Statistics

Slot: E2-TE2

Digital Assignment -1

Q1) A box contains 5 black, 7 red and 6 green balls. Three balls are drawn from this box one after another. What is the propability that the three balls are -

i) all black

ii) of different colours

iii) 2 black and 1 green color

Solution

Total no. of balls = 5 + 7 + 6 = 18

Total ways to draw 3 balls out of 18

$$^{18}C_3 = \frac{18 \times 17 \times 16}{3 \times 2 \times 1} = 816$$

(i) P(3 black)

No. of ways to choose 3 out of 5 black

$$^{5}C_3 = \frac{5 \times 4 \times 3}{3 \times 2 \times 1} = 10$$

$$P(3\ \text{black}) = \frac{^{5}C_3}{^{18}C_3} = \frac{10}{816} = \frac{5}{408} = 0.0123$$

(ii) P(all different)

$$P(\text{Different colours}) = \frac{^{5}C_1 \times {^{7}C_1} \times {^{6}C_1}}{^{18}C_3}$$

$$= \frac{5 \times 7 \times 6}{816} = \frac{35}{136} = 0.2574$$

(iii) P(2 black, 1 green)

$$P(2\text{black}, 1\text{green}) = \frac{^{5}C_2 \quad ^{6}C_1}{^{18}C_3}$$

$$= \frac{18 \times 6}{816} = \frac{5}{68} = 0.0735$$

Q2) A card is drawn from a standard deck of 52 cards, and without replacing it, a second card is drawn. If the first card is a heart. What is the probability of second card being a heart?

Solution

Total cards = 52

Hearts in deck = 13

$$P(\text{Second card is heart}/\text{First card is heart}) = \frac{12}{51}$$

$$= \frac{4}{17} = 0.2353$$

Q3) A urn contains 10 white and 3 black balls. Another urn contains 3 white and 5 black balls. Two balls are drawn at random from the first urn and placed in second urn, and then one ball is taken at random from it later. What is the probability that it is white ball?

Solution

First Urn, 10 white + 3 black = 13 balls

Second Urn, 3 white + 5 black = 8 balls

Both balls drawn are white

Choose 2 white balls from 10 - $^{10}C_2 = \frac{10 \times 9}{2 \times 1} = 45$

Choose any 2 balls from 13 = $^{13}C_2 = \frac{13 \times 12}{2 \times 1} = 78$

Probability of drawing 2 white balls = $\frac{^{10}C_2}{^{13}C_2} = \frac{45}{78} = \frac{15}{26} = 0.5769$

One white and one black ball

$$P(1 \text{ white}, 1 \text{ black}) = \frac{^{10}C_1 \, ^{3}C_1}{^{13}C_2} = \frac{10 \times 3}{78} = \frac{5}{13} = 0.3646$$

Both balls are black

$$P(2 \text{ black}) = \frac{^{3}C_2}{^{13}C_2} = \frac{3}{78} = \frac{1}{26}$$

If both transferred balls are white –

P(white from Urn 2 / 2 white transferred) $= \frac{5}{10} = \frac{1}{2}$

If 1 white & 1 black is transferred –

P(white from Urn 2 / 1-white, 1 black transferred) $= \frac{4}{10} = \frac{2}{5}$

If both transferred balls are black –

P(white from Urn 2 / 2 black transferred) $= \frac{3}{10}$

Total Probability of Drawing a white ball from Urn 2

$$P(\text{white ball}) = \frac{15}{26} \times \frac{1}{2} + \frac{5}{13} \times \frac{2}{5} + \frac{1}{26} \times \frac{3}{10}$$

$$= \frac{15}{52} + \frac{10}{65} + \frac{2}{260} = \frac{59}{130} = 0.4538$$

Q4) Find the value of a, $P(X<3)$, $P(1.3 \leq X \leq 6.7)$, $P(\frac{1}{2} < X < \frac{15}{2} / X > 3)$, the cummulative distribution function, the SD, $E(2X \pm 3)$ and $var(2X \pm 3)$ of the discrete random varible (X) withe the following probability distribution

