CONTINUOUS INJECTIONS BETWEEN THE PRODUCTS OF TWO CONNECTED NOWHERE REAL LINEARLY ORDERED SPACES

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1. Introduction

Let $f : X_0 \times X_1 \to Y_0 \times Y_1$ be a function. We say that f is *coordinate-wise* if and only if there exist $i < 2$, $g_0 : X_i \to Y_0$, and $g_1 : X_{1-i} \to Y_1$ such that for every $\langle x_0, x_1 \rangle \in X_0 \times X_1$, $f(x_0, x_1) = \langle g_0(x_i), g_1(x_{1-i}) \rangle$.

There are many homeomorphisms from $\mathbb{R}^2$ onto $\mathbb{R}^2$ that are not coordinate-wise. For example, $f(x, y) = \langle x - y, x + y \rangle$.

However, Eda and Kamijo proved the following theorem that this is not necessarily the case when we replace $\mathbb{R}$ by other connected linearly ordered spaces.

Theorem 1.1 (Eda and Kamijo). *Let K be a connected linearly ordered space such that for a dense set of $x \in K$, either $\mathrm{cf}(x)$ or $\mathrm{ci}(x)$ is uncountable. Here $\mathrm{cf}(x)$ denotes the cofinality of x and $\mathrm{ci}(x)$ the coinitiality of x. Then, for every $n < \omega$, every homeomorphism $f : K^n \to K^n$ is coordinate-wise.*

They asked if it can be extended to, for example, the cut-completion of an Aronszajn line. In this article, we shall answer this question positively with some other improvements.

Definition 1.2. A linearly ordered space L is *nowhere real* if and only if it is uncountable, but no uncountable convex set is separable.

Fact 1.3. Let K be a nowhere real linearly ordered space. Then, the closure of any countable subset of K is nowhere dense.

Theorem 1.4. *Let K_0, K_1, L_0, L_1 be connected nowhere real linearly ordered spaces. Then, every continuous injection $f : K_0 \times K_1 \to L_0 \times L_1$ is coordinate-wise.*

Thus, these four connected linearly ordered spaces may or may not be different, and the function only needs to be a continuous injection, instead of a homeomorphism.

The proof is done by set-theoretic argument using countable elementary substructures. It is quite different from the argument of Eda and Kamijo, which is much more topological. Meanwhile, many interesting lemmas are

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proved about functions between the product of two connected nowhere real linearly ordered spaces through countable elementary substructures.

The author also shows this theorem can be extended to the product of any finite number of connected nowhere real linearly ordered spaces. An article about this result is now in preparation.

2. Connected linearly ordered spaces

In this section, we shall write known basic facts on connected linearly ordered spaces. We shall use the standard interval notations very often such as for a linearly ordered set K and $a \leq b$ both in K,

$$(a, b) = \{ x \in K : a < x < b \}$$
$$[a, b] = \{ x \in K : a \leq x \leq b \}$$

Lemma 2.1. *Let K be a connected linearly ordered space. Then, for every non-empty subset A of K, if A has an upper bound in K, then A has the least upper bound in K.*

Proof. Let $A \subseteq K$ be a non-empty set that has an upper bound. Suppose that A does not have the least upper bound. Let U be the set of all upper bounds of A. Then, it is easy to see that U is clopen, and hence K is not connected, which is a contradiction. □(Lemma 2.1)

By using the same argument as the proof to show every bounded subset of $\mathbb{R}$ is compact, we can show the following lemma.

Definition 2.2. Let K be a linearly ordered space. We say that $A \subseteq K$ is *bounded* if and only if there exist $a, b \in K$ such that $A \subseteq [a, b]$. for every $x \in X$, $a \leq x \leq b$.

Lemma 2.3. *Every bounded closed subset of a connected linearly ordered space is compact.*

By using this lemma, we can show the following.

Lemma 2.4. *Let K and L be connected linearly ordered spaces and $g : K \to L$ a continuous function. Let $a < b$ be both in K. Then, there exist maximum and minimum values of g on $[a, b]$.*

We can also see the Intermediate Value Theorem for continuous functions from a connected linearly ordered spaces to a connected linearly ordered space.

Lemma 2.5. *Let K and L be connected linearly ordered spaces, and $g : K \to L$ a continuous function. Let $a < b$ be both in K. If $z \in L$ is between $g(a)$ and $g(b)$, then there exists a $c \in (a, b)$ such that $g(c) = z$.*

Proof. Without loss of generality, we may assume $g(a) < g(b)$. Let $c = \sup \{ x \in K : a \leq x \wedge \forall x' \in (a, x)(g(x') < z) \}$. Clearly, we have $a < c < b$. We shall show that $g(c) = z$.

Claim 1. $g(c) \geq z$.

$\vdash$ For every $x \in (a, c)$, by the definition of c, there exists $c' \in (x, c)$ such that for every $x' \in (a, c')$, $g(x') < z$. Hence $g(x) < z$.

Now suppose $g(c) < z$. Then, there exists $b' > c$ such that for every $x \in (c, b')$, $g(x) < z$. Then, for every $x \in (a, b')$, $f(x) < z$. Thus, $c \geq b'$, which is a contradiction. $\dashv$ (Claim 1)

Claim 2. $g(c) \leq z$.

$\vdash$ Suppose $g(c) > z$. Then, there exist $c' < c$ such that for every $x \in (c', c)$, $g(c) > z$. So, $c \leq c'$, which is a contradiction. $\dashv$ (Claim 2)

$\square$(Lemma 2.5)

3. Continuous functions from a linearly ordered space to another

Throughout this section, let K and L be any connected linearly ordered sets, $g : K \to L$ a continuous function, and M a countable elementary submodel of $H(\theta)$ with $K, L, f \in M$ for some regular cardinal θ with $\mathcal{P}(\mathcal{P}(K \cup L)) \in H(\theta)$.

Definition 3.1. Let $J(K, M)$ be the set of all $x \in \operatorname{int}(K \setminus M)$ such that $\inf(K \cap M) < x < \sup(K \cap M)$.

For every $x \in K$, let

$$\begin{aligned} \eta(K, M, x) &= \sup \{ y \in K \cap M : y < x \} \\ \zeta(K, M, x) &= \inf \{ y \in K \cap M : y > x \} \\ I(K, M, x) &= (\eta(K, M, x), \zeta(K, M, x)) \\ B(K, M, x) &= \{ \eta(K, M, x), \zeta(K, M, x) \} \end{aligned}$$

if they exist. Note that if $x \in J(K, M)$, then all of them exist.

