PERFECTLY NORMAL NONREALCOMPACT SPACES UNDER THE PROPER FORCING AXIOM

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Abstract. We analyze the behavior of a perfectly normal nonrealcompact space (ω_1, τ) on ω_1 such that for every $\gamma < \omega_1$, γ is τ-open and $\gamma+\omega$ is τ-closed under the Proper Forcing Axiom. We show that there exists a club subset D of ω_1 such that for a stationary subset of $\delta \in \mathrm{acc}(D)$, for all τ-open neighborhood N of $\delta + n$, there exists $\eta < \delta$ such that for all $\xi \in D \cap [\eta, \delta)$, $N \cap \xi$ is unbounded in ξ.

1. Introduction

E. Hewitt introduced the following property of topological spaces in [4] though he originally called such spaces *Q-spaces*.

Definition 1.1. Let X be a Tychonoff space. We say that a subset Y of X is a *zero-set* if and only if there exists a continuous function $f : X \to \mathbb{R}$ such that $Y = f^{\leftarrow}\{0\}$. A filter on the set of all zero-sets in X is called a *z-filter* on X. The maximal z-filter on X is called a *z-ultrafilter* on X.

We say that X is *realcompact* if and only if every z-ultrafilter on X with the countable intersection property is *fixed*, i.e. the total intersection is nonempty. A filter is called free if and only if it is not fixed.

The significance of realcompact spaces is demonstrated by the following theorem proved by Hewitt: For all realcompact spaces X and Y, $C(X)$ and $C(Y)$ are isomorphic if and only if X and Y are homeomorphic.

There are very few examples of perfectly normal nonrealcompact spaces. Notice that in a perfectly normal space, every closed set is a zero-set. One of the examples is the discrete space whose cardinality is measurable, but it requires a large cardinal axiom and also has a large size. Another example is an Ostaszewski space, which is characterized as an uncountable, regular, countably compact, noncompact space whose closed sets are either countable or co-countable. Such a space was first constructed from $\diamondsuit$ by A. J. Ostaszewski in [7]. The cardinality of this space is $\aleph_1$, so it has the smallest possible cardinality. While constructions by using other assumptions are known, axioms beyond ZFC are required. In particular, $\mathrm{MA} + \neg\,\mathrm{CH}$ refutes the existence of Ostaszewski spaces.

Date: December 31, 2023.

Key words and phrases. Pefectly normal; realcompact; Proper Forcing Axiom.

R. L. Blair asked if MA $+\neg$ CH implies that every perfectly normal space of cardinality less than the first measurable cardinal is realcompact. This problem was solved negatively by F. Hernández-Hernández and the author in [3]. In this paper, they constructed a model of MA $+\neg$ CH in which there exists a perfectly normal nonrealcompact space of size $\aleph_1$. They used the topology $\tau(\vec{C})$ associated with a ladder system $\vec{C}$ to provide an example. This class of topologies is found to be useful to construct other counterexamples. See [5] and [6] for more details. All known examples of perfectly normal nonrealcompact spaces are of one of these three types: discrete spaces of the first measurable cardinality or greater, Ostaszewski spaces, and ω_1 equipped with the topology associated with ladder systems.

Further, Blair and E. K. van Douwen [2] asked if there exists a ZFC example of a perfectly normal nonrealcompact space of cardinality less than the first measurable cardinal. This problem remains open even now. In fact, it is not known whether there is any model of ZFC in which there is no such space with local compactness or first countability. Note that Z .Balogh [1] proved that MA $+\neg$ CH implies that

(1) there exists no locally connected locally compact perfectly normal nonrealcompact space of cardinality $<2^{\aleph_0}$, and
(2) there exists no locally countable locally compact perfectly normal nonrealcompact space of cardinality $<2^{\aleph_0}$.

A particularly important special case of the problem of Blair and van Douwen is whether there exists such a space of size $\aleph_1$ under a stronger forcing axiom such as the Proper Forcing Axiom (PFA) or Martin's Maximum(MM). In [5], the author showed that assuming PFA, when a ladder sytem $\vec{C}$ satisfies a minor condition, $(\omega_1, \tau(\vec{C}))$ cannot be pefectly normal nonrealcompact space.

In this paper, we shall analyze a perfectly normal nonrealcompact space (ω_1, τ) on ω_1 such that

(1) for all $\gamma < \omega_1$, $\gamma + \omega$ is τ-closed, and
(2) for all $\gamma < \omega_1$, γ is τ-open.

Notice that every topology associated with a ladder system satisfies these properties. While we cannot rule out the possibility that there is a perfectly normal nonrealcompact space satisfying these conditions, we proved that if such a space exists, then it has some strong properties.

The structure of this paper is as follows. In Section 2, we shall prove some general properties of perfectly normal nonrealcompact spaces. In Section 3, we shall consider the particular class of topologies we consider in this paper and prove a lemma that will be used in the later section. In Section 4, we shall prove the main theorem. In Section 5, we shall derive some corollaries of the main theorem.

Most of the notations in this paper are standard. We shall use the interval notations on ordinals frequently. For example, $[\alpha, \beta) = \{\, \gamma \mid \alpha \leq \gamma < \beta \,\}$.

