APC 523
Problem Set 4
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Problem 1: Elliptic BVP

(a)

Using matrix inverse method for the case $N = 64$, we get the following solution:

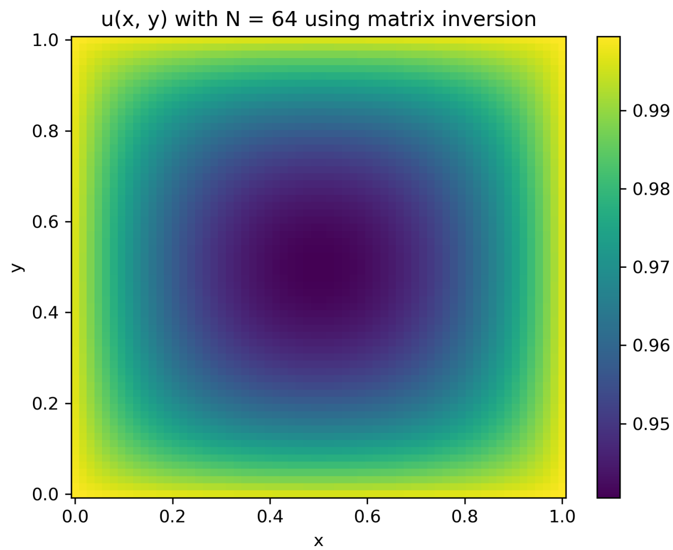
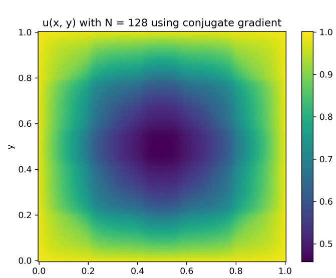


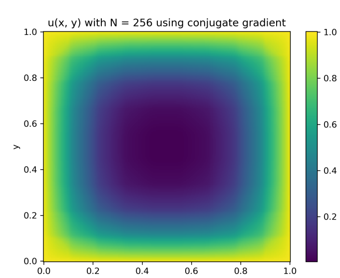
Figure 1: Solution to the nonlinear elliptic BVP $\nabla^2 u - u^4 = 0$ using matrix inversion when $N = 64$

(b)

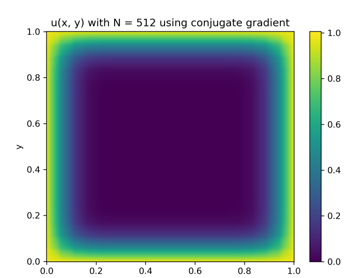
Next, I used the conjugate gradient method for the cases $N = 128, 256, and 512$. The solutions are summarized in the figures below:



(a) $N = 128$



(b) $N = 256$



(c) $N = 512$

Figure 2: Solution to the nonlinear elliptic BVP $\nabla^2 u - u^4 = 0$ using conjugate gradient

Problem 2: Beam Warming Scheme

The beam-warming scheme is written as:

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} + a\frac{3u_i^n - 4u_{i-1}^n + u_{i-2}^n}{2h} = \frac{a^2\Delta t}{2}\frac{u_i^n - 2u_{i-1}^n + u_{i-2}^n}{h^2}$$

Taylor expanding in time we get: $u_i^{n+1} = u_i^n + \Delta t\frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3}$

Using the advection equation, we can replace time derivatives with space derivative:

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} \quad \text{and} \quad \frac{\partial^3 u}{\partial t^3} = -a^3\frac{\partial^3 u}{\partial x^3} \quad \text{and this gives us} \quad \frac{u_i^{n+1}-u_i^n}{\Delta t} = \frac{\partial u}{\partial t} + \frac{a^2\Delta t}{2}\frac{\partial^2 u}{\partial x^2} - \frac{a^3\Delta t^2}{6}\frac{\partial^3 u}{\partial x^3} + \cdots$$