Clearly, $I(K, M, x)$ is a convex set. $\eta(K, M, x)$ is either an element of M or a limit point of $K \cap M$ from below (i.e. for every $x' < x$, there exists a $y \in K \cap M$ such that $x' < y < x$). Similarly, $\zeta(K, M, x)$ is either an element of M or a limit point of $K \cap M$ from above (i.e. for every $x' > x$, there exists a $y \in K \cap M$ such that $x < y < x'$).

Lemma 3.2. *Let $x \in J(K, M)$. Then, either $\eta(K, M, x) \notin M$ or $\zeta(K, M, x) \notin M$.*

Proof. Suppose not, i.e. $\eta(K, M, x) \in M$ and $\zeta(K, M, x) \in M$. By the elementarity of M, since K is connected, there exists an $x' \in K \cap M$ such that $\eta(K, M, x) < x' < \zeta(K, M, x)$. This is a contradiction by the definition of $\eta(K, M, x)$ and $\zeta(K, M, x)$. $\square$(Lemma 3.2)

We can prove the following lemma, which means that the supremum (and the infimum) of g on $I(K,M,\bar{x})$ is the value of g at one of the endpoints of $I(K,M,\bar{x})$.

Lemma 3.3. *Let $\bar{x} \in J(K,M)$. Then, there exist $x_{\sup}, x_{\inf} \in B(K,M,\bar{x})$ such that*

$$g(x_{\sup}) = \sup\{\, g(x) : x \in I(K,M,\bar{x})\,\}$$
$$g(x_{\inf}) = \inf\{\, g(x) : x \in I(K,M,\bar{x})\,\}$$

In particular, if $g(\eta(K,M,\bar{x})) = g(\zeta(K,M,\bar{x}))$, then g is constant on $I(K,M,\bar{x})$.

Proof. We shall show the existence of $x_{\sup}$ since essentially the same proof works for $x_{\inf}$. Let $a = \eta(K,M,x)$, $b = \zeta(K,M,x)$, and $v = \sup\{\, g(x) : x \in I(K,M,\bar{x})\,\}$. Suppose $g(a) \neq v$ and $g(b) \neq v$.

If $a \in M$, let $a' = a$. Otherwise, a is a limit point of $K \cap M$ from below. Let $a' \in K \cap M$ be so that $a' < a$ and for every $x \in [a',a]$, $g(x) \neq v$. Similarly, if $b \in M$, let $b' = b$. Otherwise, let $b' \in K \cap M$ be so that $b < b'$ and for every $x \in [b,b']$, $g(x) \neq v$. Then, we have $a', b' \in K \cap M$ and for every $x \in [a',b']$, if $g(x) = v$, then $a < x < b$.

Since $a', b' \in M$ and there exists an $x \in [a',b']$ such that $g(x) = b$, there exists such an $x \in M$. By the previous paragraph, we have $a < x < b$. By the definition of a and b, It implies $x \notin M$. This is a contradiction. $\square$(Lemma 3.3)

Lemma 3.4. *Let $x \in J(K,M)$ with $g(x) \in M$. Then, g is constant on $I(K,M,x)$.*

Proof. Let $v = g(x)$, $a = \eta(K,M,x)$, and $b = \zeta(K,M,x)$.

By Lemma 3.3, it suffices to show that $g(a) = g(b) = v$. We shall show $g(a) = v$. Suppose not. If $a \in M$, let $a' = a$. Otherwise, let $a' \in K \cap M$ be so that $a' < a$ and for every $y \in [a',a]$, $g(y) \neq v$. Let $x' = \inf\{\, y \in K : y \geq a' \text{ and } g(y) = v\,\}$. Then, $a' \in M$ and $x' \leq x$. So, we have $a' \leq x' \leq x_0$. This is a contradiction to the definition of a'. $\square$(Lemma 3.4)

Lemma 3.5. *Let $\bar{x} \in J(K,M)$. If $\eta(K,M,\bar{x}) \notin M$ and $g(\eta(K,M,\bar{x})) \in M$, then g is constant on $I(K,M,\bar{x})$. Similarly, if $\zeta(K,M,\bar{x}) \notin M$ and $g(\zeta(K,M,\bar{x})) \in M$, then g is constant on $I(K,M,\bar{x})$.*

Proof. Let $a = \eta(K,M,\bar{x})$, $b = \zeta(K,M,\bar{x})$ and $v = g(a)$. Suppose that g is not constant on $I(K,M,\bar{x})$. Then, $g(b) \neq g(a) = v$. Let $b' \in K \cap M$ such that $b' \geq b$ and for every $x \in [b,b']$, $g(x) \neq v$. Let x_0 be the supremum of all $x \in K$ such that $x \leq b'$ and $g(x) = v$. Then, we have $x_0 \in M$. Since $g(a) = v$, we have $x_0 \geq a$. By the definition of b', we have $x_0 < b$. The only element of $[a,b)$ that can be in M is a. So, $a = x_0 \in M$. $\Box$(Lemma 3.5)

Lemma 3.6. *Let $\bar{x} \in J(K,M)$ with $g(\bar{x}) \in J(L,M)$. Then, both $g(\eta(K,M,\bar{x}))$ and $g(\zeta(K,M,\bar{x}))$ belong to $B(L,M,g(x))$.*

Proof.

Claim 1. For every $x \in I(K,M,\bar{x})$, $g(x) \notin M$.

$\vdash$ Suppose that there exists an $x \in I(K,M,\bar{x})$ such that $g(x) \in M$. Then, by Lemma 3.4, g is constant on $I(K,M,\bar{x})$ and hence $g(\bar{x}) = g(x) \in M$. This is a contradiction $\dashv$ (Claim 1)

Thus, for every $x \in I(K,M,\bar{x})$, we have $g(x) \in I(L,M,g(\bar{x}))$. So, $g(\eta(K,M,\bar{x})) \in \mathrm{cl}(I(L,M,g(\bar{x})))$. But since $\eta(K,M,\bar{x})) \in \mathrm{cl}(K \cap M)$, we also have $g(\zeta(K,M,\bar{x})) \in \mathrm{cl}(L \cap M)$. Therefore, $g(\eta(K,M,\bar{x}))$ is either $\eta(L,M,g(\bar{x}))$ or $\zeta(L,M,g(\bar{x}))$. Similarly for $g(\zeta(K,M,\bar{x}))$. $\Box$(Lemma 3.6)

4. $f : K_0 \times K_1 \to L$

Throughout this section, we assume that K_0, K_1, L are nowhere real connected linear orders, $f : K_0 \times K_1 \to L$ is a continuous function, M is a countable elementary substructure of $H(\theta)$ with $K_0, K_1, L \in M$, for a regular cardinal θ with $\mathcal{P}(\mathcal{P}(K_0 \cup K_1 \cup L)) \in H(\theta)$.

