has a maximum element. Let K'' be the concatenation of K' and $\mathbb{R}$. Then, K'' is a connected LOTS extending K' that has no minimum or maximum element. Thus, K'' is as we desired. □

We start discussing the product of LOTS. We shall use the following multi-dimensional order and

Definition 2.4. Let n be positive integer, and let K_i be a linearly ordered topological space for each $i < n$. Let $x, y \in \prod_{i<n} K_i$. Define

(i) $x < y$ if and only if for every $i < n$, $x(i) < y(i)$
(ii) $x \leq y$ if and only if for every $i < n$, $x(i) < y(i)$,
(iii) $(x, y) = \{ z \in \prod_{i<n} K_i \mid x < z < y \}$, and
(iv) $[x, y] = \{ z \in \prod_{i<n} K_i \mid x \leq z \leq y \}$.

We shall use π_j to mean the projection to j-th coordinate, i.e.,

$$\pi_j(\langle x_0, x_1, \ldots, x_{n-1} \rangle) = x_j$$

Definition 2.5. Let n be positive integer, and let K_i be a linearly ordered topological space for each $i < n$. Let $i < n$, and $v \in K_i$. We shall define $\sigma(i, v)$ to be the function from $\vec{K}$ into $\vec{K}$ defined by for every $j < n$,

$$\sigma(i, v)(x)(j) = \begin{cases} x(j) & \text{if } j \neq i \\ v & \text{if } j = i \end{cases}$$

So, $\sigma(i, v)(x)$ is obtained by replacing or adding the i-th component of x with v.

The following fact, which can be interpreted as the intermediate value theorem with multi-dimensional domain, was proved in [4].

Fact 2.6. Let n be a positive integer, K_i a connected LOTS for each $i < n$, and L a connected LOTS. Let $a, b \in \prod_{i<n} K_I$ with $a < b$ and $g : [a, b] \to L$ a continuous function. Then, $g^{\to}[a, b]$ is convex.

3. The condensation

We shall rely on the lemmas proved by the author in [4]. In this section, we shall define notations and present the results.

3.1. **A single LOTS.**

In this subsection, we fix

- a compact connected LOTS K with at least two points,
- a regular cardinal $\theta \geq \theta_K$, and
- a countable elementary substructure M of $H(\theta)$ with $K \in M$.

Definition 3.1. Define a relation $\sim_{K,M}$ on K by for all $a, b \in K$, $a \sim_{K,M}$ if and only if

- if $a \leq b$, then $[a, b] \cap M$ is finite, and
- if $b < a$, then $[b, a] \cap M$ is finite.

When K and M are clear from the context, we shall write $\sim$ to mean $\sim_{K,M}$.

$\sim_{K,M}$ is shown to be an equivalence relation in [4]. As usual, we write $[a]_{K,M}$ to mean the $\sim_{K,M}$-equivalence class of a. We shall write $[a]$ for $[a]_{K,M}$ when K and M are clear from the context. It is easy to observe that for every $a \in K$, $[a]$ is a convex subset of K.

Definition 3.2. Define $\varphi(K, M) = \{ [a] \mid a \in K \}$. We shall define a linear order on $\varphi(K, M)$ by $[a] \leq [b]$ if and only if $a \leq b$. This is well-defined as $[a]$ and b are convex. The topology on $\varphi(K, M)$ is given by the order topology.

The following facts were proved in [4].

Fact 3.3 (T. Ishiu [4])**.** For every $x \in K$,

- $|[x] \cap M| \leq 1$,
- $[x]$ is closed and hence compact in K,
- if $\min[x]$ is not a minimum element of $K \cap M$, then $\min[x]$ is a limit point of $K \cap M$ from below, and
- if $\max[x]$ is not a maximum element of $K \cap M$, then $\max[x]$ is a limit point of $K \cap M$ from above.

$\varphi(K, M)$ has several nice properties.

Fact 3.4 (T. Ishiu [4])**.**

- $\{ [x] \mid x \in K \cap M \}$ is dense in $\varphi(K, M)$. In particular, $\varphi(K, M)$ is separable.
- $\varphi(K, M)$ is connected.
- $\varphi(K, M)$ is order-isomorphic to $[0, 1]$.

Definition 3.5. Let $x \in K$. We shall define $C(x)$ to be the set of all $c \in K$ such that for each $i < n$, $c = \min[x]$, $c = \max([x])$, or $c \in [x] \cap M$.

The following lemma is not proved in [4], but will be used in the following sections.

Lemma 3.6. *For every $a, b \in K \cap M$, if $a < b$, there exists $x \in K \cap M$ such that $a \leq x \leq b$ and x is a limit point of $K \cap M$ from below. The same statement holds when we replace 'from below' by 'from above'.*

Proof. Let $a, b \in K \cap M$ with $a < b$. Since K is connected, it is easy to build an increasing sequence $\langle x_k \mid k < \omega \rangle$ in (a, b). Without loss of generality, we may assume $\langle x_k \mid k < \omega \rangle \in M$. Let $x = \sup \{ x_k \mid k < \omega \}$. Trivially, x belongs to M and is a limit point of $K \cap M$ from below. Similarly, when we replace 'from below' by 'from above'. $\square$

Lemma 3.7.

(i) Let $x \in K$ be a limit point of $K \cap M$ from below, and $a < x$. Then, there exists $a' \in K \cap M$ such that $a < a' < x$ and a' is a limit point of $K \cap M$ from below.

(ii) Let $x \in K$ be a limit point of $K \cap M$ from above, and $b > x$. Then, there exists $b' \in K \cap M$ such that $x < b' \leq \max[b'] < b$.

Proof. We shall show (i). (ii) can be proved by a similar method.

