BROUWER'S FIXED-POINT THEOREM ON THE FINITE PRODUCT OF COMPACT LINEARLY ORDERED TOPOLOGICAL SPACES

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1. Introduction

A linearly ordered topological space (LOTS) is a linearly ordered set topologized by the order topology. See Definition 2.1 for the precise definition. Clearly, the most important example of a LOTS is $\mathbb{R}$. It is known that a LOTS is homeomorphic to a subspace of $\mathbb{R}$ if and only if it is separable. So, the finite product of compact connected LOTS can be considered as a generalization of a subspace of an Euclidean space. Since every separable LOTS is homeomorphic to a subset of $\mathbb{R}$, the finite product of separable LOTS is homeomorphic to a subset of a Euclidean space.

However, the finite product of nonseparable LOTS behaves in a very different way from Euclidean spaces. L. B. Treybig proved the following theorem:

Theorem 1.1 (Treybig [8])**.** *If X and Y are infinite Hausdorff spaces and $X \times Y$ is a continuous image of a compact LOTS, then both X and Y are measurable.*

From this theorem, we can prove the following corollary.

Corollary 1.2. *Let K_1 and K_2 be nonseparable LOTS. Then, $K_1 \times K_2$ is not measurable*

The Mardešić Conjecture, which can be interpreted as the extension of Theorem 1.1 to the product of any finite product of infinite Hausdorff spaces, was proved by G. Martínez-Cervantes and G. Plebanek in [4].

Another demonstration of how different these two types of spaces are is the following theorem, proved by the author in [2]. Here, we say that a topological space is *nowhere separable* if and only if there is no nonempty open set that is separable.

Theorem 1.3 (T. Ishiu [2])**.** *Let n be a positive integer. Let K_i and L_i be connected nowhere separable LOTS. If $f : \prod_{i<n} K_i \to \prod_{i<n} L_i$ is a continuous injective function, then f is coordinate-wise, namely there exists a*

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bijection $h : n \to n$ and a function $\tau_i : K_{h(i)} \to L_i$ for each $i < n$ such that for all $x \in \prod_{i<n} K_i$ and $k < n$,

$$f(x)(i) = \tau_i(x(h(i)))$$

Namely, the i-th coordinate of $f(x)$ depends only on the $h(i)$-th coordinate of x.

It is easy to observe that Euclidean spaces of dimension $\geq$2 have many autohomeomorphisms that are not coordinate-wise, such as rotations of less than 90°. Thus, Theorem 1.3 demonstrate how different Euclidean spaces and the finite product of nowhere separable LOTS are.

However, we can also prove some properties on Euclidean spaces are shared by the finite product of nowhere separable LOTS by using Theorem 1.3. Recall Brouwer's Fixed-point theorem, whose statement is as follows.

Theorem 1.4 (L. E. J. Brower [1])**.** *Every continuous function f from nonempty compact convex subset of a Euclidean space to itself has a fixed point.*

Meanwhile, by Theorem 1.3, it is easy to show that every function f from the finite product of compact connected nowhere separable LOTS to itself has a fixed point.

The following theorem, often called the Poincaré-Miranda Theorem, was conjectured by H. Poincaré in 1883 and proved by C. Miranda in 1940.

Theorem 1.5 (C. Miranda [5])**.** *Let n be a positive integer. Let $f_i : [-1,1]^n \to \mathbb{R}$ be a continuous function for each $i < n$ such that for each $i < n$ and every $x \in [-1,1]^n$, if $x(i) = -1$, then $f_i(x) < 0$ and if $x(i) = 1$, then $f_i(x) > 0$. Then, there exists $x \in [-1,1]^n$ such that for every $i < n$, $f(x)(i) = 0$.*

A natural variation of this theorem for the finite product of compact connected nowhere separable LOTS can be easily shown by Theorem 1.3.

In this paper, we shall prove variations of Brouwer's fixed-point theorem and the Poincaré-Miranda Theorem for the finite product of any compact connected LOTS. So, it covers the original versions and cases for the finite product of nowhere separable LOTS, and more. Note that we rely on the original versions to prove these generalizations, so they do not provide new ways to prove the original.

In the proofs of both theorems, the use of elementary substructures is essential. The author used elementary substructures to analyze the finite product of nonseparable LOTS in [2], [3], and several other papers. In this paper, we shall use them in a slightly different way so that it can even cover separable LOTS. Let n be a positive integer. For each $i < n$, let K_i and L_i be a compact connected LOTS. Suppose f is a continuous function from $\prod_{i<n} K_i$ into $\prod_{i<n} L_i$. Let M be a countable elementary substructure of $H(\theta)$ for some sufficiently large regular cardinal θ. In Sections 3, 4, and 5, we shall define a function $\varphi(f, M)$. The domain and codomain of $\varphi(f, M)$

is homeomorphic to $[0,1]^n$, so we can apply the original forms of Brouwer's fixed-point theorem and the Poincaré-Miranda Theorem. If we choose an appropriate M, this is sufficient to prove both theorems for f.

We believe that the method of this paper can be applied to prove more generalizations of theorems on Euclidean spaces to the finite product of connected LOTS. In particular, we conjecture that the Invariance of Domain Theorem can be proved for the finite product of connected LOTS, namely every continuous injection from the finite product of connected LOTS into the finite product of connected LOTS is an open map.

I dedicate this paper to my father, who had never lost genuine interest in new things until his last day.

2. Preliminary

In this paper, we shall identify an n-tuple with a function whose domain $\{0, 1, \ldots, n-1\}$. So, $x(i)$ denotes the i-th component of x. We shall use π_i to denote the i-th projection, i.e., $\pi_i(x) = x(i)$.

Recall that the order topology on a linearly ordered set is defined as follows:

Definition 2.1. Let K be a linearly ordered set. Define a set $\mathcal{B}_K$ of subsets of K by $U \in \mathcal{B}_K$ if and only if

(i) $U = (a, b)$ for some $a, b \in K$ with $a < b$,
(ii) $U = [\min K, b)$ for some $b \in K$ if $\min K$ exists, and
(iii) $U = (a, \max K]$ for some $a \in K$ if $\max K$ exists.

The topology generated by $\mathcal{B}_K$ is called the *order topology on* K. A linearly ordered set equipped with the order topology is called a *linearly ordered topological space*, which is often abbreviated as LOTS.

Fact 2.2. Let K be a LOTS. Define a set $\mathcal{S}_K$ of subsets of K by $U \in \mathcal{S}_K$ if and only if

(i) $U = \{x \in K \mid x < b\}$ for some $b \in K$, or
(ii) $U = \{x \in K \mid x > a\}$ for some $a \in K$.