Taylor expanding in space we can write:

$$u_{i+1}^n = u_i^n + h\frac{\partial u}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u}{\partial x^2} + \frac{h^3}{3!}\frac{\partial^3 u}{\partial x^3} + \frac{h^4}{4!}\frac{\partial^4 x}{\partial x^4} + \cdots \qquad u_{i-1}^n = u_i^n - h\frac{\partial u}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u}{\partial x^2} - \frac{h^3}{3!}\frac{\partial^3 u}{\partial x^3} + \frac{h^4}{4!}\frac{\partial^4 x}{\partial x^4} - \cdots$$

$$u_{i-2}^n = u_i^n - 2h\frac{\partial u}{\partial x} + \frac{4h^2}{2!}\frac{\partial^2 u}{\partial x^2} - \frac{8h^3}{3!}\frac{\partial^3 u}{\partial x^3} + \frac{16h^4}{4!}\frac{\partial^4 x}{\partial x^4} - \cdots$$

Combining these we can write:

$$a\frac{3u_i^n - 4u_{i-1}^n + u_{i-2}^n}{2h} = a\frac{\partial u}{\partial x} - a\frac{h^2}{3}\frac{\partial^3 u}{\partial x^3} + \cdots \quad \text{and} \quad \frac{a^2\Delta t}{2}\frac{u_i^n - 2u_{i-1}^n + u_{i-2}^n}{h^2} = \frac{a^2\Delta t}{2}\frac{\partial^2 u}{\partial x^2} - \frac{a^2\Delta t h}{2}\frac{\partial^3 u}{\partial x^3} + \cdots$$

Plugging everything into the original equation, we get.

$$\frac{\partial u}{\partial t} + \cancel{\frac{a^2\Delta t}{2}\frac{\partial^2 u}{\partial x^2}} - \frac{a^3\Delta t^2}{6}\frac{\partial^3 u}{\partial x^3} + a\frac{\partial u}{\partial x} - a\frac{h^2}{3}\frac{\partial^3 u}{\partial x^3} = \cancel{\frac{a^2\Delta t}{2}\frac{\partial^2 u}{\partial x^2}} - \frac{a^2\Delta t h}{2}\frac{\partial^3 u}{\partial x^3}$$

$$\text{or,} \quad \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \frac{a^3\Delta t^2}{6}\frac{\partial^3 u}{\partial x^3} + \frac{ah^2}{3}\frac{\partial^3 u}{\partial x^3} - \frac{a^2\Delta t h}{2}\frac{\partial^3 u}{\partial x^3}$$

$$\text{or,} \quad \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \frac{ah^2}{6}(c^2 - 3c + 2)\frac{\partial^3 u}{\partial x^3} + O(h^3)$$

the modified equation

Problem 3: Conserved Energy for the Verlet Scheme

(a)

For the advection equation $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = 0$, if the analytical solution is $u(t,x,y) = u(0, x-at, y-bt)$

then let $v = x - at$ and $w = y - bt$, which gives us $\begin{cases} \frac{\partial v}{\partial x} = 1 \text{ and } \frac{\partial v}{\partial t} = -a \\ \frac{\partial w}{\partial y} = 1 \text{ and } \frac{\partial w}{\partial t} = -b \end{cases}$

Using chain rule, we can write:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial x} = \frac{\partial u}{\partial v}\cdot 1 = \frac{\partial u}{\partial v}$$

also,

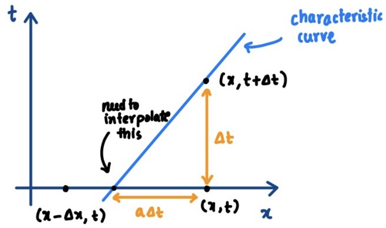
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial y} = \frac{\partial u}{\partial w}\cdot 1 = \frac{\partial u}{\partial w}$$

and

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial t} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial t} = -a\frac{\partial u}{\partial v} - b\frac{\partial u}{\partial w}$$

therefore, these values satisfy the advection equation hence $u(0, x-at, y-bt)$ is a true solution