Let $x \in K$ be a limit point of $K \cap M$ from below and $a < x$. Then, there exist $a_1, a_2 \in K \cap M$ such that $a < a_1 < a_2 < x$. By Lemma 3.6, there exists $a' \in K \cap M$ such that $a_1 < a' < a_2$ and a' is a limit point of $K \cap M$ from below. $\square$

3.2. **Finite products.** In this subsection, we fix

- a positive integer n,
- a compact connected LOTS K_i for each $i < n$,
- a regular cardinal $\theta \geq \theta_{\langle K_i | i<n \rangle}$, and
- a countable elementary substructure M of $H(\theta)$ with $\langle K_i \mid i < n\rangle \in M$.

Let $\vec{K} = \prod_{i<n} K_i$.

Definition 3.8. Define a relation $\sim_{\vec{K},M}$ on $\vec{K}$ by $x \sim_{\vec{K},M} y$ if and only if for every $i < n$, $x(n) \sim_{K_i,M} y(i)$. Clearly, $\sim_{\vec{K},M}$ is an equivalence relation on $\vec{K}$. For every $x \in \vec{K}$, let $[x]_{\vec{K},M}$ be the $\sim_{\vec{K},M}$-equivalence class of x. When $\vec{K}$ and M are clear from the context, we shall omit them and write $\sim$ and $[x]$.

Let $\varphi(\vec{K}, M)$ be the set of all $\sim_{\vec{K},M}$-equivalence classes, i.e., $\varphi(\vec{K}, M) = \{ [x] \mid x \in \vec{K} \}$. We can identify $\varphi(\vec{K}, M)$ with $\prod_{i<n} \varphi(K_i, M)$. The topology of $\varphi(\vec{K}, M)$ is the product topology on $\prod_{i<n} \varphi(K_i, M)$.

When $\vec{K}$ and M are clear from the context, we shall omit them.

By Fact 3.4, we can see the following.

Fact 3.9 (T. Ishiu [4])**.** $\varphi(\vec{K}, M)$ is homeomorphic to $[0, 1]^n$.

Definition 3.10. For every $x \in \vec{K}$, we define $C(\vec{K}, M, x) = \prod_{i<n} C(x(i))$. When $\vec{K}$ and M are clear from the context, we shall omit them.

The author [4] proved $C(x)$ has the following characterization.

Fact 3.11 (T. Ishiu [4])**.** For every $x \in \vec{K}$, $C(x) = [x] \cap \mathrm{cl}(\vec{K} \cap M)$.

3.3. **A function.** In this subsection, we fix

- two positive integers n and m,
- compact connected LOTS K_i for each $i < n$,
- compact connected LOTS L_j for each $j < m$
- a continuous function$f : \prod_{i<n} K_i \to \prod_{j<m} L_j$,
- a regular cardinal $\theta \geq \theta_{\langle K_i|i<n\rangle,\langle L_j|j<m\rangle,f}$, and
- a countable elementary substructure M of $H(\theta)$ with $\langle K_i \mid i < n\rangle, \langle L_j \mid j < m\rangle, f \in M$.

Let $\vec{K} = \prod_{i<n} K_i$ and $\vec{L} = \prod_{j<m} L_j$.

The following two facts were proved in [4].

Fact 3.12 (T. Ishiu [4])**.** For every $x \in \vec{K}$ and $j < m$, $\min{(\pi_j \circ f)}^{\to}[x] = \min{(\pi_j \circ f)}^{\to}C(x)$ and $\max{(\pi_j \circ f)}^{\to}[x] = \max{(\pi_j \circ f)}^{\to}C(x)$

Fact 3.13 (T. Ishiu [4])**.** For each $x \in \vec{K}$, $f^{\to}[x] \subseteq [f(x)]$.

Definition 3.14. Define a function $\varphi(f, M) : \varphi(\vec{K}, M) \to \varphi(\vec{L}, M)$ by

$$\varphi(f, M)(x) = [f(x)]$$

By the previous fact, this function is well-defined.

$\varphi(f, M)$ is shown to have nice properties.

Fact 3.15 (T. Ishiu [4]).

- $\varphi(f, M)$ is continuous.
- $\varphi(\vec{K}, M)$ and $\varphi(\vec{L}, M)$ are isomorphic to $[0,1]^n$, and
- $\varphi(f, M)$ can be identified with a continuous function from $[0,1]^n$ into $[0,1]^n$.

4. New lemmas

In this section, we shall prove several lemmas that have not been proved in [4]. Throughout this section, we fix

- Two positive integers n and m,
- a compact connected LOTS K_i for each $i < n$,
- a compact connected LOTS L_j for each $j < m$,
- a function $f : \prod_{i<n} K_i \to \prod_{j<m} L_j$,
- a regular cardinal $\theta \geq \theta_{\langle K_i | i<n\rangle, \langle L_j | j<m\rangle, f}$, and
- a countable elementary substructure M of $H(\theta)$ with $\langle K_i \mid i < n\rangle, \langle L_j \mid j < m\rangle, f \in M$

Let $\vec{K} = \langle K_i \mid i < n\rangle$ and $\vec{L} = \langle L_j \mid j < m\rangle$.

