Algorithms: Greedy Method

Minimum Spanning Tree

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Greedy Algorithms: Principles

- A greedy algorithm works in phases.
- At each phase:
  - You take the best you can get right now, without regard for future consequences.
  - You hope that by choosing a local optimum at each step, you will end up at a global optimum.

Greedy Algorithms: Principles

- Suppose you want to count out a certain amount of money, using the fewest possible notes/ coins.
- At each step, take the largest possible note/ coin that does not overshoot.

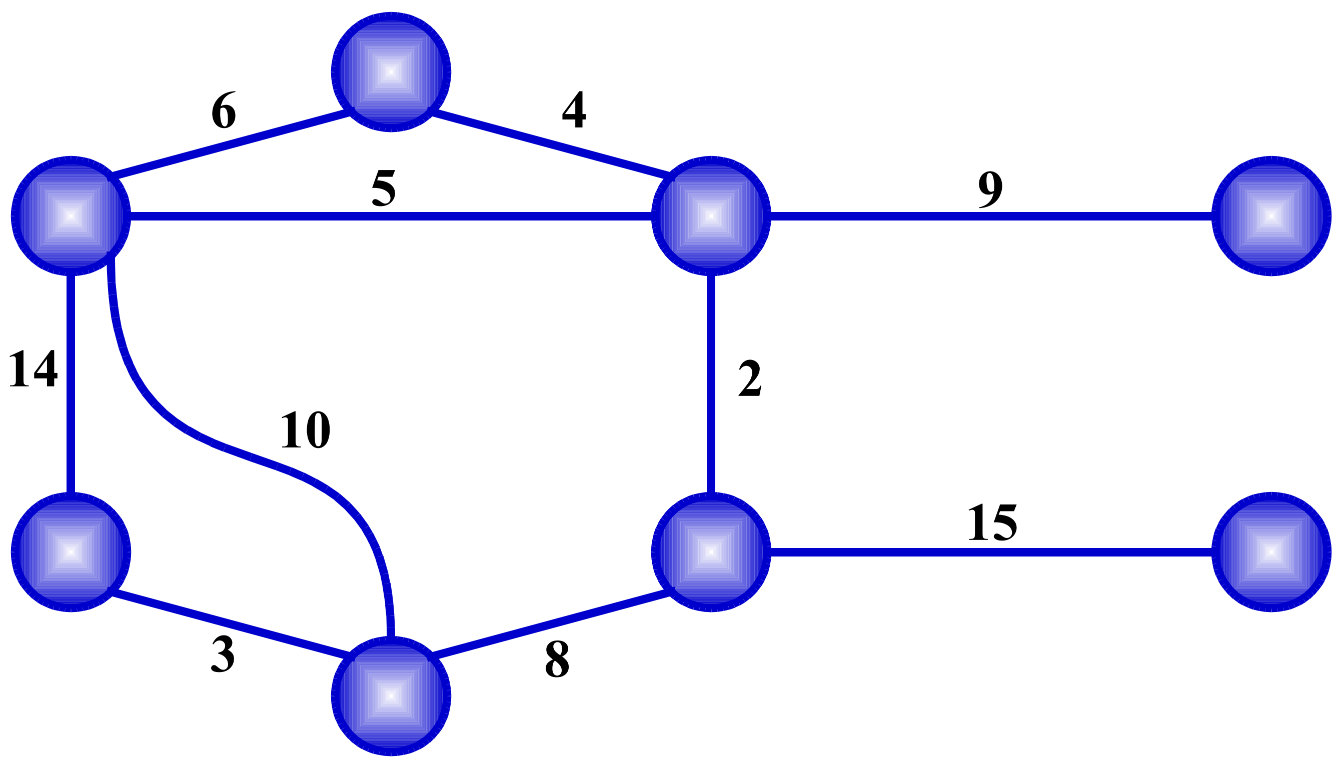
- Example: To make Tk. 177/-, you,
  - Choose a Tk. 100/- note,
  - Choose a Tk. 50/- note,
  - Choose a Tk. 20/- note,
  - Choose a Tk. 5/- coin,
  - Choose a Tk. 2/- coin.

Greedy Algorithms: Failures

- In some (fictional) monetary system, “Tonkas” come in 1 Tonkas, 7 Tonkas, and 10 Tonkas notes.
- Using a greedy algorithm to count out 15 Tonkas.
- You would get a 10 Tonkas piece and five 1 Tonkas pieces.
  - This requires six coins.
- A better solution would be to use two 7 Tonkas pieces and one 1 Tonkas piece.
  - This only requires three coins.
- The greedy algorithm results in a solution, but not in an optimal solution

