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Discrete Maths

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Homework Assignment #2

Problem 1: $S = \binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$

Proof by strong induction:

Basis case: Let $n_0 = 1$

$$S(n_0) = \frac{n(n-1)}{2}$$

$$S(1) = \frac{1(1-1)}{2} = \frac{1 \times 0}{2} = 0$$

Induction hypothesis:

For any $k \in \mathbb{N}$, $k \geq n_0$

$S(n)$ is true for $S(1), S(2), \ldots, S(k)$

we take $k+1$ blocks,

$$k+1 = x + (k+1-x) \qquad 1 \leq x \leq k$$

$$\text{Total sum} = x(k+1-x) + S(x) + S(k+1-x)$$

$$= kx + x - x^2 + \frac{x(x-1)}{2} + \frac{(1+1-x)(k+\cancel{1}-x-\cancel{1})}{2}$$

$$= \frac{\cancel{2kx} + \cancel{2x} - \cancel{2x^2} + \cancel{x^2 - x} + k^2 + k - \cancel{kx} - \cancel{kx} - \cancel{x} + \cancel{x^2}}{2}$$

$$= \frac{k^2 + k}{2} = \frac{k(k+1)}{2} = \frac{(k+1)[(k+1)-1]}{2}$$

$$= S(k+1)$$

Problem 2: Prove that $\sqrt{2}$ is not a rational number.

Proof by contradiction: Assume $\sqrt{2}$ is rational, then show the contradiction.

$\sqrt{2}$ is rational so $\sqrt{2} = \frac{n}{m}$ where $n, m \in \mathbb{Z}$, $m \geq 1$ and n, m are in lowest terms.

$$2 = \frac{n^2}{m^2} \Rightarrow 2m^2 = n^2$$

This shows that n^2 is even, therefore n is even, so $2 | n$, $n = 2a$

$n^2 = 2m^2 \Rightarrow (2a)^2 = 2m^2 \Rightarrow 4a^2 = 2m^2 \Rightarrow 2a^2 = m^2$

so m^2 is even, therefore m is even

so $2 | m$, $m = 2b$

If n is even and m is also even, this shows that $\frac{n}{m}$ is not in its lowest terms.

This is a contradiction to our assumption.

Hence $\sqrt{2}$ must be irrational.

Problem 3: Exercize 22.12 of section 22.

Prove: For every positive integer n, the tower of Hanoi puzzle (with n disks) can be solved in $2^n - 1$ moves.

Proof by induction:

P(n): The minimum number of moves needed to move n disks from one dowel to another dowel in the Tower of Hanoi puzzle is $2^n - 1$, is true for all natural numbers n.

Basis: P(1) is true because if there is one disk, then only 1 move is required. $2^1 - 1 = 2 - 1 = 1$

Inductive Step: Given P(k) is true, prove P(k+1) is true.

P(k): The minimum number of moves needed to move k disks is $2^k - 1$.

Let the puzzle have k+1 disks.

Prove that minimum number of moves needed to move k+1 disks is $2^{k+1} - 1$.

> If we have k+1 disks, to move the top k disks from the first dowel to second dowel, $2^k - 1$ moves are needed.

> To move the largest disk from first dowel to third dowel, 1 move needed.

> To move k disks from second dowel to third dowel, $2^k - 1$ moves needed.

$$\text{Total number of moves} = 2(2^k - 1) + 1$$
$$= 2 \cdot 2^k - 2 + 1 \Rightarrow 2 \cdot 2^k - 1 \Rightarrow 2^{k+1} - 1$$

Thus, we have shown that if P(k) is true, then P(k+1) is also true, combined with the base case, the proposition is true for every positive integer n.

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Problem 4:

a) $\forall n \in \mathbb{N}, x_n > 3$

Basis case: $n=0$, $x_0 > 3 \Rightarrow 4 > 3$.

Induction hypothesis: Let $n=k$, $x_k > 3$.
~~Assume~~ Prove $x_{k+1} > 3$.

Induction Step:

$$x_{k+1} = \frac{2x_k^2 - 3}{x_k + 2} > 3$$

$$\Rightarrow \frac{2x_k^2 - 3}{x_k + 2} - 3 > 0$$

$$\Rightarrow \frac{2x_k^2 - 3 - 3x_k - 6}{x_k + 2} > 0$$

$$\Rightarrow 2x_k^2 - 3x_k - 9 > 0$$

$$\Rightarrow 2x_k(x_k - 3) + 3(x_k - 3) > 0$$

$$\Rightarrow (2x_k + 3)(x_k - 3) > 0$$

Therefore, as $x_k > 3$, $2x_k + 3 > 3$ and $x_k - 3 > 3$
so $(2x_k + 3)(x_k - 3) > 3$
hence $x_{k+1} > 3$

If proposition is true for $n=0$ and true for $n=k$ and $n=k+1$, it must be true for all $n \in \mathbb{N}$.

Problem 4:

2) $\forall n \in \mathbb{N},\ x_{n+1} - 3 > \frac{3}{2}(x_n - 3)$

$$\frac{2x_n^2 - 3}{x_n + 2} - 3 > \frac{3}{2}(x_n - 3)$$

$$\frac{4x_n^2 - 6}{x_n + 2} - 6 > 3(x_n - 3)$$

$$\frac{4x_n^2 - 6}{3(x_n + 2)} - 2 > x_n - 3$$

$$\frac{4x_n^2 - 6}{3(x_n + 2)} - (x_n - 1) > 0$$

$$4x_n^2 - 6 - 3(x_n + 2)(x_n - 1) > 0$$

$$4x_n^2 - 6 - 3(x_n^2 + 2x_n - x_n - 2) > 0$$

$$4x_n^2 - 6 - 3(x_n^2 + x_n - 2) > 0$$

$$4x_n^2 - 6 - 3x_n^2 - 3x_n + 6 > 0$$

$$x_n^2 - 3x_n > 0$$

$$x_n^2 > 3x_n$$

$$x_n > 3$$

$$\Rightarrow x_n(x_n - 3) > 0$$