ECE157B - Communications Systems Lab-2 [Spring 2020]

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Contents

Chapter 1

Lecture 3: Range and Resolution – Signal Processing 2

1.1 Using wireless channel phase to estimate breathing and heart rate

The wireless channel is a function of signal wavelength λ and distance travelled $2d(t)$ (considering Tx and Rx are co-located and the target is present at a distance $d(t)$):

$$h = \frac{\lambda}{d(t)} \exp(j2\pi \frac{2d(t)}{\lambda}) \tag{1.1}$$

The phase of wireless channel is wrapped around 2π.

$$\phi = 2\pi \frac{2d(t)}{\lambda} \bmod 2\pi \tag{1.2}$$

Figure: Phase with distance

Due to phase wrapping, the solution for d is not unique. For instance, if there is a solution $d = \alpha$, then there are many other possible solutions $\alpha + \frac{\lambda}{2}, \alpha + \frac{2\lambda}{2}, \ldots, \alpha + \frac{3\lambda}{2}$.

1.2 Requirements: Phase accuracy, wavelength, and sampling rate

Suppose breathing distance is 10 cm and heart rate distance is 0.4 cm. The breathing and heart rate varies in the range $0 - 1.5$ Hz.

Wavelength: The wavelength should be large enough to avoid phase ambiguity caused by phase wrapping. In particular, the phase wrapping which occurs at diatance $\lambda/2$ should capture all the variation due to the breathing (which is 10 cm). Therefore, $\lambda/2 > 10cm \Rightarrow \lambda > 20$ cm.

Phase accuracy: The phase should be accurately measured with high resolution so as to be able to resolve small distance variations caused by the heart rate. The heart rate distance variations is $\pm$.2mm. So, the phase accuracy needs to be $2\pi \times 2d_H/\lambda$ radians $= 0.72^o$ for $\lambda = 20$ cm and $d_H = 0.2$ mm (maximum distance variation due to heart rate).

Sampling rate: The nyquist sampling rate should be twice the maximum frequency present in the distance signal. Since, maximum heart rate can be 1.5 Hz. the sampling rate should be $f_s^d > 2 \times 1.5 = 3$Hz. For a robust estimate, we choose $f_s^d = 60$ Hz.

1.2.1 Revision: Wireless channel

We will understand how we can estimate the wireless channel from a simple radar system. Consider the transmitter transmits a signal $x(t)$ with a single tone at carrier frequency f_c as follow

$$x(t) = \exp(j2\pi f_c t) \tag{1.3}$$

The receiver receives an attenuated and delayed copy the transmitted signal. The attenuation is proportional to $\hat{A} = \frac{d(t)}{\lambda}$ and the delay is proportional to $\tau = \frac{2d(t)}{c}$, where $\lambda = \frac{c}{f_c}$ is the signal wavelength, $d(t)$ is the target distance and c is speed of light. The received signal $y(t)$ is

$$y(t) = \widehat{A}\exp(j2\pi f_c(t-\tau)) \tag{1.4}$$

The received signal gets sampled at the ADC with sampling rate f_s and sampling time $Ts = 1/f_s$. We receive the digital samples of the received signal as

$$\begin{aligned} y(nTs) &= \widehat{A}\exp(j2\pi f_c(nTs-\tau)) \\ &= \widehat{A}\exp(j2\pi f_c nTs)\exp(-j2\pi f_c\tau) \\ &= \widehat{A}\exp(j2\pi f_c nTs)\exp(-j2\pi\frac{2d(nT_s)}{\lambda}) \end{aligned} \tag{1.5}$$

Now, we need to estimate the channel phase to get $d(t)$ from which we can extract breathing and heart rate. The channel estimation follows

$$\hat{h} = y(nTs)\exp(-j2\pi f_c nTs) \tag{1.6}$$

The above operation is possible at the receiver only if the Tx and Rx are perfectly time synchronized at the symbol level. We get

$$\hat{h} = \widehat{A}\exp(-j2\pi\frac{2d(nTs)}{\lambda}) \tag{1.7}$$

1.3 Meeting the breathing and heart rate requirements

1.3.1 Accuracy of phase measurements

We achieve 0.7^o stringent phase accuracy by taking advantage of high carrier frequency and averaging out the measurement noise over millions of samples.

For carrier frequency $f_c = 1.5GHz$, we choose the nyquist sampling rate $f_s = 2f_c = 3GHz$ which is 500 millions times faster compared to the required 60 Hz sampling rate for breathing/heart rate estimation. It is because breathing/heart rate varies very slowly compared to the center frequency of wireless communication system. Taking this phenomenon as an advantage, we take average over 500 million samples to reduce the impact of random noise.

How averaging helps achieving accuracy in phase measurements? Typically, ADC can cause an error of 6^o (0.1 radians) when they sample the received signal at $3GHz$. That means instead of measuring accurate phase $e^{j\phi}$, the ADC reports erroneous phase $e^{j\phi\pm0.05}$. For $Ts = 1/3GHz = 0.33ns$, the averaging of the received signal over 500M samples helps reducing the ADC noise as follow:

$$\begin{aligned} \hat{h}_{avg} &= \frac{1}{500M}\sum_{n=-250M}^{250M}\hat{h}(n) + \text{noise} \\ &= \frac{1}{500M}\sum_{n=-250M}^{250M} y(nTs)\exp(-j2\pi f_c nTs) + \text{noise} \\ &\approx \widehat{A}\exp(-j2\pi\frac{2d(0)}{\lambda}) + \frac{1}{500M}\sum_{n=-250M}^{250M}\text{noise} \end{aligned} \tag{1.8}$$

The averaging reduces the noise magnitude and thus reduces the error in phase measurement to below 0.7^o requirement.

Tradeoff: phase accuracy vs sampling rate. There is a tradeoff between phase measurement accuracy and distance sampling rate f_s^d. Averaging helps in improving phase accuracy, but, if we do more averaging, we compromise on the distance sampling rate. For instance, we average over 500M ADC samples to get $f_s^d = 60$ Hz. However, if we average over higher number of ADC samples for a fixed ADC sampling rate, we get lessor distance samples at our disposal. For instance, averaging over 1000M ADC samples (ADC running at 3GHz), we get only 30 Hz distance sampling rate.

ToDo Figure: time-frequency plot of $d(t)$.

1.4 Revision: FFT

Take a continuous physical signal with certain continuous Fourier transform. Study the effect of time windowing, time sampling, and frequency sampling. Revisit properties of FFT.

Chapter 2

Lecture 4: Multiple wavelength – breathing and heart rate

2.1 Primer: DTFT vs DFT

DTFT (Discrete Time Fourier Transform) is defined for discrete time signal (of arbitrary length) whose frequency response is continuous and periodic with a period of 2π.

$$\mathrm{DTFT}(x[n]) = X(f) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j2\pi f n} \tag{2.1}$$

On the other hand, DFT (Discrete Fourier Transform) is defined for a **periodic**[1] discrete time signal with period N whose frequency response is also **discrete** and periodic with period N.

$$\mathrm{DFT}(x[n]) = X[k] = x[n] e^{-j2\pi \frac{nk}{N}} \tag{2.2}$$

Note that

- Fourier transform is defined for a continuous time signal with continuous frequency response.
- For DTFT, we sample the signal in time domain (equivalent to multiplication by an infinite impulse train), resulting in a periodic frequency response (because of convolution with an infinite impulse train). Note the frequency response is still continuous.
- For DFT, we sample in both time and frequency domain, so the signal has to be periodic in both domains. DFT frequency response can be obtained from DTFT frequency response by sampling at $f = \frac{k}{N}$, for $k = 0, \ldots, N-1$.

Another way to remember DTFT and DFT is that: given a finite discrete time signal, if we infinitely zeropad the signal, we get DTFT. If we periodically extend the signal, we get DFT. FFT (Fast Fourier Transform) is just a fast implementation of DFT.

Because of its continuous frequency response, we cannot represent DTFT in Matlab. But, the trick is, if we highly zeropad the signal, we get close to a continuous frequency response, thus, resembling a DTFT frequency response.

[1] periodic signals are always of infinite length, but can be represented by one period.

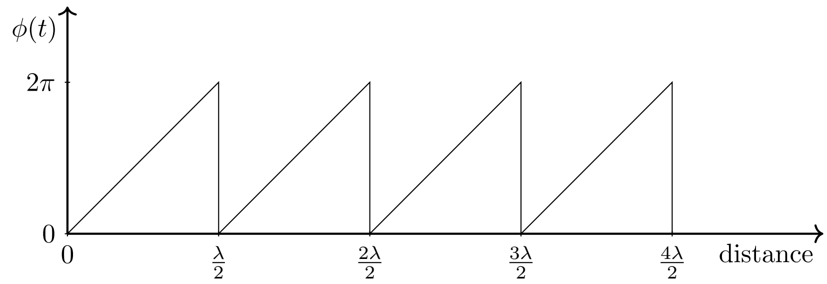

Figure 2.1: Phase $\phi(d)$ varies proportional to distance with wrapping of 2π.

2.2 Phase of wireless channel estimate

The phase $\phi(d)$ term of wireless signal is periodic and repeats itself in $[0, 2\pi]$ after distance d equal to multiples of $\lambda/2$. Therefore, as long as the variations of d is confined within $\lambda/2$, we can predict these variations from the phase signal. But, note that we cannot predict the propagation distance d_0 by only using the phase $\phi(d)$ because of the phase wrapping. Later, we will see how to measure the propagation distance by collecting multiple measurements.

$$\begin{aligned}\phi(d) &= 2\pi\frac{2d}{\lambda}\ mod\ 2\pi \\ &= 2\pi k + 2\pi\frac{2d}{\lambda} \text{ for } k \in [-\infty, \infty]\end{aligned} \tag{2.3}$$

2.3 Dealing with multiple reflections

Until now, we dealt with a single reflection arriving at the receiver. Ideally, the receiver gets multiple reflections from different targets. The channel response of two reflections from targets located at distances $d_1(t)$ and $d_2(t)$ is represented as follows:

$$\begin{aligned}h(t) &= h_1(t) + h_2(t) \\ &= \frac{\lambda}{d_1}\exp(-j2\pi\frac{2d_1(t)}{\lambda}) + \frac{\lambda}{d_2}\exp(-j2\pi\frac{2d_2(t)}{\lambda})\end{aligned} \tag{2.4}$$

For better understanding, we represent the wireless channel using a phaser diagram. In a phaser diagram, the length of an arrow represents the magnitude and the angle from x-axis represents the phase term. The wireless channel is represented as a sum of two signal with different magnitude and phases in the following phaser diagram.

Now, the question is how do we separate the two signal which are added together in the channel response. Key idea is to add more equations to solve for multiple unknown variables.

Approach 1: One idea is to observe the channel at two time instants t_1 and t_2 and subtract the two channel $h(t_1)$ and $h(t_2)$

$$\begin{aligned}h(t_2) - h(t_1) &= \frac{\lambda}{d_1(t_2)}\exp(-j2\pi\frac{2d_1(t_2)}{\lambda}) - \frac{\lambda}{d_1(t_1)}\exp(-j2\pi\frac{2d_1(t_1)}{\lambda}) \\ &\approx \frac{\lambda}{d_1}\left(\exp(-j2\pi\frac{2d_1(t_2)}{\lambda}) - \exp(-j2\pi\frac{2d_1(t_1)}{\lambda})\right)\end{aligned} \tag{2.5}$$

Here, we made an important observation that the reflections from static objects such as table, chair etc will be stationary. On the other hand, the reflections from a person will be time-varying because of the variation in

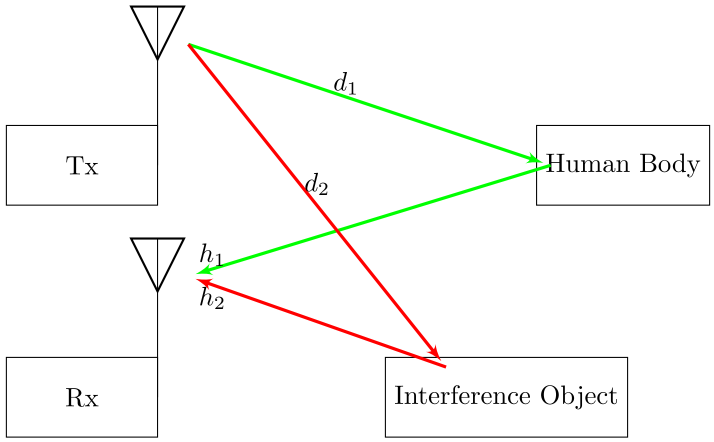

Figure 2.2: Radar capturing two reflections at distance d_1 and d_2.

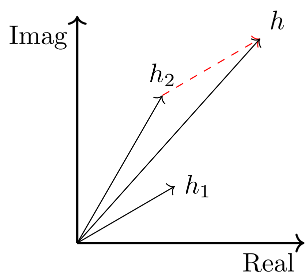

Figure 2.3: Phaser diagram

distance due to breathing and heart rate. Thus $d_2(t_1) = d_2(t_2)$, and the effect of second unwanted reflection is gone. However, what we get is the difference of signal at two time instants from which it is extremely hard to extract the breathing and heart rate.

Approach 2: Another approach is to use the channel estimates at multiple frequencies (or wavelengths) to add more equations. For instance, we get two equations by observing channel at two wavelengths λ_1 and λ_2 to solve for two variables $d_1(t)$ and $d_2(t)$:

$$h^{\lambda_1} = \frac{\lambda_1}{d_1}\exp(-j2\pi\frac{2d_1(t)}{\lambda_1}) + \frac{\lambda_1}{d_2}\exp(-j2\pi\frac{2d_2(t)}{\lambda_1}) \tag{2.6}$$

$$h^{\lambda_2} = \frac{\lambda_2}{d_1}\exp(-j2\pi\frac{2d_1(t)}{\lambda_2}) + \frac{\lambda_2}{d_2}\exp(-j2\pi\frac{2d_2(t)}{\lambda_2}) \tag{2.7}$$

Now, that we have two equations and two variables, mathematically, we can solve for $d_1(t)$ and $d_2(t)$. Try it out yourself.