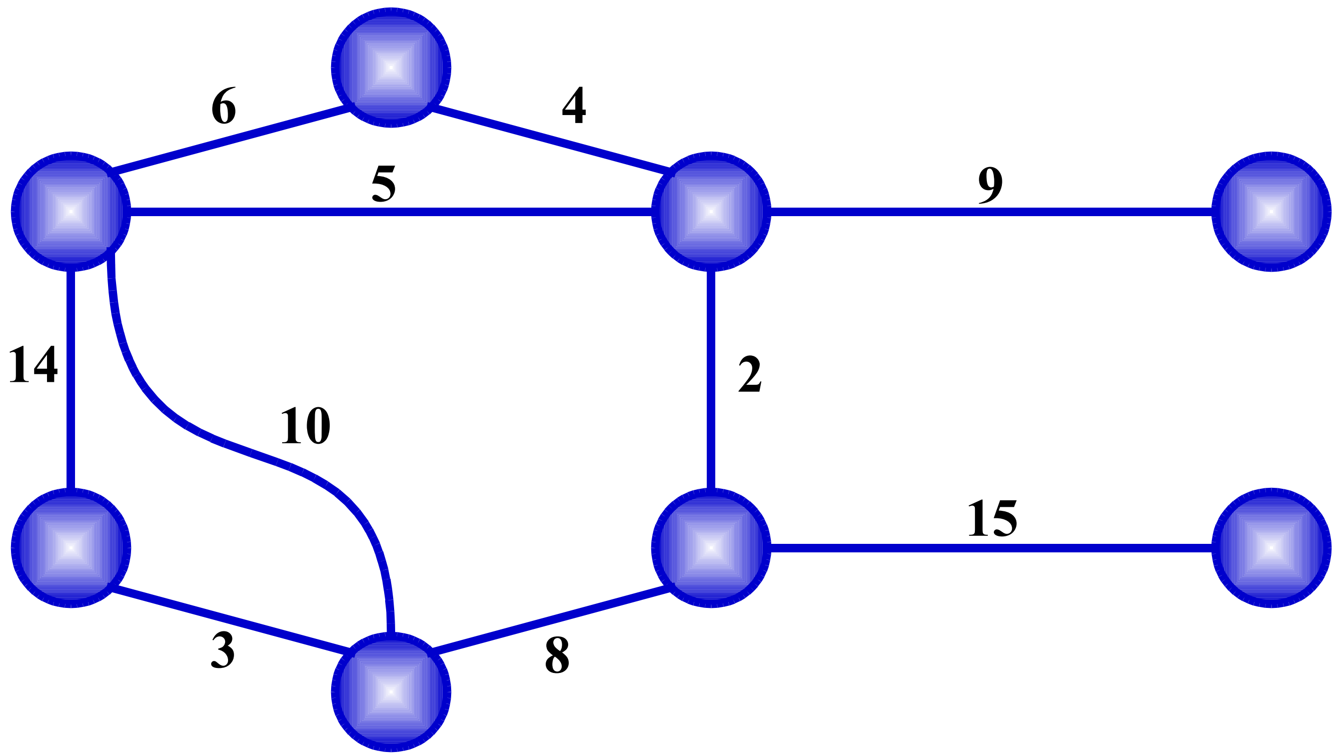
Minimum Spanning Tree

- Problem: given a connected, undirected, weighted graph:



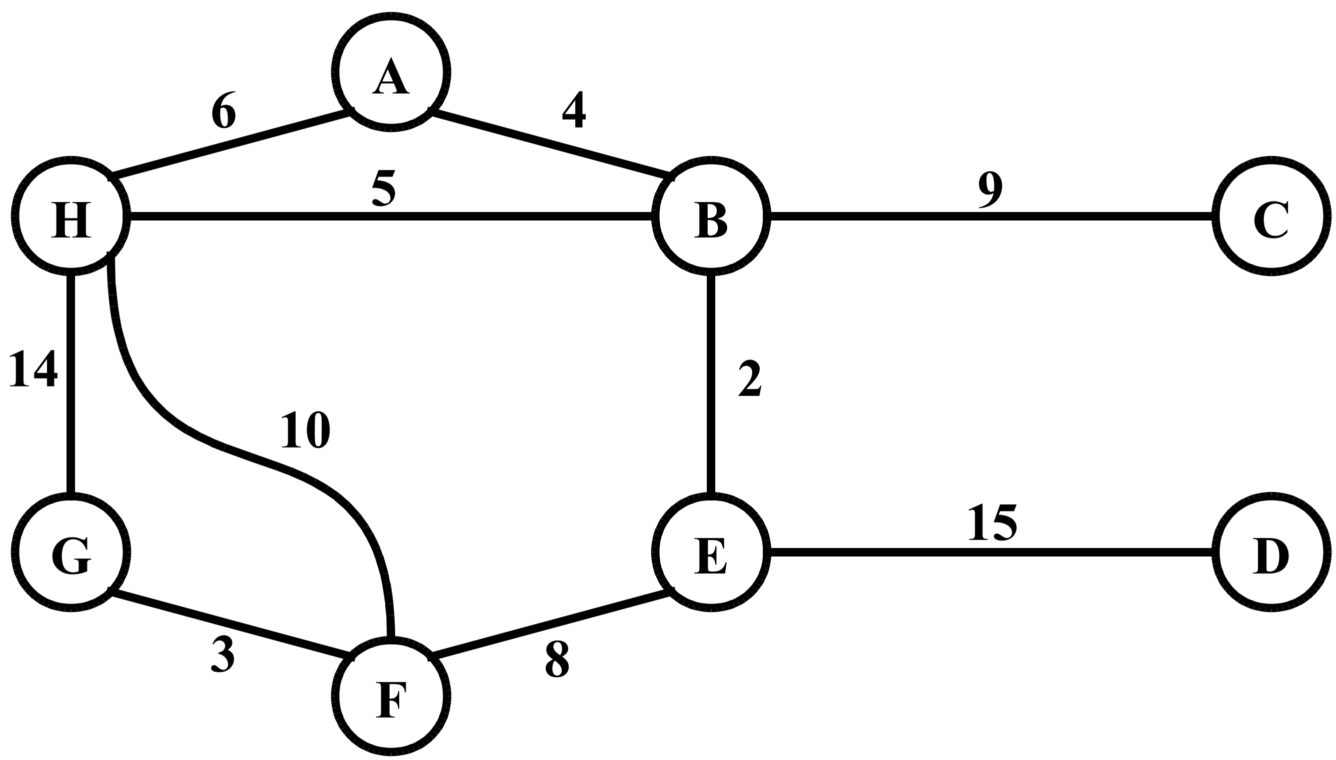
Minimum Spanning Tree

- Problem: given a connected, undirected, weighted graph, find a *spanning tree* using edges that minimize the total weight