Definition 1.2. For every ordinal α, define the *limit part of* α to be the unique limit ordinal $\mathrm{lp}(\alpha)$ such that $\alpha = \mathrm{lp}(\alpha) + n$ for some $n < \omega$. This n is also unique and called the *finite part of* α, denoted by $\mathrm{fp}(\alpha)$.

Lim denotes the class of all limit ordinals, Lim^2 the class of all limits of limit ordinals, and Lim^3 the class of all limits of ordinals in Lim^2.

The following notations were introduced by S. Shelah.

Definition 1.3. Let X be a set of ordinals. We say that α is an *accumulation point* if and only if $\alpha \in X$ and α is a limit point of X. We write $\mathrm{acc}(X)$ for the set of all accumulation points of X and $\mathrm{nacc}(X) = X \setminus \mathrm{acc}(X)$.

We shall use the following notion in Section 4

Definition 1.4. A sequence $\langle M_\alpha : \alpha < \eta \rangle$ is called a *tower* if and only if

(1) (increasing) for every $\beta < \alpha < \eta$, $M_\gamma \subseteq M_\beta$,
(2) (continuous) for every $\alpha < \eta$, if α is a limit ordinal, then $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$, and
(3) for every $\alpha < \eta$ with $\alpha + 1 < \eta$, $\langle M_\beta : \beta \leq \alpha \rangle \in M_{\alpha+1}$.

2. Perfectly normal, nonrealcompact spaces

Let (ω_1, τ) be a perfectly normal nonrealcompact space. Let $\bar{\mathcal{F}}$ be a free z-ultrafilter with countable intersection property on (ω_1, τ). Define $\mathcal{F}$ to be the filter over ω_1 by $X \in \mathcal{F}$ if and only if there exists $F \in \bar{\mathcal{F}}$ such that $F \subseteq X$. Clearly, $\mathcal{F}$ is countably complete.

Lemma 2.1. *For every $X \in \breve{\mathcal{F}}$, there exists an F_σ-set $X' \in \breve{\mathcal{F}}$ such that $X \subseteq X'$.*

Proof. Let $X \in \breve{\mathcal{F}}$. Then, there exists $F \in \bar{\mathcal{F}}$ such that $X \cap F = \emptyset$. In particular, F is τ-closed. Since (ω_1, τ) is perfectly normal, there exists a countable set $\{ F_n \mid n < \omega \}$ of τ-closed sets such that $\bigcup_{n<\omega} F_n = \omega_1 \setminus F$. Let $X' = \bigcup_{n<\omega} F_n$. □

Lemma 2.2. *For every $F \in \mathcal{F}$, $\mathrm{der}_\tau(F) \in \mathcal{F}$.*

Proof. Let $F \in \mathcal{F}$. Without loss of generality, we may assume that F is τ-closed and hence $F \in \bar{\mathcal{F}}$. To show $\mathrm{der}_\tau(F) \in \mathcal{F}$, suppose otherwise. Since $\mathrm{der}_\tau(F)$ is τ-closed and $\mathrm{der}_\tau(F) \notin \mathcal{F}$, we have $\mathrm{der}_\tau(F) \notin \bar{\mathcal{F}}$. Since $\bar{\mathcal{F}}$ is z-ultrafilter, there exists a τ-closed set $F' \in \bar{\mathcal{F}}$ such that $F' \cap \mathrm{der}_\tau(F) = \emptyset$. Without loss of generality, we may assume $F' \subseteq F$. Then, $F' \cap \mathrm{der}_\tau(F') \subseteq F' \cap \mathrm{der}_\tau(F) = \emptyset$. So F' is a τ-closed discrete set. Thus, every subset of F' is τ-closed. So, $\bar{\mathcal{F}}$ restricted to F' is an ultrafilter. This is a contradiction since $|F'| = \aleph_1$. □

Lemma 2.3. *For every $X \in \mathcal{F}^+$, $\mathrm{Cl}_\tau(X) \in \mathcal{F}$.*

Proof. Let $X \in \mathcal{F}^+$. Since $\bar{\mathcal{F}}$ is a z-ultrafilter, either $\mathrm{Cl}_\tau(X) \in \bar{\mathcal{F}}$ or $\omega_1 \setminus \mathrm{Cl}_\tau(X) \in \bar{\mathcal{F}}$. If $\omega_1 \setminus \mathrm{Cl}_\tau(X) \in \bar{\mathcal{F}}$, then since $X \cap (\omega_1 \setminus \mathrm{Cl}_\tau(X)) = \emptyset$, we have $X \notin \mathcal{F}^+$. This is a contradiction. Thus, $\mathrm{Cl}_\tau(X) \in \bar{\mathcal{F}} \subseteq \mathcal{F}$. □

Lemma 2.4. *No $\mathcal{F}$-positive set is τ-discrete.*

Proof. Let $X \in \mathcal{F}^+$. Since $X \in \mathcal{F}^+$, by Lemma 2.3, we have $\mathrm{Cl}_\tau(X) \in \mathcal{F}$. Since $X \in \mathcal{F}^+$ and $\mathrm{Cl}_\tau(X) \in \mathcal{F}$, we have $X \cap \mathrm{Cl}_\tau(X) \in \mathcal{F}^+$. So, X is not τ-discrete. $\square$