2.4 Estimating propagation delay (Alignment method)

In this section, we go back to the scenario where we had only one reflection from a person. Recall that from the phase measurement $\phi(t)$, we can only predict the distance variations which in turn will give us the breathing rate and heart rate by FFT based methods. In this section, we are interested in estimating the distance d_0 of the person from the radar system. Note that $d_0 = c\tau$, where c is the speed of light and τ is the propagation delay. Therefore, estimating d_0 is equivalent to estimating the propagation delay τ. We

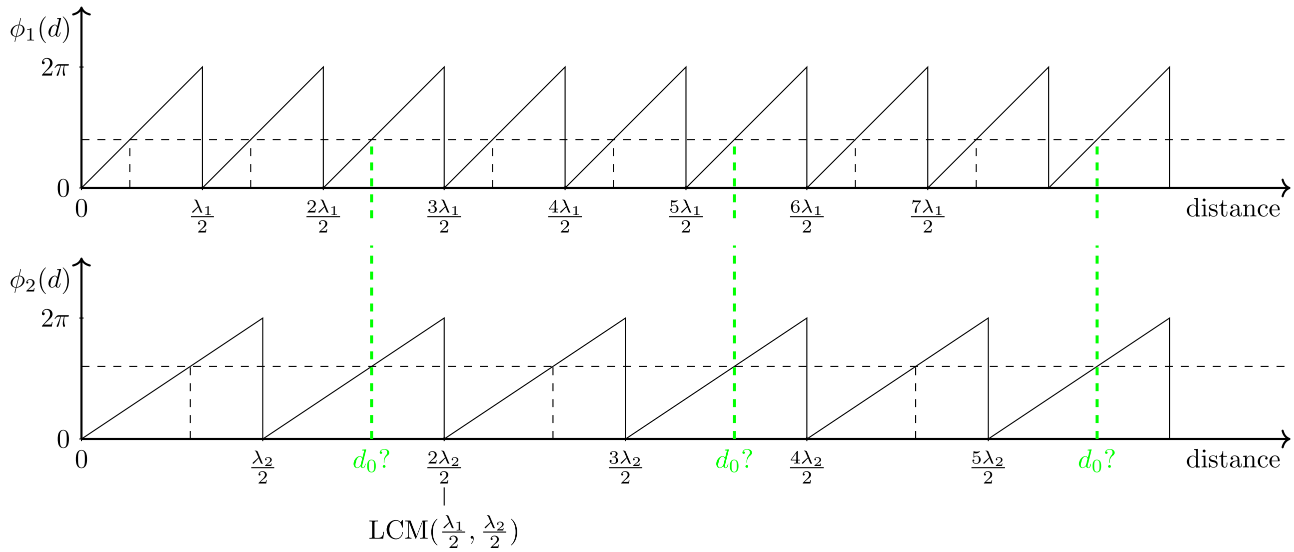

Figure 2.4: Phase $\phi_1(d)$ and $\phi_2(d)$ are obtained for two different frequencies (wavelength λ_1 and λ_2). The figures shows that if $d_0 < \text{LCM}(\frac{\lambda_1}{2}, \frac{\lambda_2}{2})$, we have a unique solution for d_0.

rewrite the wireless channel phase $\phi = 2\pi f\tau \bmod 2\pi$ as $2\pi\frac{d_0}{\lambda} \bmod 2\pi$. We can use the methods used for estimating propagation delay to also estimate the distance d_0.

Problem Definition: Estimate the distance d_0 from the following phase measurement (the factor of $2d_0$ corresponds to twice the distance travelled by wireless signal reflection).

$$\phi(d) = 2\pi\frac{2d_0}{\lambda} \bmod 2\pi \tag{2.8}$$

Issue: Because of modulus 2π, the phase is wrapped around $[0, 2\pi]$. This is a classic example of aliasing. It means that for a given value of observed phase, we can have multiple solutions of d_0. In other words, the solution for d_0 will be unique only if we have the constraint that $d_0 < \lambda/2$. Can we extend this range for which we get unique value of d_0?

Solution: We observe different phase values at different frequencies (or wavelengths). Suppose, we get ϕ_1 and ϕ_2 for two wavelengths λ_1 and λ_2 respectively:

$$\phi_1 = 2\pi\frac{2d_0}{\lambda_1} \bmod 2\pi \tag{2.9}$$

$$\phi_2 = 2\pi\frac{2d_0}{\lambda_2} \bmod 2\pi \tag{2.10}$$

The possible solution for d_0 is some multiple of $\lambda_1/2$ plus a small perturbation. Similarly, d_0 is also some (different) multiple of $\lambda_2/2$ as:

$$d_0(\lambda_1) = k_1\lambda_1/2 + \delta_1, \quad \delta_1 \in [0, \lambda_1/2] \tag{2.11}$$

$$d_0(\lambda_2) = k_2\lambda_2/2 + \delta_2, \quad \delta_2 \in [0, \lambda_2/2] \tag{2.12}$$

where the integer multiples k_1 and k_2 are unknown. The unique solution for d_0 should satisfy $d_0(\lambda_1) = d_0(\lambda_1)$. With this new constraint, we can see from Figure 2.4 that the solution for d_0 is unique if the following condition is satisfied:

$$d_0 < \text{LCM}(\lambda_1/2, \lambda_2/2) \tag{2.13}$$

This alignment method is famously known as Chinese remainder theorem.

Example 1: Consider two phase measurements at 1 GHz and 1.5 GHz respectively. The corresponding values $\lambda_1/2 = 15$cm and $\lambda_2/2 = 10$cm. The least common multiple LCM$(10, 15) = 30$ cm (See Figure 2.4). Therefore, we can uniquely estimate d_0 without aliasing if the distance is constraint to $0 < d_0 < 30$ cm.

Improvement: Can we improve the un-aliased range of possible values of d_0? The maximum value an LCM can take is the product of two number. So, we can improve the range if we choose multiple wavelengths which are co-prime. Note the LCM of two co-primes is their product.

Example 2: Consider two frequencies $f_1 = 1.5$ GHz and $f_2 = 1.6$ GHz. Calculate $\lambda_1 = c/2f_1 = 10$ cm and similarly $\lambda_2 = 9.3$ cm which are co-primes. For this scenario, LCM is $10 * 9.3 = 93$ cm. So as long as $0 < d < 93$ cm, we can accurately identify d.

The alignment approach mentioned above works only for estimating distance of a single target. Does it generalizes to estimating two distances d_1 and d_2 corresponding to two different target locations under the constraint:

$$0 < d_1 < d_2 < 93cm \tag{2.14}$$

Next we discuss tricks to get the estimate of d_1 and d_2?

2.5 Use multiple wavelengths to estimate multiple distances

Now, consider reflections from two objects placed at distances d_1 and d_2.

$$h(\lambda) = \sum_{i=1}^{2} \frac{\lambda}{d_i} \exp(-j2\pi \frac{2d_i}{\lambda}) \tag{2.15}$$

To estimate distances d_i, we make an important observation. The expression of $h(\lambda)$ looks similar to the expression of DTFT in Equation (2.1). Compare $x[n]$ and $X(f)$ in Equation (2.1) with $\frac{\lambda}{d_i}$ and $h(\lambda)$ in the above equation.

Therefore, we can do operation similar to IFFT of $h(\lambda)$ to get the distance profile $f(d)$ as follows:

$$f(d) = \sum_{\lambda=\lambda_1}^{\lambda_2} h(\lambda) \exp(j2\pi \frac{2d}{\lambda}) \tag{2.16}$$

The distance profile $f(d)$ can be evaluated at different d values. Can you see that we will get peaks for $d = d_1$ and $d = d_2$.

Proof. Substitute the value of $h(\lambda_1)$ and $h(\lambda_2)$ from (2.15) to (3.9).

$$\begin{aligned} f(d) &= h(\lambda_1) \exp(j2\pi \frac{2d}{\lambda_1}) + h(\lambda_2) \exp(j2\pi \frac{2d}{\lambda_2}) \\ &= \sum_{i=1}^{2} \frac{\lambda_1}{d_i} \exp(-j2\pi \frac{2d_i}{\lambda_1}) \exp(j2\pi \frac{2d}{\lambda_1}) + \sum_{i=1}^{2} \frac{\lambda_2}{d_i} \exp(-j2\pi \frac{2d_i}{\lambda_2}) \exp(j2\pi \frac{2d}{\lambda_2}) \end{aligned} \tag{2.17}$$

Evaluate the expression at $d = d_1$

$$\begin{aligned} f(d_1) &= \frac{\lambda_1}{d_1} + \frac{\lambda_1}{d_2} \exp\left(j2\pi \frac{2(d_1 - d_2)}{\lambda_1}\right) + \frac{\lambda_2}{d_1} + \frac{\lambda_2}{d_2} \exp\left(j2\pi \frac{2(d_1 - d_2)}{\lambda_2}\right) \\ &= \frac{(\lambda_1 + \lambda_2)}{d_1} + \frac{\lambda_1}{d_2} \exp\left(j2\pi \frac{2(d_1 - d_2)}{\lambda_1}\right) + \frac{\lambda_1}{d_2} \exp\left(j2\pi \frac{2(d_1 - d_2)}{\lambda_2}\right) \end{aligned} \tag{2.18}$$

Note that the component $\frac{\lambda_1}{d_1}$ and $\frac{\lambda_2}{d_1}$ adds up constructively while the other exponential terms may not add constructively (or they may even completely cancel each other). $\square$

Todo: Figure $f(d)$ vs d.

2.6 Range and Resolution for multiple wavelength

Range: We have already established the range of values of d for which we get a unique solution without aliasing. That is all the distances d_i should be less than the least common multiple of the wavelengths:

$$\text{Range} = \text{LCM}(\frac{\lambda_1}{2}, \frac{\lambda_2}{2}) \tag{2.19}$$

Resolution: Resolution is defined as how close the two distances d_1 and d_2 can be so that they can be distinguished in the distance profile $f(d)$. Think of the analogy with the FFT. Recall you can resolve two frequencies if the frequency samples are close enough and the resolution is the difference between two frequency samples i.e., $f_1 - f_2$, or equivalently $\approx \frac{1}{\lambda_1} - \frac{1}{\lambda_2}$. With this intuition the distance resolution is:

$$\text{Resolution} \approx \text{GCD}(\frac{\lambda_1}{2}, \frac{\lambda_2}{2}) \tag{2.20}$$

Chapter 3

Lecture 5: Scaling up to multiple wavelengths

3.1 Two step Process

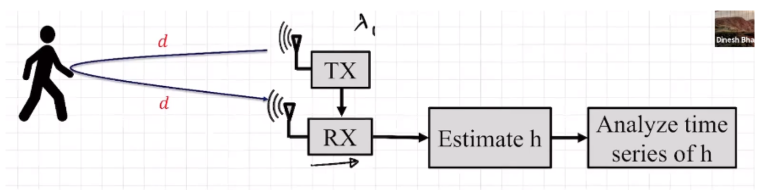

Figure 3.1: Two step process. The first step is channel estimation and second step is time series analysis. We optimize the first step for sampling rate and accuracy and the second step for range and resolution.

3.1.1 First step: Estimate the channel

$$h(\lambda, t) = \frac{\lambda}{d(t)} \exp\left(-j2\pi \frac{2d(t)}{\lambda}\right) \tag{3.1}$$

In general, we get channel for different time instants t_j and for different wavelengths λ_i.

$$h(\lambda_i, t_j) = \frac{\lambda_i}{d(t_j)} \exp\left(-j2\pi \frac{2d(t_j)}{\lambda_i}\right) \tag{3.2}$$

Recall from Lecture 3, how we estimate the channel. A simplified approach is to transmit a single tone signal at carrier frequency f_c as follow:

$$x(t) = \exp(j2\pi f_c t) \tag{3.3}$$

The received signal is delayed by $\tau = \frac{2d}{c}$ and attenuated approximately by a factor of $\frac{\lambda}{d}$.

$$\begin{aligned} y(t) &= \frac{\lambda}{d} x(t-\tau) \\ &= \frac{\lambda}{d} \exp(j2\pi f_c (t-\tau)) \end{aligned} \tag{3.4}$$

We sample the received signal at time samples nTs:

$$y(nTs) = \frac{\lambda}{d} \exp(j2\pi f_c (nTs - \tau)) \tag{3.5}$$

Channel estimation:

$$\hat{h} = y(nTs)\exp(-j2\pi f_c nTs) = \frac{\lambda}{d}\exp(-j2\pi f_c \tau) \tag{3.6}$$

Thus, we estimate the channel for multiple time instants t_j. We repeat the process at multiple carrier frequencies and independently estimate the channel for multiple wavelengths λ_i. In later lectures, we will optimize channel estimating from the context of FMCW and OFDM system. Recall from Lecture 3 that in this step, we optimize for sampling rate (how fast can we sample) and phase measurement accuracy. We have also discussed a tradeoff between sampling rate and accuracy.

3.1.2 Second step: Time series analysis

In this step, we take the channel estimate and estimate breathing/heart rate as we discussed in Lecture 1 and Lecture 2. We study what is the correct λ values to satisfy the range and resolution constraints. Recall the range and resolution depends on the wavelength of the first step. We will discuss it in detail in the following sections.

3.2 Two wavelength range and resolution - Multipath problem

We used two wavelength channel to improve the range and resolution to resolve two reflections from two different targets.

The multipath channel is represented as:

$$h(\lambda) = \frac{\lambda}{d_1}\exp(-j2\pi\frac{2d_1(t)}{\lambda}) + \frac{\lambda}{d_2}\exp(-j2\pi\frac{2d_2(t)}{\lambda}) \tag{3.7}$$

Suppose we independently measure the channel at two wavelengths λ_1 and λ_2. Therefore, we have the measurements $h(\lambda_1)$ and $h(\lambda_2)$ from which we get the distance profile $f(d)$ as follows:

$$f(d) = \sum_{\lambda=\lambda_1}^{\lambda_2} h(\lambda)\exp(j2\pi\frac{2d}{\lambda}) \tag{3.8}$$

How is $f(d)$ equivalent to a Fourier transform? We can write $\frac{2d}{\lambda} = \frac{2d}{c}\frac{c}{\lambda}$. Notice the equivalence of $\tau = 2d/c$ with time or delay domain and $f_i = c/\lambda_i$ with frequency in a standard (inverse) Fourier transform. To emphasize it, we represent $f(d) = f(\tau)$ as follow:

$$f(\tau) = \sum_{f=f_1}^{f_2} h(\lambda)\exp(j2\pi\tau f) \tag{3.9}$$

The distance profile will show two peaks at distances d_1 and d_2. And thus, we can resolve the two distances caused by multipath effect.