(b) - 1

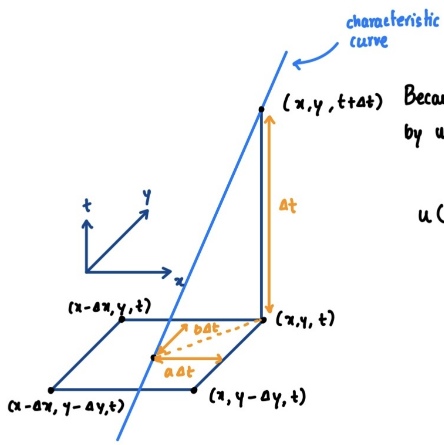


Because of the characteristic curve, we can find the value of u_i^{n+1} or $u(t+\Delta t, x)$ by using linear interpolation:

$$u(t+\Delta t, x) = \frac{a\Delta t}{\Delta x} u(x-\Delta x, t) + \left(1 - \frac{a\Delta t}{\Delta x}\right) u(t,x)$$

$$\text{or, } u_i^{n+1} = (1-\mu) u_i^n + \mu u_{i-1}^n$$

(b) - 2



Because of the characteristic curve, we can find the value of $u_{i,j}^{n+1}$ or $u(t+\Delta t, x, y)$ by using the bilinear interpolation formula:

$$u(t+\Delta t, x, y) = \frac{b\Delta t}{\Delta y}\left[\frac{a\Delta t}{\Delta x} u(t, x-\Delta x, y-\Delta y) + \left(1-\frac{a\Delta t}{\Delta x}\right) u(t, x, y-\Delta y)\right]$$

$$+ \left(1 - \frac{b\Delta t}{\Delta y}\right)\left[\frac{a\Delta t}{\Delta x} u(t, x-\Delta x, y) + \left(1-\frac{a\Delta t}{\Delta x}\right) u(t, x, y)\right]$$

$$= \left(1-\frac{b\Delta t}{\Delta y}\right)\left(1-\frac{a\Delta t}{\Delta x}\right) u(t,x,y) + \left(1-\frac{b\Delta t}{\Delta y}\right)\frac{a\Delta t}{\Delta x} u(t, x-\Delta x, y)$$

$$+ \frac{b\Delta t}{\Delta y}\left(1-\frac{a\Delta t}{\Delta x}\right) u(t, x, y-\Delta y) + \frac{b\Delta t}{\Delta y}\frac{a\Delta t}{\Delta x} u(t, x-\Delta x, y-\Delta y)$$

$$\therefore \quad u_{i,j}^{n+1} = (1-\mu)(1-\upsilon) u_{i,j}^n + \mu(1-\upsilon) u_{i-1,j}^n + \upsilon(1-\mu) u_{i,j-1}^n + \mu\upsilon u_{i-1,j-1}^n$$

(b) - 3

Taylor expanding in time we get: $u_{i,j}^{n+1} = u_{i,j}^n + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3 u}{\partial t^3} + \cdots$

$$= u_{i,j}^n + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\left(-a^2 \frac{\partial^2 u}{\partial x^2} - b^2 \frac{\partial^2 u}{\partial y^2} - 2ab\frac{\partial^2 u}{\partial x \partial y}\right) + \cdots$$

Now, Taylor expanding in space we get:

$$u_{i-1,j}^n = u_{i,j}^n - \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2!}\frac{\partial^2 u}{\partial x^2} - \frac{\Delta x^3}{3!}\frac{\partial^3 u}{\partial x^3} + \cdots \qquad u_{i,j-1}^n = u_{i,j}^n - \Delta y \frac{\partial u}{\partial y} + \frac{\Delta y^2}{2!}\frac{\partial^2 u}{\partial y^2} - \frac{\Delta y^3}{3!}\frac{\partial^3 u}{\partial y^3} + \cdots$$

$$u_{i-1,j-1}^n = u_{i,j}^n - \Delta x \frac{\partial u}{\partial x} - \Delta y \frac{\partial u}{\partial y} + \frac{1}{2}\left(\Delta x^2 \frac{\partial^2 u}{\partial x^2} + \Delta y^2 \frac{\partial^2 u}{\partial y^2} + 2\Delta x \Delta y \frac{\partial^2 u}{\partial x \partial y}\right) - \cdots$$