Lemma 4.1. *Let $\bar{x} \in J(K_0,M)$ and $\bar{y} \in \mathrm{cl}(K_1 \cap M)$. Suppose that $f(\bar{x},\bar{y}) \in J(L,M)$. Then, $f(\eta(K_0,M,\bar{x}),\bar{y}) \in B(L,M,f(\bar{x},\bar{y}))$ and $f(\zeta(K_0,M,\bar{x}),\bar{y}) \in B(L,M,f(\bar{x},\bar{y}))$.*

Proof. Let $a = \eta(K_0,M,\bar{x})$ and $b = \zeta(K_0,M,\bar{y})$. Let $B = B(L,M,f(\bar{x},\bar{y}))$.

First, suppose $\bar{y} \in M$. Then the map $x \mapsto (x,\bar{y}$ is a function from K_0 into L lying in M. By Lemma 3.6, So, $f(a,\bar{y}) \in B$ and $f(b,\bar{y}) \in B$.

Now, suppose $\bar{y}$ is a limit point of $K_1 \cap M$. Since $|B| \leq 2$ and $f(\bar{x},\bar{y}) \in J(L,M)$, there exists an open neighborhood U of $\bar{y}$ in K_1 such that

$$\begin{aligned} |f^{\rightarrow}(\{\, a \,\} \times U) \cap B| &\leq 1 \\ |f^{\rightarrow}(\{\, b \,\} \times U) \cap B| &\leq 1 \\ f^{\rightarrow}(\{\, \bar{x} \,\} \times U) &\subseteq I(L,M,f(\bar{x},\bar{y})) \end{aligned}$$

Let $y \in U \cap M$. Then, we have $f(\bar{x},y) \in I(L,M,\bar{x},\bar{y}) \subseteq J(L,M)$. By the first part of the proof, we have $f(a,y) \in B$ and $f(b,y) \in B$. By the definition of U, there exist $v, w \in B$ such that for every $y \in U \cap M$, $f(a,y) = v$ and $f(b,y) = w$. Since $\bar{y} \in \mathrm{cl}(K_1 \cap M)$, we have $f(a,\bar{y}) = v$ and $f(b,\bar{y}) = w$. Therefore, $f(a,\bar{y}), f(b,\bar{y}) \in B$. $\Box$(Lemma 4.1)

Lemma 4.2. *Let $\bar{x} \in J(K_0, M)$ and $\bar{y} \in \mathrm{cl}(K_1 \cap M)$. Then, there exist $x_{\sup}, x_{\inf} \in B(K_0, M, \bar{x})$ such that*

$$\sup\{\, f(x, \bar{y}) : x \in I(K_0, M, \bar{x}) \,\} = f(x_{\sup}, \bar{y})$$
$$\inf\{\, f(x, \bar{y}) : x \in I(K_0, M, \bar{x}) \,\} = f(x_{\inf}, \bar{y})$$

In particular, $x \mapsto f(x, \bar{y})$ is constant on $I(K_0, M, \bar{x})$ if and only if $f(\eta(K_0, M, \bar{x}), \bar{y}) = f(\zeta(K_0, M, \bar{x}), \bar{y})$.

Proof. If $\bar{y} \in M$, then $x \mapsto f(x, \bar{y})$ is a function lying in M. So, Lemma implies the conclusion.

Suppose $\bar{y}$ is a limit point of $K_1 \cap M$. Without loss of generality, we may assume that $\bar{y}$ is a limit point of $K_1 \cap M$ from below. Let $a = \eta(K_0, M, \bar{x})$, $b = \zeta(K_0, M, \bar{x})$, and $v = \sup\{\, f(x, \bar{y}) : x \in I(K_0, M, \bar{x}) \,\}$. Suppose $f(a, \bar{y}) < v$ and $f(b, \bar{y}) < v$. Then, there exist $x_0 \in (a, b)$ such that $f(x_0, \bar{y}) = v$. Then, there exist $v_0 \in L$ be so that $\max\{\, f(a, \bar{y}), f(b, \bar{y}) \,\} < v_0 < v$. Then, there exists $c \in K_1 \cap M$ such that $c < \bar{y}$ and for every $y \in [c, \bar{y}]$, $f(a, y) < v_0$, $f(b, y) < v_0$, and $f(x_0, y) > v_0$. However, by Lemma 3.3, since the map $x \mapsto f(x, y)$ belongs to M, $f(x_0, y) \leq \min\{\, f(a, y), f(b, y) \,\}$. This is a contradiction. □(Lemma 4.2)

Lemma 4.3. *Let $\bar{x} \in J(K_0, M)$ and $\bar{y} \in \mathrm{cl}(K_1 \cap M)$. If $f(\eta(K_0, M, \bar{x}), \bar{y}) \in M$ and $\eta(K_0, M, \bar{x}) \notin M$, then for every $x \in I(K_0, M, \bar{x})$, $f(x, \bar{y}) = f(\bar{x}, \bar{y})$.*

Proof. Let $a = \eta(K_0, M, \bar{x})$ and $v = f(a, \bar{y})$. Suppose that $a \notin M$ and $v \in M$ and there exists an $x \in I(K_0, M, \bar{x})$, $f(x, \bar{y}) \neq f(\bar{x}, \bar{y})$.