3.2.1 Example two-wavelength channel with single path

Example: Consider one channel tap (no multipath) at delay τ and magnitude zero.

$$h(\tau) = \delta(t-\tau) \tag{3.10}$$

The frequency response will be

$$H(\lambda) = e^{j2\pi\frac{c}{\lambda}\tau} \tag{3.11}$$

The response $H(\lambda)$ is continuous in λ and acquires infinite bandwidth. In practice, we can measure this channel at some finite set of frequencies. Suppose, we measure the channel at two wavelengths λ_1 and λ_2.

$$\hat{H} = e^{j2\pi\frac{c}{\lambda_1}\tau}\delta(\lambda-\lambda_1) + e^{j2\pi\frac{c}{\lambda_2}\tau}\delta(\lambda-\lambda_2) \tag{3.12}$$

The inverse FFT will give

$$\begin{aligned}\hat{h}(t) &= e^{j2\pi\frac{c}{\lambda_1}(t-\tau)} + e^{j2\pi\frac{c}{\lambda_2}(t-\tau)} \\ &= e^{j2\pi\frac{c}{\lambda_1}(t-\tau)}\left[1 + e^{j2\pi(\frac{c}{\lambda_2}-\frac{c}{\lambda_1})(t-\tau)}\right]\end{aligned} \tag{3.13}$$

We see that $\hat{h}(t)$ is maximum at $t = \tau$ and goes down to zero at $t = \tau \pm 1/B$ where $B = f_2 - f_1$ is the bandwidth of the system. Moreover, $\hat{h}(t)$ is periodic with period $\frac{1}{B}$. The delay $\tau = \frac{2d(t)}{c}$ has variations due to breathing/heart rate because this distance $d(t)$ is slowly varying with time.

3.2.2 Range and resolution of two-wavelength system

When can the reflections be separated? We can clearly separate two reflections if $\tau_2 - \tau_1 > \frac{1}{B}$.
What is the range/aliasing? The range is $\frac{1}{B}$, because the $\hat{h}(t)$ is periodic with period $\frac{1}{B}$. So, if $\tau > \frac{1}{B}$, it will fall into second period and so we cannot detect it because of the periodicity of $\frac{1}{B}$.

3.3 Extending to multiple wavelengths range and resolution

Now consider a system with $N+1$ frequencies $f_1, f_2, \ldots, f_{N+1}$ (N frequency bins). The range improves by N times, but the resolution remains the same.

$$\textbf{Range/Aliasing: } = \frac{N}{B} \tag{3.14}$$

The resolution remains the same as

$$\textbf{Resolution: } = \frac{1}{B} \tag{3.15}$$

In case of multi-path, we see multiple sincs in delay domain added together. We can separate these sincs if they are apart by delay higher than the resolution value. Increasing the bandwidth improves the resolution and so the ability to separate multi-paths.

Note that if the sinc is continuous in time domain, it will require infinite storage to capture a continuous signal. So, we need to appropriately sample it. Sampling in time domain is equivalently to multiplying by an infinite impulse train. What sampling rate should we choose?

3.4 Example two-wavelength channel without multipath

Two-wavelength system: Figure 3.2 shows two-wavelength system at $f_1 = 1.5$ GHz and $f_2 = 1.6$ GHz. The bandwidth is $B = f_2 - f_1 = 0.1$GHz. We get the range $= 1/B = 10$ns and resolution $= 1/B = 10$ns. Note here that the range is so low that we cannot even observe signal for one complete sinc-width that is $2/B = 20ns$.
Multi-wavelength system: Figure 3.3 shows multi-wavelength system with 21 wavelengths $f_1, \ldots, f_{N+1}$ and so we have $N = 20$ frequency bins. We choose $f_1 = 1.5$ GHz and $f_2 = 3.5$ GHz. The bandwidth is $B = f_{N+1} - f_1 = 2$GHz. We get the range $= N/B = 10$ns and resolution $= 1/B = 0.5$ns. Note the sinc width is $2/B = 1ns$.

3.5 Example two-wavelength channel with multipath

Now, consider a wireless channel with two reflections with delays τ_1 and τ_2. In Figure 3.4, we consider $\tau_1 = 6.7$ns and $\tau_2 = 7.2$ns. We have $\tau_2 - \tau_1 = 0.5$ns which is equal to the resolution and so the two peaks are barely separable. In Figure, 3.5, we consider $\tau_1 = 6.7$ns and $\tau_2 = 11.7$ns. In this case, the resolution is good enough to separate two peaks, but the second peak has delay higher than the range (10 ns) value. And so, we cannot uniquely identify the two peaks.

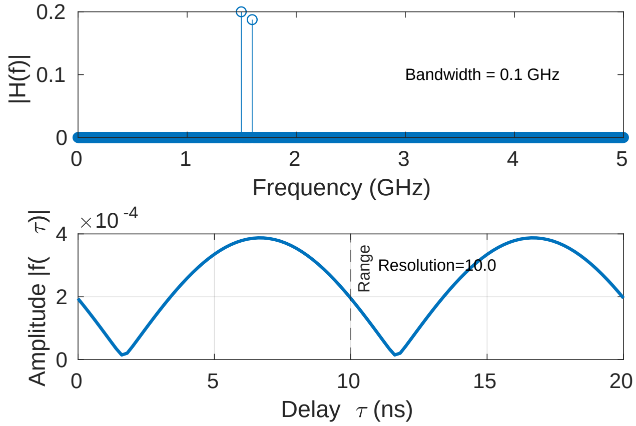

Figure 3.2: Two-wavelength system

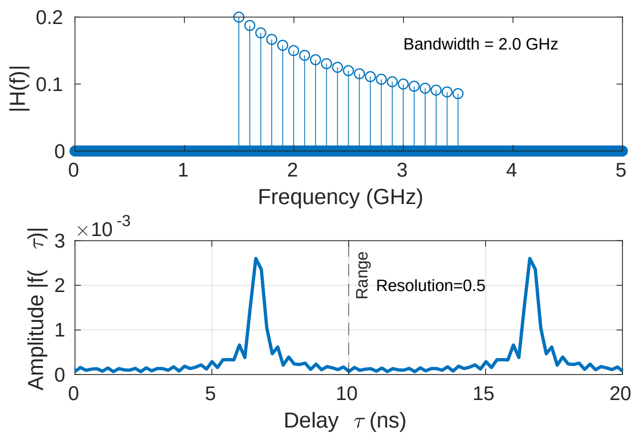

Figure 3.3: Multi-wavelength system

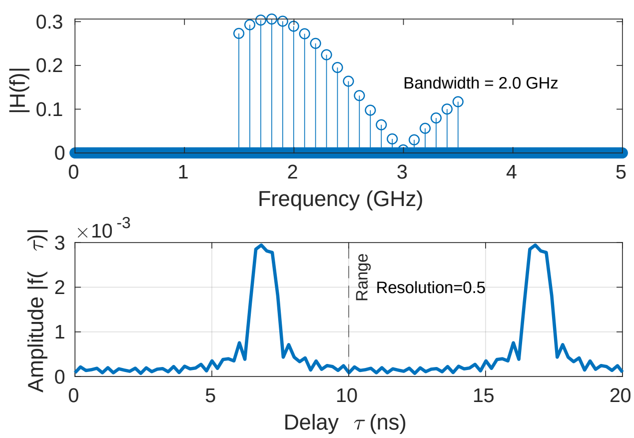

Figure 3.4: Two-wavelength system: Two-paths at $d_1 = 1$m ($\tau_1 = 6.7$ns) and $d_2 = 1.075$m ($\tau_2 = 7.2$ns)

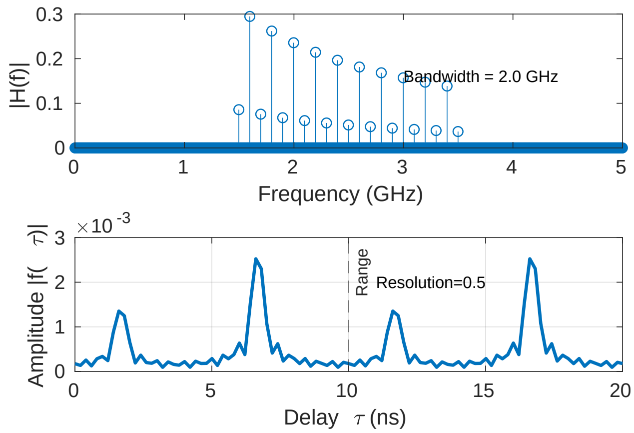

Figure 3.5: Multi-wavelength system: Two-paths at $d_1 = 1$m ($\tau_1 = 6.7$ns) and $d_2 = 1.75$m ($\tau_2 = 11.7$ns)

Chapter 4

Lecture 6: Schedule of Multi-tone for sensing

4.1 Recap: Multi-wavelength system

A multi-wavelength system of bandwidth B and N frequency bins provides a range of $\frac{N}{B}$ and resolution of $\frac{1}{B}$ in time-of-flight. We know $\tau = \frac{2d}{c}$. So, the corresponding range and resolution for distance signal is

$$\text{Range: } d \sim \frac{cN}{B} \tag{4.1}$$

$$\text{Resolution: } d \sim \frac{c}{2B} \tag{4.2}$$

Range determines the ability to uniquely estimate the distance of targets and resolution determines the ability to distinguish multipath signal.

4.2 Scheduling: How do we use multiple tones?

We first measure the channel $h(t_j, \lambda_i)$ for all time t_j ($j = 0, \ldots, M-1$) and wavelengths λ_i ($i = 1, \ldots, N$). Then, for a fixed time t_0, we take IFFT of the channel across all wavelengths to get the distance profile:

$$f_0(\tau) = \sum_{\lambda_i=\lambda_1}^{\lambda_N} h(t_0, \lambda_i) \exp(j2\pi \frac{c}{\lambda_i} \tau) \tag{4.3}$$

where $\tau = \frac{2d}{c}$ is the time-of-flight. This operation is essentially IFFT with appropriate variables. We repeat the above IFFT for all time instants t_j, to get M distance profiles $f_0(\tau), \ldots, f_{M-1}(\tau)$. Fundamentally, both τ and t_j represents time, but we can think of τ as displaced time.
Example: Consider the required sampling frequency for breathing/heart rate is $f_s = 60$ Hz. For every 1/60 sec, we get one IFFT. We perform a total of M such IFFTs over M time instants.
Question: How do we measure the channel $h(t_j, \lambda_i)$ for all time t_j and wavelengths λ_i with the least overhead?
Solution: A simplistic approach is presented as follows. We sequentially transmit multiple tones at wavelengths $\lambda_1, \ldots, \lambda_N$ in a round-robin way. For each wavelength, we independently estimate the channel $h(t_j, \lambda_i)$ as we have seen in the last lecture.

$$h(t_0, \lambda_1) = \int_{t=0}^{t=1/60} y(t) \exp(-j2\pi f_1 t) \tag{4.4}$$

In discrete domain, it can be written as summation:

$$h(t_0, \lambda_1) = \sum_{n=0}^{N_{max}} y(t) \exp(-j2\pi f_1 n T_s) \tag{4.5}$$

where N_{max} is the maximum number of samples we can use depending on the ADC sampling rate (e.g. 1GHz) and the required sampling rate of distance signal (e.g. 60 Hz). For example $N_{max} = \frac{f_s^{ADC}}{f_s} = \frac{1GHz}{60} \approx 60K$. The process is repeated for time intervals $[0, \frac{1}{f_s}], [\frac{1}{f_s}, \frac{2}{f_s}], \ldots, [\frac{M-1}{f_s}, \frac{M}{f_s}]$ for $f_s = 60$Hz and in this way, we get $h(t_j, \lambda_i)\ \forall i, j$.

Fundamental Limitation: How long do we need to send signal at λ_1 before switching to next wavelength λ_2? Which one of range or the resolution is the deciding factor? Since the range is $\frac{N}{B}$, it does not make sense to scan for wavelength λ_1 for more than $\frac{N}{B}$ time. But, in reality, we need to stay at a wavelength for a much longer time to ensure that the channel can be accurately estimated for that wavelength. This poses a fundamental limitation of our sequential round-robin sensing approach. Can we scan multiple wavelengths at the same time to overcome this limitation?

4.3 Using multiple tones concurrently

Concurrent multi-tone sensing uses multiple wavelengths at the same time to estimate the channel for all the wavelengths jointly. However, the challenge for estimating channel at wavelength say λ_1 is that the signal at wavelengths $\lambda_2, \lambda_3, \ldots$ will cause interference.

4.3.1 Two-wavelength concurrent channel estimation

We transmit a two-tone signal $x(t)$

$$x(t) = e^{j2\pi f_1 t} + e^{j2\pi f_2 t} \tag{4.6}$$

and receive a delayed version $y(t)$

$$y(t) = x(t - \tau) \tag{4.7}$$

For simplicity, assume the channel magnitude is one and only the phase varies because of time-of-flight τ. We next describe channel estimation for wavelength λ_1:

$$\begin{aligned} h(t_0, \lambda_1) &= \int_t x(t-\tau) e^{-j2\pi f_1 t} \\ &= \int_t e^{j2\pi f_1 (t-\tau)} e^{-j2\pi f_1 t} + e^{j2\pi f_2 (t-\tau)} e^{-j2\pi f_1 t} \\ &= \int_t e^{-j2\pi f_1 \tau} + \int_t e^{j2\pi (f_2 - f_1) t} e^{-j2\pi f_2 \tau} \end{aligned} \tag{4.8}$$

The first term is desired channel at λ_1 while the second term is corruption or interference term. Our goal is to make the second term zero. We write:

$$h(t_0, \lambda_1) = \int_t e^{-j2\pi f_1 \tau} + e^{-j2\pi f_2 \tau} \left(\int_t e^{j2\pi (f_2 - f_1) t} \right) \tag{4.9}$$

When is the integration (or summation) of a sinusoidal signal zero? It is zero when we sum over a complete period of the wave. The period of a sinusoidal signal at tone $f_2 - f_1$ is $\frac{1}{f_2 - f_1}$. Therefore, if we integrate over a period of $\frac{1}{f_2 - f_1}$, we can get rid of the interference term. This is called the condition of orthogonality.