Lemma 2.5. *Let $F \in \mathcal{F}$ be τ-closed. Then, $(F, \tau \restriction F)$ is perfectly normal and nonrealcompact.*

Proof. Since F is a τ-closed subspace of a perfectly normal space (ω_1, τ), $(F, \tau \restriction F)$ is perfectly normal. Let $\bar{\mathcal{F}}'$ be the z-filter on $(F, \tau \restriction F)$ defined by for every closed subset C of $(F, \tau \restriction F)$, $C \in \bar{\mathcal{F}}'$ if and only if $C \in \mathcal{F}$. Then, it is easy to see that $\bar{\mathcal{F}}'$ is a z-ultrafilter with countable intersection property, whose total intersection is empty. $\square$

3. Our special case

Suppose that (ω_1, τ) is a perfectly normal nonrealcompact space such that

(1) for all $\gamma < \omega_1$, $\gamma + \omega$ is τ-closed, and
(2) for all $\gamma < \omega_1$, γ is τ-open.

$\bar{\mathcal{F}}$ and $\mathcal{F}$ are defined as in the previous section.

Lemma 3.1. *Let $\gamma < \omega_1$ and $\zeta < \mathrm{lp}(\gamma)$. Then, $[\zeta + \omega, \mathrm{lp}(\gamma)) \cup \{\gamma\}$ is a τ-open neighborhood of γ.*

Proof. Let $\gamma < \omega_1$ and $\zeta < \mathrm{lp}(\gamma)$. By assumption, $\gamma + 1 = [0, \gamma]$ is τ-open. Since $\zeta + \omega$ is τ-closed, $[0, \gamma] \setminus (\zeta + \omega) = [\zeta + \omega, \gamma]$ is τ-open. Since τ is Hausdorff, for every $n < \mathrm{fp}(\gamma)$, $\{\mathrm{lp}(\gamma) + n\}$ is τ-closed. So,

$$[\zeta + \omega, \mathrm{lp}(\gamma)) \cup \{\gamma\} = [\zeta + \omega, \gamma] \setminus \{\mathrm{lp}(\gamma) + n \mid n < \mathrm{fp}(\gamma)\}$$

is τ-open. $\square$

Lemma 3.2. *If D is a club subset of $\omega_1 \cap \mathrm{Lim}$, then $B(D) \in \mathcal{F}$.*

Proof. First, we shall show that $B(D)$ is τ-closed. Let γ be a τ-limit point of $B(D)$. Then, for every $\zeta < \mathrm{lp}(\gamma)$, since $[\zeta + \omega, \mathrm{lp}(\gamma)) \cup \{\gamma\}$ is a τ-open neighborhood of γ, we have $[\zeta+\omega, \mathrm{lp}(\gamma)) \cap B(D) \neq \emptyset$. If there exists $\zeta < \mathrm{lp}(\gamma)$ such that $\zeta + \omega = \mathrm{lp}(\gamma)$, this is a contradiction. Suppose not, i.e. for every $\zeta < \mathrm{lp}(\gamma)$, $\zeta + \omega < \mathrm{lp}(\gamma)$. Then, $B(D) \cap \mathrm{lp}(\gamma)$ is unbounded in $\mathrm{lp}(\gamma)$. So, $D \cap \mathrm{lp}(\gamma)$ is unbounded in $\mathrm{lp}(\gamma)$. Since D is closed, we have $\mathrm{lp}(\gamma) \in D$ and hence $\gamma \in B(D)$. So, $B(D)$ is τ-closed.

Now, we shall show $B(D) \in \mathcal{F}$. Suppose not. Since $B(D)$ is τ-closed and $B(D) \notin \mathcal{F}$, there exists $F \in \bar{\mathcal{F}}$ such that $F \cap B(D) = \emptyset$ and F is τ-closed. It is easy to build a sequence $\langle F_m \mid m < \omega \rangle$ of subsets of F such that $\bigcup_{m<\omega} F_m = F$ and for each consecutive pair δ, δ' of D and for every $m < \omega$, $|F_m \cap [\delta, \delta')| \leq 1$.

Claim 3.2.1. For every $m < \omega$, F_m is τ-closed discrete.

Proof. Suppose not, ie. there exists a τ-limit point γ of F_m. Recall that for every consecutive pair δ, δ' of D, $|F_m \cap [\delta, \delta')| \leq 1$. So, $\gamma \in B(D)$. Since $F_m \subseteq F$ and F is τ-closed, we have $\gamma \in F$. Thus, $\gamma \in B(D) \cap F$. This is a contradiction. □

By Lemma 2.4, for every $m < \omega$, $F_m \in \breve{\mathcal{F}}$. Since $\mathcal{F}$ is countably complete, $F = \bigcup_{m<\omega} F_m \in \breve{\mathcal{F}}$. This is a contradiction. □

4. Under Proper Forcing Axiom

Throughout this section, we shall assume PFA in addition to the assumption in the previous section.