Then, $\mathcal{S}_K$ is a subbasis for the topology of K. Namely, the topology of K is the smallest topology on K that contain $\mathcal{S}_K$.

See [6] for more information about the order topology. We shall remind the readers of the basic facts about LOTS.

Fact 2.3. Let K be a LOTS.

(i) If K is connected, then K is self-dense and has the least upper-bound property.
(ii) If K is separable and connected, then K is order-isomorphic to a convex subset of $\mathbb{R}$, i.e. one of $\{0\}$, $[0,1]$, $[0,1)$, $(0,1]$, and $(0,1)$.
(iii) If K is connected and has both endpoints, then K is compact.
(iv) If K is compact, then K has both endpoints.

The following lemma is used to simplify proofs in the later sections.

Lemma 2.4. *Let K be a LOTS, $x \in K$, and U an open neighborhood of x. Then, there exist $a, b \in K$ such that $[a, b] \subseteq U$ and $[a, b]$ is a neighborhood of x, i.e. $[a, b]$ contains an open neighborhood of x.*

Proof. Since K is regular, there exists an open neighborhood U' of x such that $\mathrm{cl}(U') \subseteq U$. Without loss of generality, we may assume that $U' \in \mathfrak{B}_K$. Notice that in all cases of the definition of $\mathfrak{B}_K$, it is easy to see that there exists $a, b \in K$ such that $\mathrm{cl}(U') = [a, b]$. So, $U' \subseteq [a, b] = \mathrm{cl}(U') \subseteq U$ $\square$

We shall use the following notations that are generalized to the product of LOTS.

Definition 2.5. Let n be positive integer, and let K_i be a linearly ordered topological space for each $i < n$. For each $x, y \in \prod_{i<n} K_i$, define

- $x < y$ if and only if for every $i < n$, $x(i) < y(i)$
- $x \leq y$ if and only if for every $i < n$, $x(i) < y(i)$, and
- $[x, y] = \{ z \in \prod_{i<n} K_i \mid x \leq z \leq y \}$.

For every nonempty subset S of $\prod_{i<n} K_i$, define $\inf S$, $\sup S$, $\min S$, and $\max S$ to be the elements of $\prod_{i<n} K_i$ defined by for all $i < n$,

$$
\begin{aligned}
(\inf S)(i) &= \inf \{ x(i) \mid x \in S \} \\
(\sup S)(i) &= \sup \{ x(i) \mid x \in S \} \\
(\min S)(i) &= \min \{ x(i) \mid x \in S \} \\
(\max S)(i) &= \max \{ x(i) \mid x \in S \}
\end{aligned}
$$

if they exist.

Note that in case that K_i is compact and connected for every $i < n$, then both $\inf S$ and $\sup S$ exist for all nonempty subset S of $\prod_{i<n} K_i$. If S is compact, both $\min S$ and $\max S$ exist.

To describe a sufficiently large regular cardinal in the context, we shall define the following notation.

Definition 2.6. Let $S_0, \ldots, S_{n-1}$ be any set. Let $\theta_{S_0,\ldots,S_{n-1}}$ be the least regular cardinal θ such that $\mathcal{P}(S_0 \cup S_1 \cup \cdots \cup S_{n-1}) \in H(\theta)$.

3. The condensation $\varphi(K, M)$

In this section, we shall fix a compact connected LOTS K, a regular cardinal $\theta \geq \theta_K$, and countable elementary substructure M of $H(\theta)$. Note that K has both endpoints. Assume K has at least two points. Then, we shall define a condensation $\varphi(K, M)$ of K. We will not give a general definition of condensations, but the readers may refer to [7] for more information. This condensation plays a critical role in this paper.

Definition 3.1. Define a relation $\sim_{K,M}$ on K by for all $a, b \in K$ with $a \leq b$, $a \sim_{K,M} b$ if and only if $[a, b] \cap M$ is finite.

When K and M are clear from the context, we shall write $\sim$ to mean $\sim_{K,M}$.

Lemma 3.2. *$\sim$ is an equivalence relation.*

Proof. It is trivial to see that $\sim$ is reflexive and symmetric. To show that $\sim$ is transitive, let $a, b, c \in K$ be so that $a \sim b$ and $b \sim c$. We shall show that $a \sim c$. The only nontrivial cases are $a < b < c$ and $c < b < a$. Without loss of generality, we may assume $a < b < c$. Then,

$$[a, c] \cap M = ([a, b] \cap M) \cup ([b, c] \cap M)$$

Since $a \sim b$ and $b \sim c$, $[a, b] \cap M$ and $[b, c] \cap M$ are finite. Thus, $[a, c] \cap M$ is finite and hence $a \sim c$. □

Definition 3.3. For each $a \in K$, we shall write $[a]_{K,M}$ to mean the $\sim_{K,M}$-equivalence class of a. When K and M are clear from the context, we simply write $[a]$ to mean $[a]_{K,M}$.

It is easy to see that for every $a \in K$, $[a]$ is a convex subset of K.

Definition 3.4. Let $\varphi(K, M) = \{ [a] \mid a \in K \}$. We shall define a linear order on $\varphi(K, M)$ by $[a] \leq [b]$ if and only if $a \leq b$. This is well-defined as $[a]$ and $[b]$ are convex. The topology on $\varphi(K, M)$ is given by the order topology.

It is easy to observe that $\varphi(K, M)$ is a condensation in the sense of [7].

Lemma 3.5. *For every $x \in K$, $|[x] \cap M| \leq 1$.*

Proof. Suppose not, i.e., there exist $a, b \in [x] \cap M$ with $a < b$. Since K is connected, $[a, b]$ is infinite. Since $a, b \in M$, by the elementarity of M, $[a, b] \in M$. So, $[a, b] \cap M$ is also infinite. However, we assumed $a, b \in [x]$ and hence $a \sim b$. By definition, it means $[a, b] \cap M$ is finite. This is a contradiction □

Lemma 3.6. *For every $x \in K$, $[x]$ is closed and hence compact in K.*

Proof. Let $x \in K$. Since K has both endpoints, both $\inf[x]$ and $\sup[x]$ exist. Since $[x]$ is convex, it suffices to show that $\inf[x], \sup[x] \in [x]$. We shall show $\inf[x] \in [x]$. The proof of $\sup[x] \in [x]$ is similar.