4.3.2 Multi-wavelength concurrent channel estimation

Now what if we have multiple frequencies at $f_1, f_2, \ldots, f_N$ and the total bandwidth is $B = (f_N - f_1)$. Also, the frequency spacing would be $f_2 - f_1 = \frac{B}{N-1}$. The transmit signal carrying multiple tones is

$$x(t) = \sum_{i=1}^{N} e^{j2\pi f_i t} \tag{4.10}$$

The received signal is

$$y(t) = x(t - \tau) \tag{4.11}$$

We extract the channel $h(t_0\lambda_1)$ for wavelength λ_1 as follows:

$$\begin{aligned} h(t_0, \lambda_1) &= \sum_{t=0}^{T_p} y(t) e^{-j2\pi f_i t} \\ &= \sum_{t=0}^{T_p} e^{-j2\pi f_1 \tau} + e^{-j2\pi f_2 \tau} \left(\sum_{t=0}^{T_p} \sum_{i=2}^{N} e^{j2\pi (f_i - f_1) t} \right) \end{aligned} \tag{4.12}$$

In this case, there will be one intended signal and $N-1$ interference terms. How to make all the interference terms zero? Using insights from two-tone analysis, we get that the time-period of signal should be a multiple of $\frac{1}{f_2-f_1}, \ldots, \frac{1}{f_N-f_1}$. Thus, the orthogonality condition is to integrate over a period T_p found by:

$$\begin{aligned} T_p &= \mathrm{LCM}\left\{ \frac{1}{f_2 - f_1}, \ldots, \frac{1}{f_N - f_1} \right\} \\ &= \mathrm{LCM}\left\{ \frac{N-1}{B}, \frac{N-1}{2B}, \frac{N-1}{(N-1)B} \right\} \\ &= \frac{N-1}{B} \\ &\approx \frac{N}{B} \end{aligned} \tag{4.13}$$

We make the approximation for large N. Surprisingly, the time window $T_p = \frac{N}{B}$ that we use to average the signal turns out to be the range. We already have the range constraint on the time-of-flight as $\tau < \frac{N}{B}$. Therefore, we are not wasting more than $\frac{N}{B}$ time to get one snapshot of the channel for all the wavelengths! Over time, we get multiple such snapshots at regular time intervals $\frac{N}{B}, \frac{2N}{B}, \frac{3N}{B}, \ldots, \frac{MN}{B}$.
Food for thought: What if we do not have equally spaced frequency samples. How does the solution changes?

4.4 Problem with this architecture?

Problem: Since the transmitter (Tx) and the receiver (Rx) are co-located, there is a strong interference/leakage from the Tx to the Rx.
Solution1: Threshold the time-of-flight: Remove the Tx leakage by thresholding its short time-of-flight (ToF), while the desired reflected signal has a higher ToF. However, this approach works only for short duration transmit pulses, and won't work in our case where we integrate the received signal (sum of sinusoidals) over a long time window. Why we don't use a short pulse? Because a short pule in time will require a large bandwidth and in turn a large sampling rate ADC, which are expensive, bulky, and power hungry.
Solution2: Use Full-duplex radios to cancel the Tx leakage using self-interference techniques.

We mathematically formulate the Tx leakage issue for a single tone system. We transmit

$$x(t) = e^{j2\pi f_1 t} \tag{4.14}$$

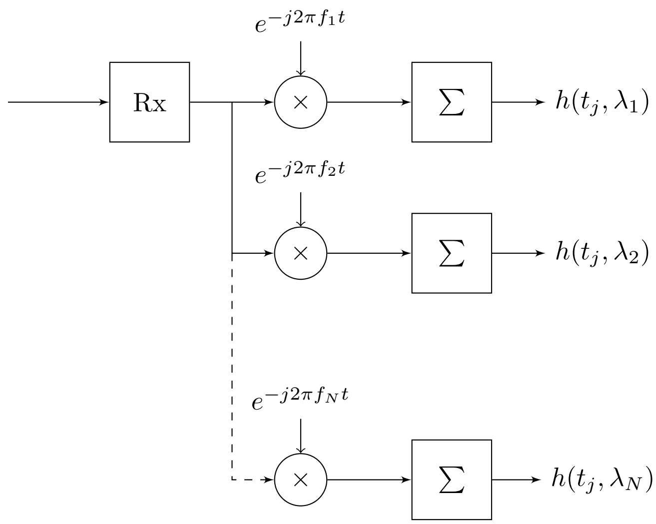

Figure 4.1: Architecture for concurrent multi-wavelength receiver: The receiver demodulates the received signal $y(t)$ using multiple frequencies $e^{-j2\pi f_i t}$ and sum each of the output over $\frac{N}{B}$ samples to get $h(t_0, \lambda_i)$.

The received signal along with the Tx leakage (with ToF ϵ) is

$$y(t) = \alpha_1 x(t-\epsilon) + \alpha_2 x(t - \frac{2d}{c}) \tag{4.15}$$

The Tx leakage is usually $1000\times$ stronger compared to the desired signal, i.e., $\alpha_1 \approx 1000 \times \alpha_2$. The channel estimation gives

$$y(t)e^{-j2\pi f_1 t} = \alpha_1 e^{-j2\pi f_1 \epsilon} + \alpha_2 e^{-j2\pi f_1 \frac{2d}{c}} \tag{4.16}$$

We can see why it is hard to remove the leakage term. But it can be solved through Full Duplex techniques to a good extent through bulky and expensive hardware. Similar issue due to multi-path reflections arriving at a later time and adding up to the direct path signal. How to solve multipath interference issue?
Solution3: FMCW (next lecture)

Final thoughts: We can reduce the hardware cost by demodulating at a single carrier frequency and perform digital processing to extract the signal. Students are encouraged to read Wideo paper posted on Piazza.

Chapter 5

Lecture 7: FMCW short periods of each tone of the multi-tone

5.1 Recap: multi-tone sensing

In the last lecture, we discussed how we use multi-wavelength sensing to estimate the channel $h(t_j, \lambda_i)$ for $j = 1, \dots M$ time instants and $i = 1, \dots, N$ wavelengths. We formulated the interference problem and found that an integration/summation over time windows $[\frac{N}{B}, \frac{2N}{B}], [\frac{2N}{B}, \frac{3N}{B}], \dots, [\frac{(M-1)N}{B}, \frac{MN}{B}]$ recovers interference free channel for all time instants $t_0, t_1, \dots, t_{M-1}$ and wavelengths $\lambda_1, \dots, \lambda_N$. We generally don't use time window $[0, \frac{N}{B}]$ to account for the time-of-flight τ, which we assume satisfies the range constraint $\tau < \frac{N}{B}$.

5.2 Is there some other approach to the Tx leakage problem?

We studied the problem of Tx leakage that causes a very strong interference to the desired signal at the receiver. The Tx leakage appears after a ToF of ϵ and the desired signal appears at a ToF of $\frac{2d}{c}$.

Figure 7.1 (ToDo). Caption: Time-frequency plots of Tx and Rx signal with Tx leakage for multiple ***concurrent*** wavelengths.

Figure 7.2 (ToDo). Caption: (a) Frequency-time plot of ***staircase*** transmit and receive waveform. The received signal has Tx leakage and the weak reflected signal is shifted in time by $\frac{2d}{c}$. (b) Equivalent time series with two wavelengths.

5.2.1 Intuition: frequency and time are related

We mathematically formulate the Tx leakage problem for a multi-wavelength system. We transmit a signal $x(t)$ with two concurrent tones as

$$\begin{aligned} x(t) &= x_{\lambda_1}(t) + x_{\lambda_2}(t) \\ &= \exp(j2\pi f_1 t) + \exp(j2\pi f_2 t) \end{aligned} \tag{5.1}$$

The received signal would also have two tones and each tone contains the Tx leakage and the reflected signal as follows:

$$\begin{aligned} y(t) &= y_{\lambda_1}(t) + y_{\lambda_2}(t) \\ &= \alpha_1 x_{\lambda_1}(t-\epsilon) + \alpha_2 x_{\lambda_1}(t - \frac{2d}{c}) + \alpha_1 x_{\lambda_2}(t-\epsilon) + \alpha_2 x_{\lambda_2}(t - \frac{2d}{c}) \end{aligned} \tag{5.2}$$

We recover the channel corresponding to the first tone λ_1 as follows:

$$[y_{\lambda_1}(t) + y_{\lambda_2}(t)] x_{\lambda_1}(t)^* = \alpha_1 \exp(j2\pi f_1 \epsilon) + \alpha_2 \exp(-j2\pi f_1 \frac{2d}{c}) \tag{5.3}$$

We can separate the two signal (Tx leakage and reflected signal) using ToF thresholds for the staircase waveforms.

5.3 FMCW Introduction and Intuition

Till now we considered stair case function. We saw the intuition how it can suppress the interference. We next discuss FMCW waveform. FMCW consider continuous transmission of a smooth tone continuously changing frequency linearly with time. It is easy to implement on a hardware.

Figure 7.3 (ToDo): Caption: Shows FMCW signal $x(t)$ and received signal $y(t)$.

In FMCW, the transmit frequency varies linearly with time as $f = f_0 + kt$, where k is the slope value. The channel estimation intuitively follows:

$$\int_t y(t)x(t)^* = \int_t y(t)e^{-j2\pi f_1 t} \tag{5.4}$$

The received signal has two components, one due to leakage at time shifts of $\tau_1 = \epsilon$ and another at $\tau_2 = \frac{2d}{c}$. Since time and frequency are linearly related, we experience a shift in frequency as well by Δf_1 and Δf_2 respectively.

$$\begin{aligned} \Delta f_1 &= k\epsilon \\ \Delta f_2 &= k\frac{2d}{c} \end{aligned} \tag{5.5}$$

We will write formal maths later, but intuitively we get the channel as:

$$h = e^{j2\pi\Delta f_1 t} + e^{j2\pi\Delta f_2 t} \tag{5.6}$$

The first term corresponds to the Tx leakage and the second term corresponds to the desired signal. How can we filter out the Tx leakage and only keep the desired signal?

Let T is total time of the chirp. It is of the order of 10μsec. Also assume the system has a bandwidth B and the chirp varies over the frequencies from f_0 to $f_0 + B$. Take a simplified case where we are looking at a time window from $\frac{2d}{c}$ to T and analyse the multiplication during this time. We can extract the desired signal using this time-window. A window in time is hard to implement. Instead, we implement a window in frequency domain by utilizing the linear time-frequency relationship. We apply a blocker filter that only keeps frequencies of interest (β) as follows:

$$\text{Blocker filter: } \Delta f_1 < \beta \tag{5.7}$$

The blocker filter will block every frequency component too close $x(t)$ (such as Tx leakage signal). The filter is actually a band pass filter with an upper cutoff frequency as well.

Figure 7.4 (ToDo): Block diagram with a band pass filter.

Figure 7.5 (ToDo): Many chirps over a long time interval.

The blocker filter has following properties:

1. The blocker filter is relative to $x(t)$, i.e., it blocks the frequencies that are closer to frequency of $x(t)$ at time t.
2. Single tone effect for instantaneous time: We take a small time window at time t to make sure there is no inter-tone interference.
3. HPF to reject strong nearby reflection. To make sure no nearby signal reduces the dynamic-range/sensitivity or causes strong interference.

Finally, after applying the blocker filter, we estimate the frequency shift Δf_2 due to the reflected signal and extract the ToF by utilizing the linear relationship:

$$\Delta f_2 = k\tau = k\frac{2d}{c} \tag{5.8}$$

5.4 FMCW Math: How do we recover the time-of-flight

The transmitted signal for a chirp signal $x(t)$ is given by

$$x(t) = \exp(j2\pi(f_0 t + \frac{k}{2}t^2)) \tag{5.9}$$

The equation of line is

$$f = f_0 + kt \tag{5.10}$$

Verify the above linearity relation using (5.9): The phase is given by

$$\phi = f_0 t + \frac{k}{2}t^2 \tag{5.11}$$

Derivative of phase with respect to time gives the frequency

$$f = d(\phi)/dt = f_0 + \frac{k}{2}2t = f_0 + kt \tag{5.12}$$

Thus, the frequency-time relationship is verified.

The received signal from one reflected path is given by

$$y(t) = \alpha x(t - \tau) \tag{5.13}$$

We perform the channel estimation as follows:

$$\begin{aligned} h(t) = y(t)x(t)^* &= \alpha x(t-\tau)x(t)^* \\ &= \alpha \exp(j2\pi f_0(t-\tau) + \frac{k}{2}(t-\tau)^2) \times \exp(-j2\pi(f_0 t + \frac{k}{2}t^2)) \\ &= \exp(-j2\pi f_0 \tau)\exp(j2\pi\frac{k}{2}\tau^2)\exp(-j2\pi\frac{k}{2}2t\tau) \\ &= A(\tau)\exp(-j2\pi k\tau t) \end{aligned} \tag{5.14}$$

Note that $A(\tau)$ is independent of time t, but depends on initial frequency f_0 and delay τ. We can drop $\frac{k}{2}\tau^2$ term because it is very small.

$$\begin{aligned} A(\tau) &= \alpha \exp(-j2\pi f_0\tau)\exp(j2\pi\frac{k}{2}\tau^2) \\ &\approx \alpha \exp(-j2\pi f_0 \tau) \end{aligned} \tag{5.15}$$

The most important thing to note is that the frequency Δf at which $y(t)x(t)^*$ is oscillating is given by

$$\Delta f = k\tau \tag{5.16}$$

Thus, by measuring the frequency Δf, we can recover the time-of-flight τ. In the next lecture, we will learn how to make use of these ideas to estimate the breathing and heart rate and also find out the range and resolution requirements for accurate estimation.

Chapter 6

Lecture 8: FMCW range and resolution

6.1 Recap: FMCW channel estimate

In the last lecture, we obtained an expression of channel estimate for an FMCW system. We get the frequency of oscillation $k\tau$ and amplitude $A(\tau)$ as a function of $f_0\tau$, where the notation is: k as chirp rate, f_0 as starting frequency, and τ as time-of-flight.

$$h(t) = y(t)x(t)^* = A(\tau)\exp(-j2\pi k\tau t) \tag{6.1}$$

$$A(\tau) = \alpha\exp(-j2\pi f_0\tau)\exp(j2\pi\frac{k}{2}\tau^2) \tag{6.2}$$

6.2 Meeting Range and Resolution requirements

Example: We explain through an example how to design for FMCW parameters k, B, f_0 to satisfy the range and resolution requirements.

- **Range and resolution:** Take bandwidth $B = 2$GHz. We get a resolution $\frac{1}{B} \sim 0.5ns$ which corresponds to $d_0 = 0.5$ft[1]. Suppose we want a range of 100 ft. Therefore, we choose the chirp length of $T = 100$ns. By using these range and resolution constraints, we design for the chirp rate k. Recall the linear time-frequency relationship $f = f_0 + kt$. We get

$$k = \frac{B}{T} = \frac{2GHz}{100ns} = 2\times 10^{16}Hz/sec \tag{6.3}$$

- **Phase accuracy:** Now, assume we choose the starting frequency $f_0 = 5$GHz. The phase of $A(\tau)$ is $f_0\tau$, so it completes a full cycle at $\tau = \frac{1}{f_0} = 0.2ns$. Thus, one 360^o rotation in time happens after 0.2 ns. If we get phase accurate to 6^o, then we get time accuracy of $\frac{0.2}{360/6} = 0.0033ns$ which gives distance accuracy of 0.0033 ft (or 1 mm).