Lemma 4.1. *For every $x \in \vec{K}$, $f^{\to}C(x) \subseteq C(f(x))$*

Proof. Let $c \in C(x)$. By Fact 3.13, since $c \sim x$, we have $f(c) \sim f(x)$, i.e. $f(c) \in [f(x)]$. By Lemma 3.11, we have $c \in \mathrm{cl}(\vec{K} \cap M)$. Since f is continuous and injective, we have $f(c) = \mathrm{cl}(f^{\to}(\vec{K} \cap M)) \subseteq \mathrm{cl}(\vec{L} \cap M)$. Thus, $f(c) \in [f(x)] \cap \mathrm{cl}(\vec{L} \cap M)$ By Lemma 3.11 we have $f(c) \in C(f(x))$. □

Lemma 4.2. *For every $\bar{y} \in \varphi(\vec{L}, M)$, $\varphi(f, M)^{\leftarrow}\bar{y}$ is finite.*

Proof. Suppose not, i.e., $\varphi(f, M)^{\leftarrow}\bar{y}$ is infinite. Let $\langle \bar{x}_k \mid k < \omega\rangle$ be a distinct enumeration of $\varphi(f, M)^{\leftarrow}\bar{y}$. For each $k < \omega$, let $x_k \in \vec{K}$ be so that $[x_k] = \bar{x}_k$. By Lemma 4.1, we have $f^{\to}C(x_k) \subseteq C(f(x))$. Since $C(f(x))$ is finite, there exist $k, l < \omega$ such that $f^{\to}C(x_k) \cap f^{\to}C(x_l) \neq \varnothing$. So, there exist $c \in C(x_k)$ and $c' \in C(x_l)$ such that $f(c) = f(c')$. Since f is injective, it implies $c = c'$. Thus, $x_k \sim x_l$ and hence $\bar{x}_k = [x_k] = [x_l] = \bar{x}_l$. This is a contradiction. □

Lemma 4.3. *Let $x_0 \in \vec{K}$ and $i_0 < n$ with $\min \vec{K} < \min[x_0]$ and $\max[x_0] < \max \vec{K}$. Suppose that there exist $j_0 < m$ and $q_0 \in L_{j_0}$ such that for every $c \in C(x)$ with $c(i_0) = \min[x_0(i_0)]$, $f(c)(j_0) = q_0$. Then, for every $x \in [x_0]$ with $x(i_0) = \min[x_0(i_0)]$, $f(x)(j_0) = q_0$. This holds when $\min[x(i_0)]$ is replaced by $\max[x(i_0)]$.*

Proof. Suppose not, i.e., there exists $x_1 \in [x_0]$ with $x_1(i_0) = \min[x_0(i_0)]$ such that $f(x_1)(j_0) \neq q_0$. Since $f^{\to}[x_0] \subseteq [f(x_0)]$ and $|[f(x_0)] \cap \mathrm{cl}(L_{j_0} \cap M)| \leq 3$, by Fact 2.6, without loss of generality, we may assume $f(x_1)(j_0) \notin \mathrm{cl}(L_{j_0} \cap M)$.

Let W be an open neighborhood of $f(x_1)(j_0)$ such that $W \cap \mathrm{cl}(L_{j_0} \cap M) = \varnothing$. Then, there exists $a', b' \in \vec{K}$ such that $a' < x_1 < b'$ such that $(\pi_{j_0} \circ f)^{\to}(a', b') \subseteq W$.

Since $C(q_0)$ is finite, there exists an open neighborhood W' of q_0 such that $W' \cap C(q_0) = \{ q_0 \}$. For each $c \in C(x)$ with $c(i_0) = \min[x_0(i_0)]$, there exists $a_c, b_c \in \vec{K}$ such that $a_c < c < b_c$ and $(\pi_{j_0} \circ f)^{\to}(a_c, b_c) \subseteq W'$. Notice that for every $c \in C(x)$ with $c(i_0) = \min[x_0(i_0)]$, we have $a_c(i_0) < c(i_0) = \min[x_0(i_0)]$.

Let

$$p_0 = \max(\{ a'(i_0) \} \cup \{ a_c(i_0) \mid c \in C(x) \wedge c(i_0) = \min[x_0(i_0)] \})$$

By Lemma 3.7, there exists $p \in K_{i_0} \cap M$ such that $p_0 < \min[p]$ and $p < \min[x_0(i_0)]$. Let $x_1' = \sigma(i_0, p)(x_1)$.

Claim 1. $f(x_1')(j_0) \sim q_0$.

Proof. By definition, $x_1 \in (a', b')$. It is also easy to see $x_1' \in (a', b')$. So, $f(x_1)(j_0), f(x_1')(j_0) \in (\pi_{j_0} \circ f)^{\to}(a', b')$. Since (a', b') is connected, so is $(\pi_{j_0} \circ f)^{\to}(a', b')$. Since $(\pi_{j_0} \circ f)^{\to}(a', b') \subseteq W$, we have $(\pi_{j_0} \circ f)^{\to}(a', b') \cap \mathrm{cl}(L_{j_0} \cap M) = \varnothing$. So, $f(x_1)(j_0) \sim f(x_1')(j_0)$ Since $x_1 \in [x_0]$, we have $x_0 \sim x_1$ and $q_0 \sim f(x_0)(j_0) \sim f(x_1)(j_0)$. Therefore, $f(x_1')(j_0) \sim q_0$. ☐

Claim 2. For every $c \in C(x_1')$, $f(c)(j_0) = q_0$.

Proof. Let $c \in C(x_1')$. By Lemma 4.1, we have $f(c)(j_0) \in C(f(x_1')(j_0))$. By the previous claim, we have $f(x_1')(j_0) \sim q_0$ and hence $C(f(x_1')(j_0)) = C(q_0)$. So, $f(c)(j_0) \in C(q_0)$.