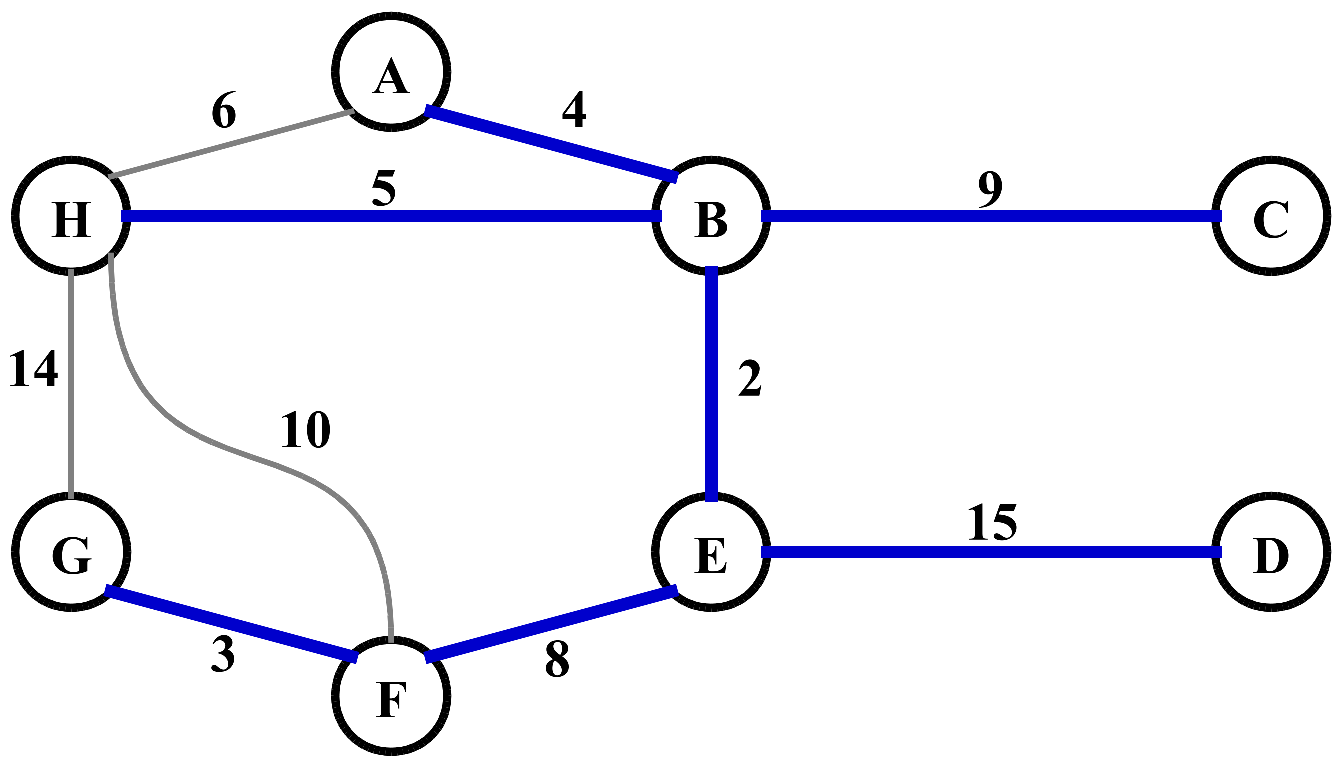
Minimum Spanning Tree

- *Which edges form the minimum spanning tree (MST) of the graph as shown below?*



Minimum Spanning Tree

- Answer:

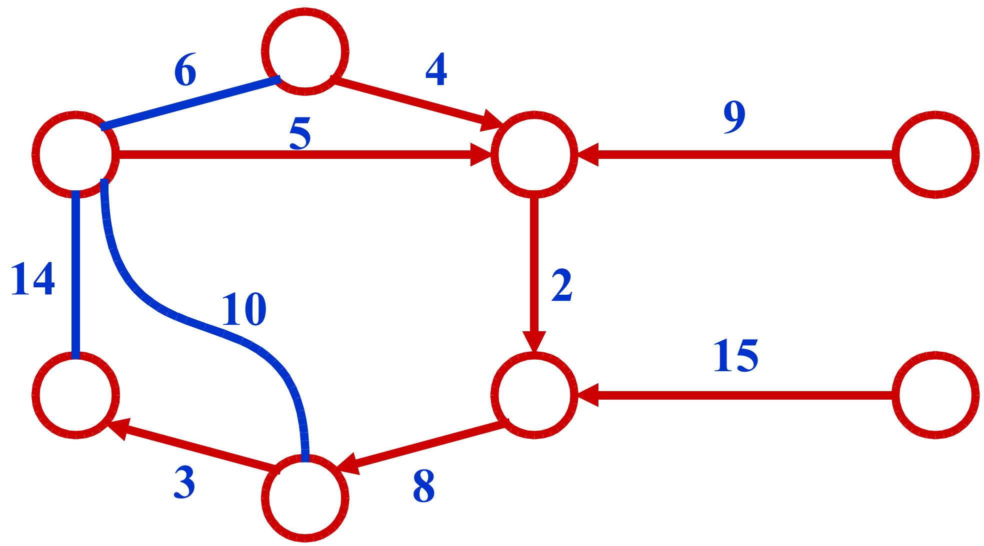


Minimum Spanning Tree

- MSTs satisfy the *optimal substructure* property: an optimal minimum spanning tree is composed of optimal minimum spanning subtrees
  - Let T be an MST of G with an edge (u, v) in the middle
  - Removing (u, v) partitions T into two trees T_1 and T_2
  - Claim: T_1 is an MST of $G_1 = (V_1,E_1)$, and T_2 is an MST of $G_2 = (V_2,E_2)$

    (*Do V_1 and V_2 share vertices? Why?*)
  - Proof: $w(T) = w(u,v) + w(T_1) + w(T_2)$

    (There can't be a better tree than T_1 or T_2. Then T would be suboptimal)



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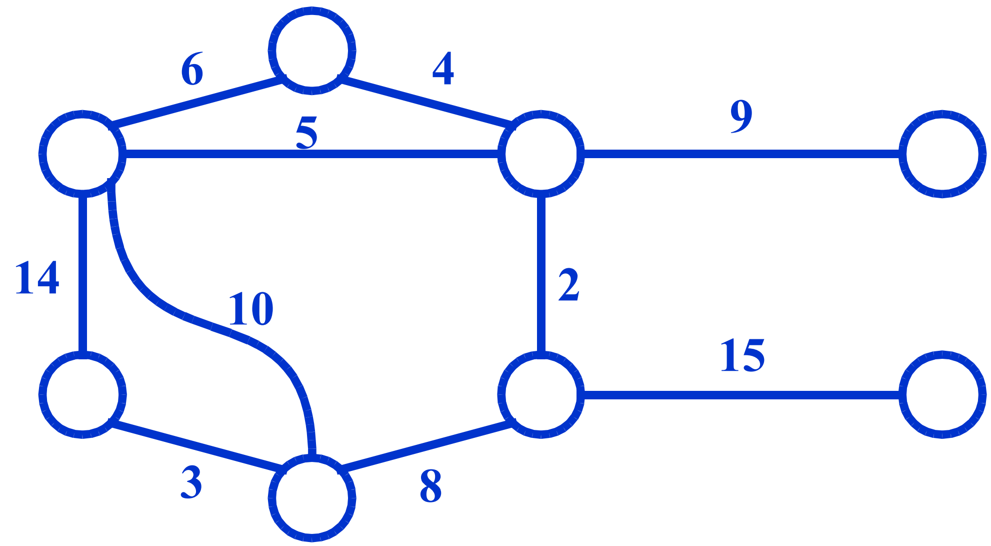