- **Second term in $A(\tau)$ is not useful:** Also look for the second term in $A(\tau)$ with phase $\frac{k}{2}\tau^2$. Suppose there is a target at $2d_0 = 10ft$. Thus, $\tau_0 = 10ns$. We get

$$\frac{k}{2}\tau^2 = \frac{2\times 10^{16}}{2}(10\times 10^{-9})^2 = 1 \tag{6.4}$$

[1]For quick calculation remember that 1 ns is roughly equivalent to 1 ft (30 cm) obtained by multiplying speed of light.

Now, if we have change in τ_0 by $1ns$ (1 ft change in distance) due to breathing to $\tau_0 = 11ns$, we get $\frac{k}{2}\tau^2 = \frac{2\times10^{16}}{2}(11\times10^{-9})^2 = 1.21$, that corresponds to just 0.21 radians change in the phase, therefore, we conclude that the second term in $A(\tau)$ doesn't change with variations in τ. In contrast, the first term in $A(\tau)$ provides much better resolution (1 mm resolution as we have seen in the previous example).

6.3 Recovery algorithm

Now that we have designed for FMCW parameters k, B, f_0 to satisfy the range and resolution requirements, we would explore how can we actually recover the breathing and heart rate signal.

6.3.1 Intuition

Recall the expression of distance profile that we obtained for a concurrent wavelength system.

$$f(d) = \sum_{\lambda=\lambda_1}^{\lambda_N} h(\lambda)e^{j2\pi\frac{d}{\lambda}} \tag{6.5}$$

How does it extend to FMCW? We have channel estimate $h(t) = y(t)x(t)^*$. We obtain the distance profile $f(d)$ as follows:

$$\begin{aligned} f(d) &= \int_t h(t)e^{j2\pi\frac{d}{\lambda(t)}} \\ &= \sum_{nT_s} h(nT_s)e^{j2\pi\frac{d}{\lambda(nT_s)}} \end{aligned} \tag{6.6}$$

Note the equivalence of (6.5) and (6.6). They are both essentially IFFT operation. We will prove (6.6) formally in the next subsection.

Figure 8.1 (ToDo): f vs t and lambda vs t.

6.3.2 Math: Formulation of the recovery signal

Lets recall the calculation as $h(t)$.

$$h(t) = y(t)x(t)^* \tag{6.7}$$

We rewrite the channel estimate in a simplified form assuming there is a single reflection at ToF τ_0:

$$\begin{aligned} h(t) &= A(\tau_0)\exp(-j2\pi k\tau_0 t) \\ &= \alpha\exp(-j2\pi(f_0\tau_0 + k\tau_0 t)) \\ &= \alpha\exp(-j2\pi\tau_0(f_0 + kt)) \\ &= \alpha\exp(-j2\pi\tau_0 f_r(t)) \end{aligned} \tag{6.8}$$

where $f_r(t) = f_0 + kt$ is the instantaneous frequency. Think of $h(t)$ as a function of frequency. We can call it $h(f_r(t))$ for better understanding.

$$h(t) \sim h(f_r(t)) \tag{6.9}$$

The goal is to use IFFT insights to recover the ToF τ_0. We use the following DTFT relation and evaluate $f(\tau)$ for a continuous values of τ.

$$\begin{aligned} f(\tau) &= \int_t h(t)\exp(j2\pi\tau f_r(t)) \\ &= \int_t h(t)\exp(j2\pi\tau\frac{c}{\lambda(t)}) \end{aligned} \tag{6.10}$$

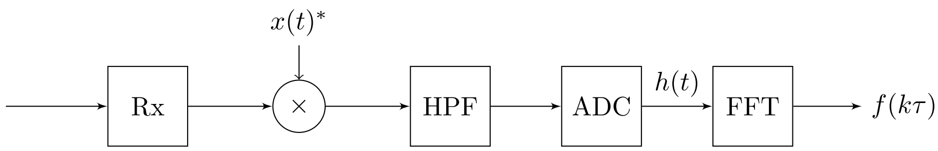

Figure 6.1: Architecture for FMCW receiver: The receiver demodulates the received signal $y(t)$ using the transmitted signal $x(t)$. It applies HPF to remove the Tx leakage. Then ADC provides digital samples at certain sampling rate to give channel $h(t)$ Finally, rangeFFT is applied to get the ToF profile $f(k\tau)$.

We hope to see a peak of $f(\tau)$ at $\tau = \tau_0$ and in this way we can estimate τ_0. Equivalently, if a person is at d distance. We have the total distance travelled by wireless signal as $2d = c\tau$. We get the distance profile as follows:

$$f(d) = \int_t h(t) \exp(j2\pi \frac{2d}{\lambda(t)}) \tag{6.11}$$

6.4 Range and Resolution for multiple reflections

Now consider we have L reflections with ToF τ_i, for $i = 1, \ldots, L$. The multi-path channel $h(t)$ is written as:

$$h(t) = \sum_{\tau_i} A(\tau_i) \exp(-j2\pi k \tau_i t) \tag{6.12}$$

In this case we get the ToF profile $f(\tau)$ as

$$f(\tau) = \int_t h(t) \exp(j2\pi k \tau t) \tag{6.13}$$

The receiver architecture of FMCW is explained in Figure 6.1.

Suppose we have two paths at $k\tau_1$ and $k\tau_2$. When can we separate these two components?

6.4.1 Resolution: Separation in frequency domain

Ideally, if we can observe $h(t)$ for infinite time, we can get delta functions at τ_1 and τ_2 after taking IFFT. But, in practice, we observe $h(t)$ over a time window T. We know that windowing in one domain will result in convolution with corresponding sinc functions in the other domain. Therefore, we actually see two sincs in $f(\tau)$ at $k\tau_1$ and $k\tau_2$. Recall, we can separate the two peaks if they are far apart by equivalent of half of the sinc width. Since, we observed $h(t)$ over a time window T, the sinc width in IFFT domain is $\frac{2}{T}$. Thus, we get the resolution as follows:

The resolution is $k\tau \sim \frac{1}{T}$. Recall we had $k = \frac{B}{T}$. Therefore,

$$\textbf{Resolution: } \tau \sim \frac{1}{B} \tag{6.14}$$

Since $\tau = \frac{2d}{c}$. We get resolution in distance as

$$\textbf{Resolution: } d \sim \frac{c}{2B} \tag{6.15}$$

Example1: Consider bandwidth is $B = 2$GHz. Can we resolve two objects separated by 15 cm?
Ans: The two-way distance of $2 \times 15 = 30$ cm (1 ft) requires delay resolution less than $1ns$. (Remember $1ft$ always corresponds to $1ns$). For $B = 2$GHz, we get the delay resolution $\frac{1}{B} = \frac{1}{2e9} = 0.5ns$. Thus, 2GHz bandwidth can easily distinguish two objects separated by 15 cm.

6.4.2 Range: How far a reflection can be uniquely measured

Assume the channel $h(t)$ is sampled with a sampling period T_s. So, the ToF profile $f(\tau)$ will be periodic with period $1/T_s$. Thus the range is $k\tau \sim 1/T_s$, or $\tau \sim \frac{1}{kT_s}$. Since we know $kT = B$, we get $\tau \sim \frac{T}{BT_s}$. Finally, using $T = NT_s$, we get

$$\textbf{Range: } \tau \sim \frac{N}{B} \tag{6.16}$$

Similarly for distance, using $\tau = \frac{2d}{c}$, we get:

$$\textbf{Range: } d \sim \frac{cN}{2B} \tag{6.17}$$

Example2: Take the FMCW time-period $T = 100$ns and assume $N = 10$ samples. Thus, the sampling period would be $T_s = \frac{T}{N} = 10$ns and sampling rate $f_s = 1/T_s = 100$ MHz. Take $B = 2$ GHz and $f_0 = 5GHz$. What is the resolution and range we can get?
Ans: We get resolution $\frac{1}{B} = 0.5$ ns. That's $10/.5 = 20$ times fine-grained compared to what we get from the sampling rate. Thus, even if the sampling rate is small, we can distinguish the reflections which are very close. Moreover, the distance resolution is $d = c\tau/2 = 0.5/2ft = 7.5cm$. Thus, we can separate two reflections which are as close as 7.5 cm.

6.4.3 Accuracy for heart rate

Resolution and accuracy are two different things. Resolution is how we can separate things. Accuracy is how accurate we can get the one desired peak. Suppose we estimate the heart rate from channel amplitude $A(\tau) = \exp(j2\pi f_0 \tau)$. If we have a phase accuracy of $\phi \sim 6^o$, we can detect small movement due to heart rate as accurate as $\tau \sim \frac{1}{f_0}\frac{\phi}{360^o} = \frac{1}{5GHz}\frac{1}{60} = 0.0033ns$ (1mm).

Chapter 7

Lecture 9: FMCW breathing and heart rate

7.1 Recap: Range and resolution requirements

In last lecture, we estimate the channel $h(t)$ which consists of ToF τ_i for $i = 1, \ldots, L$ reflections.

$$h(t) = \sum_i a_i \exp(-j2\pi f_0 \tau_i) \exp(-j2\pi k \tau_i t) \tag{7.1}$$

We also obtain the ToF profile by IFFT operation (also called RangeFFT):

$$\begin{aligned} f(k\tau) &= \int_{t=0}^{T} h(t) \exp(j2\pi k \tau t) \\ &= \sum_i a_i \exp(-j2\pi f_0 \tau_i) \int_{t=0}^{T} \exp(j2\pi k(\tau - \tau_i)t) \\ &= \sum_i A(\tau_i) \mathrm{sinc}(Tk(\tau - \tau_i)) \\ &= \sum_i A(\tau_i) \mathrm{sinc}(B(\tau - \tau_i)) \end{aligned} \tag{7.2}$$

where T is the time window of integration corresponding to one chirp duration. We observe that $f(k\tau)$ consists of multiple sinc functions shifted in ToF domain by τ_i, have the width of sinc equal to the bandwidth $B = Tk$, and magnitude proportional to $A(\tau_i) = a_i \exp(-j2\pi f_0 \tau_i)$.

We conclude with a discrete version of channel:

$$h(nT_s) = \sum_i a_i \exp(-j2\pi f_0 \tau_i) \exp(-j2\pi k \tau_i n T_s) \tag{7.3}$$

and discrete ToF profile:

$$f(k\tau) = \sum_{t=0}^{nT_s} h(nT_s) \exp(j2\pi k \tau n T_s) \tag{7.4}$$

7.1.1 More insights on range, resolution, and accuracy

Resolution: The time resolution is $\frac{1}{2B}$, which corresponds to distance resolution of $d \sim \frac{c}{4B}$. It means that we can distinguish two multipaths if they are separated by a distance more than the distance resolution.

Range: The range is $\frac{N}{B}$ in time-of-flight. The distance range is $d \sim \frac{c}{2B}N$ which means we can resolve a path uniquely if all the distances are within the range. Suppose we fix the bandwidth B to satisfy the resolution requirement. Then, note that the range can be increased by increasing the number of frequency samples N.
Accuracy: In fact one wavelength is good to estimate the breathing and heart rate. For instance, we have seen $f_c = 1.5\ GHz$ is enough. But for a robust system (under multipath and noise), we need multiple wavelengths.

7.1.2 FFT based formulation for recovering the $f(d)$

Let the equivalent frequency is $f_m = k\tau_m$. The IFFT operation is

$$f(k\tau_m) = \sum_{n=0}^{N} h(nT_s)\exp(j2\pi k\tau_m nT_s) \tag{7.5}$$

The range of τ_m is $\tau_m \sim (0, \frac{N}{B})$. The separation between two consecutive delay samples τ_1 and τ_2 would be $\tau_2 - \tau_1 = \frac{1}{B}$.

7.2 Breathing and heart rate variation and understanding application level specs applied to FMCW

Example: Suppose the breathing and heart rate causes peak-to-peak variations of $d_B = 10$cm and $d_H = 1$ mm. The breathing rate is 0.2Hz (12 bpm) and heart rate is 1Hz (60 bpm). What these specs translates to the application level requirement of FMCW?
Answer: We assume the environment is stationary between the time interval $[0, T]$. From a FMCW system with sampling interval T_s, we get one measurement of $d(t)$ every $T = NT_s$ time samples. Recall that we take RangeFFT for every chirp i.e., we get $f(k\tau_m, 0), f(k\tau_m, T), f(k\tau_m, 2T), \ldots$ and so on for all the chirps.

Distance resolution to get breathing rate: Assume the center frequency $f_0 = 5GHz$. Bandwidth is $B = 2Ghz$ and chirp duration $T = 100$ns. We get distance resolution $d \sim \frac{c}{2B} = 7.5$ cm. Thus, we can separate two objects separated by 7.5 cm. In fact, we can also get breathing rate from here because the variations due to breathing is 10 cm. But the variations due to heart rate in below the ToF resolution. In the next lecture, we will see how can we estimate the heart-rate.

Sampling rate: The Nyquist sampling rate for the distance signal is $f_s = 2f_H = 2$ Hz. Therefore, we want $\frac{1}{T} > f_s = 2Hz$. Since $T = 100$ ns, we get $\frac{1}{T} = 10$ MHz. We want Nyquist sampling rate of 2Hz but we are getting 10 MHz, much more than what we need.

Therefore the most challenging specs is the bandwidth, which defines the ToF resolution. In practice, we need a very high resolution for certain application. For instance, medical examination of double breathing requires a very high measurement resolution.

7.3 Recover breathing and heart rate from FMCW

Due to limited bandwidth, the ToF resolution is limited. The issue is a high discretizing of $k\tau$ bins. We only have N ToF samples from $k\tau_1$ to $k\tau_N$. How do we get more fine-grained $k\tau$?

Super-precision to get accurate breathing: We can get more ToF samples by upsampling and interpolation. Especially, between every consecutive ToF samples say τ_m and τ_{m+1}, we get R additional ToF samples $\tau_m, \tau_m + \Delta, \tau_m + 2\Delta, \ldots, \tau_m + R\Delta$. Where $\Delta = \frac{\tau_{m+1} - \tau_m}{R}$ and $\tau_m + R\Delta = \tau_{m+1}$. The above process is equivalent to zero-padding the channel $h(t)$ and then taking a longer point FFT (In this case taking $N \times R$ point FFT). The zero-padding will give DTFT-like response which is more like continuous signal in ToF domain.

$$\tilde{h}(nT_s) = \begin{cases} h(nT_s), & \text{if } n \leq N. \\ 0, & \text{if } N < n \leq NR. \end{cases} \tag{7.6}$$

$$f(k\tau) = \sum_{n=0}^{NR} \tilde{h}(nT_s) \exp(-j2\pi k\tau nT_s) \tag{7.7}$$

In this way, we get super-precision ToF profile. With such a super-precision profile, we can estimate the breathing rate precisely. We emphasize that the above super-precision process can help in getting precise or more accurate measurements of any signal that is slowly varying with time such as breathing. However, that does not change the resolution. Recall the resolution is defined as the ability to distinguish two signals concurrently (such as user and wall reflection).