$X = x$:	0	1	2	3	4	5	6	7	8
$p = P(X=x)$:	a	3a	5a	7a	9a	11a	13a	15a	17a

Solution

Total Probability $= 1$

$$a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$$

$$81a = 1 \Rightarrow \boxed{a = \frac{1}{81}}$$

$$P(X<3) = P(X=0) + P(X=1) + P(X=2)$$

$$= a + 3a + 5a = 9a = \frac{9}{81} = \frac{1}{9} = 0.1111$$

$P(1.3 \leq X \leq 6.7) = P(2 \leq X \leq 6)$

$= 5a + 7a + 9a + 11a + 13a = 45a$

$= \frac{45}{81} = \frac{5}{9} = 0.5555$

$P(1/2 < X < 15/2 \,/\, X > 3) = P(1 \leq X < 15/2 \,/\, X > 3)$

$= \frac{P((1 < X < 7) \cap (X > 3))}{P(X > 3)}$ $\quad X = 4, 5, 6, 7$

$X = \cancel{4}, 5, 6, 7, 8$

$= \frac{33a}{65a} = \frac{33}{65} = 0.5077$

Cummulative Distribution Function (CDF)

$$F(x) = \begin{cases} 1/81 & x = 0 \\ 4/81 & x \leq 1 \\ 1/9 & x \leq 2 \\ 16/81 & x \leq 3 \\ 25/81 & x \leq 4 \\ 4/9 & x \leq 5 \\ 49/81 & x \leq 6 \\ 64/81 & x \leq 7 \\ 1 & x \leq 8 \end{cases}$$

$E(X) = 0 \times a + 1 \times 3a + 2 \times 5a + 3 \times 7a + 4 \times 9a + 5 \times 11a + 6 \times 13a + 7 \times 15a + 8 \times 17a$

$= 444a = \frac{444}{81} = \frac{148}{27}$

$E(X^2) = 0 + 3a + 20a + 63a + 144a + 275a + 468a + 735a + 1088a$

$= 2796a = \frac{2796}{81} = \frac{932}{27}$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{932}{27} - \left(\frac{148}{27}\right)^2 = \frac{932}{27} - \frac{21904}{729}$$

$$\sigma^2 = 4.4719$$

$$SD = \sqrt{Var(X)}$$

$$SD = \sigma = 2.1147$$

$E(2x \pm 3)$ and $Var(2x \pm 3)$

$$E(2x \pm 3) = 2E(x) \pm 3 = 2 \times \frac{148}{27} \pm 3 = 10.96 \pm 3$$

$$E(2x \pm 3) = \frac{296 + 81}{27} = \frac{377}{27}$$

$$E(2x - 3) = \frac{296 - 81}{27} = \frac{215}{27}$$

$$Var(2x \pm 3) = 4\, Var(x)$$

$$= 4 \times \frac{3260}{729} = \frac{13040}{729} = 17.888$$

Q5) Suppose X is continuous R.V. with the probability density function given by $f(x) = \begin{cases} kxe^{-\lambda x} & ; \quad x \geq 0,\ \lambda > 0 \\ 0 & ; \quad \text{otherwise} \end{cases}$

Find k, $P(X<5)$, $P(1 \leq X \leq 100 \mid X \leq 5)$, the cummulative distribution function and the SD of X.

Solution

$$\int_0^\infty f(x)\,dx = 1 \qquad \int_0^\infty kxe^{-\lambda x}\,dx = 1 \Rightarrow k \cdot \frac{1}{\lambda^2} = 1$$

$$k = \lambda^2$$

$$P(X<5) = \int_0^5 \lambda^2 x e^{-\lambda x}\,dx$$

$$= 1 - (1 + 5\lambda)e^{-5\lambda}$$

$$P(X \leq 1) = \int_0^1 \lambda^2 x e^{-\lambda x}\,dx = 1 - (1+\lambda)e^{-\lambda}$$

$$P(1 \leq X \leq 100 \mid X \leq 5) = (1 - (1+5\lambda)e^{-5\lambda}) - (1 - (1+\lambda)e^{-\lambda})$$

Cummulative Distribution Function (CDF)

$$F(x) = \int_0^{\lambda} \lambda^2 t e^{-\lambda t}\, dt$$

$$= 1 - (1 + \lambda x) e^{-\lambda x}$$

SD of X

$$E(X) = \int_0^{\infty} x f(x)\, dx = \int_0^{\infty} x \lambda^2 x e^{-\lambda x} dx = \frac{2}{\lambda}$$

$$E(X^2) = \int_0^{\infty} x^2 f(x)\, dx = \int_0^{\infty} x^2 \lambda^2 x e^{-\lambda x} dx = 6/\lambda^2$$

$$var(X) = E(X^2) - [E(X)]^2 = \frac{6}{\lambda^2} - \left(\frac{2}{\lambda}\right)^2 = \frac{2}{\lambda^2}$$

$$SD = \sigma = \sqrt{var(X)} = \frac{\sqrt{2}}{\lambda}$$

Q6) 2D RV (X, Y) has the joint probability function given by $P(X = x, Y = y) = k(3x + 5y)$, for $x = 0, 1, 2, 3$ and $y = 0, 1$.