For each $n < \omega$, define $X_n = \{ \delta + n \mid \delta \in \omega_1 \cap \mathrm{Lim} \}$. Then, trivially $\bigcup_{n<\omega} X_n = \omega_1 \in \mathcal{F}$. So, there exists $n < \omega$ such that $X_n \in \mathcal{F}^+$. Fix such an n.

Theorem 4.1. *There exists a club subset D of ω_1 such that for a stationary subset of $\delta \in \mathrm{acc}(D)$, for all τ-open neighborhood N of $\delta + n$, there exists $\eta < \delta$ such that for all $\xi \in D \cap [\eta, \delta)$, $N \cap \xi$ is unbounded in ξ.*

Proof. Suppose not, i.e.

(∗) for all club subset D of ω_1, there exists a club subset E of $\mathrm{acc}(D)$ such that for each $\delta \in E$, there exists a τ-open neighborhood N of $\delta + n$ such that for every $\eta < \delta$, there exists $\xi \in D \cap [\eta, \delta)$ such that $N \cap \xi$ is bounded in ξ.

Let P be the set of all functions p such that for some $\delta \in \omega_1 \cap \mathrm{Lim}$, $p : X_n \cap \delta \to \omega$ and for each $k < \omega$, $p^{\leftarrow}\{ k \}$ is $\tau \restriction X_n$-closed discrete, namely $p^{\leftarrow}\{ k \}$ has no τ-limit point that belongs to X_n. This δ is called the *height of* p and denoted by $\mathrm{ht}(p)$. P is ordered by extension.

Claim 4.1.1. Let $p \in P$, $\delta \in \omega_1 \cap \mathrm{Lim}$, and $q : X_n \cap \delta \to \omega$ such that q extends p. Suppose that for all $k < \omega$ and $\gamma \in X_n \cap [\mathrm{ht}(p) + \omega, \mathrm{ht}(q) + \omega)$, γ is not a τ-limit point of $q^{\leftarrow}\{ k \}$. Then, $q \in P$.

Proof. Let p, δ, q satisfy the assumption. It suffices to show that for all $k < \omega$ and $\gamma \in X_n$, γ is not a τ-limit point of $q^{\leftarrow}\{ k \}$.

If $\gamma \in [\mathrm{ht}(p) + \omega, \mathrm{ht}(q) + \omega)$, then this follows from the assumption. Suppose that $\gamma \geq \mathrm{ht}(q) + \omega$. Since $\mathrm{ht}(q) + \omega$ is τ-closed and $q^{\leftarrow}\{ k \} \subseteq \mathrm{dom}(q) \subseteq \mathrm{ht}(q) + \omega$, γ is not a τ-limit point of $q^{\leftarrow}\{ k \}$.

Suppose $\gamma < \mathrm{ht}(p) + \omega$. Since $p \in P$, γ is not a τ-limit point of $p^{\leftarrow}\{ k \}$. So, there exists a τ-open neighborhood N of γ such that $p^{\leftarrow}\{ k \} \cap N \subseteq \{ \gamma \}$. Without loss of generality, we may assume $N \subseteq \mathrm{lp}(\gamma) \cup \{ \gamma \}$. Since $\mathrm{lp}(\gamma) \leq \gamma < \mathrm{ht}(p) + \omega$ and $\mathrm{ht}(p) + \omega$ is the least limit ordinal greater than $\mathrm{ht}(p)$, we have $\mathrm{lp}(\gamma) \leq \mathrm{ht}(p)$. So, $N \subseteq \mathrm{ht}(p) \cup \{ \gamma \}$. Thus,

$$\begin{aligned} q^{\leftarrow}\{\,k\,\}\cap N &= (q^{\leftarrow}\{\,k\,\}\cap(\mathrm{ht}(p)\cup\{\,\gamma\,\}))\cap N \\ &\subseteq ((q^{\leftarrow}\{\,k\,\}\cap\mathrm{ht}(p))\cup\{\,\gamma\,\})\cap N \\ &= (p^{\leftarrow}\{\,k\,\}\cup\{\,\gamma\,\})\cap N \\ &\subseteq \{\,\gamma\,\}. \end{aligned}$$

So, γ is not a τ-limit point of $q^{\leftarrow}\{\,k\,\}$. ☐

Claim 4.1.2. Let $\langle p_m \mid m<\omega\rangle$ be a decreasing sequence in P. Let $q=\bigcup_{m<\omega} p_m$ and $\delta=\sup_{m<\omega}\mathrm{ht}(p_m)$. Suppose that for all $k<\omega$, $\delta+n$ is not a τ-limit point of $q^{\leftarrow}\{\,k\,\}$. Then, $q\in P$.

Proof. Let $\langle p_m \mid m<\omega\rangle$, q, and δ be as in the assumption. Clearly, $q: X_n\cap\delta\to\omega$. We shall show that for every $k<\omega$ and $\gamma\in X_n$, γ is not a τ-limit point of $q^{\leftarrow}\{\,k\,\}$. Let $k<\omega$ and $\gamma\in X_n$.