Suppose $\inf[x] \notin [x]$, i.e., $\inf[x] \not\sim x$. Then, $[\inf[x], x] \cap M$ is infinite. Then, there exists $a \in (\inf[x], x)$ such that $|[a, x] \cap M| \geq 2$. By Lemma 3.5, we have $a \not\sim x$. Since $[x]$ is convex, we have $(a, x] \subseteq [x]$. So, $a \leq \inf[x]$. This is a contradiction to the assumption $a \in (\inf[x], x)$. □

By the previous lemma, we can see that for every $x \in K$, $\min[x]$ and $\max[x]$ always exist.

Lemma 3.7. *Let $x \in K$. Then,*

(i) If $\min[x]$ is not a minimum element of K, then $\min[x]$ is a limit point of $K \cap M$ from below.

(ii) If $\max[x]$ *is not a maximum element of* K*,* $\max[x]$ *is a limit point of* $K \cap M$ *from above.*

Proof. We shall focus on (i) as (ii) can be shown in the same way.

Suppose that $x \in K$ and $\inf[x]$ is not a minimum element of K. Let $a \in K$ be so that $a < \inf[x]$. Trivially, we have $a \notin [x]$ and hence $a \not\sim x$. Thus, $[a, x] \cap M$ is infinite. Let $a' \in [a, x] \cap M$ with $a' \neq a, x$. Then, clearly $a < a' < x$. So, $\inf[x]$ is a limit point of $K \cap M$ from below. $\square$

Lemma 3.8. $\{ [c] \mid c \in K \cap M \}$ *is dense in* $\varphi(K, M)$*. In particular,* $\varphi(K, M)$ *is separable.*

Proof. Let $a, b \in K$ with $[a] < [b]$. Namely, $a \not\sim b$ and $a < b$. Since $a \not\sim b$, $[a, b] \cap M$ is infinite. Since $|[a] \cap M| \leq 1$ and $|[b] \cap M| \leq 1$, we have $([a, b] \cap M) \setminus ([a] \cup [b]) \neq \varnothing$. Let c be an element of this set. Then, $c \in M$, $a \leq c \leq b$, $a \not\sim c$ and $c \not\sim b$. So, $[a] < [c] < [b]$. $\square$

Lemma 3.9. $\varphi(K, M)$ *is connected.*

Proof. Suppose not, i.e., there exist nonempty open sets U and W in $\varphi(K, M)$ such that $U \cap W = \varnothing$ and $U \cup W = \varphi(K, M)$. Let $[a] \in U$ and $[b] \in W$. Without loss of generality, we may assume $a < b$. Define

$$U' = \bigcup_{x \in U} x \qquad\qquad W' = \bigcup_{x \in W} x$$

Clearly, a is a lower bound of $[a, b] \cap W'$. Let $c = \inf([a, b] \cap W')$. Since K is connected, such a c exists. By the definition of U and W, we have either $[c] \in U$ or $[c] \in W$.

Case 1. $[c] \in U$.

Since U is open in $\varphi(K, M)$, there exist $a', b' \in K$ such that $[a'] < [c] < [b']$ and $[[a'], [b']] \subseteq U$. So, $[c, b'] \subseteq U'$. However, $c = \inf([a, b] \cap W')$. This is a contradiction.

Case 2. $[c] \in W$

Since W is open in $\varphi(K, M)$, there exist $a', b' \in K$ such that $[a'] < [c] < [b']$ and $[[a'], [b']] \subseteq W$. So, $[a', c] \subseteq W'$. However, $c = \inf([a, b] \cap W')$. This is a contradiction. $\square$

Lemma 3.10. $\varphi(K, M)$ *is order-isomorphic to* $[0, 1]$.

Proof. By Lemma 3.8 and Lemma 3.9, $\varphi(K, M)$ is a separable connected LOTS. By Fact 2.3, $\varphi(K, M)$ is order-isomorphic to a convex subset of $\mathbb{R}$. Since $|K| \geq 2$, we have $|\varphi(K, M)| \geq 2$. Since K has both endpoints, so does $\varphi(K, M)$. $\square$

4. $g : \prod_{i<n} K_i \to L$

In this section, we assume that

- n is a positive integer,

- for each $i < n$, K_i is a compact connected LOTS
- L is a compact connected LOTS,
- $g : \prod_{i<n} K_i \to L$ is a continuous function,
- $\theta \geq \theta_{\langle K_i | i<n \rangle, L, g}$ is a regular cardinal, and
- M is a countable elementary substructure of $H(\theta)$ with $g \in M$,

Let $\vec{K} = \prod_{i<n} K_i$.

Lemma 4.1. *Let $x \in \vec{K}$ and U an open neighborhood of x. Then, there exists $a, b \in \vec{K}$ such that $[a, b] \subseteq U$ and $[a, b]$ is a neighborhood of x, i.e. $[a, b]$ contains an open neighborhood of x.*

Proof. Without loss of generality, we may assume that U is a basic open set, i.e. $U = \prod_{i<n} \pi_i{}^{\to} U$ and for every $i < n$, $\pi_i{}^{\to} U$ is an open neighborhood of $x(i)$. For each $i < n$, since $\pi_i{}^{\to} U$ is an open neighborhood of $x(i)$, by Lemma 2.4, there exist $a(i), b(i) \in K_i$ such that $[a(i), b(i)] \subseteq \pi_i{}^{\to} U$ and $[a(i), b(i)]$ is a neighborhood of $x(i)$. Now it is easy to see that $[a, b] \subseteq U$ and $[a, b]$ is a neighborhood of x. $\square$

Lemma 4.2. *Let $a, b \in \vec{K}$ be so that $a < b$. Then, $g^{\to}[a, b]$ is convex.*

Proof. Let $z_1, z_2 \in g^{\to}[a, b]$ with $z_1 < z_2$. Let $z \in (z_1, z_2)$. We shall show that there exists $x \in [a, b]$ such that $g(x) = z$.

Since $z_1, z_2 \in g^{\to}[a, b]$, there exists $x_1, x_2 \in [a, b]$ such that $g(x_1) = z_1$ and $g(x_2) = z_2$. Without loss of generality, we may assume $x_1 < x_2$. We shall define a sequence $\langle y_i \mid i \leq n \rangle$ as follows.

$$y_i(j) = \begin{cases} x_2(j) & \text{if } j < i \\ x_1(j) & \text{if } j \geq i \end{cases}$$

Notice that $y_0 = x_1$, $y_n = x_2$, and for every $i < n$, $y_i(j) \neq y_{i+1}(j)$ if and only if $i = j$. Let i be the maximum such that $g(y_i) \leq z$. Since $g(y_0) = g(x_1) = z_1 < z$, such a i exists. Since $g(y_n) = g(x_2) = z_2 > z$, we have $i < n$. So, $g(y_{i+1}) > z$. We can apply the ordinary Intermediate Value Theorem to prove that there exists $x \in \vec{K}$ such that $y_i(i) \leq x(i) \leq y_{i+1}(i)$ and for every $j < n$ with $j \neq i$, $x(i) = y_i(i)$. Clearly, we have $x \in [a, b]$. $\square$

Definition 4.3. Let $x \in \vec{K}$. We shall define $C(x)$ to be the set of all $c \in \vec{K}$ such that for each $i < n$, $c(i) = \min[x(i)]$, $c(i) = \max([x(i)])$, or $c(i) \in [x(i)] \cap M$.