Define $d = \sigma(i_0, \min[x_0(i_0)])(c)$. Then, $a_d < c < b_d$. Thus, $f(c)(j_0) \in W'$. Therefore, $f(c)(j_0) \in W' \cap C(q_0) = \{ q_0 \}$. So, $f(c)(j_0) = q_0$. ☐

By Fact 3.12, the previous claim implies that for every $x' \in [x_1']$, $f(x')(j_0) = q_0$. In particular, $f(x_1')(j_0) = q_0$. This is a contradiction. ☐

Lemma 4.4. *Suppose f is injective. Let $x \in \vec{K}$ with $\min \vec{K} < \min[x]$ and $\max[x] < \max \vec{K}$, and $i_0 < n$. Then, there exists $j_0 < m$ such that $\pi_{j_0} \circ f$ is constant on $\{ x' \in [x] \mid x'(i_0) = \min[x(i_0)] \}$.*

Proof. Since $\min \vec{K} < \min[x]$, $\min[x(i_0)]$ is a limit point of $K_{i_0} \cap M$ from below. So, there exists an increasing sequence $\langle x_k(i_0) \mid k < \omega \rangle$ in $K_{i_0} \cap M$ such that $\sup \{ x_k(i_0) \mid k < \omega \} = min[x(i_0)]$. For every $i < n$ with $i \neq i_0$, let $x_k(i) = x(i)$. Then, $\langle x_k \mid k < \omega \rangle$ is a sequence in $\vec{K}$. Since $\langle x_k(i_0) \mid k < \omega \rangle$ is an increasing sequence in $K_{i_0} \cap M$, $\langle [x_k(i_0) \mid k < \omega] \rangle$ is an increasing sequence in $\varphi(K_{i_0}, M)$. So, $\langle [x_k] \mid k < \omega \rangle$ is a distinct convergent sequence in $\varphi(\vec{K}, M)$.

For each $k < \omega$, let $y_k = f(x_k)$ and $y = f(x)$. By thinning this out, we may assume that for each $j < n$, $\langle y_k(j) \mid k < \omega \rangle$ is constant or strictly monotone. So, $\langle [y_k] \mid k < \omega \rangle$ is a convergent sequence in $\varphi(\vec{L}, M)$. Since x is the limit point of $\{ x_k \mid k < \omega \}$, $f(x) = y$ is the limit point of $\langle y_k \mid k < \omega \rangle$. Thus, $[y]$ is the limit point of $\langle [y_k] \mid k < \omega \rangle$. By Lemma 4.2, since $\langle [x_k] \mid k < \omega \rangle$ is distinct, $\langle [y_k] \mid k < \omega \rangle$ cannot be constant. So, there exists $j_0 < n$ such that $\langle [y_k(j_0)] \mid k < \omega \rangle$ is strictly monotone.

Now, we shall show that for every $x' \in [x]$ with $x'(i_0) = \min[x(i_0)]$, $\pi_{j_0} \circ f(x') = \pi_{j_0} \circ f(x)$, i.e., $f(x')(j_0) = f(x)(j_0)$. For each $k < \omega$, define $x'_k \in \vec{K}$ by

$$x'_k(i) = \begin{cases} x_k(i) & \text{if } i = i_0 \\ x'(i) & \text{if } i \neq i_0 \end{cases}$$

Claim 1. For every $k < \omega$, $x'_k \sim x_k$.

Proof. Let $i < n$. We shall show that $x'_k(i) \sim x_k(i)$. If $i = i_0$, it is trivial since $x'_k(i_0) = x_k(i_0)$. If $i \neq i_0$, we have $x'_k(i) = x'(i) \sim x(i) = x_k(i)$. □

So, for every $k < \omega$, we have $[x'_k] = [x_k]$. Thus, $f(x'_k) \in \varphi(f, M)[x'_k] = \varphi(f, M)[x_k] = [y_k] = [f(x_k)]$. So, $f(x_k), f(x'_k) \in [y_k]$.

Recall that $\langle [y_k(j_0)] \mid k < \omega \rangle$ is strictly monotone and $[y(j_0)]$ is its limit point. Suppose that it is strictly increasing. Then, since $f(x_k)(j_0), f(x'_k)(j_0) \in [y_k(j_0)]$, $\min[y(j_0)]$ is the limit point of $\{ f(x_k)(j_0) \mid k < \omega \}$ and $\{ f(x'_k)(j_0) \mid k < \omega \}$. Since $\{ f(x_k)(j_0) \mid k < \omega \}$ converges to $f(x)(j_0)$ and $\{ f(x'_k)(j_0) \mid k < \omega \}$ converges to $f(x')(j_0)$, we have $f(x)(j_0) = f(x')(j_0)$.

□

Lemma 4.5. *Suppose f is injective. Let $x_0 \in \prod_{i<n} K_i$, $i_0 < n$, and $j_0 < m$ be so that $f(x_0)(j_0) \notin \mathrm{cl}(L_{j_0} \cap M)$. Suppose that $\pi_{j_0} \circ f$ is constant on $\{ x \in [x_0] \mid x(i_0) = \min[x_0(i_0)] \}$, i.e. x_0, i_0, j_0 satisfy the conclusion of Lemma 4.4. Let q_0 be the constant value of $\pi_{j_0} \circ f$ on $\{ x \in [x_0] \mid x(i_0) = \min[x_0(i_0)] \}$. Then, there exists a sequence $\langle U_i \mid i < n \wedge i \neq i_0 \rangle$ such that for each $i < n$ with $i \neq i_0$, U_i is a basic open neighborhood of $x_0(i_0)$ and for every $x \in \vec{K}$, if $x(i_0) = \min[x_0(i_0)]$ and for every $i < n$ with $i \neq i_0$, $x(i) \in U_i$, then $f(x)(j_0) = q_0$.*

Proof. Since $f(x_0)(j_0) \notin \mathrm{cl}(L_{j_0} \cap M)$ and L_{j_0} is regular, there exists an open neighborhood W' of $f(x_0)(j_0)$ such that $\mathrm{cl}(W') \cap \mathrm{cl}(L_{j_0} \cap M) = \varnothing$. Since $\pi_{j_0} \circ f$ is continuous, there exists a basic open neighborhood U' of x_0 such that $\pi_{j_0} \circ f^{\rightarrow} U' \subseteq W'$. Since $\pi_{j_0} \circ f^{\rightarrow} \mathrm{cl}(U') \subseteq \mathrm{cl}(W')$, we have $\pi_{j_0} \circ f^{\rightarrow} \mathrm{cl}(U') \cap \mathrm{cl}(L_{j_0} \cap M) = \varnothing$.