(i) Find the value of k

(ii) Find all the marginal and conditional distributions of X & Y

(iii) Find the prob. distribution of Z, mean & variance of Z, where $Z = X + Y$

Solution

(i) $$\sum_{x=0}^{3} \sum_{y=0}^{1} P(X = x, Y = y) = 1$$

$$\sum_{x=0}^{3} \sum_{y=0}^{1} k(3x + 5y) = 1$$

For $x = 0$: $k(3\times0 + 5\times0) + k(3\times0 + 5\times1) = k(0) + k(5) = 5k$

For $x = 1$: $k(3\times1 + 5\times0) + k(3\times1 + 5\times1) = k(3) + k(8) = 11k$

For $x = 2$: $k(3\times2 + 5\times0) + k(3\times2 + 5\times1) = k(6) + k(11) = 17k$

For $x = 3$: $k(3\times3 + 5\times0) + k(3\times3 + 5\times1) = k(9) + k(14) = 23k$

$$5k + 11k + 17k + 23k = 56k$$

$$k = 1/56$$

(ii) Marginal Distribution

$$P(X=x) = \sum_{y=0}^{1} P(X=x, Y=y)$$

For $X=0$, $P(X=0) = k(0) + k(5) = 5k = 5 \times \frac{1}{56} = \frac{5}{56} = 0.0893$

For $X=1$, $P(X=1) = k(3) + k(8) = 11k = \frac{11}{56} = 0.1964$

For $X=2$, $P(X=2) = 17k = \frac{17}{56} = 0.3036$

For $X=3$, $P(X=3) = 23k = \frac{23}{56} = 0.4107$

Marginal Distribution of Y

$$P(Y=y) = \sum_{x=0}^{3} P(X=x, Y=y)$$

$P(X=0) = k(0) + k(3) + k(6) + k(9) = 18k = \frac{18}{56} = \frac{9}{28} = 0.3214$

$P(X=1) = k(5) + k(8) + k(11) + k(14) = 38k = \frac{19}{28} = 0.6786$

Conditional Distributions of X given Y

$$P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}$$

For $(Y=0)$

$P(X=0 \mid Y=0) = \frac{k(0)}{P(Y=0)} = 0$

$P(X=1 / Y=0) = \frac{k(3)}{P(Y=0)} = \frac{3k}{9/28} = \frac{28}{168} = \frac{1}{6} = 0.1667$

$P(X=2 \mid Y=0) = \frac{k(6)}{P(Y=0)} = \frac{6k}{9/28} = \frac{56}{168} = \frac{1}{3} = 0.3333$

$P(X=3 / Y=0) = \frac{k(9)}{P(Y=0)} = \frac{9k}{9/28} = \frac{28}{168} = \frac{1}{2} = 0.5$

For $(Y=1)$

$P(X=0 \mid Y=1) = \frac{k(5)}{P(Y=1)} = \frac{5k}{19/28} = \frac{5 \times 28}{19 \times 56} = \frac{5}{38} = 0.1316$

$P(X=1 / Y=1) = \frac{k(8)}{P(Y=1)} = \frac{8k}{19/28} = \frac{8}{38} = \frac{4}{19} = 0.2105$

$$P(X=2 \mid Y=1) = \frac{k(11)}{P(Y=1)} = \frac{11k}{19/28} = \frac{11}{38} = \cancel{\frac{7}{19}}\, 0.2895$$

$$P(X=3 \mid Y=1) = \frac{k(14)}{P(Y=1)} = \frac{14k}{19/28} = \frac{14}{38} = \frac{7}{19} = 0.3684$$