Phase to get accurate heart-rate: Note that until now, we were looking at the peak of $f(k\tau)$ to infer the ToF variations due to breathing. For detecting the heart-rate using this approach, we need very high accuracy in ToF which is limited by the phase noise. Another and more accurate way to estimate small variation in ToF is to look at the phase term $A(\tau)$.

$$A(\tau) = a_i \exp(-j2\pi f_0 \tau_i) \tag{7.8}$$

Example: consider phase noise 6^o, and start frequency $f_0 = 5$GHz. We did calculation in previous lecture and got ToF accuracy of 0.0033 ns, which corresponds to 1 mm distance accuracy required for the heart rate.

Chapter 8

Lecture 10: Heart-rate recovery with phase of FMCW waveform

8.1 Recap: Super-precision ToF estimate

To get an accurate ToF τ_0 corresponding to the user, we can increase the precision of RangeFFT response $f(k\tau)$ in ToF domain by interpolation methods. In other words, if we get a dense samples (by adding many samples between every τ_m and τ_{m+1}) in ToF doamin, we can resolve for the correct value of user's location $\tau_0 = \frac{2d}{c}$ more precisely to many decimal places. To do this, recall from your DSP class that interpolation is performed by inserting zeros in between the samples in τ domain (upsampling) and then applying an appropriate low-pass filter (reconstruction filter) for perfect reconstruction. This process is equivalent to zero-padding the channel $h(nT_s)$ and then taking a longer length FFT to get ToF profile $f(k\tau)$.

8.2 Breathing and heart-rate recovery using FMCW

Recall the expression of FMCW multi-path channel with L paths.

$$h(nT_s) = \sum_{i=1}^{L} a_i \exp(-j2\pi f_0 \tau_i) \exp(-j2\pi k \tau_i n T_s) \tag{8.1}$$

We repeat the RangeFFT operation on all the chirps to get the following ToF response across time-frames or chirps:

$$\begin{aligned} f(k\tau, \ell T) &= \sum_{n=1}^{N} h(\ell T + nT_s) \exp(j2\pi k\tau(\ell T + nT_s)) \\ &= \sum_{i=1}^{L} A(\tau_i) \mathrm{sinc}(B(\tau - \tau_i)) \end{aligned} \tag{8.2}$$

where the period of one chirp is $T = NT_s$ with a total of N samples in a chirp. Also, we observe many chirps over a duration of $\max(\ell T) = T_{max}$ chirps (The number of chirps would be $\frac{T_{max}}{T}$).

8.2.1 FFT-bin based analysis for breathing rate recovery

What happens to the ToF for a person due to breathing? The ToF changes gradually over multiple chirps. This results in the shifting of the sinc in the ToF domain. With super-precision, we can get fine-grained

ToF bins and recover the ToF variations due to breathing by looking at the ToF corresponding to maximum amplitude $\max\{A(\tau_0)\}$.

How do we recover the minute variations due to heart-rate (order of 1 mm) accurately?

Issue: The resolution in ToF is $\frac{c}{2B} = 7.5$ cm for bandwidth $B = 2$ GHz. So we cannot observe small change on ToF of the order of 1 mm. On top of that the amplitude measurements are noisy and may not give correct ToF due to noise. So, we cannot use the amplitude for heart-rate. Therefore, we rely on the phase of the signal $A(\tau_i)$:

$$A(\tau_i) = a_i \exp(-j2\pi f_0 \tau_i) \tag{8.3}$$

Recall, we take the RangeFFT response $f(k\tau_0, \ell T)$ that corresponds to the user with ToF τ_0 and for all the chirps $\ell T = [T, 2T, \ldots, T_{max}]$, where T_{max} is the overall observation time for all the chirps.

Chapter 9

Lecture 11: Remove static multi-path for breath-rate detection

9.1 Modeling of the static reflection with FMCW

The discrete channel estimate is:

$$h(nT_s) = \sum \frac{\lambda}{d_i} \exp(-j2\pi f_0 \tau_i) \exp(j2\pi k \tau_i n T_s) \tag{9.1}$$

We repeat the RangeFFT operation on all the chirps to get the following ToF response across time-frames or chirps:

$$\begin{aligned} f(k\tau, \ell T) &= \sum_{n=1}^{N} h(\ell T + nT_s) \exp(j2\pi k\tau(\ell T + nT_s)) \\ &= \sum_{i=1}^{L} \frac{\lambda}{d_i} \exp(-j2\pi f_0 \tau_i) \mathrm{sinc}(B(\tau - \tau_i)) \end{aligned} \tag{9.2}$$

9.1.1 Subtraction based approach to remove static multipath

Consider a special case with two paths

$$f(k\tau, \ell T) = \frac{\lambda}{d_1} \exp(-j2\pi f_0 \tau_1) \mathrm{sinc}(B(\tau - \tau_1)) + \frac{\lambda}{d_2} \exp(-j2\pi f_0 \tau_2) \mathrm{sinc}(B(\tau - \tau_2)) \tag{9.3}$$

where the distance of user d_1 is

$$d_1 = d_0 + d_B \sin() + d_H \sin() \tag{9.4}$$

Assume d_0 is stationary for a static user. Also assume d_2 is stationary due to static multipath reflection.

We perform the following subtraction

$$f(k\tau, T) - f(k\tau, 0) = \frac{\lambda}{d_1(1)} \exp(-j2\pi f_0 \tau_1(1)) \mathrm{sinc}(B(\tau - \tau_1(1))) - \frac{\lambda}{d_1(0)} \exp(-j2\pi f_0 \tau_1(0)) \mathrm{sinc}(B(\tau - \tau_1(0))) \tag{9.5}$$

where $\frac{\lambda}{d_1(1)}$ term and sinc term will make a small change while exponential term makes a major change because of high start frequency f_0. So we can make the following approximation:

$$
\begin{aligned}
g(k\tau, T) &= f(k\tau, T) - f(k\tau, 0) \\
&= \frac{\lambda}{d_1(1)} \text{sinc}(B(\tau - \tau_1(1))) \Big[\exp(-j2\pi f_0 \tau_1(1)) - \exp(-j2\pi f_0 \tau_1(0)) \Big] \\
&= \frac{\lambda}{d_1(1)} \text{sinc}(B(\tau - \tau_1(1))) \exp(-j2\pi f_0 \tau_1(1)) \Big[1 - \exp(-j2\pi f_0 (\tau_1(1) - \tau_1(0))) \Big]
\end{aligned}
\tag{9.6}
$$

The {1-exponential} term varies over [0,2] when the ToF varies as $2\pi f_0(\tau(1) - \tau(0)) \sim [0, \pi]$
More generally for a given chirp ℓ,

$$
\begin{aligned}
g(k\tau, \ell T) &= f(k\tau, \ell T) - f(k\tau, (\ell - 1)T) \\
&= \frac{\lambda}{d_1(\ell)} \text{sinc}(B(\tau - \tau_1(\ell))) \exp(-j2\pi f_0 \tau_1(\ell)) \Big[1 - \exp\big(- j2\pi f_0 \big(\tau_1(\ell) - \tau_1(\ell - 1)\big)\big) \Big]
\end{aligned}
\tag{9.7}
$$

4:20

Problem: What is the problem with this kind of subtraction?

Ideally, subtracting in DTFT domain removes the multipath, but subtracting the FFT causes some residues of multipath. Is there any other ways of subtraction of multipath?

Solution: Yes. We can subtract the multipath in channel domain directly. We write the generalized channel $h(nT_s, \ell T)$ as follows:

$$
h(nT_s, \ell T) = \sum_i \frac{\lambda}{d_i(\ell)} \exp(-j2\pi f_0 \tau_i(\ell)) \exp(j2\pi k \tau_i(\ell) n T_s) \tag{9.8}
$$

We do the subtraction in h domain as

$$
\begin{aligned}
g_h(nT_s, \ell T) &= f(nT_s, \ell T) - f(nT_s, (\ell - 1)T) \\
&= \sum_i \frac{\lambda}{d_i(\ell)} \exp(-j2\pi f_0 \tau_i(\ell)) \exp(j2\pi k \tau_i(\ell) n T_s) \\
&\quad - \sum_i \frac{\lambda}{d_i(\ell - 1)} \exp(-j2\pi f_0 \tau_i(\ell - 1)) \exp(j2\pi k \tau_i(\ell - 1) n T_s)
\end{aligned}
\tag{9.9}
$$

This subtraction will remove the multipath and the signal corresponding to the user (at τ_1) will be left out. Also, note the term $\frac{\lambda}{d_i(\ell)}$ and $\exp(j2\pi k \tau_i(\ell) n T_s)$ varies slowly over time and can be assumed to be constant for two chirp periods.

$$
\begin{aligned}
g_h(nT_s, \ell T) = \frac{\lambda}{d_i(\ell)} &\exp(-j2\pi f_0 \tau_i(\ell)) \exp(j2\pi k \tau_i(\ell) n T_s) \times \\
&\Big[1 - \exp(-j2\pi f_0 (\tau_1(\ell) - \tau_1(\ell - 1))) \Big]
\end{aligned}
\tag{9.10}
$$

1. Subtraction in any domain $\Rightarrow$
2. Amplitude has the information similar to the phase previously.

9.2 Recovery of breathing and heart-rate post subtraction

After we perform subtraction of channel and then take RangeFFT, we get a single sinc at $k\tau_1$. The amplitude of sinc will change depending on {1-exp} function and the phase will also change. We observe the change in phase for all the chirps (for $\ell = 0, 1, \ldots, \ell_{max}$) and extract the breathing and heart rate using FFT-based techniques we learned in early lectures.

Chapter 10

Lecture 12: DopplerFFT

10.1 Modeling of dynamic reflection with FMCW

When the merson moves with a constant speed v, the distance (or ToF) varies as

$$d(t) = d_0 + v_0 t + d_B \cos() + d_H \cos() \tag{10.1}$$

The actual distance and velocity is d and v, so that $d(t) = d + vt$.
The continuous channel estimate is:

$$h(t) = \sum \alpha_i \exp(-j2\pi f_0 \tau_i(t)) \exp(j2\pi k \tau_i(t) t) \tag{10.2}$$

$$h(t) = \sum \alpha_i \exp(-j2\pi f_0 \frac{d_0 + v_0 t}{c}) \exp(j2\pi k (\frac{d_0 + v_0 t}{c}) n T_s) \tag{10.3}$$

The second term has second order variations

$$h(nT_s, \ell T) = \alpha \exp(-j2\pi f_0 \frac{d_0 + v_0 \ell T}{c}) \exp(j2\pi k (\frac{d_0 + v_0 \ell T}{c}) n T_s) \tag{10.4}$$

For $n = 0, \ell = 0$

$$h(0,0) = \sum \alpha_i \exp(-j2\pi f_0 \frac{d_0}{c}) \tag{10.5}$$

For $n = 1, \ell = 1$

$$h(T_s, T) = \sum \alpha_i \exp(-j2\pi f_0 \frac{d_0 + v_0 T}{c}) \exp(j2\pi k (\frac{d_0 + v_0 T}{c}) n T_s) \tag{10.6}$$

After taking RangeFFT

$$f(k\tau, \ell T) = \alpha \exp(-j2\pi f_0 \frac{d_0 + v_0 \ell T}{c}) \mathrm{sinc}(k(\tau - \frac{d_0 + v_0 \ell T}{c})) \tag{10.7}$$

In general, when the multi-paths are present. Assume each multipath object is at a distance d_i in the same direction from the radar and all the objects are moving away from the radar with velocity v_i, for $i = 0, \ldots, L-1$. The multi-path channel gives the following RangeFFT response:

$$f(k\tau, \ell T) = \sum_i \alpha_i \exp(-j2\pi f_0 \frac{d_i + v_i \ell T}{c}) \mathrm{sinc}(k(\tau - \frac{d_i + v_i \ell T}{c})) \tag{10.8}$$

10.2 Invoking the second FFT to understand the velocity/motion

Now, we take a second FFT along the chirps $\ell T = [0.T, \ldots, MT]$, we get the following FFT response:

$$H(k\tau, \nu) = \sum_{\ell T = 0.T}^{MT} f(k\tau, \ell T) \exp(+j2\pi f_0 \frac{\nu \ell T}{c}) \tag{10.9}$$

Note that the equivalent frequency in the above FFT is $\frac{f_0 \nu}{c}$.

Note that we take the above FFT of $f(k\tau, \ell T)$ for a particular value of τ which corresponds to a peak in $f(k\tau, \ell T)$. Recall that the signal $f(k\tau, \ell T)$ looks like a sinc in τ domain and the peak of this sinc would correspond to the correct ToF of the user. But, this sinc will move along τ slowly over time as the user walks away from the radar with velocity v. Due to the user motion, the peak will shift to the next τ sample and so we cannot take FFT over a long time window and this harms our ability to estimate the heart and breathing rate.

For now assume the second term $\mathrm{sinc}(k(\tau - \frac{d_i + v_i \ell T}{c}))$ is stationary

We expend $H(k\tau, \ell T)$ as

$$H(k\tau, \ell T) = \sum_i \alpha_i \exp\left(-j2\pi f_0 \frac{d_i}{c}\right) \mathrm{sinc}\left(\frac{f_0}{c}(v - v_i)\right) \mathrm{sinc}\left(k(\tau - \frac{d_i + v_i \ell T}{c})\right) \tag{10.10}$$

10.3 Range and Resolution of the Doppler measurement

The frequency of second FFT is $\frac{f_0 \nu}{c}$. Since we are taking M point FFT over a window of $\ell T \in [0, MT]$, we get the following Doppler resolution in frequency response

$$\begin{aligned} \textbf{Doppler Resolution: } \frac{f_0 \nu}{c} &\sim \frac{1}{MT} = \frac{F_s}{NM} \\ \nu &\sim \frac{c}{f_0 MT} \end{aligned} \tag{10.11}$$

where we used the relation $T = NT_s$ and $F_s = \frac{1}{T_s}$. Recall F_s is ADC sampling rate, N is number of samples in one chirp and M is total number of chips considered. Similarly, we can get the range for the Doppler ν as

$$\begin{aligned} \textbf{Doppler Range: } \frac{f_0 \nu}{c} &\sim \frac{1}{T} = \frac{F_s}{N} \\ \nu &\sim \frac{c}{f_0 T} \end{aligned} \tag{10.12}$$

Example: Given we observe the channel for total duration of $T = 100\mu$sec and the sampling period is $T_s = 50$nsec. Find the range and resolution for both distance(delay) and Doppler. Breathing rate = 0.2Hz and heart-rate = 1Hz.