Since $C(f(x_0)(j_0))$ is finite and L_{j_0} is regular, there exists an open neighborhood W'' of q_0 such that $|\mathrm{cl}(W'') \cap C(f(x)(j_0))| \leq 1$.

Claim 1. $\mathrm{cl}(W'') \cap C(f(x_0)(j_0)) = \{ q_0 \}$

Proof. Let $c \in C(x_0)$ with $c(i_0) = \min[x_0(i_0)]$. Then, by Lemma 4.1, $q_0 = f(c)(j_0) \in C(f(x)(j_0))$. By assumption, $f(\tilde{x})(j_0) = f(c)(j_0) = q_0$. So, $q_0 \in \mathrm{cl}(W'') \cap C(f(x)(j_0))$. Since $|\mathrm{cl}(W'') \cap C(f(x)(j_0))| \leq 1$, it implies $\mathrm{cl}(W'') \cap C(f(x)(j_0)) = \{ q_0 \}$. □

By assumption, for every $x \in [x_0]$ with $x(i_0) = \min[x_0(i_0)]$, we have $f(x)(j_0) = q_0$. So, there exists an open neighborhood U_x of x such that $\pi_{j_0} \circ f^{\to} U_x \subseteq W''$. Since $\min \vec{K} < \min[x_0]$ and $\max[x_0] < \max \vec{K}$, there exists $a_x, b_x \in \vec{K}$ such that $a_x < x < b_x$ and $[a_x, b_x] \subseteq U_x$. Since $\{ x \in [x_0] \mid x(i_0) = \min[x_0(i_0)] \}$ is compact, there exists a finite subset F of $\{ x \in [x_0] \mid x(i_0) = \min[x_0(i_0)] \}$ such that $\{ x \in [x_0] \mid x(i_0) = \min[x_0(i_0)] \} \subseteq \bigcup_{x \in F}(a_x, b_x)$.

Let $i < n$ be with $i \neq i_0$.

Claim 2. For each $i < n$ with $i \neq i_0$, neither $\{ a_x(i) \mid x \in F \wedge a_x(i) < \min[x_0(i)] \}$ nor $\{ b_x(i) \mid x \in F \wedge \max[x_0(i)] < b_x(i) \}$ is empty.

Proof. Let $c = \min[x_0]$. Since $\{ x \in [x_0] \mid x(i_0) = \min[x_0(i_0)] \} \subseteq \bigcup_{x \in F} U_x$, there exists $x \in F$ such that $c \in U_x$. So, for every $i < n$ with $i \neq i_0$, we have $a_x(i) < c(i) = \min[x_0(i)]$. Thus, $\{ a_x(i) \mid x \in F \wedge a_x(i) < \min[x_0(i)] \} \neq \varnothing$

Define $d \in [x_0]$ by

$$d(i) = \begin{cases} \max[x_0(i)] & \text{if } i \neq i_0 \\ \min[x_0(i)] & \text{if } i = i_0 \end{cases}$$

Since $\{ x \in [x_0] \mid x(i_0) = \min[x_0(i_0)] \} \subseteq \bigcup_{x \in F} U_x$, there exists $x \in F$ such that $d \in U_x$. So, for every $i < n$ with $i \neq i_0$, we have $\max[x_0(i)] = d(i) < b_x(i)$. Thus, $\{ b_x(i) \mid x \in F \wedge \max[x_0(i)] < b_x(i) \} \neq \varnothing$ $\square$

For each $i < n$ with $i \neq i_0$, let

$$a'(i) = \max \{ a_x(i) \mid x \in F \wedge a_x(i) < \min[x_0(i)] \}$$
$$b'(i) = \min \{ b_x(i) \mid x \in F \wedge \max[x_0(i)] < b_x(i) \}$$

Claim 3. Let $x \in \vec{K}$ be such that $x(i) = \min[x_0(i_0)]$ and for every $i < n$ with $i \neq i_0$, $a'(i) < x(i) < b'(i)$. Then, there exists $y \in F$ such that $x \in (a_y, b_y)$.

Proof. Let $x \in (a', b')$ with $x(i_0) = \min[x_0(i_0)]$. Define $x' \in [x_0]$ by for each $i < n$,

$$x'(i) = \begin{cases} \min[x_0(i_0)] & \text{if } i = i_0 \\ x(i) & \text{if } x'(i) \in [x_0(i)] \\ \min[x_0(i)] & \text{if } x(i) < \min[x_0(i)] \\ \max[x_0(i)] & \text{if } x(i) > \max[x_0(i)] \end{cases}$$

Then, by the definition of F, there exists $y \in F$ such that $x' \in (a_y, b_y)$. We shall show that $x \in (a_y, b_y)$ by proving for each $i < n$, $a_y(i) < x(i) < b_y(i)$. Since $a_y < y < b_y$, we have $a_y(i_0) < y(i_0) < b_y(i_0)$. By the definition of F, we have $y(i_0) = \min[x_0(i_0)]$. Since $x(i_0) = \min[x_0(i_0)] = y(i_0)$, we have $a_y(i_0) < x(i_0) < b_y(i_0)$.

Let $i < n$ with $i \neq i_0$.