(iii) Probability Distribution, Mean & Variance of $Z = X+Y$

$$P(Z=0) = P(X=0, Y=0) = 0$$

$$P(Z=1) = P(X=0, Y=1) = 5k = 5/56$$

$$P(Z=2) = P(X=1, Y=0) + P(X=1, Y=1) = 3k + 8k = \frac{11}{56} = 0.1964$$

$$P(Z=3) = P(X=2, Y=0) + P(X=2, Y=1) = 6k + 11k = \frac{17}{56} = 0.3036$$

$$P(Z=4) = P(X=3, Y=0) + P(X=3, Y=1) = 9k + 14k = \frac{23}{56} = 0.4107$$

Mean of Z

$$E(Z) = \Sigma\, z P(Z=z) = 1 \times \frac{5}{56} + 2 \times \frac{11}{56} + 3 \times \frac{17}{56} + 4 \times \frac{23}{56}$$

$$E(Z) = \frac{5}{56} + \frac{22}{56} + \frac{51}{56} + \frac{92}{56}$$

$$= \frac{170}{56} = \frac{85}{26}$$

$$\approx 3.04$$

Variance of Z

$$Var(Z) = \sigma_z^2 = E(Z^2) - [E(Z)^2]$$

$$E(Z^2) = \Sigma\, z^2 P(Z=z) = 1^2 * \frac{5}{56} + 2^2 \times \frac{11}{56} + 3^2 \times \frac{17}{56} + 4^2 \times \frac{23}{56}$$

$$E(Z^2) = \frac{5}{56} + \frac{44}{56} + \frac{153}{56} + \frac{368}{56} = \frac{570}{56} = \frac{285}{28}$$

$$var(Z) = \frac{285}{28} - \left(\frac{85}{28}\right)^2 = \frac{285}{28} - \frac{7225}{784} = \frac{285}{28} - \frac{7225}{784}$$

$$= \frac{7980 - 7225}{784} = \frac{755}{784} = 0.9630$$

Q7) Two RV X and Y have following joint probability density function

$$f_{xy}(x,y) = \begin{cases} 2-x-y, & 0 \le x \le 1, 0 \le y \le 1) \\ 0, & \text{otherwise} \end{cases}$$

Find : i) $P(X > 1/2)$, $P(Y < 1/2)$ and $P(X > 1/2 \mid Y < 1/2)$.

ii) the marginal probability density function of X and Y,

(iii) the conditional density functions of X and Y,

(iv) the variances var(X) and var(Y)

(v) the covariances between X and Y

Solution

(i) $P(X > 1/2) = \int_{1/2}^{1}\int_{0}^{1} (2-x-y)\,dy\,dx = \left[(2-x)y - y^2/2\right]_0^1 = \left[1.5x - \frac{x^2}{2}\right]_{1/2}^{1}$

$= 0.375$

$P(Y < 1/2) = \int_0^1\int_0^{1/2} (2-x-y)\,dy\,dx = \left[(2-x)y - y^2/2\right]_0^{1/2} = \left(\left(1-\frac{x}{2}\right) - \frac{1}{8}\right)$

$= 0.625$

$P(X > 1/2 \mid Y < 1/2) = \int_{1/2}^{1}\int_{0}^{1/2} (2-x-y)\,dy\,dx$

$= \int_{1/2}^{1} \left(\left(1-\frac{x}{2}\right) - \frac{1}{8}\right) dx = 0.25$

$$P(X > 1/2 \mid Y < 1/2) = \frac{P(X > 1/2 \cap Y < 1/2)}{P(Y < 1/2)} = \frac{0.25}{0.625} = 0.4$$

(ii) Marginal Probability Density Functions of X and Y

$f_X(x) = \int_0^1 (2-x-y)\,dy$

$= \frac{3}{2} - x \quad$ for $0 \le x \le 1$

$f_Y(y) = \int_0^1 (2-x-y)\,dx$

$= \frac{3}{2} - y \quad$ for $0 \le y \le 1$

(iii) Conditional Density function, for $0 \leq x, y \leq 1$

$$f_{X/Y}(x/y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{2-x-y}{f_Y(y)}$$

$$= \frac{2-x-y}{3/2-y}$$

$$f_{Y/X}(y/x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{2-x-y}{f_X(x)}$$

$$= \frac{2-x-y}{3/2-x}$$

(iv) Variance var(X) and var(Y)

$$E(X) = \int_0^1 x f_X(x)\,dx = \int_0^1 x(3/2-x)\,dx = 5/12$$

$$E(X^2) = \int_0^1 x^2 f_X(x)\,dx = \int_0^1 x^2(3/2-x)\,dx = 1/4$$

$$\text{var}(X) = E(X^2) - [E(X)]^2$$

$$= \frac{1}{4} - (5/12)^2 = \frac{11}{144}$$

$$= 0.0763$$

$$E(Y) = \int_0^1 y f_Y(y)\,dy = \int_0^1 y(3/2-y)\,dy = 5/12$$

$$E(Y^2) = \int_0^1 y^2 f_Y(y)\,dy = \int_0^1 y^2(3/2-y)\,dy = 1/4$$

$$\text{var}(Y) = E(Y^2) - [E(Y)]^2$$

$$= \frac{1}{4} - (5/12)^2 = \frac{11}{144}$$

$$= 0.0763$$

(V) Covariance between X and Y

$$\operatorname{cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$E(X,Y) = \int_0^1\int_0^1 xy\, f_{XY}(x,y)\, dy\, dx$$