1. Range delay is $\frac{c}{2B} \frac{T}{T_s} = 15000$ cm.

2. Resolution of delay is $\frac{c}{2B} = 7.5$ cm.

3. Doppler range is $\frac{f_s}{N} = \frac{20MHz}{2000} = 10$KHz

4. Doppler resolution is $\frac{f_s}{NM} = 10/M$KHz (what is M?)

10.3.1 Advantage of subtraction-based modeling

The breathing and heart rate are close to DC leakage. Therefore, the sinc spread of DC will harm the estimate of breathing rate. We emphasize that the subtracting methods we discussed in the last lecture are indeed very powerful technique to remove the sinc spread of the DC component or any undesired slowly varying signal. There are many robust cancellation techniques based on subtraction method. For instance, RP-MUSIC technique is used to successively subtract all the sinc leakages. These iterative cancellation techniques also takes into account the correlation between delay and Doppler of multiple reflections and so they are robust. The details on these techniques are out of scope of this class.

Chapter 11

Lecture 13: Velocity model based Dynamic Multi-path suppression

Consider a person moving with velocity v. The distance signal varies as:

$$d = d_P + d_B \sin(2\pi f_B t) \tag{11.1}$$

where

$$d_P = d_0 + v_0 t \tag{11.2}$$

The true velocity is time varying due to breathing. Therefore, the ?? is not just $\frac{f_0 v_0}{c}$

$$\frac{\partial d}{\partial t} = v_0 + d_B(2\pi f_B)\cos(2\pi f_B t) \tag{11.3}$$

Suppressing motion term in this Fourier analysis which give nice model for v_0

If you do not suppress the v_0 term, then the phase measurement will have an additional velocity term. That velocity is not just a constant v_0, but time-varying velocity. So, we have to take FFT, suppress the velocity and take IFFT again to get rid of distortion due to velocity and then recover the breathing/heart-rate.

11.1 Modeling the breathing and heart-rate with the velocity model

We first write Mclarian series of a signal $f(x)$ is given by

miss here

$$d_B \sin(2\pi f_B t) = d_B(2\pi f_B t - \frac{(2\pi f_B t)^3}{3!} + \cdots) \tag{11.4}$$

The derivative is

$$\frac{\partial}{\partial t}(d_B \sin(2\pi f_B t)) = d_B(2\pi f_B) - \frac{3(2\pi f_B t)^2}{3!} \tag{11.5}$$

Assume there is only one reflection from the user moving with velocity v_0. The rangeFFT response is

$$f(k\tau, \ell T) = \alpha_1 \exp(-j2\pi f_0 \frac{2d}{c})\text{sinc}(k(\tau - \frac{2d}{c})) \tag{11.6}$$

11.1.1 harmonics of breathing and heart-rate

What is the Fourier transform of the following term

$$\alpha_1 \exp(-j4\pi f_0 d_1) \exp(-j4\pi f_0 d_B(2\pi f_B t)) \exp(+j4\pi f_0 d_B \frac{(2\pi f_B t)^3}{3!})$$

Take the term in the exponent and take its derivative. The derivative will give us the time-varying frequency of this signal

$$\frac{\partial}{\partial T}(4\pi f_0 \frac{d_1 + d_B \sin(2\pi f_B t)}{c}) = \frac{4\pi f_0 d_B}{c} \cos(2\pi f_B t)(2\pi f_B) \tag{11.7}$$

These are the frequencies at which we will see the peaks in the frequency domain. We can further extend the cos() term to identify all the frequency harmonics:

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \tag{11.8}$$

Therefore,

$$\frac{4\pi f_0 d_B f_B}{c} - \frac{4\pi f_0 d_B f_B}{c} \frac{(2\pi f_B t)^2}{2!} \tag{11.9}$$

This looks like equivalent to a slow-varying FMBC process??

11.1.2 Difference in human motion compared to breathing and heart-rate

Typical speed of human motion is 1m/s. The breahing and heart-rate causes 3cm/s and 3mm/s variations. Can we make use of this order of magnitude difference in speed?

$$\frac{\partial d}{\partial t} = v_0 + d_B 2\pi f_b \cos(2\pi f_B t) + d_H(2\pi f_H) \cos(2\pi f_H t) \tag{11.10}$$

The periodic term can be modeled and the velocity can be recovered with kalman filtering methods.

11.1.3 Kalman Filter

The Kalman motion model consists of

$$f(d(t)) = d_0 + v(t)t + a(t)t^2/2 + \cdots \tag{11.11}$$

The velocity is also a time model

$$v_n(t) = v(t-1) + \alpha_1 v(t-2) + \cdots \tag{11.12}$$

and we have some constraints such as

$$\begin{aligned} &0 < v(t) < 1m/s \\ &0 < a(t) < \text{ max acceleration} \end{aligned} \tag{11.13}$$

Figure: Architecture with Kalman Filter

After Kalman filtering, we remove the distortion due to human velocity. We then take IFFT to get velocity free $f(k\tau)$ from which we can recover the breathing and heart-rate.

Chapter 12

Alternate FMCW systems

12.1 Quadratic chirp problem

For quadratic chirp, we transmit the signal:

$$x(t) = \exp(-j2\pi(f_0 t + \frac{k}{3}t^3)) \tag{12.1}$$

The received signal is

$$y(t-\tau)x(t)^* = \exp(j2\pi f_0(t-\tau) + \frac{k}{3}(t-\tau)^3)\exp(-j2\pi f_0 t - \frac{k}{3}t^3) \tag{12.2}$$

$$= \exp(-j2\pi f_0\tau - 2\pi k t^2\tau + 2\pi k\tau^2 t - 2\pi\frac{k}{3}\tau^3) \tag{12.3}$$

Look at the t^2 term. Take derivative to get the frequency of oscillation $f = 2k\tau t$ which varies with time. It has a slope of $2k\tau$. If you plot the spectrogram of this chirp, you get a linear curve with slope $2k\tau$. Find this slope and estimate τ. The other components are not dominant. We can understand by considering some values. For instance, for $\tau = 1$ns, we have $\tau^2 = 10^{-18}$ sec, which is much smaller compared to τ.

Chapter 13

Lecture 15: Introduction to space transform

13.1 Limitation of current modeling for human motion

Until now, we have looked at the radial distance say d_r and radial velocity say v_r. The previous techniques fails when two person are at the same radial distance. The idea is to utilize the fact that the two person can be at different angular locations. We can distinguish the two person by finding their respective direction. We will also discuss the case when a person may also move in a non radial direction with certain velocity.

13.2 Location and Tracking driven modeling

We use multiple antennas at the receiver to estimate the direction (angle) of the user.

13.2.1 Signal received at the multiple receiver from a point source

Consider a point source (e.g. reflector) from which signal is coming to multiple receivers. Assume the source is at far-field, i.e., the signal is coming as parallel waves to the receivers. Assume the signal is coming at an angle θ measured from the axis normal to the array. How do we estimate θ from the received signal?

Assume the signal travels a distance d from Tx to the source, the distance of signal traveled at the receivers is a function of θ as follows:

$$\begin{aligned}
&\mathbf{Rx1:}\ d + d_1\\
&\mathbf{Rx2:}\ d + d_2 = d + d_1 + s\sin(\theta)\\
&\mathbf{Rx}N_r\mathbf{:}\ d + d_{N_r} = d + d_1 + (N_r - 1)s\sin(\theta)
\end{aligned} \tag{13.1}$$

In general ith receiver gets the signal that has travelled a distance of $d_i = d_1 + (i-1)s\sin(\theta)$ from the source for $i = 1, 2, \ldots, N_r$ and s is the spacing between the receiver antennas (assume uniform spacing).

The wireless channel at each receiver can be written as

$$\begin{aligned}
h_i(\lambda) &= \alpha \exp(-j2\pi\frac{d + d_i}{\lambda})\\
&= \alpha \exp(-j2\pi\frac{d + d_1}{\lambda})\exp(-j2\pi\frac{(i-1)s\sin(\theta)}{\lambda})
\end{aligned} \tag{13.2}$$

To estimate θ, we apply the following space transform for a range of values of ϕ:

$$R(\phi) = \sum_{i=1}^{N_r} h_i(\lambda)\exp(+j2\pi(i-1)\frac{s\sin(\phi)}{\lambda}) \tag{13.3}$$

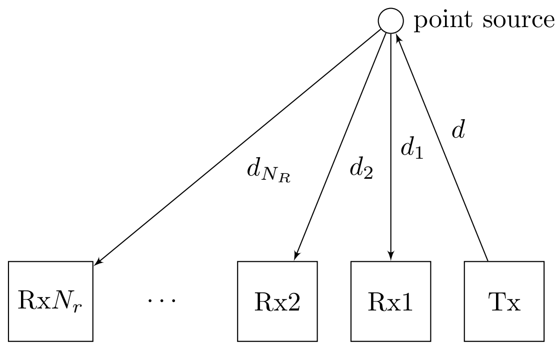

Figure 13.1: Architecture with one Tx and r receivers (Rx). Each receiver (antenna) receive the signal that travels different distances $d + d_i$ for $i = 1, 2, \ldots, N_r$ receivers.

where we will observe peak at $\phi = \theta$ and thus we can estimate the AoA θ. Note that there are better ways to estimate AoA using MUSIC algorithm and its variants. We cover those techniques in ECE257B class.

13.2.2 FMCW based modeling at multiple receiver

Recall the channel estimate for FMCW system has the following expression:

$$h(nT_s) = \alpha \exp(-j2\pi f_0 \tau) \exp(-j2\pi k\tau t) \tag{13.4}$$

where $\tau = \frac{d+d_1}{c}$ at Rx1. Similarly, at Rx2, we have $\tau = \frac{d+d_2}{c}$. We write the channel estimate at Rx1 and Rx2 as

$$h_1(nT_s) = \alpha \exp(-j2\pi f_0 \frac{d+d_1}{c}) \exp(-j2\pi k \frac{d+d_1}{c} nT_s) \tag{13.5}$$

$$h_2(nT_s) = \alpha \exp(-j2\pi f_0 \frac{d+d_2}{c}) \exp(-j2\pi k \frac{d+d_2}{c} nT_s) \tag{13.6}$$

We can obtain the phase difference between channel at Rx2 and Rx1 as follow

$$\frac{h_2(nT_s)}{h_1(nT_s)} = \exp(-j2\pi f_0 \frac{d_2-d_1}{c}) \exp(-j2\pi k \frac{d_2-d_1}{c} nT_s) \tag{13.7}$$

where we know that $d_2 - d_1 = s\sin(\theta)$. If the antenna spacing s is of the order of the wavelength ($s \approx \lambda$), so $s\sin(\theta) < \lambda$. The small change in distance across antennas causes a small change in the ToF ($\Delta\tau = \frac{d_2-d_1}{c}$) as well. For a small change in ToF, the second term in the channel doesn't change much, but the first term may change significantly.

Example: Consider $f_0 = 5$GHz, $B = 1$GHz.

The wavelength would be $\lambda = \frac{c}{f_0} = 6$cm. Therefore, if $d_2 - d_1 \approx \lambda$, we see a significant change in the first term of channel. Let's figure out why the resolution of the second term in the channel is low. The resolution (in distance) of the second term is $\frac{c}{kNT_s} = \frac{c}{B} = 30$cm, which is not enough to capture small change in distance across two antennas.

13.3 Recovery of Angle of Arrival (AoA)

We rewrite the expression for channel at ith antenna as

$$h_i(nT_s) = \exp(-j2\pi f_0 \tau - j2\pi f_0 \frac{(i-1)s\sin(\theta)}{c}) \exp(-j2\pi k\tau nT_s - j2\pi k \frac{(i-1)s\sin(\theta)}{c} nT_s) \tag{13.8}$$

which we can approximate by ignoring the changes in the second term:

$$h_i(nT_s) = \exp(-j2\pi f_0\tau - j2\pi f_0\frac{(i-1)s\sin(\theta)}{c})\exp(-j2\pi k\tau nT_s) \tag{13.9}$$

Take the spatial transform across all the receive antennas to obtain the AoA profile $R(\theta)$ as follows:

$$R(\theta) = \sum_{i=1}^{N_r} h_i(nT_s)\exp(+j2\pi f_0\frac{(i-1)s\sin(\theta)}{c}) \tag{13.10}$$

For ease of understanding, the location of antennas in space $[0, s, 2s, \ldots, (N_r - 1)s]$ is equivalent to time-domain and the θ-space ($\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$) is equivalent to the frequency-domain. Therefore, we can take advantage of FFT based techniques to get the AoA profile.

In Eq. (13.10), we considered a transform based on the first term in the channel and ignored the second term because that had small variations across antennas. Now, we consider both channel terms and write the following transform.

$$R'(\theta) = \sum_{i=1}^{N_r} h_i(nT_s)\exp(+j2\pi f_0\frac{(i-1)s\sin(\theta)}{c})\exp(+j2\pi k\frac{(i-1)s\sin(\theta)}{c}nT_s) \tag{13.11}$$

13.4 Joint estimation of AoA and ToF

In previous lectures, we have studied how we recover the ToF-profile using FFT based techniques:

$$\begin{aligned} f_i(k\tau) &= \sum_{nT_s=1}^{NT_s} h_i(nT_s)\exp(+j2\pi k\tau nT_s) \\ &= \exp(-j2\pi f_0\tau_i)\sin(k(\tau - \tau_i)) \end{aligned} \tag{13.12}$$

where $\tau_i = \tau + \frac{s\sin(\theta)}{c}$ at the ith antenna.

We can now take another transform over all the antennas to recover the AoA-ToF profile $R(k\tau, \theta)$ as follows:

$$R(k\tau, \theta) = \sum_{i=1}^{N_r} f_i(k\tau)\exp(-j2\pi(i-1)\frac{s\sin(\theta)}{\lambda}) \tag{13.13}$$

Can we define a function $R'(k\tau, \theta)$, which doesn't make an assumption that $\frac{is\sin(\theta)}{c}$ is insignificant and so the changes in the second term of the channel cannot be ignored.

Note that both the information of ToF and AoA is captured in the variations of τ. If we measure τ, we can get both AoA and ToF. In fact, it is a 3-dimensional inter-tuning if we consider Doppler as well.