Notice that this definition is different from the one in [2] and [3]. Note that by Lemma 3.5, $|[x(i)] \cap M| \leq 1$. So, for each $i < n$, there are at most three possible value of $c(i)$ for $c \in C(x)$. Thus, $|C(x)| \leq 3^n$. In particular, it is finite.

Lemma 4.4. *For every $x \in \vec{K}$, $C(x) = [x] \cap \mathrm{Cl}(\vec{K} \cap M)$.*

Proof. First, we shall show that $C(x) \subseteq [x] \cap \mathrm{Cl}(\vec{K} \cap M)$. Let $c \in C(x)$. By the definition of $C(x)$, it is easy to see $c \in [x]$. To show $c \in \mathrm{Cl}(\vec{K} \cap M)$, let U be an open neighborhood of c. Then, there exist $a, b \in \vec{K}$ such that $[a, b] \subseteq U$ and $[a, b]$ contains an open neighborhood of c.

We shall define $y \in \vec{K} \cap M$ as follows. Let $i < n$. If $c(i) \in M$, then let $y(i) = c(i)$. Suppose not. Then, either $c(i) = \min[x(i)]$ or $c(i) = \max[x(i)]$.

Case 1. $c(i) = \min[x(i)]$

Since $c(i) \notin M$, we have $c(i) \neq \min K_i$. Since $[a(i), b(i)]$ contains an open neighborhood of $c(i)$, we have $a(i) < c(i)$. By Lemma 3.7, $c(i)$ is a limit point from below of $K_i \cap M$. Let $y(i) \in (a(i), c(i)) \cap M$.

Case 2. $c(i) = \max[x(i)]$

By the same agrument as in the previous case, there exists $y(i) \in (c(i), b(i)) \cap m$.

This completes the definition of y. Clearly we have $y(i) \in [a(i), b(i)] \cap M$ for every $i < n$, and hence $y \in [a, b] \cap M \subseteq U \cap M$. Thus, $c \in \mathrm{Cl}(\vec{K} \cap M)$.

Conversely, let $c \in [x] \cap \mathrm{Cl}(\vec{K} \cap M)$. Let $i < n$. We shall show $c(i) = \min[x(i)]$, $c(i) = \max[x(i)]$, or $c(i) \in M$. If $c(i) \in M$, we are done. Suppose not. Since $c \in \mathrm{Cl}(\vec{K} \cap M)$, we have $c(i) \in \mathrm{Cl}(K_i \cap M)$. Since $c(i) \in [x(i)] \cap \mathrm{Cl}(K_i \cap M)$, it is easy to see $c(i) = \min[x(i)]$ or $c(i) = \max[x(i)]$. □

Definition 4.5. Define a relation $\sim_{\vec{K},M}$ on $\vec{K}$ by $x \sim_{\vec{K},M} y$ if and only if for every $i < n$, $x(i) \sim_{K_i,M} y(i)$. Let $\varphi(\vec{K}, M)$ be the set of all $\sim_{\vec{K},M}$-equivalence classes. It can be identified with $\prod_{i<n} \varphi(K_i, M)$. The topology on $\varphi(\vec{K}, M)$ is given as the product topology on $\prod_{i<n} \varphi(K_i, M)$. The subscript on $\sim$ will be omitted when they are clear from the context.

Lemma 4.6. *Let $x \in \vec{K}$. For each $c \in C(x)$, let U_c be an an open neighborhood of c in $\vec{K}$. Then, there exist $a, b \in \vec{K} \cap M$ such that*

(i) $a \leq \min[x] \leq \max[x] \leq b$,
(ii) for every $i < n$, if $\min[x(i)] \neq \min K_i$, then $a(i) < \min[x(i)]$,
(iii) for every $i < n$, if $\max[x(i)] \neq \max K_i$, then $b(i) > \max[x(i)]$, and
(iv) for every $y \in [a, b] \cap M$, there exists $c \in C(x)$ such that $y \in U_c$.

Proof. Define $\eta, \zeta \in \vec{K}$ by for each $i < n$, $\eta(i) = \min[x(i)]$ and $\zeta(i) = \max[x(i)]$. Notice that for each $i < n$, if $\eta(i) \neq \min K_i$, then $\eta(i)$ is a limit point of $K_i \cap M$ from below, and if $\zeta(i) \neq \max K_i$, then $\zeta(i)$ is a limit point of $K_i \cap M$ from above.

For each $c \in C(x)$, since U_c is an open neighborhood of c in $\vec{K}$, by Lemma 4.1, there exists $a_c, b_c \in \vec{K}$ such that $[a_c, b_c] \subseteq U_c$ and $[a_c, b_c]$ is a neighborhood of c.

We shall define $a, b \in \vec{K} \cap M$ so that for every $i < n$,

- $a(i) \leq \eta(i) \leq \zeta(i) \leq b(i)$, and

- for every $c \in C$,
  - if $c(i) = \eta(i)$, then $a_c(i) \leq a(i)$, and
  - if $c(i) = \zeta(i)$, then $b(i) \leq b_c(i)$.