Since there exists an $x \in I(K_0, M, \bar{x})$, $f(x, \bar{y}) \neq f(\bar{x}, \bar{y})$, there exists an $\bar{x}' \in I(K_0, M, \bar{x})$ such that $f(\bar{x}', \bar{y}) \in J(L, M)$. Set $v' = f(\bar{x}', \bar{y})$. Let U be an open neighborhood of $\bar{y}$ in K_1 such that for every $y \in U$, $f(\bar{x}', y) \in I(L, M, v')$ and $|B(L, M, v') \cap f^{\rightarrow}(\{\, a \,\} \times U)| = 1$. Since $\bar{y} \in \mathrm{cl}(K_1 \cap M)$, there exists a $y \in U \cap M$. Then, we have $f(\bar{x}', y) \in I(L, M, v')$. Hence, $f(a, y) \in B(L, M, v')$. Since both $f(a, y)$ and $f(a, \bar{y})$ belong to $B(L, M, v') \cap f^{\rightarrow}(\{\, a \,\} \times U)$, we have $f(a, y) = f(a, \bar{y}) \notin M$. By Lemma 3.5, the map $x \mapsto f(x, y)$ is constant on $I(K_0, M, \bar{x})$. However, we have $f(\bar{x}', y) \in I(L, M, v')$ and hence $f(a, y) \neq f(\bar{x}', y)$. This is a contradiction. □(Lemma 4.3)

Lemma 4.4. *Let $\bar{x} \in J(K_0, M)$, and $\bar{y} \in K_1$ be such that $f(\bar{x}, \bar{y}) \in J(L, M)$.*

(i) *If $\bar{y}$ is a limit point of $K_1 \cap M$ from below, then there exists a $\bar{y}' \in K_1$ such that $\bar{y}' < \bar{y}$ and for every $y \in [\bar{y}', \bar{y}]$, $f(\eta(K_0, M, \bar{x}), y) = f(\eta(K_0, M, \bar{x}), \bar{y})$, and $f(\zeta(K_0, M, \bar{x}), y) = f(\zeta(K_0, M, \bar{x}), \bar{y})$.*

(ii) *If $\bar{y}$ is a limit point of $K_1 \cap M$ from above, then there exists a $\bar{y}' \in K_1$ such that $\bar{y}' > \bar{y}$ and for every $y \in [\bar{y}, \bar{y}']$, $f(\eta(K_0, M, \bar{x}), y) = f(\eta(K_0, M, \bar{x}), \bar{y})$, and $f(\zeta(K_0, M, \bar{x}), y) = f(\zeta(K_0, M, \bar{x}), \bar{y})$.*

Proof. We shall prove only (i) as a similar proof works for (ii).

Let $a = \eta(K_0, M, \bar{x})$, $b = \zeta(K_0, M, \bar{x})$, and $v = f(\bar{x}, \bar{y})$. Since $f(a, \bar{y}) \in \operatorname{cl}(L \cap M)$ and $f(\bar{x}, \bar{y}) \notin \operatorname{cl}(L \cap M)$, the map $y \mapsto f(x, \bar{y})$ is not constant on $[a, b]$. By Lemma 4.2, we can see $f(a, \bar{y}) \neq f(b, \bar{y})$.

Then, there exists $y_1 \in K_1$ such that $y_1 < \bar{y}$ and

$$|B(L, M, v) \cap f^{\rightarrow}(\{\, a \,\} \times [y_1, \bar{y}])| = 1$$
$$|B(L, M, v) \cap f^{\rightarrow}(\{\, b \,\} \times [y_1, \bar{y}])| = 1$$
$$f^{\rightarrow}(\{\, \bar{x} \,\} \times [y_1, \bar{y}]) \subseteq I(L, M, v)$$

Claim 1. For every $y \in [y_1, \bar{y}] \cap M$, $f(a, y) = f(a, \bar{y})$, and $f(b, y) = f(b, \bar{y})$.

$\vdash$ Let $y \in [y_1, \bar{y}] \cap M$. Then, by the definition of y', $f(\bar{x}, y) \in I(L, M, v)$. By Lemma 4.1, we have $f(a, y) \in B(L, M, f(\bar{x}, y)) = B(L, M, v)$ and $f(b, y) \in B(L, M, f(\bar{x}, y)) = B(L, M, v)$. Hence, since $y_1 < y < \bar{y}$, we have $f(a, y) = f(a, \bar{y})$ and $f(b, y) = f(b, \bar{y})$. $\dashv$ (Claim 1)

Now, let $y \in [y_1, \bar{y}]$. We shall show that $f(a, y) = f(a, \bar{y})$ and $f(b, y) = f(b, \bar{y})$. If $y \in M$, then we are done. Suppose not. Let $c = \eta(K_1, M, y)$ and $d = \zeta(K_1, M, y)$. Note that both exist since $y_1 < y < \bar{y}$, $\bar{y}$ is a limit point of $K_1 \cap M$ from above, and $y_1 \in M$. By the previous claim, we have $f(a, c) = f(a, d) = f(a, \bar{y})$ and $f(b, c) = f(b, d) = f(b, \bar{y})$. By Lemma 4.2, we can show that the maps $y' \mapsto f(a, y')$ and $y' \mapsto f(b, y')$ are both constant. So, $f(a, y) = f(a, \bar{y})$ and $f(b, y) = f(b, \bar{y})$. $\square$(Lemma 4.4)

Lemma 4.5. *Let $\bar{x} \in J(K_0, M)$ and $\bar{y} \in J(K_1, M)$. Let $a = \eta(K_0, M, \bar{x})$, $b = \zeta(K_0, M, \bar{x})$, $c = \eta(K_1, M, \bar{y})$, and $d = \zeta(K_1, M, \bar{y})$. Let $v \in L \setminus f^{\rightarrow}(\partial(I(K_0, M, \bar{x}) \times I(K_1, M, \bar{y})))$.*

Then, there exist $a', b' \in K_0 \cap M$ and $c', d' \in K_1 \cap M$ such that $a' \leq a < b \leq b'$, $c' \leq c < d \leq d'$, and for every $\langle x, y \rangle \in [a', b'] \times [c', d']$, if $f(x, y) = v$, then $\langle x, y \rangle \in (a, b) \times [c, d]$.

Proof. Since $\partial(I(K_0, M, \bar{x}) \times I(K_1, M, \bar{y})$ is closed, for each $\langle x, y \rangle \in \partial(I(K_0, M, \bar{x}) \times I(K_1, M, \bar{y}))$, there exists an open neighborhood $U_{x,y}$ of $\langle x, y \rangle$ such that $v \notin f^{\rightarrow} \operatorname{cl}(U_{x,y})$. Without loss of generality, we may assume that there exist $a_{x,y}, b_{x,y} \in K_0$ and $c_{x,y}, d_{x,y} \in K_1$ such that $U_{x,y} = (a_{x,y}, b_{x,y}) \times (c_{x,y}, d_{x,y})$. Notice that $\partial(I(K_0, M, \bar{x}) \times I(K_1, M, \bar{y})$ is compact. So, there are $\langle x_0, y_0 \rangle, \ldots, \langle x_{n-1}, y_{n-1} \rangle$ such that $\partial(I(K_0, M, \bar{x}) \times I(K_1, M, \bar{y}) \subseteq U_{x_0,y_0} \cup \ldots \cup U_{x_{n-1},y_{n-1}}$.