Plugging all of this into the CTU scheme and simplifying we get:

not sure about this

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = \frac{a\Delta x}{2}(1-\mu)\frac{\partial^2 u}{\partial x^2} + \frac{b\Delta y}{2}(1-\upsilon)\frac{\partial^2 u}{\partial y^2} + O(\Delta x^2, \Delta y^2)$$

(c)

For the CTU method, the results and errors for the $N = 128, 256, \& 512$ cases are shown in the figures below:

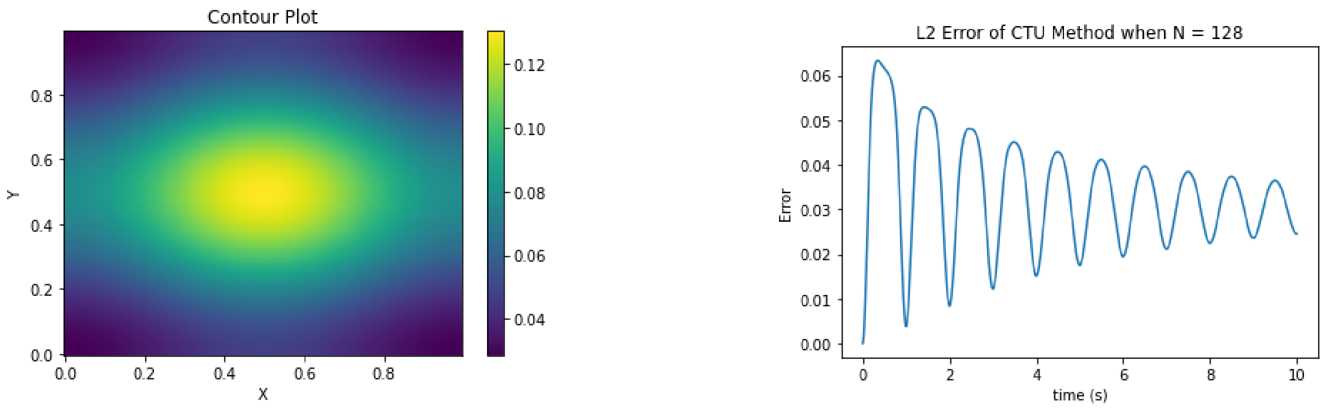


Figure 3: CTU Solution to the advection equation when $N = 128$

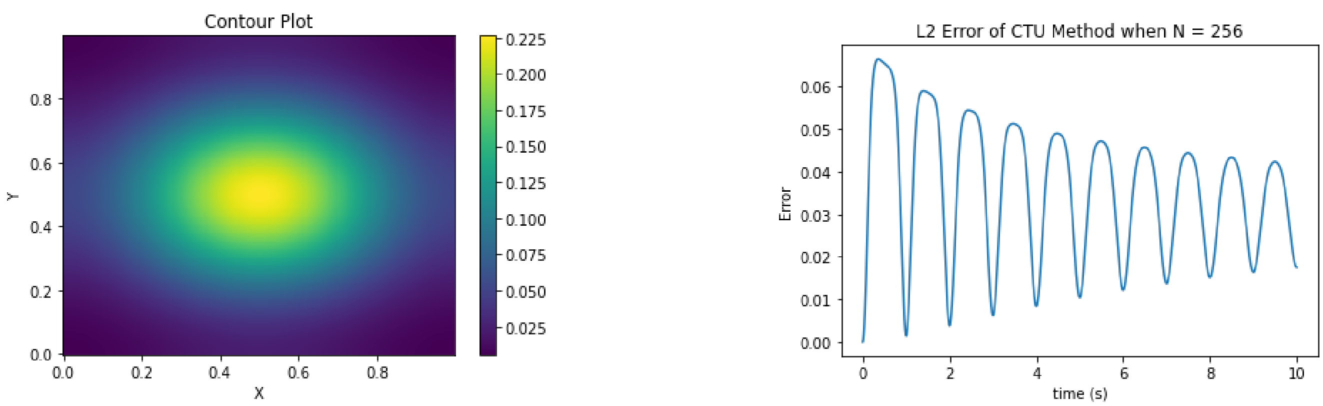


Figure 4: CTU Solution to the advection equation when $N = 256$

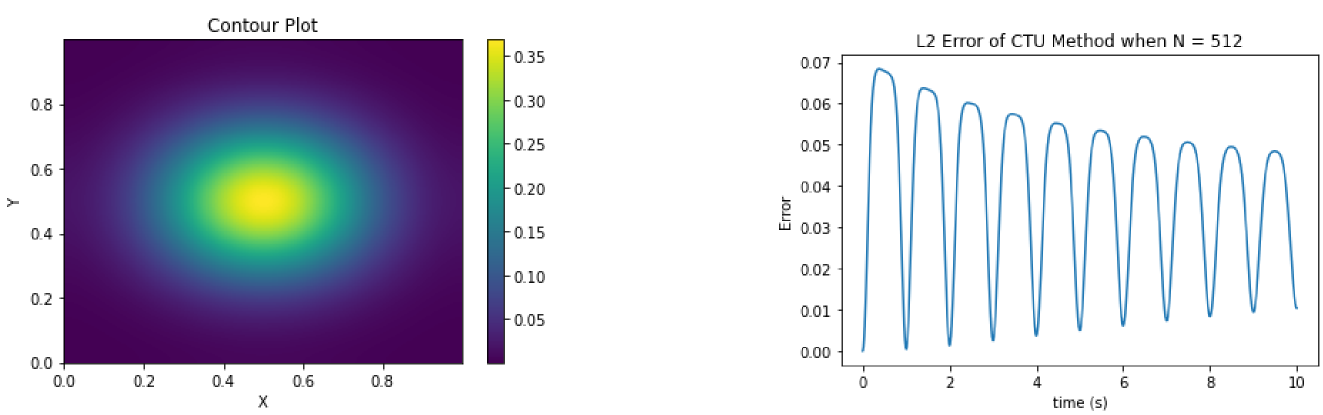


Figure 5: CTU Solution to the advection equation when $N = 512$

For the Modified LW method, the results and errors for the $N = 128, 256, \& 512$ cases are shown in the figures below:

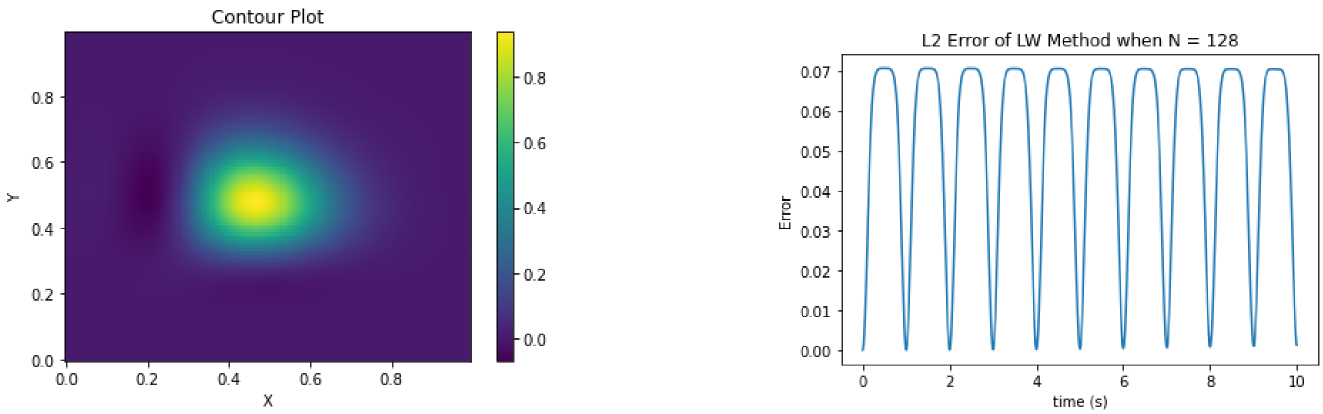


Figure 6: Modified LW Solution to the advection equation when $N = 128$

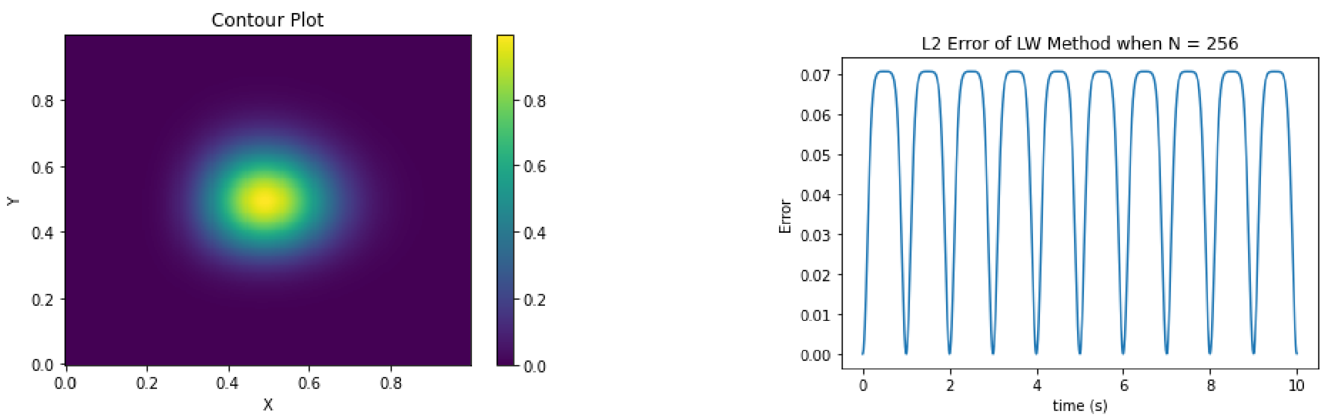


Figure 7: Modified LW Solution to the advection equation when $N = 256$

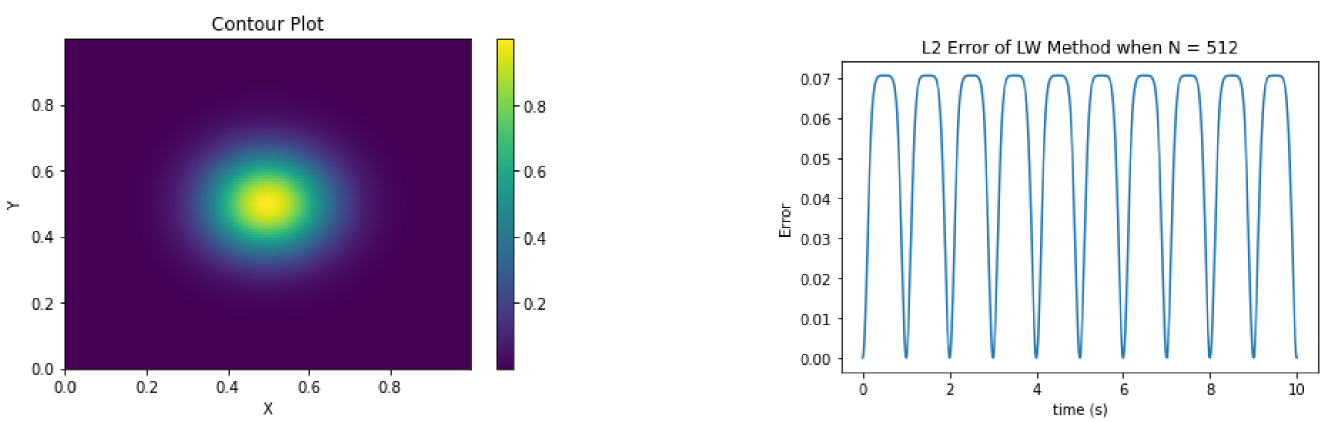


Figure 8: Modified LW Solution to the advection equation when $N = 512$

Note that while the python script `p3c.py` is included and can be run in the command line for all the cases of domain size N, the cases where $N = 256$ and $N = 512$ take several hours to run, and I was unable to make the script more efficient. All the resulting plots are included in the `p3c plots` folder.