Run on example graph

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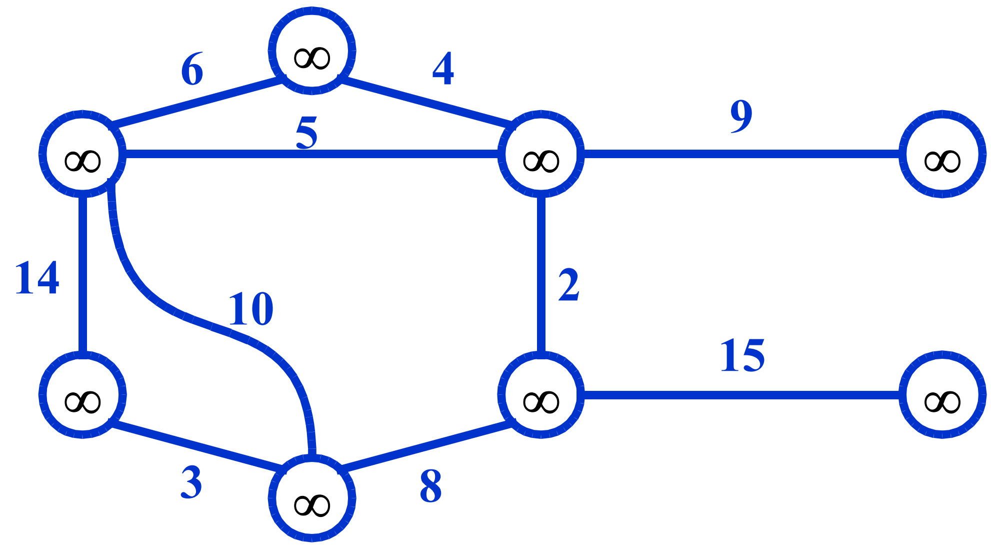

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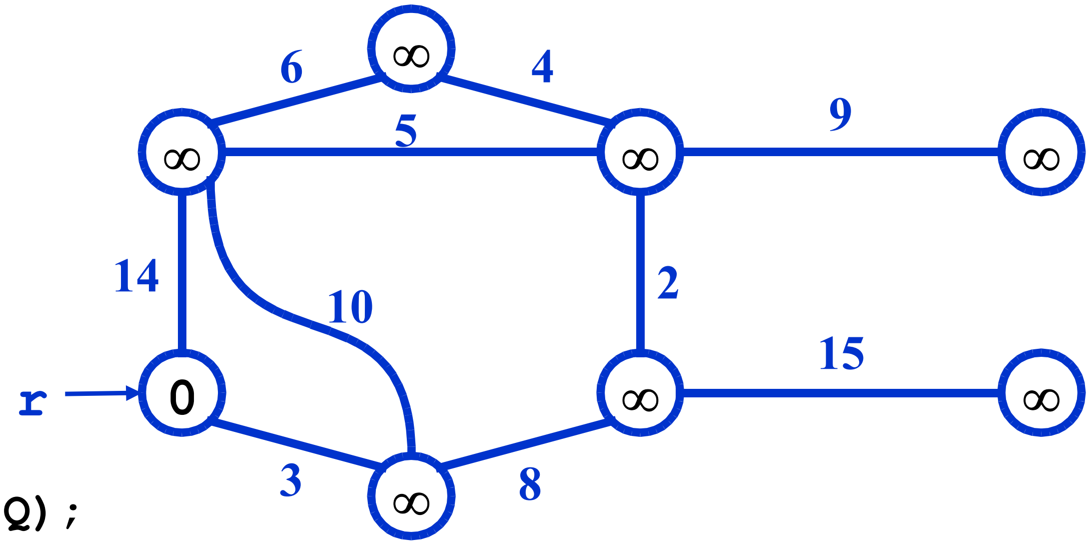

Pick a start vertex r

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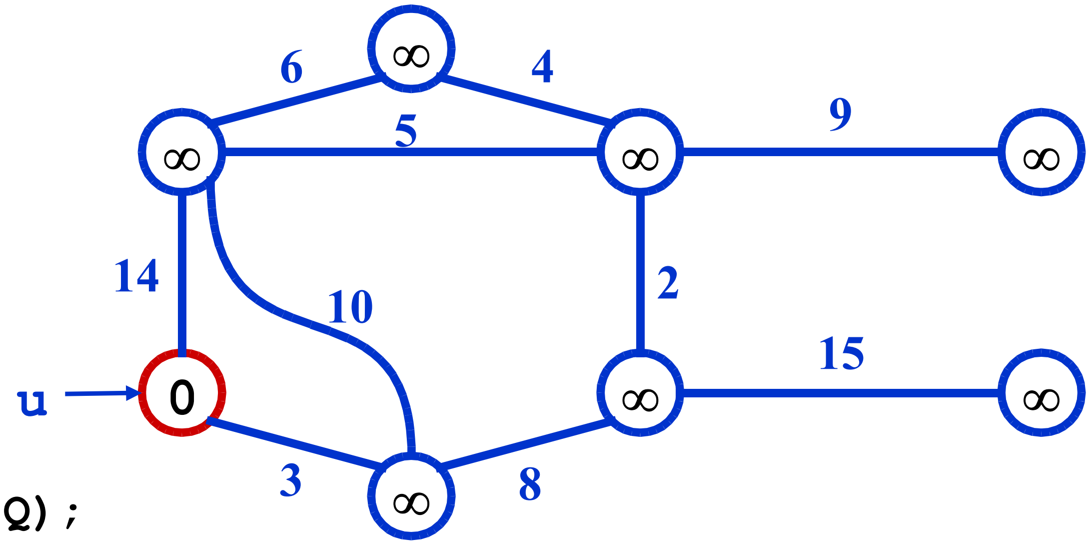