Let $a' \in K_0 \cap M$ be so that $a' \leq a$ and for every $i < n$, if $a_{x_i,y_i} < a$, then $a_{x_i,y_i} < a'$. Let $b' \in K_0 \cap M$ be so that $b \leq b'$ and for every $i < n$, if $b < b_{x_i,y_i}$, then $b' < b_{x_i,y_i}$. Let $c' \in K_1 \cap M$ be so that $c' \leq c$ and for every $i < n$, if $c_{x_i,y_i} < c$, then $c_{x_i,y_i} < c'$. Let $d' \in K_1 \cap M$ be so that $d \leq d'$ and for every $i < n$, if $d < d_{x_i,y_i}$, then $d < d_{x_i,y_i}$. Then, it is easy to see that $[a', b'] \times [c', d'] \subseteq ((a, b) \times (c, d)) \cup U_{x_0,y_0} \cup \cdots \cup U_{x_{n-1},y_{n-1}}$. So, if $\langle x, y \rangle \in [a', b'] \times [c', d']$ and $f(x, y) = v$, then $\langle x, y \rangle \in (a, b) \times (c, d)$.

$\square$(Lemma 4.5)

Lemma 4.6. *Let* $\bar{x} \in J(K_0, M)$ *and* $\bar{y} \in J(K_1, M)$*. Then,* $\sup f^{\to}(I(K_0, M, \bar{x}) \times I(K_1, M, \bar{y})) = \max f^{\to}(B(K_0, M, \bar{x}) \times B(K_1, M, \bar{y}))$*. Similarly, for infimum.*

Proof. Let $a = \eta(K_0, M, \bar{x})$, $b = \zeta(K_0, M, \bar{x})$, $c = \eta(K_1, M, \bar{y})$, and $d = \zeta(K_1, M, \bar{y})$. Then, we have $I(K_0, M, \bar{x}) \times I(K_1, M, \bar{y}) = (a, b) \times (c, d)$, and $B(K_0, M, \bar{x}) \times B(K_1, M, \bar{y}) = \{ \langle a, c\rangle, \langle a, d\rangle, \langle b, c\rangle, \langle b, d\rangle \}$. Let $v = \sup f^{\to}((a, b) \times (c, d))$.

Claim 1. $v \in f^{\to}(\partial(I(K_0, M, \bar{x}) \times I(K_1, M, \bar{y})))$

$\vdash$ Suppose not, i.e. $v \notin f^{\to}(\partial(I(K_0, M, \bar{x}) \times I(K_1, M, \bar{y})))$. Then by Lemma 4.5, there exist $a', b' \in K_0 \cap M$ and $c', d' \in K_1 \cap M$ with $a' \leq a < b \leq b'$ and $c' \leq c < d \leq d'$ such that for every $\langle x, y\rangle \in [a', b'] \times [c', d']$, if $f(x, y) = v$, then $\langle x, y\rangle \in (a, b) \times (c, d)$. Note that there exists $\langle x, y\rangle \in [a', b'] \times [c', d']$ such that $f(x, y) = v$. By the elementarity of M¡ there exists such an $\langle x, y\rangle \in M$. By the definition of a', b', c' and d', we have $\langle x, y\rangle \in (a, b) \times (c, d)$. This is a contradiction to the definition of a, b, c, and d. $\dashv$ (Claim 1)

Let $\langle x_0, y_0\rangle \in \partial(I(K_0, M, \bar{x}) \times I(K_1, M, \bar{y}))$ be so that $f(x_0, y_0) = v$. Then, at least one of $x_0 = a$, $x_0 = b$, $y_0 = c$ and $y_0 = d$ holds. For example, suppose $y_0 = c$. Then, $v = \sup\{ f(x, c) : x \in I(K_0, M, \bar{x}) \}$. By Lemma 4.2, we have either $f(a, c) = v$ or $f(b, c) = v$. By a similar argument, we can see that $v \in \{ f(a, c), f(b, c), f(a, d), f(b, d) \}$ in other three cases, too. $\square$(Lemma 4.6)

Lemma 4.7. *Let* $\bar{x} \in J(K_0, M)$ *and* $\bar{y} \in J(K_1, M)$ *be such that* $f(\bar{x}, \bar{y}) \in M$*. Then, for every* $x \in I(K_0, M, \bar{x})$ *and* $y \in I(K_1, M, \bar{y})$*,* $f(x, y) = f(\bar{x}, \bar{y})$*.*

Proof. Let $v = f(\bar{x}, \bar{y})$. Let $a = \eta(K_0, M, \bar{x})$, $b = \zeta(K_0, M, \bar{x})$, $c = \eta(K_1, M, \bar{y})$, and $d = \zeta(K_1, M, \bar{y})$.

Case 1. $a, c \in M$

Then, b is a limit point of $K_0 \cap M$ from above, and d is a limit point of $K_1 \cap M$ from above.

Claim 1. $f(b, d) = v$.

$\vdash$ It suffices to show that for every $b' \in K_0$ and $d' \in K_1$, if $b < b'$ and $d < d'$, then there exist $x \in K_0 \cap M$ and $y \in K_1 \cap M$ such that $b < x < b'$, $d < y < d'$, and $f(x, y) = v$. Let $b' \in K_0$ and $d' \in K_1$ with $b < b'$ and $d < d'$. Since b is a limit point of $K_0 \cap M$ from above and d is a limit point of $K_1 \cap M$ from above, there exist $b'' \in K_0 \cap M$ and $d'' \in K_1 \cap M$ such that $b < b'' < b'$ and $d < d'' < d$. Since '$\exists x \in K_0 \exists y \in K_1(a < x <$

$b'' \wedge c < y < d'' \wedge f(x,y) = v)$' holds in V, it also holds in M. So, there exist $x \in K_0 \cap M$ and $y \in K_1 \cap M$ such that $a < x < b''$, $c < y < d''$, and $f(x,y) = v$. However, by the definition of b and d, we have $b < x$ and $d < y$. $\dashv$ (Claim 1)

By Lemma 4.3, since $d \notin M$ and $f(b,d) = v \in M$, for every $y \in [c,d]$, $f(b,y) = v$. In particular, $f(b,c) = v$.