Let $i < n$. If $\eta(i) = \min K_i$, let $a(i) = \eta(i)$. Then, for every $c \in C$ with $c(i) = \eta(i)$, we have $a_c(i) \leq c(i) = \eta(i) = a(i)$. Suppose not, i.e., $\eta(i) > \min K_i$. For every $c \in C(x)$ with $c(i) = \eta(i)$, since $[a_c(i), b_c(i)]$ is an open neighborhood of $c(i)$ and $c(i) = \eta(i) > \min K_i$, we have $a_c(i) < c(i) = \eta(i)$. So, $\max\{a_c(i) \mid c \in C(x) \wedge c(i) = \eta(i)\} < \eta(i)$. Since $\eta(i)$ is a limit point of $K_i \cap M$ from below, there exists $a(i) \in K_i \cap M$ such that $\max\{a_c(i) \mid c \in C(x) \wedge c(i) = \eta(i)\} < a(i) < \eta(i)$. Similarly, if $\zeta(i) = \max K_i$, let $b(i) = \zeta(i)$. Then, for every $c \in C$ with $c(i) = \zeta(i)$, we have $b(i) = \zeta(i) = c(i) \leq b_c(i)$. Suppose not, i.e., $\zeta(i) < \max K_i$. Then, $\min\{\, b_c \mid c \in C(x) \wedge c(i) = \zeta(i) \,\} > \zeta(i)$. Since $\zeta(i)$ is a limit point of $K_i \cap M$ from above, there exists $b(i) \in K_i \cap M$ such that $\zeta(i) < b(i) < \min\{\, b_c \mid c \in C(x) \wedge c(i) = \zeta(i) \,\}$.

We shall show that a and b satisfy the conclusion. (i)–(iii) are clear. To see (iv), let $y \in [a, b] \cap M$. Define $c \in C(x)$ by

$$c(i) = \begin{cases} \eta(i) & \text{if } y(i) \leq \eta(i) \\ \zeta(i) & \text{if } y(i) \geq \zeta(i) \\ y(i) & \text{if } \eta(i) < y(i) < \zeta(i) \end{cases}$$

Notice that if $\eta(i) < y(i) < \zeta(i)$, then $y(i) \in [x(i)] \cap M$. So, we can see $c \in C(x)$.

We shall show $y \in U_c$. It suffices to show $y \in [a_c, b_c]$. Let $i < n$. If $c(i) = \eta(i)$, then

$$a_c(i) \leq a(i) \leq y(i) \leq \eta(i) = c(i) \leq b_c(i)$$

If $c(i) = \zeta(i)$, then

$$a_c(i) \leq c(i) = \zeta(i) \leq y(i) \leq b(i) \leq b_c(i)$$

If $c(i) = y(i)$, then since $a_c(i) \leq c(i) \leq b_c(i)$, we have $a_c(i) \leq y(i) \leq b_c(i)$. Therefore, $y \in [a_c, b_c]$. □

Notice that the conclusion of the previous lemma implies that $[[a], [b]]$ is a closed neighborhood of $[x]$ in $\varphi(\vec{K}, M)$.

Lemma 4.7. *For all $x \in \vec{K}$, if $g(x) \in M$, then there exists $c \in C(x)$ such that $g(c) = g(x)$.*

Proof. Let $x \in \vec{K}$ with $g(x) \in M$. Let $p = g(x)$. Suppose that the conclusion does not hold, i.e., for all $c \in C(x)$, $g(c) \neq p$. So, for each $c \in C(x)$, there exists an open neighborhood U_c of c such that $p \notin g^{\to} U_c$. Then, by Lemma 4.6, there exist $a, b \in \vec{K} \cap M$ such that $a \leq \min[x] \leq \max[x] \leq b$ and for every $y \in [a, b] \cap M$, there exists $c \in C(x)$ such that $y \in U_c$. Since $a, b, p \in M$, by the elementarity of M, there exists $y \in [a, b] \cap M$ such that

$g(y) = p$. By the definition of a and b, there exists $c \in C(x)$ such that $y \in U_c$. So, $p \notin g^{\to}[a_c, b_c]$ and hence $g(y) \neq p$. This is a contradiction. $\square$

Lemma 4.8. *For all $x, y \in \vec{K}$, if $x \sim y$, then we have $g(x) \sim g(y)$.*

Proof. Suppose not, i.e., $g(x) \not\sim g(y)$. Without loss of generality, we may assume $g(x) < g(y)$. Define $\eta, \zeta \in \vec{K}$ by $\eta(i) = \min[x(i)]$ and $\zeta(i) = \max[x(i)]$ for all $i < n$. By Lemma 4.2, $g^{\to}[\eta, \zeta]$ is convex and hence $[[g(x)], [g(y)]] \subseteq g^{\to}[\eta, \zeta]$. Since $g(x) \not\sim g(y)$, $[[g(x)], [g(y)]] \cap M$ is infinite.

By Lemma 4.7, for every $p \in g^{\to}[\eta, \zeta] \cap M$, there exists $c \in C(x)$ such that $g(c) = p$. So, $|g^{\to}[\eta, \zeta] \cap M| \leq |C(x)|$. Thus, $g^{\to}[\eta, \zeta] \cap M$ is finite. Since $[[g(x)], [g(y)]] \subseteq g^{\to}[\eta, \zeta]$, it implies that $[[g(x)], [g(y)]] \cap M$ is finite. This is a contradiction. $\square$

Lemma 4.9. *Let $x \in \vec{K}$. Then, $\min g^{\to}[x] = \min g^{\to}C(x)$ and $\max g^{\to}[x] = \max g^{\to}C(x)$.*

Proof. For each $i < n$, by Lemma 3.6, $[x(i)]$ is compact. Thus, $[x]$ is also compact. Hence, $g^{\to}[x]$ is a compact subset of L, so a bounded closed interval of L. In particular, $\min g^{\to}[x]$ and $\max g^{\to}[x]$ exist.

To show $\min g^{\to}[x] = \min g^{\to}C(x)$, it suffices to show that there exists $c \in C(x)$ such that $g(c) = \min g^{\to}[x]$. Suppose that $\min g^{\to}[x] \neq \min g^{\to}C(x)$. Since $C(x) \subseteq [x]$, we have $\min g^{\to}[x] \leq \min g^{\to}C(x)$. So, $\min g^{\to}[x] < \min g^{\to}C(x)$. Since L is connected, there exists $p \in L$ such that $\min g^{\to}[x] < p < \min g^{\to}C(x)$. Then, for each $c \in C(x)$, there exist an open neighborhood U_c of c such that for every $y \in U_c$, $g(y) > p$.