Case 1. $x(i) \in [x_0(i)]$

We have $x'(i) = x(i)$ and $a_y(i) < x'(i) < b_y(i)$. So, $a_y(i) < x(i) < b_y(i)$.

Case 2. $x(i) < \min[x_0(i)]$

Then $x'(i) = \min[x_0(i)]$. Since $x' \in (a_y, b_y)$, we have $a_y(i) < x'(i) < b_y(i)$. So, $a_y(i) < \min[x_0(i)]$. Thus, $a_y(i) \leq a'(i)$. Since $x \in (a', b')$, we have $a'(i) < x(i)$. So, $a_x(i) < x(i)$. Also, $x(i) < \min[x_0(i)] = x'(i) < b_y(i)$. Thus, $a_y(i) < x(i) < b_y(i)$.

Case 3. $x(i) > \max[x_0(i)]$

Then, $x'(i) = \max[x_0(i)]$. Since $x' \in (a_y, b_y)$, we have $a_y(i) < x'(i) < b_y(i)$. So, $\max[x_0(i)] < b_y(i)$. Thus, $b'(i) \leq b_y(i)$. Since $x \in (a', b')$, we have $x(i) < b'(i)$. So, $x'(i) < b_y(i)$. Moreover, $x(i) > \max[x_0(i)] = x'(i) > a_y(i)$. Thus, $a_y(i) < x(i) < b_y(i)$. □

Claim 4. Let $x \in \vec{K}$ be such that $x(i) = \min[x_0(i_0)]$ and for every $i < n$ with $i \neq i_0$, $a'(i) < x(i) < b'(i)$. If $f(x)(j_0) \in C(f(x_0)(j_0))$, then $f(x)(j_0) = q_0$.

Proof. Let x be as in the assumption. Suppose $f(x)(j_0) \in C(f(x_0)(j_0))$ By the previous claim, there exists $y \in F$ such that $x \in (a_y, b_y) \subseteq U_y$. Recall that $(\pi_{j_0} \circ f)^{\rightarrow} U_y \subseteq W''$. So, $f(x)(j_0) \in W''$. Thus, $f(x)(j_0) \in \mathrm{cl}(W'') \cap C(f(x_0)(j_0)) = \{ q_0 \}$. Therefore, $f(x)(j_0) = q_0$. □

By Lemma 3.7, there exist $a(i), b(i) \in K_i$ such that $a'(i) < a(i) < \min[x_0(i)]$, $\max[x_0(i)] < b(i) < b'(i)$, $a(i)$ is a limit point of $K_i \cap M$ from below, and $b(i)$ is a limit point of $K_i \cap M$ from above.

For each $i < n$ with $i \neq i_0$, let $U_i = \pi_i{}^{\rightarrow} U' \cap (a(i), b(i))$. We shall show that $\langle U_i \mid i < n \wedge i \neq i_0 \rangle$ works. Clearly, for every $i < n$ with $i \neq i_0$, U_i is a basic open neighborhood of $x_0(i_0)$. Let $x \in \vec{K}$ be so that $x(i_0) = \min[x_0(i_0)]$ and for every $i < n$ with $i \neq i_0$, $x(i) \in U_i$.

Let $x' = \sigma(i_0, x_0)(x)$. Then, $x' \in U'$. So, $f(x_0)(j_0), f(x')(j_0) \in \pi_{j_0} \circ f^{\rightarrow} U'$. Since U' is connected, $\pi_{j_0} \circ f^{\rightarrow} U'$ is also connected. Since $\pi_{j_0} \circ f^{\rightarrow} U' \cap \mathrm{cl}(L_{j_0} \cap M) = \varnothing$, we have $f(x_0)(j_0) \sim f(x')(j_0)$. So, $[f(x_0)(j_0)] = [f(x')(j_0)]$.

Claim 5. For every $c \in C(x')$ with $c(i_0) = \min[x_0(i_0)]$, $f(c)(j_0) = q_0$.

Proof. Let $c \in C(x')$ with $c(i_0) = \min[x_0(i_0)]$. By Lemma 4.1, we have $f(c)(j_0) \in C(f(x')(j_0)) = C(f(x_0)(j_0))$.

We shall show that for every $i < n$ with $i \neq i_0$, $a(i) \leq c(i) \leq b(i)$. We have $x'(i) = x(i) \in U_i \subseteq (a(i), b(i))$. Notice that $\min[x'(i)]$ is the greatest element of K_i less than $x'(i)$ that is a limit point of $K_i \cap M$ from below. Since $a(i)$ is a limit point of $K_i \cap M$ from below, we have $a(i) \leq \min[x'(i)]$. Similarly, $\max[x'(i)]$ is the least element of K_i greater than $x'(i)$ that is a limit point of $K_i \cap M$ from above. Since $b(i)$ is a limit point of $K_i \cap M$ from above, we have $\max[x'(i)] \leq b(i)$.

For every $i < n$ with $i \neq i_0$, since $a'(i) < a(i)$ and $b(i) < b'(i)$, we have $a'(i) < c(i) < b'(i)$. By the previous claim, we have $f(c)(j_0) = q_0$. □

By Lemma 4.3, the previous claim implies that $f(x)(j_0) = q_0$. □

5. No continuous injection from the finite product to the finite product with a smaller dimension

This section is devoted to the proof of the main theorem.

Theorem 5.1. *Let n be a positive integer. Let K_i be a connected LOTS with at least two points for each $i < n$ and L_j a connected LOTS for each $j < n-1$. Then, there exists no continuous injective function from a non-empty open subset of $\prod_{i<n} K_i$ into $\prod_{j<n-1} L_j$. Note that when $n = 1$, we stipulate $\prod_{j<0} L_j$ to be a singleton.*

Proof. For each positive integer n, let $(*)_n$ be the statement that if K_i is a connected LOTS with at least two points and no minimum or maximum elements for each $i < n$, L_j is a connected LOTS with no minimum or maximum elements for each $j < n-1$, then there exists no continuous injection from a non-empty open subset of $\prod_{i<n} K_i$ into $\prod_{j<n-1} L_j$.