By a similar argument, we can show that for every $x \in [a,b]$, $f(x,d) = v$. In particular, $f(a,d) = v$.

Since $f(b,c) = v$, by Lemma 4.3, $f(a,b) = v$. By Lemma 4.6, for every $x \in [a,b] \times [c,d]$, $f(x,y) = v$.

Case 2. $a \in M$ and $c, d \notin M$.

Claim 2. For every $b' \in K_0$ and $c', d' \in K_1$ with $b < b'$ and $c' < c < d < d'$, there exists $\langle x, y\rangle \in K_0 \times K_1$ such that $b < x < b'$, $y \in (c',c) \cup (d,d')$, and $f(x,y) = v$.

$\vdash$ Since '$\exists \langle x,y\rangle \in K_0 \times K_1 (a < x < b' \wedge c' < y < d' \wedge f(x,y) = v)$' is true in V, there exist $\langle x,y\rangle \in (K_0 \times K_1) \cap M$ such that $a < x < b'$, $c' < y < d'$, and $f(x,y) = v$. By the definition of a and b, since $x \in M$, we have $x \notin (a,b]$ and hence $b < x < b'$. Similarly, by the definition of c and d, since $y \in M$, we have $y \notin [c,d]$, hence $y \in (c',c) \cup (d,d')$. $\dashv$ (Claim 2)

By the previous claim, it is easy to see that either $f(b,c) = v$ or $f(b,d) = v$. By the same argument as in the previous case, we can show that for every $y \in K_1$ with $c \leq y \leq d$, $f(b,y) = v$. Since $f(b,c) = f(b,d) = v$ and $b \notin M$, for every $x \in K_0 \cap M$ with $a \leq x \leq b$, we have $f(x,c) = f(x,d) = v$. In particular, we have $f(a,c) = f(a,d) = f(b,c) = f(b,d) = v$. By Lemma 4.6, for every $x \in K_0$ and $y \in K_1$ with $a \leq x \leq b$ and $c \leq y \leq d$, we have $f(x,y) = v$.

Case 3. $a, b, c, d \notin M$.

As in the previous case, we can show the following claim.

Claim 3. For every $a', b' \in K_0$ and $c', d' \in K_1$ with $a' < a < b < b'$ and $c' < c < d < d'$, there exist $x \in (a',a) \cup (b,b')$ and $y \in (c',c) \cup (d,d')$ such that $f(x,y) = v$.

It implies that at least one of $f(a,c)$, $f(a,d)$, $f(b,c)$, and $f(b,d)$ is equal to v. In all cases, we can apply Lemma 4.3 several times to show that $f(a,c) = f(a,d) = f(b,c) = f(b,d) = v$. $\Box$(Lemma 4.7)

By the previous lemma, we can easily see the following lemma.

Lemma 4.8. *Let $\bar{x} \in J(K_0, M)$ and $\bar{y} \in J(K_1, M)$. If f is not constant on $I(K_0, M, \bar{x}) \times I(K_1, M, \bar{y})$), then*

$$f^{\to}(I(K_0, M, \bar{x}) \times I(K_1, M, \bar{y})) = I(L, M, f(\bar{x}, \bar{y}))$$
$$f^{\to}(B(K_0, M, \bar{x}) \times B(K_1, M, \bar{y})) = B(L, M, f(\bar{x}, \bar{y}))$$

Proof. Suppose that f is not constant on $I(K_0, M, \bar{x}) \times I(K_1, M, \bar{y})$). Then, without loss of generality, we may assume $f(\bar{x}, \bar{y}) \in J(L, M)$. By Lemma 4.7, for every $\langle x, y\rangle \in I(K_0, M, \bar{x}) \times I(K_1, M, \bar{y})$, $f(x, y) \notin M$. Hence, we have $f^{\to}(I(K_0, M, \bar{x}) \times I(K_1, M, \bar{y})) \subseteq I(L, M, f(\bar{x}, \bar{y}))$.

It is easy to see $f^{\to}(B(K_0, M, \bar{x}) \times B(K_1, M, \bar{y})) \subseteq B(L, M, f(\bar{x}, \bar{y}))$. But if $f^{\to}(B(K_0, M, \bar{x}) \times B(K_1, M, \bar{y}))$ is a singleton, then by Lemma 4.6, f is constant on $I(K_0, M, \bar{x}) \times I(K_1, M, \bar{y})$, which is a contradiction to the assumption. So, $f^{\to}(B(K_0, M, \bar{x}) \times B(K_1, M, \bar{y})) = B(L, M, f(\bar{x}, \bar{y}))$. It follows that $f^{\to}(I(K_0, M, \bar{x}) \times I(K_1, M, \bar{y})) = I(L, M, f(\bar{x}, \bar{y}))$. □(Lemma 4.8)

5. $f : K_0 \times K_1 \to L_0 \times L_1$

Let K_0, K_1, L_0, L_1 be connected linearly ordered spaces. Let $f : K_0 \times K_1 \to L_0 \times L_1$ be an injective continuous function. Let g_0, g_1 be so that $f(x, y) = \langle g_0(x, y), g_1(x, y)\rangle$ for every $\langle x, y\rangle \in K_0 \times K_1$. Let M be a countable elementary submodel of $H(\theta)$ for some regular cardinal θ with $\mathcal{P}(\mathcal{P}(K \cup L)) \in H(\theta)$.

Lemma 5.1. *Let $\bar{x} \in J(K_0, M)$ and $\bar{y} \in J(K_1, M)$. Then, for every $i < 2$, g_i is not constant on $I(K_0, M, \bar{x}) \times I(K_1, M, \bar{y})$.*

Proof. Let $a = \eta(K_0, M, \bar{x})$, $b = \zeta(K_0, M, \bar{x})$, $c = \eta(K_1, M, \bar{y})$, and $d = \zeta(K_1, M, \bar{y})$. Suppose not, i.e. there exists $i < 2$ such that g_i is constant on $I(K_0, M, \bar{x}) \times I(K_1, M, \bar{y})$. .

Since f is injective, we have either $g_{1-i}(a, c) < g_{1-i}(b, c)$ or $g_{1-i}(a, c) > g_{1-i}(b, c)$. Without loss of generality, we may assume $g_{1-i}(a, c) < g_{1-i}(b, c)$.