If $\gamma=\delta+n$, then by the assumption, γ is not a τ-limit point of $q^{\leftarrow}\{\,k\,\}$.

Suppose $\gamma\geq\delta+\omega$. Since $\delta+\omega$ is τ-closed and $q^{\leftarrow}\{\,k\,\}\subseteq\mathrm{dom}(q)=\bigcup_{m<\omega}\mathrm{dom}(p_m)\subseteq\sup_{m<\omega}\mathrm{ht}(p_m)=\delta$, γ is not a τ-limit point of $q^{\leftarrow}\{\,k\,\}$.

Suppose $\gamma<\delta$. Then, there exists an $m<\omega$ such that $\gamma<\mathrm{ht}(p_m)$. So, $\mathrm{ht}(p_m)$ is a τ-open neighborhood of γ. Since $p_m\in P$, γ is not a τ-limit point of ${p_m}^{\leftarrow}\{\,k\,\}$. So, there exists a τ-open neighborhood N of γ such that ${p_m}^{\leftarrow}\{\,k\,\}\cap N\subseteq\{\,\gamma\,\}$ and $N\subseteq\mathrm{lp}(\gamma)\cup\{\,\gamma\,\}$. In particular, since $\mathrm{ht}(p_m)$ is a limit ordinal greater than γ, we have $N\subseteq\mathrm{ht}(p_m)$. So,

$$\begin{aligned} q^{\leftarrow}\{\,k\,\}\cap N &= q^{\leftarrow}\{\,k\,\}\cap\mathrm{ht}(p_m)\cap N \\ &= {p_m}^{\leftarrow}\{\,k\,\}\cap N \\ &\subseteq\{\,\gamma\,\} \end{aligned}$$

So, γ is not a τ-limit point of $q^{\leftarrow}\{\,k\,\}$.

Thus, for all cases, γ is not a τ-limit point of $q^{\leftarrow}\{\,k\,\}$. Thus, $q\in P$. ☐

Claim 4.1.3. For every $p\in P$, $\zeta<\omega_1$, and $k<\omega$, there exists $q\leq p$ such that $\mathrm{ht}(q)\geq\zeta$ and $q^{\leftarrow}k\subseteq\mathrm{ht}(p)$.

Proof. Let $p\in P$, $\zeta<\omega_1$, and $k<\omega$. Without loss of generality, we may assume that ζ is a limit ordinal. If $\zeta\leq\mathrm{ht}(p)$, then it is trivial. Suppose $\zeta>\mathrm{ht}(p)$. Since $\zeta<\omega_1$, $[\mathrm{ht}(p),\zeta)\cap X_n$ is countable. So, it is easy to find a function $q: X_n\cap\zeta\to\omega$ such that $q\restriction(X_n\cap\mathrm{ht}(p))=p$, $(q^{\leftarrow}k)\setminus\mathrm{ht}(p)=\emptyset$, and for every $l<\omega$, $|(q^{\leftarrow}\{\,l\,\})\setminus\mathrm{ht}(p)|\leq 1$.

It suffices to show that $q\in P$. By definition, $q: X_n\cap\zeta\to\omega$ and q extends p. By Claim 4.1.1, it suffices to show that for all $l<\omega$ and $\gamma\in X_n\setminus[\mathrm{ht}(p)+\omega,\mathrm{ht}(q)+\omega)$, γ is not a τ-limit point of $q^{\leftarrow}\{\,l\,\}$.

Let $l<\omega$ and $\gamma\in X_n\setminus[\mathrm{ht}(p)+\omega,\mathrm{ht}(q)+\omega)$. Since $p\in P$, there exists a τ-open neighborhood N of γ such that $p^{\leftarrow}\{\,l\,\}\cap N\subseteq\{\,\gamma\,\}$ and $N\subseteq\mathrm{lp}(\gamma)\cup\{\,\gamma\,\}$.

So,

$$\begin{aligned}
q^{\leftarrow}\{\,l\,\} \cap N &= ((q^{\leftarrow}\{\,l\,\} \cap \mathrm{ht}(p)) \cup (q^{\leftarrow}\{\,l\,\} \setminus \mathrm{ht}(p))) \cap N \\
&= (p^{\leftarrow}\{\,l\,\} \cup (q^{\leftarrow}\{\,l\,\} \setminus \mathrm{ht}(p))) \cap N \\
&\subseteq (p^{\leftarrow}\{\,l\,\} \cap N) \cup (q^{\leftarrow}\{\,l\,\} \setminus \mathrm{ht}(p)) \\
&\subseteq \{\,\gamma\,\} \cup (q^{\leftarrow}\{\,l\,\} \setminus \mathrm{ht}(p))
\end{aligned}$$

By the definition, $|q^{\leftarrow}\{\,l\,\} \setminus \mathrm{ht}(p)| \leq 1$. So, $q^{\leftarrow}\{\,l\,\} \cap N$ has at most two elements. Thererfore, γ is not a τ-limit point of $q^{\leftarrow}\{\,l\,\}$.