By Lemma 4.6, there exist $a, b \in \vec{K} \cap M$ such that $a \leq \min[x] \leq \max[x] \leq b$ and for every $y \in [a, b] \cap M$, there exists $c \in C(x)$ such that $y \in U_c$. Since $[a, b]$ is compact, there exists $y \in [a, b]$ such that $g(y) = \min g^{\to}[a, b]$. By the elementarity of M, since $g, a, b \in M$, we may assume $y \in M$. By the definition of a and b, there exists $c \in C(x)$ such that $y \in U_c$, By the definition of U_c, $\min g^{\to}[a, b] = g(y) > p$. Since $[x] \subseteq [a, b]$, we have $\min g^{\to}[a, b] \leq \min g^{\to}[x] < p$. This is a contradiction. $\square$

Lemma 4.10. *Let $x \in \vec{K}$ and $p \in L \cap M$.*

(i) If $g(x) > p$ and $g(x) \not\sim p$, then there exists $a, b \in \vec{K} \cap M$ such that
- $a \leq \min[x] \leq \max[x] \leq b$,
- *for every $i < n$, if $\min[x(i)] \neq \min K_i$, then $a(i) < \min[x(i)]$,*
- *for every $i < n$, if $\max[x(i)] \neq \max K_i$, then $b(i) > \max[x(i)]$, and*
- *for all $y \in [a, b]$, $g(y) > p$.*

(ii) If $g(x) < p$ and $g(x) \not\sim p$, then there exists $a, b \in \vec{K} \cap M$ such that
- $a \leq \min[x] \leq \max[x] \leq b$,
- *for every $i < n$, if $\min[x(i)] \neq \min K_i$, then $a(i) < \min[x(i)]$,*
- *for every $i < n$, if $\max[x(i)] \neq \max K_i$, then $b(i) > \max[x(i)]$, and*
- *for all $y \in [a, b]$, $g(y) < p$.*

Proof. We shall show only (i) as (ii) can be shown in the same way.

Suppose $g(x) > p$ and $g(x) \not\sim p$.

Claim 1. For every $c \in C(x)$, $g(c) > p$.

$\vdash$ By Lemma 4.8, since $x \sim c$, we have $g(x) \sim g(c)$. So, $g(c) \in [g(x)]$. Since $g(x) \not\sim p$, we have $p \notin [g(x)]$. Since $[g(x)]$ is convex and $g(x) > p$, we have $g(c) > p$. $\dashv$ (Claim 1)

So, for every $c \in C(x)$, there exist an open neighborhood U_c of c such that for every $y \in U_c$, $g(y) > p$. By Lemma 4.6, there exist $a, b \in \vec{K} \cap M$ such that

- $a \leq \min[x] \leq \max[x] \leq b$,
- for every $i < n$, if $\min[x(i)] \neq \min K_i$, then $a(i) < \min[x(i)]$,
- for every $i < n$, if $\max[x(i)] \neq \max K_i$, then $b(i) > \min[x(i)]$, and
- and for every $y \in [a, b] \cap M$, there exists $c \in C(x)$ such that $y \in U_c$.

Now, we shall prove that this works. Since $g, a, b \in M$, there exists $y \in [a, b] \cap M$ such that $g(y) = \min g^{\rightarrow}[a, b]$. By the definition of a and b, there exists $c \in C(x)$ such that $y \in U_c$. By the definition of U_c, $g(y) > p$. So, $\min g^{\rightarrow}[a, b] > p$. Thus, for every $y \in [a, b]$, we have $g(y) > p$. $\square$

Define $\varphi(g, M) : \varphi(\vec{K}, M) \to \varphi(L, M)$ by

$$\varphi(g, M)\,([x]) = [g(x)]$$

So, $[g(x)]$ exists. By Lemma 4.8, $\varphi(g, M)$ is well-defined.

Lemma 4.11. *$\varphi(g, M)$ is continuous.*

Proof. By Fact 2.2, the set of subsets of $\varphi(L, M)$ of the form $[[\min L], [p]]$ or $[[p], [\max L]]$ is a subbasis for the topology of $\varphi(L, M)$. So, it suffices to show that for every $p \in \vec{K}$, $\varphi(g, M)^{\leftarrow}([\min L], [p])$ and $\varphi(g, M)^{\leftarrow}([p], [\max L])$ are open. We shall focus on the former as the latter can be shown in the same way.

Let $x \in \vec{K}$ be so that $[x] \in \varphi(g, M)^{\leftarrow}([\min L], [p])$. So, $[g(x)] = \varphi(g, M)([x]) < [p]$. By Lemma 3.8, there exists $p' \in L \cap M$ such that $[g(x)] < [p'] < [p]$.

By Lemma 4.10, there exists $a, b \in \vec{K} \cap M$ such that

- $a \leq \min[x] \leq \max[x] \leq b$,
- for every $i < n$, if $\min[x(i)] \neq \min K_i$, then $a(i) < \min[x(i)]$,
- for every $i < n$, if $\max[x(i)] \neq \max K_i$, then $b(i) > \max[x(i)]$, and
- for all $y \in [a, b]$, $g(y) < p'$.

For each $i < n$, we shall define an open subset W_i of $\varphi(K_i, M)$ as follows. Notice that at least one of $\min[x(i)] \neq \min K_i$ and $\max[x(i)] \neq \max K_i$ is true. Define W_i by:

$$W_i = \begin{cases} ([a(i)], [b(i)]) & \text{if } \min[x(i)] \neq \min K_i \text{ and } \max[x(i)] \neq \max K_i \\ [[a(i)], [b(i)]) & \text{if } \min[x(i)] = \min K_i \\ ([a(i)], [b(i)]] & \text{if } \max[x(i)] = \max K_i \end{cases}$$

Notice that if $\min[x(i)] = \min K_i$, then $a(i) = \min K_i$ and hence $[a(i)] = \min \varphi(K_i, M)$. Similarly, if $\max[x(i)] = \max K_i$, then $b(i) = \max K_i$ and hence $[b(i)] = \max \varphi(K_i, M)$. So, W_i is an open subset of $\varphi(K_i, M)$. Thus, $W = \prod_{i<n} W_i$ is an open subset of $\varphi(\vec{K}, M)$. It is easy to see that $[x] \in W$. We shall show that $W \subseteq \varphi(g, M)^{\leftarrow}([\min L], [p])$. Let $y \in \vec{K}$ be so that $[y] \in W$. Since $W = \prod_{i<n} W_i$, for each $i < n$, we have $[y(i)] \in W_i$.

Claim 1. $y \in [a, b]$.