Claim 1. $g_{1-i}(b, c) < g_{1-i}(b, d)$

$\vdash$ Suppose not. Then, $g_{1-i}(b, c) > g_{1-i}(b, d)$. If $g_{1-i}(b, c) < g_{1-i}(a, c)$, then by Lemma 2.5, there exist $y \in (c, d)$ such that $g_{1-i}(b, y) = g_{1-i}(a, c)$. This is a contradiction to the assumption that f is injective. $\dashv$ (Claim 1)

So, we have $g_{1-i}(b, c) < g_{1-i}(b, d)$. Similarly, $g_{1-i}(b, d) < g_{1-i}(a, d) < g_{1-i}(a, c)$. Therefore, $g_{1-i}(a, c) < g_{1-i}(b, c) < g_{1-i}(b, d) < g_{1-i}(a, d) < g_{1-i}(a, c)$, which is a contradiction. □(Lemma 5.1)

Lemma 5.2. *Let $\bar{x} \in J(K_0, M)$ and $\bar{y} \in J(K_1, M)$. Let $a = \eta(K_0, M, \bar{x})$, $b = \zeta(K_0, M, \bar{x})$, $c = \eta(K_1, M, \bar{y})$, and $d = \zeta(K_1, M, \bar{y})$. Then, there exists $i < 2$ such that $g_i(a, c) = g_i(a, d)$, $g_i(b, c) = g_i(b, d)$, $g_{1-i}(a, c) = g_{1-i}(b, c)$, and $g_{1-i}(a, d) = g_{1-i}(b, d)$.*

Proof. By Lemma 5.1, for every $i < 2$, $g_i(\bar{x}, \bar{y}) \in J(L_i, M)$. Let $v = g_0(\bar{x}, \bar{y})$ and $w = g_1(\bar{x}, \bar{y})$. Let $a' = \eta(L_0, M, v)$, $b' = \zeta(L_0, M, v)$, $c' = \eta(L_1, M, w)$, and $d' = \zeta(L_1, M, w)$. By Lemma 4.8, for every $i < 2$, $g_i{}^{\rightarrow}(\{ a, b \} \times \{ c, d \}) = B(L, M, g_i(\bar{x}, \bar{y}))$, which means that $\{ f(a, c), f(a, d), f(b, c), f(b, d) \} = \{ \langle a', c' \rangle, \langle a', d' \rangle, \langle b', c' \rangle, \langle b', d' \rangle \}$. By reversing the order of L_0 and/or L_1 if necessary, we may assume $f(a, c) = \langle a', c' \rangle$.

Claim 1. If for every $i < 2$, $g_i(a, c) \neq g_i(b, c)$, then $c \in M$

$\vdash$ Suppose $c \notin M$. Then, c is a limit point of $K_1 \cap M$ from below. By Lemma 4.4, there exists $c' \in K_1$ such that $c' < c$ and for every $y \in K_1$ and $i < 2$, if $c' < y < c$, then $g_i(a, y) = g_i(a, c)$ and $g_i(b, y) = g_i(b, c)$. Clearly, this contradicts to the assumption that f is injective. $\dashv$ (Claim 1)

Claim 2. There exists $i < 2$ such that $g_i(a, c) = g_i(b, c)$.

$\vdash$ Suppose not. By the previous claim, $c \in M$. We also have $f(b, c) = \langle b', d' \rangle$. Then, we have either '$f(a, d) = \langle a', d' \rangle$ and $f(b, d) = \langle b', c' \rangle$' or vice versa. In either case, we have $g_i(a, d) \neq g_i(b, d)$ for every $i < 2$. However, by the same argument as in the previous claim, we can show $d \in M$. This is a contradiction. $\dashv$ (Claim 2)

By the same argument, there exist $j, k, l < 2$ such that $g_j(b, c) = g_j(b, d)$, $g_k(b, d) = g_k(a, d)$, and $g_l(a, d) = g_l(a, c)$. But since f is injective, it is easy to see $i = k \neq j = l$. $\square$(Lemma 5.2)

Lemma 5.3. *Let $\bar{x} \in J(K_0, M)$, $a = \eta(K_0, M, \bar{x})$, and $b = \zeta(K_0, M, \bar{x})$. Then, for every $y_0 \in J(K_1, M)$, there exists $i < 2$ such that for every $y \in K_1$ with $y_0 < y \leq \sup(K_1 \cap M)$, we have $g_i(a, y_0) = g_i(a, y)$ and $g_i(b, y_0) = g_i(b, y)$.*

Proof. Let $y_0 \in J(K_1, M)$. Then, by Lemma 5.2, there exists $i < 2$ such that for every $y \in K_1$, if $y_0 \leq y \leq \zeta(K_1, M, y_0)$, $g_i(a, y) = g_i(a, y_0)$ and $g_i(b, y) = g_i(b, y_0)$. Moreover, $g_i(a, y_0) \neq g_i(b, y_0)$. ' Now, we shall show that for every $y \in K_1$, if $y_0 \leq y < \sup(K_1 \cap M)$, then $g_i(a, y) = g_i(a, y_0)$ and $g_i(b, y) = g_i(b, y_0)$. Suppose not. Let y_1 be the infimum of $y \in K_1$ with $y \geq y_0$ such that either $g_i(a, y) \neq g(a, y_0)$ or $g_i(b, y) \neq g(b, y_0)$. Note $g_i(a, y_1) = g(a, y_0)$ and $g_i(b, y_1) = g(b, y_0)$.

Case 1. y_1 is a limit point of $K_1 \cap M$ from above.

Since $g_i(a, y_1) = g_i(a, y_0) \neq g_i(b, y_0) = g_i(b, y_1)$, there exists $x_0 \in K_1$ such that $a < x_0 < b$ and $g_i(x_0, y_1) \in J(L_i, M)$. So, by Lemma 4.4, there exists $y_2 \in K_1$ such that $y_1 < y_2$ and for every $y \in K_1$, if $y_1 \leq y \leq y_2$, then $g_i(a, y) = g_i(a, y_1)$ and $g_i(a, y) = g_i(a, y_1)$. But this is a contradiction to the definition of y_1.

Case 2. y_1 is a limit point of $K_1 \cap M$ from below, but not from above.

Let $d = \zeta(K_1, M, y_1)$. By Lemma 5.2, since $g_i(a, y_1) \neq g_i(b, y_1)$, we have for every $y \in K_1$, if $y_1 \leq y \leq d$, $g_i(a, y) = g_i(a, y_1)$ and $g_i(b, y) = g_i(b, y_1)$. This is a contradiction to the definition of y_1.