So, by Claim 4.1.1, we have $q \in P$. Therefore, $q \in P$. Trivially, $\mathrm{ht}(q) = \zeta$ and $q^{\leftarrow}k \subseteq \mathrm{ht}(p)$. □

Claim 4.1.4. P is proper.

Proof. Let θ be a regular cardinal such that $\theta > 2^{|P|}$ and $\tau, P \in H(\theta)$. Let $\langle M_\alpha \mid \alpha < \omega_1 \rangle$ be a tower of countable elementary submodels of $H(\theta)$ with $\tau, P \in M_0$. Let $D = \{\,\alpha < \omega_1 \mid M_\alpha \cap \omega_1 = \alpha\,\}$. It is easy to see that D is a club subset of ω_1.

Let θ' be a regular cardinal such that $\theta' > 2^\theta$. Let M be a countable elementary submodel of $H(\theta')$ such that $\langle M_\alpha \mid \alpha < \omega_1 \rangle \in M$. Let $p \in M \cap P$. We shall show that there exists $q \leq p$ that is (M, P)-generic. Let $\langle \mathcal{D}_m \mid m < \omega \rangle$ be an enumeration of all open dense subsets of P lying in M. Let $\delta = M \cap \omega_1$. It is easy to see that $D \in M$. Let $E \in M$ be a witness of $(*)$ for D. Then, clearly $\delta \in E$. By the definition of E, there exists a τ-open neighborhood N of $\delta + n$ such that for every $\eta < \delta$, there exists $\xi \in D \cap [\eta, \delta)$ such that $N \cap \xi$ is bounded in ξ.

We shall build a decreasing sequence $\langle p_m \mid m < \omega \rangle$ in P and two increasing sequence $\langle \alpha_m \mid m < \omega \rangle$ and $\langle \zeta_m \mid m < \omega \rangle$ in δ as follows. Let $\langle \beta_n \mid \beta < \omega \rangle$ an increasing cofinal sequence in δ. Let $p_0 = p$. By the definition of N, there exists $\alpha_0 \in D \cap \delta$ such that $\alpha_0 \geq \beta_0$, $p_0, \mathcal{D}_0 \in M_{\alpha_0}$ and $N \cap \alpha_0$ is bounded in α_0. So, there exists $\zeta_0 < \alpha_0$ such that $[\zeta_0, \alpha_0) \cap N = \emptyset$. Since $\alpha_0 \in D$, we have $M_{\alpha_0} \cap \omega_1 = \alpha_0$. So, α_0 is a limit of limit ordinals. Thus, without loss of generality, we may assume ζ_0 is a limit ordinal. By the previous claim, there exists $p'_0 \leq p_0$ be so that $p'_0 \in M_{\alpha_0}$ and $\mathrm{ht}(p'_0) \geq \zeta_0$. Let $p_1 \leq p'_0$ be so that $p_1 \in M_{\alpha_0} \cap \mathcal{D}_0$.

Now suppose that $m \in [1, \omega)$, p_m and α_l, ζ_l for all $l < m$ are defined. By the definition of E, there exists $\alpha_m \in D \cap \delta$ so that $\alpha_m \geq \beta_m$, $\alpha_{m-1} < \alpha_m$, $p_m, \mathcal{D}_m \in M_{\alpha_m}$, and $N \cap \alpha_m$ is bounded in α_m. So, there exists $\zeta_m < \alpha_m$ such that $[\zeta_m, \alpha_m) \cap N = \emptyset$. As in the previous paragraph, we may assume ζ_m is a limit ordinal. By the previous claim, there exists $p'_m \leq p_m$ such that $p'_m \in M_{\alpha_m}$, $\mathrm{ht}(p'_m) \geq \zeta_m$, and $(p'_m)^{\leftarrow}k \subseteq \mathrm{ht}(p_m)$. Let $p_{m+1} \leq p'_m$ be so that $p_{m+1} \in M_{\alpha_m} \cap \mathcal{D}_m$.

Let $q = \bigcup_{m<\omega} p_m$. It is routine to show $\sup_{m<\omega} \mathrm{ht}(p_m) = \delta$. We shall show that $q \in P$. By Claim 4.1.2, it suffices to show that for every $k < \omega$, $\delta + n$ is not a τ-limit point of $q^{\leftarrow}\{\,k\,\}$.