Red vertices have been removed from Q

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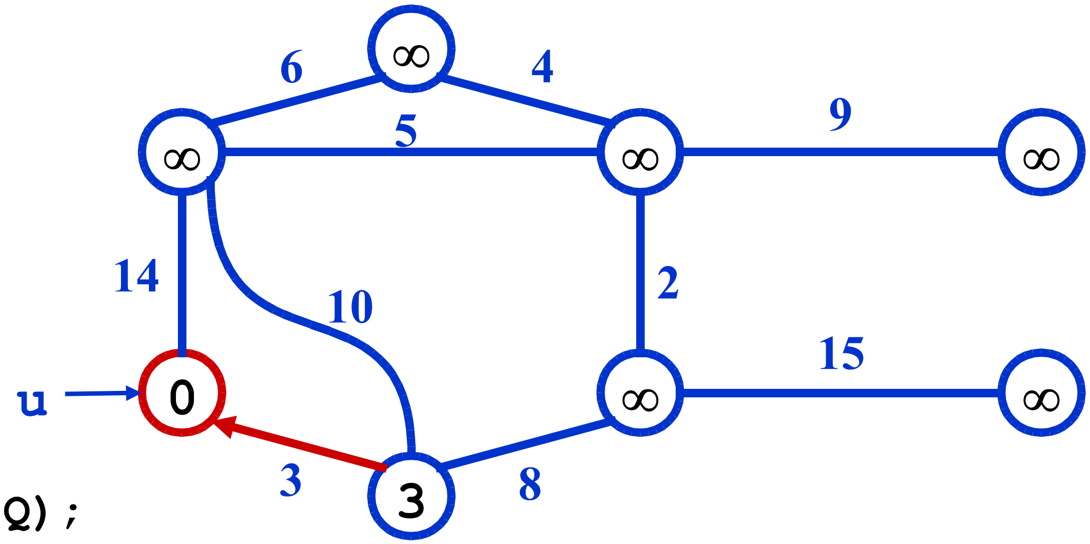

Red arrows indicate parent pointers

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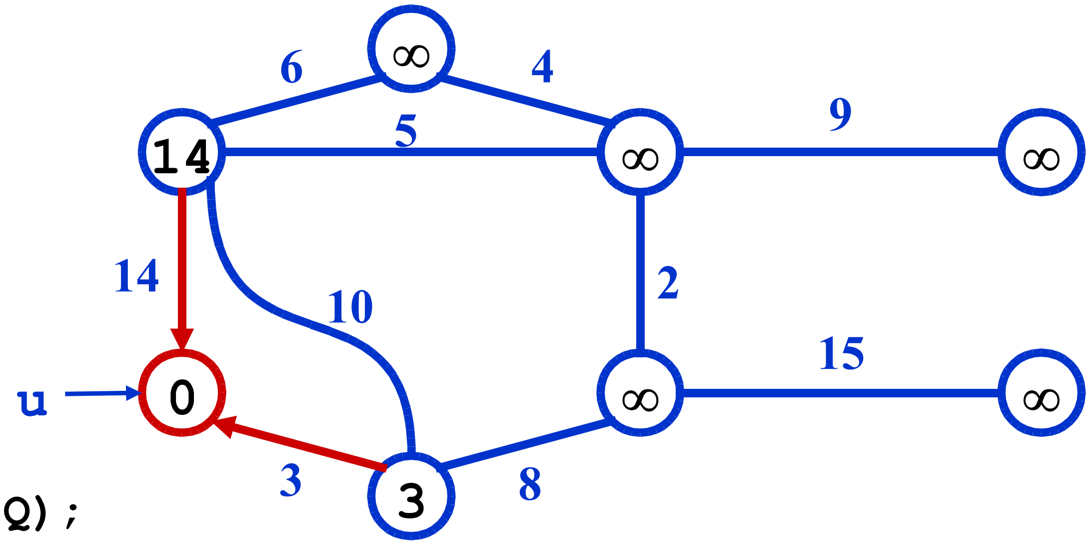

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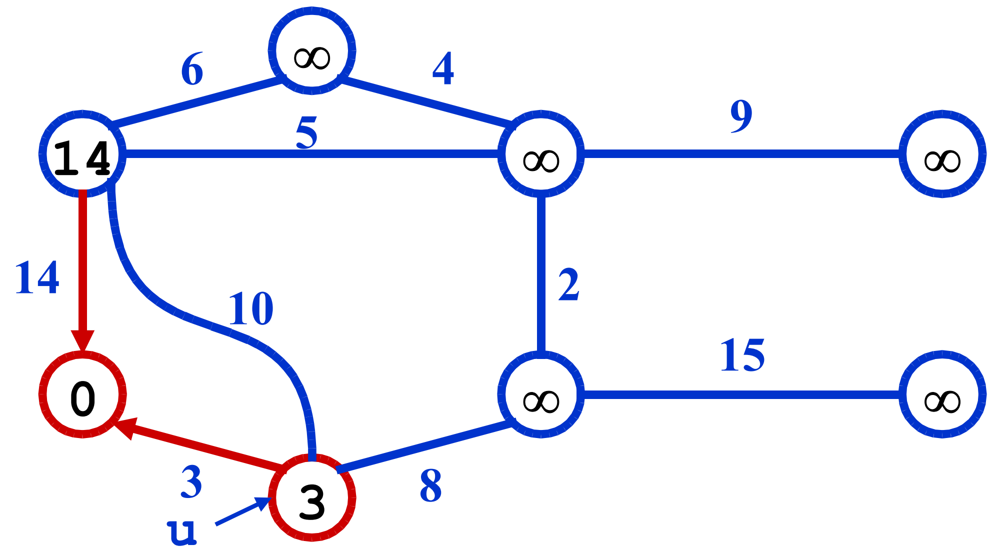