$\vdash$ We shall show for every $i < n$, $y(i) \in [a(i), b(i)]$. Let $i < n$. Then, $[y(i)] \in W_i$. If $\min[x(i)] \neq \min K_i$ and $\max[x(i)] \neq \max K_i$, then $W_i = ([a(i)], [b(i)])$ and hence $[a(i)] < [y(i)] < [b(i)]$. So, $\max[a(i)] < \min[y(i)] \leq y(i) \leq \max[y(i)] < \max[b(i)]$. If $\min[x(i)] = \min K_i$, then we have $W_i = [[a(i)], [b(i)])$ and $a(i) = \min K_i$. So, $a(i) \leq y(i)$ is trivial. Also, we have $[y(i)] < [b(i)]$ and hence $y(i) \leq \max[y(i)] < \min[b(i)] \leq b(i)$. Similarly for the case $\max[x(i)] = \max K_i$. $\dashv$ (Claim 1)

By the definition of a and b, since $y \in [a, b]$, we have $g(y) < p'$. Thus, $\varphi(g, M)[y] = [g(y)] \leq [p'] < [p]$. Namely, $[y] \in \varphi(g, M)^{\leftarrow}([\min L], [p])$. Therefore, $W \subseteq \varphi(g, M)^{\leftarrow}([\min L], [p])$. □

5. $f : \prod_{i<n} K_i \to \prod_{i<n} L_i$

In this section, we assume that

- n is a positive integer,
- for each $i < n$, K_i and L_i are compact connected LOTS,
- $f : \prod_{i<n} K_i \to \prod_{i<n} L_i$ is a continuous function,
- $\theta \geq \theta_{\langle K_i | i<n \rangle, \langle L_i | i<n \rangle, f}$ is a regular cardinal, and
- M is a countable elementary substructure of $H(\theta)$ with $f \in M$,

As in the previous section, let $\vec{K} = \prod_{i<n} K_i$ and $\vec{L} = \prod_{i<n} L_i$. For each $i < n$, let $g_i = \pi_i \circ f$. So, g_i is a continuous function from $\vec{K}$ into K_i. Define $\varphi(f, M)$ to be the function from $\vec{K}$ into $\vec{L}$ by for every $x \in \vec{K}$,

$$\varphi(f, M)(x)(i) = \varphi(g_i, M)(x)$$

Lemma 5.1. *$\varphi(f, M)$ is continuous.*

Proof. It is clear since for each $i < n$, $\varphi(g_i, M)$ is continuous. □

Lemma 5.2. *$\varphi(\vec{K}, M)$ and $\varphi(\vec{L}, M)$ are isomorphic to $[0, 1]^n$.*

Proof. For each $i < n$, by Lemma 3.10, $\varphi(K_i, M)$ is isomorphic to $[0, 1]$. Since $\varphi(\vec{K}, M)$ can be identified with $\prod_{i<n} \varphi(K_i, M)$, it is homeomorphic to $[0, 1]^n$. Similarly for $\varphi(\vec{L}, M)$. □

By the previous two lemmas, we can observe that $\varphi(f, M)$ has the same structure as a continuous function from $[0, 1]^n$ to $[0, 1]^n$. This fact plays important roles in the proofs of the main theorems.

6. The fixed-point theorem

This section is devoted to the proof of the following theorem.

Theorem 6.1. *Let n be a positive integer. Let K_i be a compact connected linearly ordered topological space for each $i < n$. Then, every continuous function $f : \prod_{i<n} K_i \to \prod_{i<n} K_i$ has a fixed point, namely there exists an $x \in \prod_{i<n} K_i$ such that $f(x) = x$.*

Proof. Let $\vec{K} = \prod_{i<n} K_i$. Suppose otherwise. Then, for every $x \in \vec{K}$, $f(x) \neq x$.

Claim 1. For every $x \in \vec{K}$, there exists an open neighborhood U of x such that $U \cap f^{\to}U = \varnothing$.

$\vdash$ Since $\vec{K}$ is Hausdorff, there exist an open neighborhood U of x and an open neighborhood W of $f(x)$ such that $U \cap W = \varnothing$. Since f is continuous, $f^{\leftarrow}W$ is an open neighborhood of x. Let $U' = U \cap f^{\leftarrow}W$. We shall show that $U' \cap f^{\to}U' = \varnothing$. Suppose not, i.e. there exists $x' \in U' \cap f^{\to}U'$. Since $x' \in f^{\to}U'$, there exists $y \in U'$ such that $f(y) = x'$. Since $y \in U' \subseteq f^{\leftarrow}W$, we have $x' \in W$. However, we also have $x' \in U' \subseteq U$ and hence $x' \in U \cap W$. This is a contradiction to the assumption $U \cap W = \varnothing$. $\dashv$ (Claim 1)

By the previous claim, for each $x \in \vec{K}$, we can take an open neighborhood U_x such that $U_x \cap f^{\to}U_x = \varnothing$. By the definition of the product topology and order topology, we may assume that $U_x = \prod_{i<n} \pi_i{}^{\to}U_x$ and for every $i < n$, $\pi_i{}^{\to}U_x$ is a convex open set in K_i. Since K_i is compact for all $i < n$, $\vec{K}$ is compact. So, there exist finitely many elements $x_0, x_1, \ldots, x_{m-1}$ such that $\vec{K} = \bigcup_{j<m} U_{x_j}$.

Let $\theta = \theta_{\vec{K}}$ and M a countable elementary submodel of $H(\theta)$ with $\vec{K}, f, U_{x_0}, \ldots, U_{x_{m-1}} \in M$. By Lemma 3.10, for every $i < n$, $\varphi(K_i, M)$ is order-isomorphic to $[0,1]$. So, $\varphi(\vec{K}, M)$ is homeomorphic to $[0,1]^n$.

By Lemma 4.11, $\varphi(f, M)$ is a continuous function from $\prod_{i<n} \varphi(K_i, M)$ into $\prod_{i<n} \varphi(K_i, M)$. By Brouwer's fixed-point theorem, there exists $y \in \prod_{i<n} \varphi(K_i, M)$ such that $\varphi(f, M)(y) = y$. Namely, for every $i < n$,

$$\varphi(\pi_i \circ f, M)(y) = y(i)$$

We shall define $x \in \vec{K}$ so that $[x] = y$ as follows. For each $i < n$, by Lemma 3.5, we have $|y(i) \cap M| \leq 1$. If $y(i) \cap M \neq \varnothing$, let $x(i)$ be the only element of $y(i) \cap M$. Otherwise, let $x(i)$ be an arbitrary element of $y(i)$.

Since $\vec{K} = \bigcup_{j<m} U_{x_j}$, there exists $j < m$ such that $x \in U_{x_j}$. So, for each $i < n$, $x(i) \in \pi_j{}^{\to}U_{x_j}$. By assumption, $\pi_i{}^{\to}U_{x_j}$ is a convex open set in K_i.

Claim 2. For each $i < n$, if $y(i) \neq \min \varphi(K_i, M)$, then $\inf \pi_i{}^{\to}U_{x_j} < \inf y(i)$.