Let $k < \omega$. Since $p_k \in P$, $\delta + n$ is not a τ-limit point of ${p_k}^{\leftarrow}\{\, k \,\}$. Thus, there exists a τ-open neighborhood of $\delta+n$ such that ${p_k}^{\leftarrow}\{\, k \,\} \cap N \subseteq \{\, \delta + n \,\}$. Since $\mathrm{ht}(p_k) \leq \delta \leq \delta+n$, we have ${p_k}^{\leftarrow}\{\, k \,\} \subseteq \mathrm{dom}(p_k) \subseteq \mathrm{ht}(p_k) \subseteq \delta + n$ and hence $\delta + n \notin {p_k}^{\leftarrow}\{\, k \,\}$. So, ${p_k}^{\leftarrow}\{\, k \,\} \cap N = \emptyset$. By induction on $m \in [k, \omega)$, we shall show that ${p_m}^{\leftarrow}\{\, k \,\} \cap N = \emptyset$. By the definition of N, we have ${p_k}^{\leftarrow}\{\, k \,\} \cap N = \emptyset$. Let $m \in [k, \omega)$ be so that ${p_m}^{\leftarrow}\{\, k \,\} \cap N = \emptyset$. By the construction, $(p'_m)^{\leftarrow}\{\, k \,\} \subseteq \mathrm{ht}(p_m)$. So, $(p'_m)^{\leftarrow}\{\, k \,\} \cap N \subseteq (p'_m)^{\leftarrow}\{\, k \,\} \cap N \cap \mathrm{ht}(p_m) = {p_m}^{\leftarrow}\{\, k \,\} \cap N = \emptyset$. Since $p_{m+1} \in M_{\alpha_m}$, we have $\mathrm{ht}(p_{m+1}) \in M_{\alpha_m} \cap \omega_1 = \alpha_m$. Since $\mathrm{ht}(p'_m) \geq \zeta_m$, we have ${p_{m+1}}^{\leftarrow}\{\, k \,\} \setminus \mathrm{ht}(p'_m) \subseteq [\zeta_m, \mathrm{ht}(p_{m+1})) \subseteq [\zeta_m, \alpha_m)$. Thus, $({p_{m+1}}^{\leftarrow}\{\, k \,\} \setminus \mathrm{ht}(p'_m)) \cap N \subseteq [\zeta_m, \alpha_m) \cap N = \emptyset$. Therefore, ${p_{m+1}}^{\leftarrow}\{\, k \,\} \cap N \subseteq \mathrm{ht}(p'_m)$. So,

$$
\begin{aligned}
{p_{m+1}}^{\leftarrow}\{\, k \,\} \cap N &= ({p_{m+1}}^{\leftarrow}\{\, k \,\} \cap \mathrm{ht}(p'_m)) \cap N \\
&= {p'_m}^{\leftarrow}\{\, k \,\} \cap N \\
&= \emptyset
\end{aligned}
$$

Thus, $q^{\leftarrow}\{\, k \,\} \cap N = \bigcup_{m<\omega} {p_m}^{\leftarrow}\{\, k \,\} \cap N = \emptyset$. Hence, $\delta + n$ is not a τ-limit point of q.

Now that we know $q \in P$, it is easy to see that q is (N, P)-generic. $\square$

By PFA, there exists a filter G on P such that for every $\zeta < \omega_1$ there exists $p \in G$ such that $\mathrm{ht}(p) \geq \zeta$. Let $g : X_n \to \omega$ be defined by $g = \bigcup G$. Since (ω_1, τ) is locally countable, it is easy to see that for every $k < \omega$, $g^{\leftarrow}\{\, k \,\}$ is a $\tau \restriction X_n$-closed discrete subset of X_n. Thus, $g^{\leftarrow}\{\, k \,\}$ is τ-discrete. By Lemma 2.4, for every $k < \omega$, $g^{\leftarrow}\{\, k \,\} \in \breve{\mathcal{F}}$. Since $\mathcal{F}$ is countably complete, we have $X_n = \bigcup_{k<\omega} g^{\leftarrow}\{\, k \,\} \in \breve{\mathcal{F}}$. This is a contradiction to the assumption $X_n \in \mathcal{F}^+$. $\square$

5. Consequences

In this section, we shall discuss some consequences of Theorem 4.1. Let D be a club subset of ω_1 as in the conclusion of theorem 4.1. So, there exists a stationary subset S of $\mathrm{acc}(D)$ such that for all $\delta \in S$ and τ-open neighborhood N of $\delta + n$, there exists $\eta < \delta$ such that for all $\xi \in D \cap [\eta, \delta)$, $N \cap \xi$ is unbounded in ξ.

Corollary 5.1. *There exist a topology σ on ω_1 and a stationary subset T of ω_1 such that (ω_1, σ) is perfectly normal and nonrealcompact and for all $\gamma \in T$ and σ-open neighborhood N of γ, there exists $\eta < \gamma$ such that for all $\xi \in [\eta, \gamma) \cap \mathrm{Lim}^2$, $N \cap \xi$ is unbounded in ξ.*

Proof. By Lemma 3.2, we have $B(D) \in \mathcal{F}$. By Lemma 2.5, $(B(D), \tau \restriction B(D))$ is perfectly normal and nonrealcompact. Let $\langle \delta_\gamma \mid \gamma < \omega_1 \rangle$ be the increasing enumeration of $B(D)$. We shall define a function $f : B(D) \to \omega_1$ by for all

$\gamma < \omega_1$ and $m < \omega$

$$f(\delta_\gamma + m) = \begin{cases} \omega\gamma & \text{if } m = n \\ \omega\gamma + n & \text{if } m = 0 \\ \omega\gamma + m & \text{if } m \neq 0 \text{ and } m \neq n \end{cases}$$

It is easy to see that f is a bijection. Define $g : \omega_1 \cap \mathrm{Lim} \to \omega_1$ by $g(\gamma) = \bar{\gamma}$ if and only if $\gamma = \omega\bar{\gamma}$. Define a topology σ on ω_1 by for all $U \subseteq \omega_1$, $U \in \sigma$ if and only if $f^{\leftarrow}U \in \tau \restriction B(D)$. Then, trivially f is a homeomorphism from $(B(D), \tau \restriction B(D))$ onto (ω_1, σ). Therefore, (ω_1, σ) is perfectly normal and nonrealcompact.