First, we shall observe that it suffices to prove $(*)_n$ for all positive integer n. Let n, K_i, and L_j be as in the assumption. Suppose that the conclusion fails, namely there exists a continuous injective function f from a non-empty open subset U of $\prod_{i<n} K_i$ into $\prod_{j<n-1} L_j$. Then, there exists a sequence $\langle U_i \mid i < n\rangle$ such that $\prod_{i<n} U_i \subseteq U$ and for each $i < n$, U_i is a non-empty basic open set in K_i. We shall define $a(i), b(i) \in U_i$ as follows. For each $i < n$, since U_i is a non-empty basic open set and U_i is connected, U_i is infinite. So, there exist $a(i) < x(i) < x'(i) < b(i)$ in U_i. Then, $(a(i), b(i))$ is a connected LOTS with at least two point and no minimum or maximum elements. For each $j < n-1$, by Lemma 2.3, there exists a connected LOTS L'_j extending L_j with no minimum or maximum element. Define $\bar{f} : \prod_{i<n}(a(i), b(i)) \to \prod_{j<n-1} L'_j$ by $\bar{f}(x) = f(x)$. Trivially, $\bar{f}$ is continuous and injective. However, this contradicts $(*)_n$.

By induction, we shall prove $(*)_n$ holds for every positive integer n. For $(*)_1$, we identified $\prod_{j<0} L_j$ as a singleton, and K_0 has at least two points. So, clearly there exists no continuous injective function from a non-empty open subset of $\prod_{i<1}$ into $\prod_{j<0} L_j$.

Let n be a positive integer such that $(*)_m$ holds for every positive integer $m < n$. Suppose that $(*)_n$ does not hold. So, there exist a connected LOTS with at least two points for each $i < n$, a connected LOTS L_j for each $j < n-1$, and a continuous injection f from a non-empty open subset U of $\prod_{i<n} K_i$ into $\prod_{j<n-1} L_j$.

Let $\vec{K} = \prod_{i<n} K_i$ and $\vec{L} = \prod_{j<n-1}$.

Case 1. There exists $a, b \in \prod_{i<n} K_i$ such that $a < b$, (a, b) is separable, and $(a, b) \subseteq U$.

Then, for every $i < n$, $(a(i), b(i))$ is a separable connected LOTS with at least two elements and no minimum or maximum elements. So, $(a(i), b(i))$ is homeomorphic to $\mathbb{R}$. Let $K'_i = (a(i), b(i))$. For each $j < n-1$, let

$L'_j = \pi_j \circ f^{\to}(a,b)$. Since (a,b) is separable and connected, so is L'_j. Thus, L'_j is homeomorphic to a subset of $\mathbb{R}$.

Let g be the restriction of f to $\prod_{i<n} K'_i$. Then, g is a continuous injection from $\prod_{i<n} K'_i$ into $\prod_{j<n-1} L'_j$. Since K'_i is homeomorphic to $\mathbb{R}$ for every $i < n$ and L'_j is homeomorphic to a subset of $\mathbb{R}$, we can construct a continuous injection from $\mathbb{R}^n$ into $\mathbb{R}^{n-1}$. However, this is impossible by the ordinary Invariance of Domain Theorem. This is a contradiction.

Case 2. For all $a, b \in \prod_{i<n} K_i$, if $a < b$ and $(a,b) \subseteq U$, then (a,b) is not separable.

Since $\prod_{i<n} K_i$ is regular, there exist $a_0, b_0 \in \prod_{i<n} K_i$ be so that $a_0 < b_0$ and $[a_0, b_0] \subseteq U$.

Let θ be a regular cardinal with $\theta \geq \theta_{\langle K_i | i<n\rangle, \langle L_j | j<n-1\rangle, f}$. Let M be a countable elementary substructure of $H(\theta)$ with $\langle K_i \mid i < n\rangle, \langle L_j \mid j < n-1\rangle, f, U, a_0, b_0 \in M$.

By assumption, (a_0, b_0) is not separable. So, there exists $i_0 < n$ such that $(a_0(i_0), b_0(i_0))$ is not separable. So, there exists $p_0 \in (a_0(i_0), b_0(i_0)) \setminus \mathrm{cl}(K_{i_0} \cap M)$. Let $x_0 \in (a_0, b_0)$ be so that $x_0(i_0) = p_0$ and for every $i < n$ with $i \neq i_0$, $x_0(i) \in M$ and $x_0(i)$ is a limit point of $K_i \cap M$ from below.

Claim 1. There exist $p_1 \in [p_0]$ and $j_0 < n-1$ such that $f(\sigma(i_0, p_1)(x_0))(j_0) \notin \mathrm{cl}(L_{j_0} \cap M)$.

Proof. Define a function $g : [a_0(i_0), b_0(i_0)] \to \vec{L}$ by for every $p \in [a_0(i_0), b_0(i_0)] \to \vec{L}$ and $j < n-1$, $g(p) = f(\sigma(i_0, p)(x_0))$.

Clearly g is continuous. We shall show that g is injective. If p and p' are both elements of $[a_0(i_0), b_0(i_0)]$, then since f is injective, $f(\sigma(i_0, p)(x_0)) \neq f(\sigma(i_0, p')(x_0))$. So, $g(p) \neq g(p')$.