Case 3. y_1 is a limit point of $K_1 \cap M$ neither from above nor from below.

Let $c = \eta(K_1, M, y_1)$ and $d = \zeta(K_1, M, y_1)$. Then, we have $g_i(a, c) = g_i(a, y_0) \neq g_i(b, y_1) = g_i(b, c)$. By Lemma 5.2, for every $y \in K_1$, if $c \leq y \leq d$, then $g_i(a, y) = g_i(a, c) = g_(a, y_0)$ and $g_i(b, y) = g_i(b, c) = g_i(b, y_0)$. This is a contradiction to the definition of y_1. □(Lemma 5.3)

Lemma 5.4. *Let $\bar{x} \in J(K_0, M)$, $a = \eta(K_0, M, \bar{x})$, and $b = \zeta(K_0, M, \bar{x})$. Then, there exists $i < 2$ such that for every $y, y' \in (\inf(K_1 \cap M), \sup(K_1 \cap M))$, we have $g_i(a, y) = g_i(a, y')$ and $g_i(b, y) = g_i(b, y')$*

Proof. Let $y_0 \in J(K_1, M)$ be arbitrary. Let $c = \eta(K_1, M, y_0)$ and $d = \zeta(K_1, M, y_0)$. By Lemma 5.2, there exists $i < 2$ such that for every $y \in [c, d]$, $g_i(a, y) = g_i(a, y_0)$ and $g_i(b, y) = g_i(b, y_0)$.

By Lemma 5.3, there exists $j < 2$ such that for every $y \in K_1$, if $y_0 \leq y < \sup(K_1 \cap M)$, then $g_j(a, y) = g_j(a, y_0)$ and $g_j(b, y) = g_j(b, y_0)$. However, since $g_i(a, y) = g_i(a, y_0)$, we have $j = i$. By using symmetry, we can also show that for every $y \in K_1$, if $\inf(K_1 \cap M) < y \leq y_0$, $g_i(a, y) = g_i(a, y_0)$ and $g_i(b, y) = g_i(b, y_0)$. □(Lemma 5.4)

6. The proof of the main theorem

Let $K_0, K_1, L_0, L_1, f, g_0, g_1$ be as in the previous section. Now we do not fix M.

Lemma 6.1. *For every $\langle x_0, y_0 \rangle \in K_0 \times K_1$, there exist $i < 2$ such that for every $y \in K_1$, $g_i(x_0, y) = g_i(x_0, y_0)$.*

Proof. Let $x_0 \in K_0$ and $y_0 \in K_1$. By way of contradiction, we assume that for every $i < 2$. there exists $z_i \in K_1$ such that $g_i(x_0, z_i) \neq g_i(x_0, y_0)$.

Let M be a countable elementary submodel of $H(\theta)$ with K_0, K_1, L_0, L_1, f, g_0, g_1, x_0, y_0, z_0, $z_1 \in M$.

By Lemma 5.4, there exists $i < 2$ such that for every $y \in K_1$, if $\inf(K_1 \cap M) < y < \sup(K_1 \cap M)$, then $g_i(x_0, y) = g_i(x_0, y_0)$. But then $g_i(x_0, z_i) = g_i(x_0, y_0)$, which is a contradiction. □(Lemma 6.1)

We can finally show the main theorem.

Proof of Theorem 1.4. By Lemma 6.1, for every $x \in K_0$, there exists $i_x < 2$ such that for every $y, y' \in K_1$, $g_{i_x}(x, y) = g_{i_x}(x, y')$. Similarly, for every $y \in K_1$, there exist $j_y < 2$ such that for every $x, x' \in K_0$, $g_{j_y}(x, y) = g_{j_y}(x', y)$. Now it suffices to show there exist $i \neq j < 2$ such that for $x \in K_0$, $i_x = i$ and for every $y \in K_1$, $j_y = j$.

Let $y \in K_1$ be arbitrary and let $j = j_y$. We shall show that for every $x \in K_0$, $i_x \neq j$. Suppose not, i.e. $i_x = j$. Let $a, b \in K_0$ be so that $a < x < b$ and $c, d \in K_1$ so that $c < y < d$. Since $j = j_y = i_x$, we have $g_j(a, y) = g_j(b, y) = g_j(x, c) = g_j(x, d) = g_j(x, y)$. Since f is injective, none of $g_{1-j}(a, y)$, $g_{1-j}(b, y)$, $g_{1-j}(x, c)$, and $g_{1-j}(x, d)$ are equal to $g_{1-j}(x, y)$. So, either there exist two of them that are greater than $g_{1-j}(x, y)$, or there exist two of them that are smaller than $g_{1-j}(x, y)$[1]. For example, suppose $g_{1-j}(a, y) > g_{1-j}(x, c) > g_{1-j}(x, y)$. By the intermediate value theorem, there exists $a' \in K_0$ such that $a < a' < x$ and $g_{1-j}(a', y) = g_{1-j}(x, c)$. Then, $f(a', y) = f(x, c)$, which is a contradiction to the injectivity of f. $\square$(Theorem 1.4)

7. Open questions

The author also prove the following theorem that extends Theorem 1.4 to any finite number of connected linearly ordered spaces though the proof is significantly more difficult and complicated.

Theorem 7.1. *Let $K_0, K_1, \ldots, K_{n-1}, L_0, L_1, \ldots, L_{n-1}$ be connected nowhere real linearly ordered spaces, and $f : \Pi_{i<n} K_i \to \Pi_{i<n} L_i$ a continuous injection. Then, f is coordinate-wise.*

A paper about this result is in preparation.

Theorem 1.4 demonstrates that the situation is totally different when we use connected nowhere real linearly ordered spaces instead of $\mathbb{R}$. Hence, we may ask the following broad question.

Question 1. What theorems about $\mathbb{R}$ can be extended to generalized linearly ordered spaces?

Considering the lemmas proved in this article, it seems very unlikely that any nice theorems about $\mathbb{R}$ can be generalized to connected linearly ordered spaces?

While we heavily relied on the linear orders in this article, we have no evidence that it is essential. It means that the following question is still wide open.

Question 2. Can we weaken the assumption that K_0, K_1, L_0, L_1 are linearly ordered spaces? For example, what if they are 1-dimensional in some sense?

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[1]You do not need to have these four points, but three among these four suffice