Proof. Suppose $y(i) \neq \min \varphi(K_i, M)$. Then, $x(i) \neq \min K_i$. Since $\pi_i{}^{\to}U_{x_j}$ is an open subset of K_i with $x(i) \in \pi_i{}^{\to}U_{x_j}$, we have $\inf \pi_i{}^{\to}U_{x_j} < x(i)$. Since

$y(i) \cap M \subseteq \{ x(i) \}$, we have $[\inf y(i), x(i)) \cap M = \varnothing$. Since $U_{x_j} \in M$, we have $\inf \pi_i{}^{\rightarrow} U_{x_j} \in M$. Hence, we have $\inf \pi_i{}^{\rightarrow} U_{x_j} < \inf y(i)$ □

Similarly, we can prove the following claim.

Claim 3. For each $i < n$, if $y(i) \neq \max \varphi(K_i, M)$, then $\sup y(i) < \sup \pi_i{}^{\rightarrow} U_{x_j}$.

Claim 4. For each $i < n$, $y(i) \subseteq \pi_i{}^{\rightarrow} U_{x_j}$.

Proof.

Case 1. $y(i) = \min \varphi(K_i, M)$
Then, $\min K_i \in y(i) \cap M$ and hence $x(i) = \min K_i$. Since $\pi_i{}^{\rightarrow} U_{x_j}$ is a convex open set, we have $\pi_i{}^{\rightarrow} U_{x_j} = [x(i), \sup \pi_i{}^{\rightarrow} U_{x_j})$. By Claim 3, we have $\sup y(i) < \sup \pi_i{}^{\rightarrow} U_{x_j}$. So,

$$\begin{aligned} y(i) &= [x(i), \sup y(i)) \\ &\subseteq [x(i), \sup \pi_i{}^{\rightarrow} U_{x_j}) = \pi_i{}^{\rightarrow} U_{x_j}. \end{aligned}$$

Case 2. $y(i) = \max \varphi(K_i, M)$

Similar to the previous case.

Case 3. $\min \varphi(K_i, M) < y(i) < \max \varphi(K_i, M)$

Then, by Claim 2, we have $\inf \pi_i{}^{\rightarrow} U_{x_j} < \inf y(i)$. By Claim 3, $\sup y(i) < \sup \pi_i{}^{\rightarrow} U_{x_j}$. So, $y(i) \subseteq \pi_i{}^{\rightarrow} U_{x_j}$ □

Recall that for every $i < n$, $x(i) \in y(i)$. By Claim 4, we have $x(i) \in \pi_i{}^{\rightarrow} U_{x_j}$. Since $U_{x_j} = \prod_{i<n} \pi_i{}^{\leftarrow} U_{x_j}$, we have $x \in U_{x_j}$. By the definition of U_{x_j}, $U_{x_j} \cap f^{\rightarrow} U_{x_j} = \varnothing$. So, $f(x) \notin U_{x_j}$. Thus, there exists $i < n$ such that $f(x)(i) \notin \pi_i{}^{\rightarrow} U_{x_j}$. Since $y(i) \subseteq \pi_i{}^{\rightarrow} U_{x_j}$, we have $(\pi_i \circ f)(x) = f(x)(i) \notin y(i)$. So,

$$\begin{aligned} \varphi(\pi_i \circ f, M)(y) &= \varphi(\pi_i \circ f, M)([x]) \\ &= [\pi_i \circ f(x)] \neq y(i) \end{aligned}$$

This is a contradiction. □

7. The Poincaré-Miranda Theorem

We shall show the following theorem in this section.

Theorem 7.1. *Let n be a positive integer. Let K_i and L_i be compact connected LOTS for each $i < n$. Define $\vec{K} = \prod_{i<n} K_i$, $\vec{L} = \prod_{i<n} L_i$, $a = \min \vec{K}$, and $b = \max \vec{K}$. Let $f : \vec{K} \to \vec{L}$ be a continuous function. Let $z \in \vec{L}$ be so that for every $i < n$ and $x \in \vec{K}$,*

- *if $x(i) = a(i)$, then $f(x)(i) < z$, and*
- *if $x(i) = b(i)$, then $f(x)(i) > z$*

Then $z \in \operatorname{ran}(f)$.

Proof. Suppose not, i.e. $z \notin \mathrm{ran}(f)$. Since $\vec{K}$ is compact, so is $\mathrm{ran}(f)$. In particular, $\mathrm{ran}(f)$ is closed. Thus, there exists an open neighborhood W of z such that $W \cap \mathrm{ran}(f) = \varnothing$.

Let $\theta = \theta_{\vec{K},\vec{L},f}$ and M a countable elementary substructure of $H(\theta)$ with $\vec{K}, \vec{L}, f, z, W \in M$. Notice that by Lemma 3.10, for every $i < n$, $\varphi(K_i, M)$ and $\varphi(L_i, M)$ are isomorphic to $[0,1]$. So, $\varphi(\vec{K}, M)$ and $\varphi(\vec{L}, M)$ are isomorphic to $[0,1]^n$. Moreover, for every $i < n$ and $x \in \vec{K}$,

- if $[x](i) = [a](i)$, then $\varphi(f, M)([x])(i) < [z]$, and
- if $[x](i) = [b](i)$, then $\varphi(f, M)([x])(i) > [z]$.

Thus, by the ordinary Poincaré-Miranda Theorem, there exists $x \in \vec{K}$ such that $\varphi(f, M)([x]) = [z]$. In particular, $f(x) \in [z]$.

Claim 1. $[z] \subseteq W$.

$\vdash$ By the definition of M, z and W belong to M. By Lemma 4.1 and the elemenarity of M, there exists $a', b' \in \vec{L} \cap M$ such that $[a', b'] \subseteq W$ and $[a', b']$ contains an open neighborhood of z. In particular, for every $i < n$, $[a'(i), b'(i)]$ contains an open neighborhood of $z(i)$.

It suffices to show that for each $i < n$, $a'(i) \leq \min[z(i)]$ and $\max[z(i)] \leq b'(i)$. If $z(i) = \min L_i$, then it is easy to see $a'(i) = \min[z(i)]$. Suppose $z(i) > \min L_i$. Since $[a'(i), b'(i)]$ contains an open neighborhood of $z(i)$, we have $a'(i) < z(i)$. Notice $(\min[z(i)], z(i)) \cap M = \varnothing$. Since $a'(i)$ belongs to M, we have $a'(i) \leq \min[z(i)]$. Similarly, we can prove $\max[z(i)] \leq b'(i)$. $\dashv$ (Claim 1)

So, $f(x) \in W$. This is a contradiction to the assumption that $W \cap \mathrm{ran}(f) = \varnothing$. $\square$

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