Let T be the set of all $\gamma \in \omega_1 \cap \mathrm{Lim}^3$ such that $\delta_{g(\gamma)} \in S$. It is easy to see that T is stationary. We shall show that for all $\gamma \in T$ and σ-open neighborhoods N of γ, there exists $\eta < \gamma$ such that for all $\xi \in [\eta, \gamma) \cap \mathrm{Lim}^2$, $N \cap \xi$ is unbounded in ξ. Let $\gamma \in T$ and N a σ-open neighborhood of γ. By the definition of T, $\gamma \in \mathrm{Lim}^3$ and $\delta_{g(\gamma)} \in S$. Then, $f(\delta_{g(\gamma)} + n) = \omega g(\gamma) = \gamma$

Since N is a σ-open neighborhood of γ, $f^{\leftarrow}N$ is a $\tau \restriction B(D)$-open neighborhood of $\delta_{g(\gamma)} + n$. Since $\delta_{g(\gamma)} \in S$, there exists $\zeta < \delta_{g(\gamma)}$ such that for all $\xi \in D \cap \big[\zeta, \delta_{g(\gamma)}\big)$, $f^{\leftarrow}N \cap \xi$ is unbounded in ξ. Since $\gamma \in \mathrm{Lim}^3$, we have $g(\gamma) \in \mathrm{Lim}^2$. Thus, there exists $\bar{\eta} \in g(\gamma) \cap \mathrm{Lim}$ such that $\delta_{\bar{\eta}} \geq \zeta$. Let $\eta = \omega\bar{\eta}$. Clearly, $g(\eta) = \bar{\eta}$.

Now, we shall show that for all $\xi \in [\eta, \gamma) \cap \mathrm{Lim}^2$, $N \cap \xi$ is unbounded in ξ. To do this, let $\varepsilon < \xi$ and we shall prove that $N \cap [\varepsilon, \xi) \neq \emptyset$. Since $\xi \in \mathrm{Lim}^2$, without loss of generality, we may assume $\varepsilon \in \mathrm{Lim}$. Since $\xi \in \mathrm{Lim}^2$, we have $g(\xi) \in \mathrm{Lim}$. Since $\eta \leq \xi$, we have $g(\eta) \leq g(\xi)$ and hence $\delta_{g(\eta)} \leq \delta_{g(\xi)}$. Since $\delta_{g(\eta)} = \delta_{\bar{\eta}} \geq \zeta$, we have $\delta_{g(\xi)} \in D \cap \big[\zeta, \delta_{g(\gamma)}\big)$. So, $f^{\leftarrow}N \cap \delta_{g(\xi)}$ is unbounded in $\delta_{g(\xi)}$. In particular, there exists $\mu \in f^{\leftarrow}N \cap \big[\delta_{g(\varepsilon)}, \delta_{g(\xi)}\big)$. So, $f(\mu) \in N$ and $f(\mu) \in [\varepsilon, \xi)$. Thus, $N \cap \xi$ is unbounded in ξ. $\square$

The space we built in the previous corollary has the following property.

Corollary 5.2. *Let (ω_1, σ) and T be the witnesses of Corollary 5.1. Then, for every σ-open set U, if $U \cap T$ is stationary, there exists $\eta < \omega_1$ such that for all $\xi \in [\eta, \omega_1) \cap \mathrm{Lim}^2$, $U \cap \xi$ is unbounded in ξ.*

Proof. Let U be a σ-open set such that $U \cap T$ is stationary. For each $\gamma \in U \cap T$, since U is σ-open, there exists a σ-open neighborhood N_γ of γ such that $N_\gamma \subseteq U$. By the definition of (ω_1, σ) and T, for each $\gamma \in U \cap T$, there exists $\eta_\gamma < \gamma$ such that for all $\xi \in [\eta_\gamma, \gamma) \cap \mathrm{Lim}^2$, $N_\gamma \cap \xi$ is unbounded in ξ. By Fodor's Lemma, there exist a stationary subset T' of $U \cap T$ and $\eta < \omega_1$ such that for every $\gamma \in T'$, $\eta_\gamma = \eta$. So, for every $\gamma \in T'$ and $\xi \in [\eta, \gamma) \cap \mathrm{Lim}^2$, $N_\gamma \cap \xi$ is unbounded in ξ. Since $N_\gamma \subseteq U$, for every $\gamma \in T'$ and $\xi \in [\eta, \gamma) \cap \mathrm{Lim}^2$, $U \cap \xi$ is unbounded in ξ. Since T' is stationary and hence unbounded in ω_1, for every $\xi \in [\eta, \omega_1)$, $U \cap \xi$ is unbounded in ξ. $\square$

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