In particular, g is not constant on $[p_0]$. So, there exists $j_0 < n-1$ such that $\pi_{j_0} \circ g$ is not constant on $[p_0]$. Since $[p_0]$ and L_{j_0} is connected, $\pi_{j_0} \circ g^{\to}[p_0]$ is infinite. Notice that $\pi_{j_0} \circ g^{\to}[p_0] \subseteq \pi_{j_0} \circ f^{\to}[x_0]$. So,

$$\begin{aligned}
\pi_{j_0} \cap g^{\to}[p_0] \cap \mathrm{cl}(L_{j_0} \cap M) &\subseteq \pi_{j_0} \circ f^{\to}[x_0] \cap \mathrm{cl}(L_{j_0} \cap M) \\
&\subseteq [f(x_0)(j_0)] \cap \mathrm{cl}(L_{j_0} \cap M) \\
&= C(f(x_0)(j_0))
\end{aligned}$$

Since $C(f(x_0)(j_0))$ is finite, $\pi_{j_0} \cap g^{\to}[p_0] \setminus \mathrm{cl}(L_{j_0} \cap M) \neq \varnothing$. Let $p_1 \in [p_0]$ be so that $g(p_1)(j_0) \notin \mathrm{cl}(L_{j_0} \cap M)$. By the definition of g, we have $f(\sigma(i_0, p_1)(x_0)) \notin \mathrm{cl}(L_{j_0} \cap M)$. □

Let $x_1 = \sigma(i_0, p_1)(x_0)$. Since $f(x_1)(j_0) \notin \mathrm{cl}(L_{j_0} \cap M)$, there exist $a', b' \in (a_0, b_0)$ be so that $a' < x_1 < b'$ and $\pi_{j_0} \circ f^{\to}(a', b') \cap \mathrm{cl}(L_{j_0} \cap M) = \varnothing$. Since $\pi_{j_0} \circ f^{\to}(a', b')$ is connected in L_{j_0}, it implies that for every $x \in (a', b')$, $f(x)(j_0) \sim f(x_1)(j_0)$.

Let $c_0 = \sigma(i_0, \min[p_0])(x_0)$. Then, $f(c_0)(j_0) \in C(f(x_0)(j_0))$. Let $q_0 = f(c_0)(j_0)$. Since $C(f(x_0))(j_0)$ is finite, there exists $a'', b'' \in (a_0, b_0)$ such that $a'' < c_0 < b''$ and $\pi_{j_0} \circ f^{\to}(a'', b'') \cap C(q_0) = \{ q_0 \}$.

We shall define $a_1, b_1 \in (a_0, b_0)$ as follows. Let $a_1(i_0) = a'(i_0)$ and $b_1(i_0) = b'(i_0)$. Let $i < n$ with $i \neq i_0$. By Lemma 3.7, there exists $a_1(i) \in K_i \cap M$ such that $\max\{\, a'(i), a''(i) \,\} < a_1(i)$ and $a_1(i)$ is a limit point of $K_i \cap M$ from below. Let $b_1(i) = x_0(i)$.

Claim 2. For every $x \in (a_1, b_1)$ and $c \in C(x)$ with $c(i_0) = \min[p_0]$, $f(c)(j_0) = q_0$.

Proof. Let x and c be as in the definition. By the definition of a_1 and b_1, we have $(a_1, b_1) \subseteq (a', b')$, and hence $x \in (a', b')$. So, $f(x)(j_0) \sim q_0$.

By Lemma 4.1, since $c \in C(x)$, we have $f(c)(j_0) \in C(f(x)(j_0)) = C(q_0)$.

For each $i < n$ with $i \neq i_0$, we have $a''(i) < a_1(i) < x(i)$. Since $a_1(i)$ is a limit point of $K_i \cap M$ from below, we have $a_1(x) \leq \min[x(i)] \leq c(i)$. We also have $c(i) < b_1(i) = x_0(i) < b''(i)$ Since $c(i_0) = \min[p_0]$, we have $a''(i) < c(i_0) < b''(i)$. So, $c \in (a'', b'')$. Thus, $f(c)(j_0) \in \pi_{j_0} \circ f^{\rightarrow}(a'', b'')$

So, $f(c)(j_0) \in \pi_{j_0} \circ f^{\rightarrow}(a'', b'') \cap C(q_0) = \{\, q_0 \,\}$. Therefore, $f(c)(j_0) = q_0$. □

Claim 3. For every $x \in (a_1, b_1)$, if $x(i_0) = \min[p_0]$, then $f(x)(j_0) = q_0$

Proof. By Lemma 4.3, this claim follows from the previous one. □

Define $\bar{f} : \prod_{i<n, i\neq i_0}(a_1(i), b_1(i)) \to \prod_{j<n-1, j\neq j_0}$ by for every $x \in \prod_{i<n,i\neq i_0}(a_1(i), b_1(i))$ and $j < n-1$ with $j \neq j_0$, $\bar{f}(x)(j) = f(\sigma(i_0, \min[p_0])(x))(j)$.

Claim 4. $\bar{f}$ is injective.

Proof. Let x and x' be distinct elements of $\prod_{i<n,i\neq i_0}(a_1(i), b_1(i))$. Since f is injective, we have $f(\sigma(i_0, \min[p_0])(x)) \neq f(\sigma(i_0, \min[p_0])(x'))$. So, there exists $j < n-1$ such that $f(\sigma(i_0, \min[p_0])(x))(j) \neq f(\sigma(i_0, \min[p_0])(x'))(j)$. By the previous claim, $f(\sigma(i_0, \min[p_0])(x))(j_0) = f(\sigma(i_0, \min[p_0])(x'))(j_0) = q_0$. Therefore, $j \neq j_0$. So, $\bar{f}(x) \neq \bar{f}(x')$. □

However, this contradicts $(*)_{n-1}$. □

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