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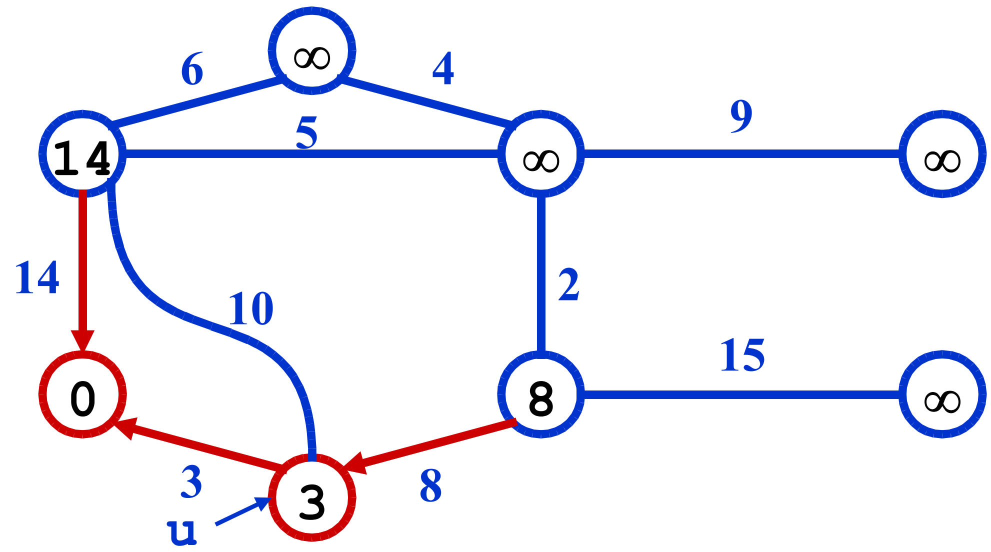

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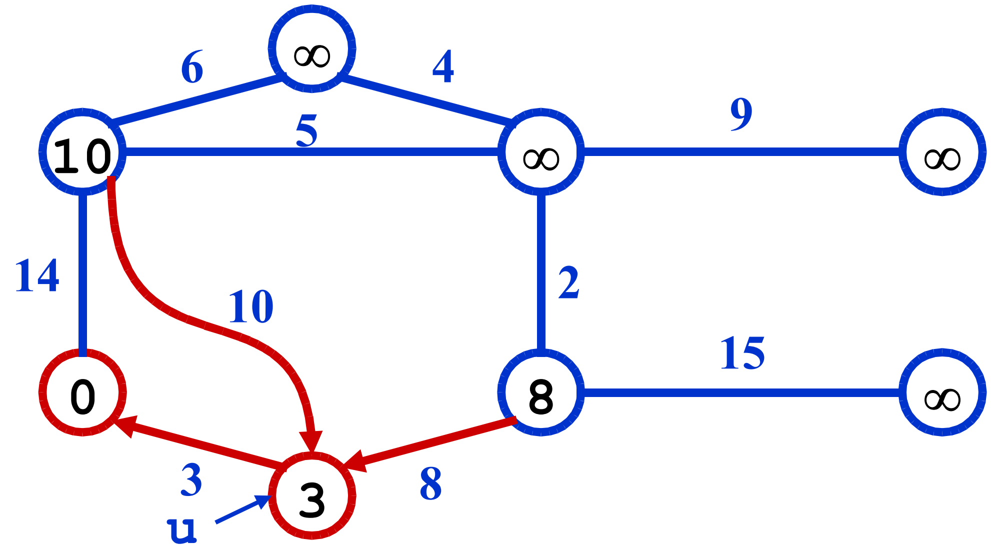

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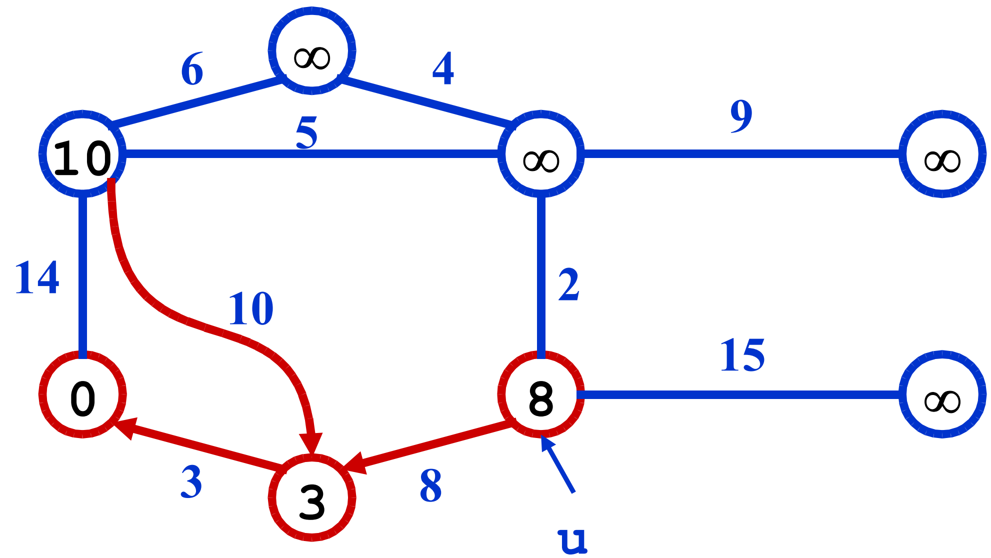

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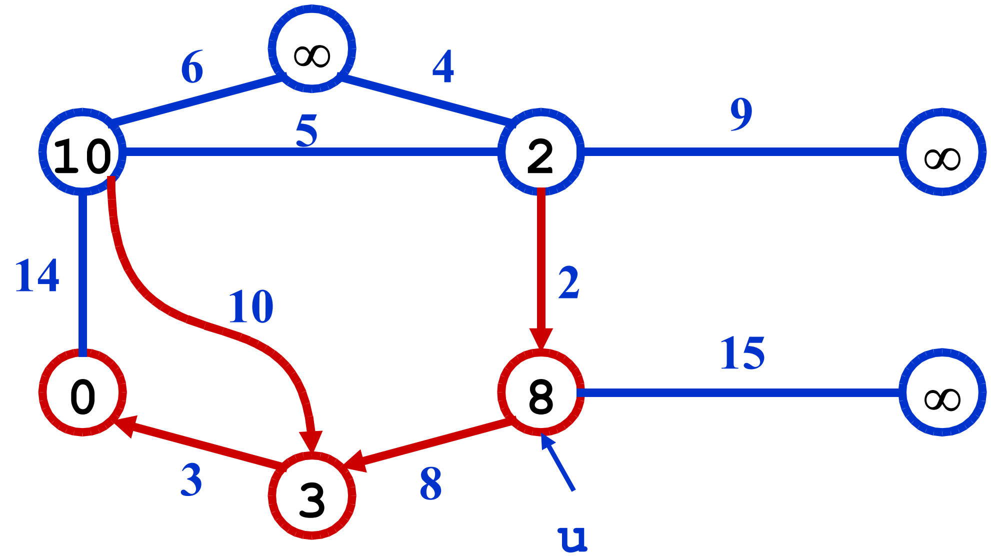