Can we make a joint estimation for τ and θ? ToDo

13.4.1 Impact of phase wrapping

We have seen that we measure θ by observing the difference in phase at multiple receivers. But, we know that the received signal phase has phase wrapping of 2π. If the variation in phase is beyond 2π then we will have an ambiguity in the estimate of θ. What spacing between receiver antennas s is required to avoid phase wrapping?

$$\textbf{Phase difference: } (\phi) = \frac{2\pi s\sin(\theta)}{\lambda} \bmod 2\pi \tag{13.14}$$

For $s = \lambda$, we get $\phi = 2\pi\sin(\theta)$. We know that $\sin(\theta) \in [-1, 1]$ for $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Thus, without wrapping the phase varies $\phi \in [-2\pi, 2\pi]$ and so the phase wrapping at 2π would definitely cause ambiguity in estimating θ in its range.

Chapter 14

Lecture 16: Range and resolution for the AoA

14.1 Recap: AoA estimation

Consider an FMCW radar with one transmit antenna and N_r receive antennas. The spacing between receive antennas is s. The target object is placed at a distance d from the radar making an angle θ from the axis normal to the array. We assume far-field, i.e., $s \ll d$. The transmitter transmits FMCW signal $x(t)$ and the received signal at ith receive antenna can be written as

$$y_i(t) = x(t - \frac{2d}{c} - \frac{(i-1)s\sin(\theta)}{c}) \tag{14.1}$$

We estimate the channel $h_i(nT_s)$ at the ith antenna using FMCW channel estimation techniques:

$$h_i(nT_s) = \exp(-j2\pi f_0(\frac{2d}{c} + \frac{(i-1)s\sin(\theta)}{c}))\exp(-j2\pi k(\frac{2d}{c} + \frac{(i-1)s\sin(\theta)}{c})nT_s) \tag{14.2}$$

We have also discussed how we recover the AoA profile by FFT-based techniques. We also obtained joint AoA-ToF profile $R(k\tau, \theta)$.

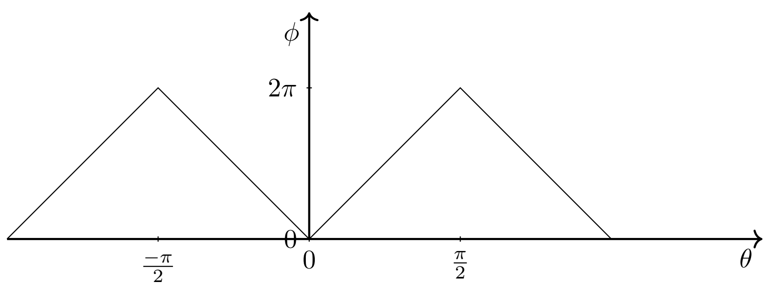

Figure 14.1: Phase $\phi(\theta)$ with wrapping of 2π for the case $s = \lambda$.

14.2 Range of the angle and aliasing condition

The additional phase at the receive antenna $i+1$ relative to the phase at antenna i is $\phi = \frac{2\pi s\sin(\theta)}{\lambda}$. The desired range of θ is $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Consider a case when $s = \lambda$. We get $\phi = \frac{2\pi\sin(\theta)}{\lambda} \bmod 2\pi$. Since $\sin(\theta) \in [-1, 1]$, we will see phase wrapping as shown in Figure 14.1. Therefore, the desired range cannot be achieved.

In general, the unaliased range of can be obtained by the following condition:

$$\begin{aligned} \textbf{AoA Range: } \frac{2\pi s\sin(\theta)}{\lambda} &= \pi \\ \Rightarrow \theta &\sim \sin^{-1}(\frac{\lambda}{2s}) \end{aligned} \tag{14.3}$$

Note that we want unaliased phase to be in the range $[-\pi, \pi]$ to get some range of AoA as $\theta \sim [-\theta_0, \theta_0]$, where $\theta_0 \sim \sin^{-1}(\frac{\lambda}{2s})$. Also, note that there is a fundamental limitation that the range must be in $[-\pi/2, \pi/2]$ for a linear array. One technique to increase the range to entire 360^o space is to use 2D arrays.

14.3 Fourier based analysis for range and resolution

We recover both ToF and AOA as follows:

$$R(k\tau, \theta) = \sum_{i=1}^{N_r} f_i(k\tau)\exp(-j2\pi\frac{(i-1)s\sin(\theta)}{\lambda}) \tag{14.4}$$

We will develop FFT-based insights from this equation to find the AoA resolution.

14.3.1 Fourier analysis-based resolution

We know that antenna space and θ-space are related by Fourier transform. The space domain varies as $\frac{(i-1)s}{\lambda}$ (equivalent to time-domain) and the θ-domain varies as $\sin(\theta)$ (which is equivalent to frequency domain). If we have a total number antennas N_r, we get resolution in θ as follows:

$$\begin{aligned} \textbf{Resolution: } \frac{s\sin(\theta)}{\lambda} &\sim \frac{1}{N_r} \\ \sin(\theta) &\sim \frac{\lambda}{N_r s} \\ \theta &\sim \sin^{-1}(\frac{\lambda}{N_r s}) \\ &= \sin^{-1}(\frac{1}{N_r}) \quad \text{for } \ s = \lambda \end{aligned} \tag{14.5}$$

Note that $\sin(\theta)$ is equivalent to frequency and has a uniform variation in its range. But, the inverse sine function is not a linear operation. Hence, the variations is θ will not be uniform. This leads to different resolution in AoA based on the actual value of AoA as shown in Figure 14.2.

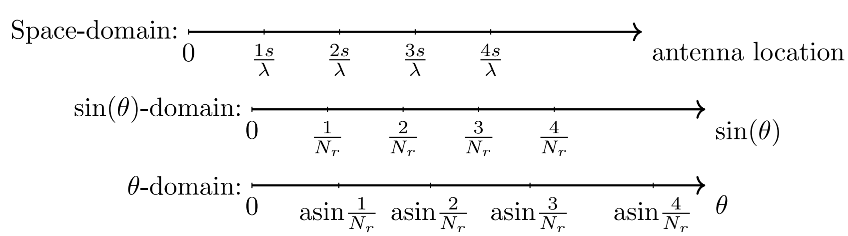

Figure 14.2: Explanation for non-uniform AoA resolution (illustrated for the case $s = \lambda$. and $N_r = 5$). The figure is roughly to the scale.

The resolution is dependent upon the actual θ_1 or AoA where a path already exist. For instance, for the case of $s = \lambda$, if the AoA is $\sin^{-1}(\frac{3}{N_r})$. The left and right resolution would be:

$$\textbf{Right resolution:} \quad \left| \sin^{-1}(\frac{4}{N_r}) - \sin^{-1}(\frac{3}{N_r}) \right| \tag{14.6}$$

$$\textbf{Left resolution:} \quad \left| \sin^{-1}(\frac{3}{N_r}) - \sin^{-1}(\frac{2}{N_r}) \right| \tag{14.7}$$

When θ is close to zero, we get a high resolution in θ, but if θ is closer to $\pi/2$, we have a poor resolution in θ. This is because the $\sin(\theta)$ function has small variation (slope close to zero) when $\theta \to \pi/2$. On the other hand, when $\theta \to 0$, we have a large change in $\sin(\theta)$ which can be measured and so we have a high resolution in estimating θ. We conclude that if a person is standing in front of the radar facing it, we have the highest resolution in AoA.

Chapter 15

Lecture 17: All on one– delay, Doppler and AoA

15.1 Joint estimation of AoA, ToF, Doppler

15.1.1 AoA for each reflection

Consider one Tx and N_r receivers forming a uniform linear array. The radar receives reflection from a person characterized by (ToF, Doppler, AoA) triplet (τ_1, v_1, θ_1).

The wireless channel estimated by the radar at antenna i and time instant nT_s is written as

$$h_i(nT_s) = \exp(-j2\pi f_0 \tau_{\text{total}}) \exp(-j2\pi k \tau_{\text{total}} nT_s) \tag{15.1}$$

where

$$\tau_{\text{total}} = \tau_1 + v_1 t + (i-1)\frac{s \sin(\theta)}{c} \tag{15.2}$$

We assume the velocity v_1 absorbs the breathing and heart-rate signal.

We take the first transform to get the ToF profile.

$$f_i(k\tau, \ell T) = \sum_{nT_s=0}^{(N-1)T_s} h_i(nT_s) \exp(+j2\pi k\tau nT_s) \tag{15.3}$$

We evaluate the above FFT-based analysis for ToF profile

$$f_i(k\tau, \ell T) = \exp(-j2\pi f_0 \tau_{\text{total}})\text{sinc}(k(\tau - \tau_{\text{total}})) \tag{15.4}$$

Expand the expression for τ_{total}, we get:

$$f_i(k\tau, \ell T) = \exp(-j2\pi f_0(\tau_1 + v_1 t + \frac{(i-1)s\sin(\theta)}{c}))\text{sinc}(k(\tau - \tau_{\text{total}})) \tag{15.5}$$

Now, we take the second transform across all the chirps for Doppler-profile

$$H_i(k\tau, \frac{f_0 \nu}{c}) = \sum_{\ell T=0}^{(M-1)T} f_i(k\tau, \ell T) \exp(+j2\pi f_0 \frac{\nu}{c} \ell T) \tag{15.6}$$

We solve the above equation

$$H_i(k\tau, \frac{f_0\nu}{c}) = \sum_{\ell T=0}^{(M-1)T} \exp(-j2\pi f_0(\tau_1 + v_1 t + \frac{(i-1)s\sin(\theta)}{c})) \exp(+j2\pi f_0 \frac{\nu}{c}\ell T)\text{sinc}(k(\tau - \tau_{\text{total}})) \tag{15.7}$$

Solving this summation given another sinc in Doppler domain ν as follows:

$$H_i(k\tau, \frac{f_0\nu}{c}) = \exp(-j2\pi f_0(\tau_1 + \frac{(i-1)s\sin(\theta)}{c}))\mathrm{sinc}(\frac{f_0(\nu - v_1)}{c})\mathrm{sinc}(k(\tau - \tau_{\text{total}})) \tag{15.8}$$

Range and resolution in 2-dimensional delay-Doppler space

Suppose two objects with delay-Doppler values (τ_1, v_1) and (τ_2, v_2). For simplicity assume $\tau_2 > \tau_1$ and $v_2 > v_1$. What is the resolution in delay-Doppler domain to separate the two objects?

If we only look at delay domain, the resolution is based on distance $\tau_2 - \tau_1$. If we only look at the Doppler domain, then the Doppler resolution is based on distance $v_2 - v_1$. But, if we look in 2D delay-Doppler domain, we are looking at 2D distance $\sqrt{(\tau_2 - \tau_1)^2 + (v_2 - v_1)^2}$. Definitely the diagonal distance is longer than individual 1D distances and so we have achieved higher resolution is separating the two objects.

Food for thought: Can we quantify the 2D resolution mathematically?

15.1.2 Third transform for AoA

We now take the third FFT along the antenna array to get the AoA-profile as follow

$$R(k\tau, f_0\nu, \sin(\theta)) = \sum_{i=1}^{N_r} H_i(k\tau, \frac{f_0\nu}{c})\exp(+j2\pi\frac{(i-1)s\sin(\theta)}{\lambda}) \tag{15.9}$$

We evaluate the above transform

$$Q(k\tau, f_0\nu, \sin(\theta)) = \sum_{i=1}^{N_r}\left[\exp(-j2\pi f_0(\tau_1 + \frac{(i-1)s\sin(\theta)}{c}))\exp(+j2\pi\frac{(i-1)s\sin(\theta)}{\lambda})\right]\mathrm{sinc}(\frac{f_0(\nu - v_1)}{c})\mathrm{sinc}(k(\tau - \tau_{\text{total}})) \tag{15.10}$$

We obtain the following joint AoA-ToF-Doppler profile.

$$Q(k\tau, f_0\nu, \sin(\theta)) = \exp(-j2\pi f_0\tau_1)\mathrm{sinc}(\frac{(i-1)s\sin(\theta)}{\lambda})\mathrm{sinc}(\frac{f_0(\nu - v_1)}{c})\mathrm{sinc}(k(\tau - \tau_{\text{total}})) \tag{15.11}$$

15.2 Another approach

todo

$$Q(k\tau, f_0\nu, \sin(\theta)) = \sum_{\ell T=0}^{(M-1)T} R(k\tau, \ell T, \sin(\theta))\exp(+j2\pi\frac{f_0\nu}{c}t) \tag{15.12}$$

We get a 3-dimensional profile form where we can

$$\sum_{(\tau_i, \nu_i, \theta_i)} \exp(-j2\pi f_0\tau_1)\mathrm{sinc}(\frac{(i-1)s\sin(\theta)}{\lambda})\mathrm{sinc}(\frac{f_0(\nu - v_1)}{c})\mathrm{sinc}(k(\tau - \tau_{\text{total}})) \tag{15.13}$$

15.2.1 Breathing and heart rate

For breathing and heart rate, we take $R()$ and not $Q()$.

$$R(k\tau, \frac{s\sin(\theta)}{\lambda}, \ell T) = \exp(-j2\pi f_0(\tau_1 + v_1 t))\mathrm{sinc}(\frac{s(\sin(\theta) - \sin(\theta_1))}{\lambda})\mathrm{sinc}(k(\tau - \tau_1)) \tag{15.14}$$

We evaluate $R(k\tau, \frac{s\sin(\theta)}{\lambda}, \ell T)$ at the person's location $\tau = \tau_1$ and $\theta = \theta_1$ as

$$R(k\tau_1, \frac{s\sin(\theta_1)}{\lambda}, \ell T) = \exp(-j2\pi f_0(\tau_1 + v_1 t)) \tag{15.15}$$

We see that the sinc terms disappeared and we get a time varying signal corresponding to the person's location. Next steps are clear from previous lectures: Extract the phase of this signal and take FFT to find out peaks at person's breathing and heart-rate.

The above analysis is done for a single person and in the absence of multipath. It is straightforward to extract the signal corresponding to every person by first separating them by their 2D location and extracting the phase signal corresponding to their location. We can also remove the impact of multipath and undesired reflections by noting the fact that multipath signal has different AoA and ToF compared to that corresponding to the person.

Chapter 16

Lecture 20:

16.1 Reduce to 3D transform when AoA and AoD are the same

We only have 3 Tx and 4 Rx, we get low accuracy and low resolution. How do we mitigate this problem?

Assume the object is far away from the radar and so approximately AoA and AoD are the same, i.e., $\theta_{Tx} = \theta_{Rx}$. How can we leverage it to improve the resolution and accuracy?

If the object is not moving, then Tx can provide better resolution. What if the object is moving?

Bibliography