$$= \int_0^1 y \left[\int_0^1 (2x - x^2 - xy)\, dx\right] dy$$

$$= \int_0^1 y\left(1 - \tfrac{1}{3} - \tfrac{y}{2}\right) dy$$

$$= \int_0^1 y\left(\tfrac{2}{3} - \tfrac{y}{2}\right) dy$$

$$= \tfrac{2}{3}\int_0^1 y\, dy - \tfrac{1}{2}\int_0^1 y^2\, dy$$

$$= \tfrac{2}{3}\left[\tfrac{y^2}{2}\right]_0^1 - \tfrac{1}{2}\left[\tfrac{y^3}{3}\right]_0^1$$

$$= \tfrac{1}{6}$$

$$\operatorname{cov}(X,Y) = \frac{1}{6} - \frac{5}{12} \times \frac{5}{12} = \frac{-1}{144}$$

$$= -6.944 \times 10^{-3}$$

Q8) If X represents the outcome when a fair die is tossed, then find the moment generating function of X and from hence find E(X) and var(X)

Solution

MGF, $M_X(t) = E(e^{tx})$

possible outcomes of X are 1, 2, 3, 4, 5, 6 each with the probability $1/6$

So, $M_x(t) = \sum_{x=1}^{6} P(X=x).\, e^{tx}$

$$= \tfrac{1}{6}\left(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}\right)$$

Taking first derivative of $M_x(t)$ at $t=0$,

$$E(X) = M_x'(0)$$

$$M_x(t) = \frac{1}{6}(e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t})$$

at $t=0$

$$M_x'(0) = \frac{1}{6}(1+2+3+4+5+6) = \frac{1}{6} \times 21 = 3.5$$

$$E(X) = 3.5$$

Now,

$$Var(X) = E(X^2) - [E(X)]^2$$

$$M_x''(t) = \frac{1}{6}(e^t + 4e^{2t} + 9e^{3t} + 16e^{4t} + 25e^{5t} + 36e^{6t})$$

at $t=0$,

$$M_x''(0) = \frac{1}{6}(1+4+9+16+25+36) = \frac{1}{6} \times 91$$

$$= 15.1667$$

So, $E(X^2) = 15.1667$

$$var(X) = 15.1667 - (3.5)^2 = 15.1667 - 12.25$$

$$= 2.9167$$

Q9) Compute the coefficient of correlation between the following two set of measures X and Y and hence obtain the eq^n of regression lines

X	Y	X^2	Y^2	XY
65	67	4225	4489	4355
67	68	4489	4624	4556
66	68	4356	4624	4488
71	70	5041	4900	4970
67	64	4489	4096	4288
70	67	4900	4489	4690
68	72	4624	5184	4896
69	70	4761	4900	4830
$\Sigma X =$ 538	$\Sigma Y =$ 486	$\Sigma X^2 =$ 36214	$\Sigma Y^2 =$ 36842	$\Sigma XY =$ 42686

$$\bar{X} = \frac{538}{8} = 67.25$$

$$\bar{Y} = \frac{486}{8} = 60.75$$

$$r = \frac{n \Sigma XY - \Sigma X \Sigma Y}{\sqrt{(n\Sigma X^2 - (\Sigma X)^2)(n \Sigma Y^2 - (\Sigma Y)^2)}}$$

$$= \frac{8 \times 42686 - 538 \times 486}{\sqrt{(8 \times 36214 - 538^2)(8 \times 36842 - 486^2)}}$$

$$r = 20.18$$

$$b_{yx} = r \frac{\sigma_y}{\sigma_x} = \frac{79200}{268} \approx 297.0$$

X on Y

$$X - \bar{X} = r \frac{\sigma_x}{\sigma_y} (Y - \bar{Y})$$

$$X = 1.36\,Y - 15.35$$

Y on X

$$Y - \bar{Y} = r \frac{\sigma_y}{\sigma_x} (X - \bar{X})$$

$$Y = 297X - 13969.5$$