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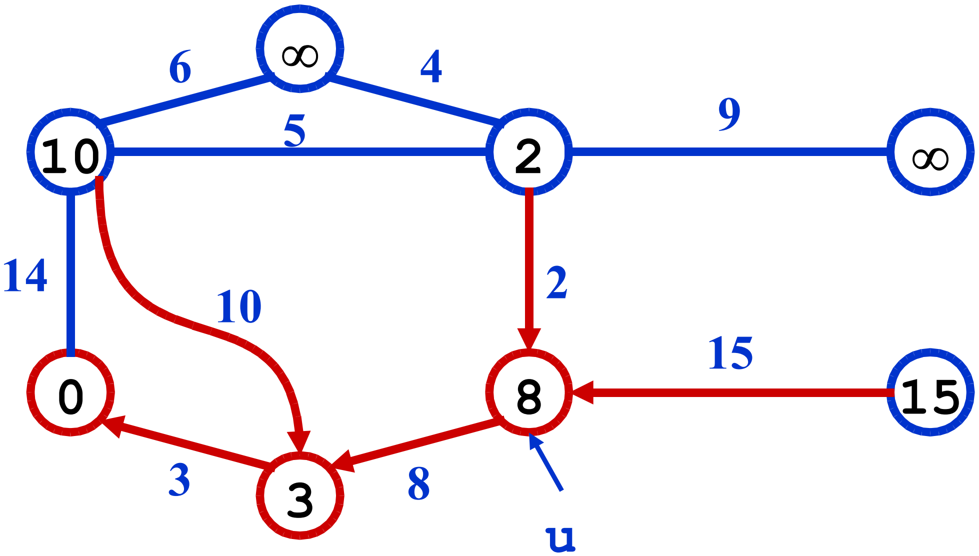

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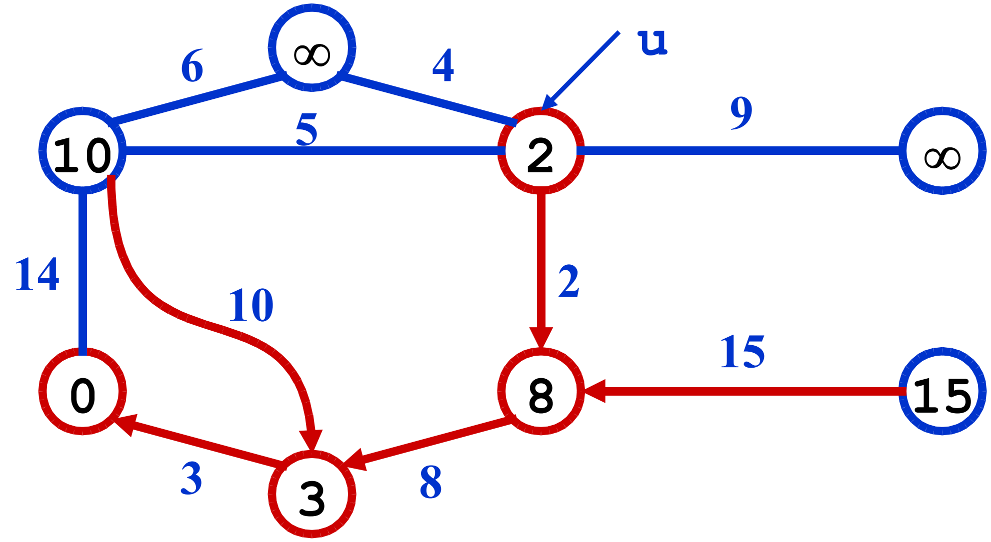

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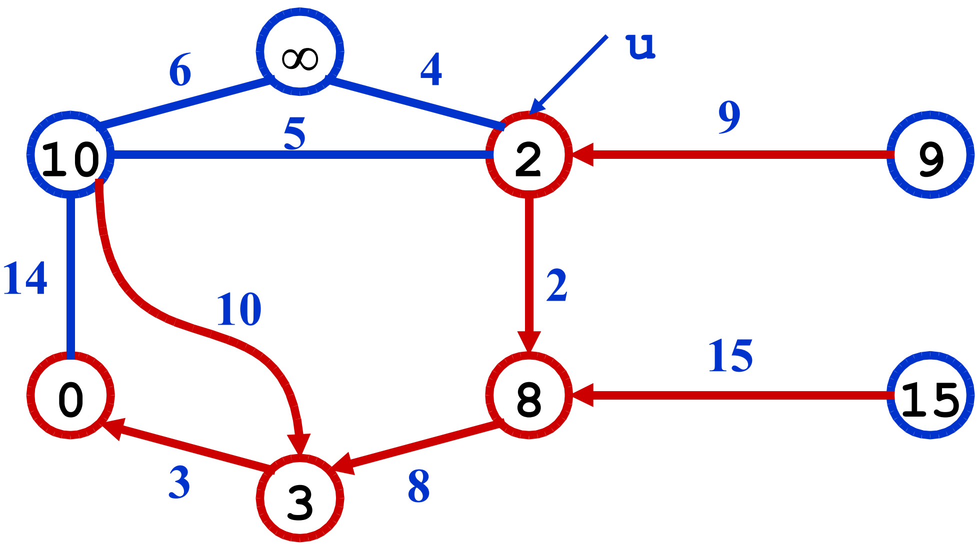

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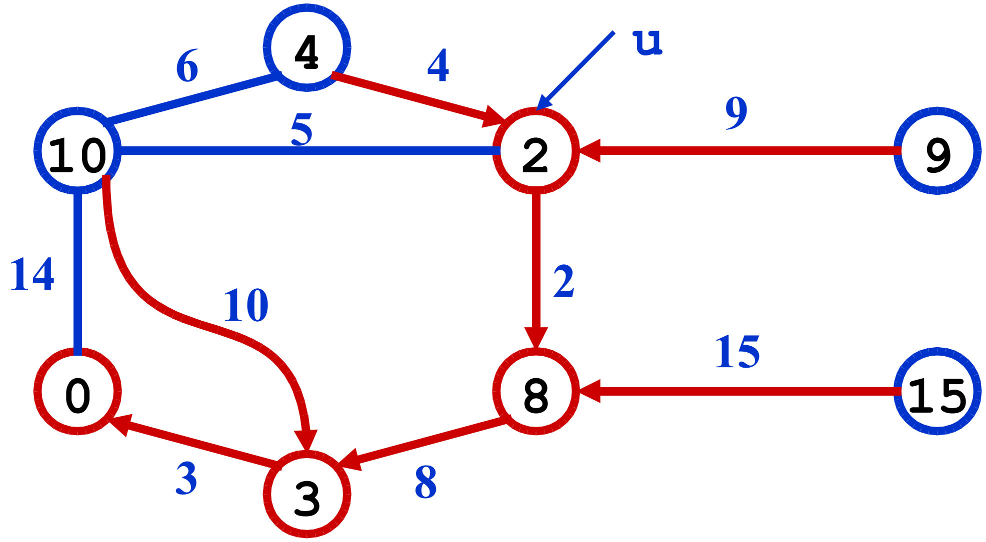

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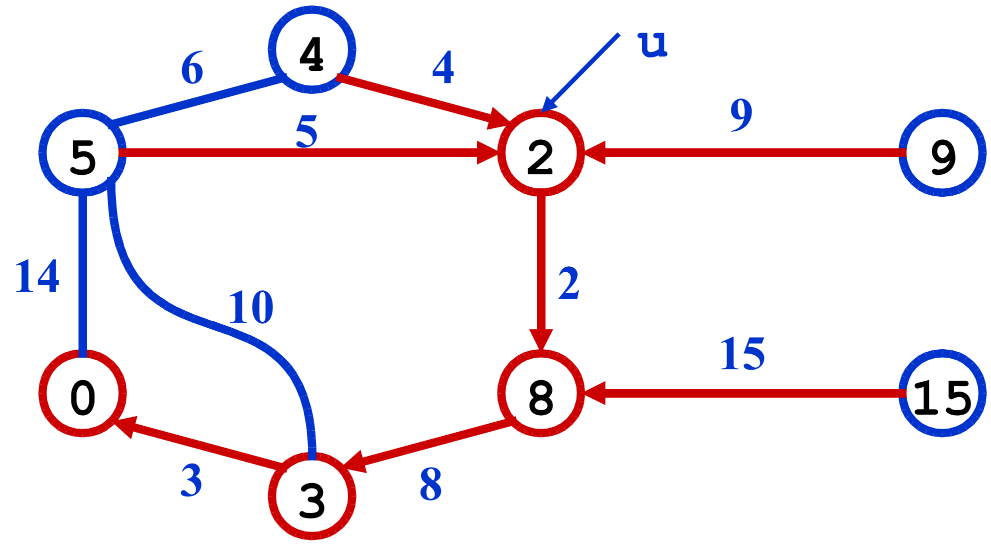

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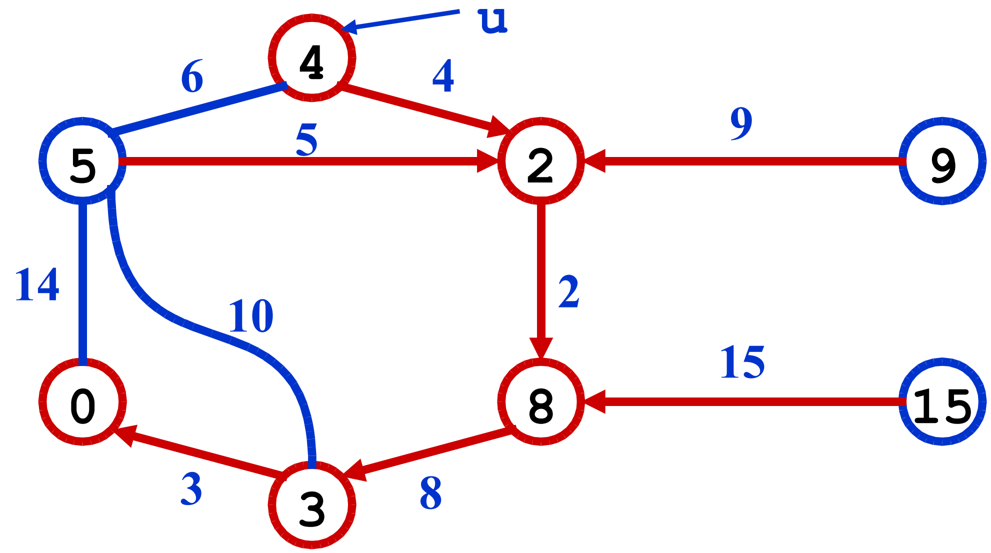

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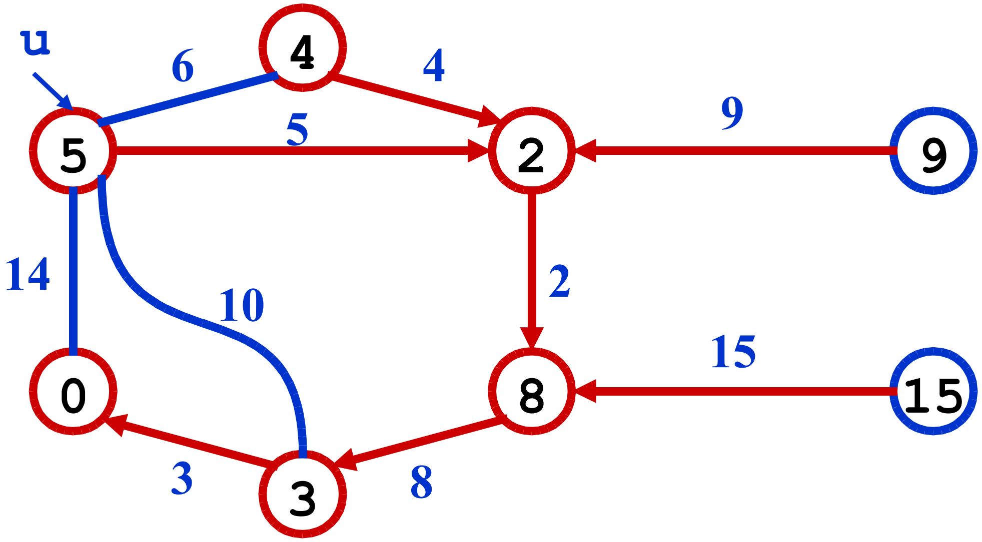

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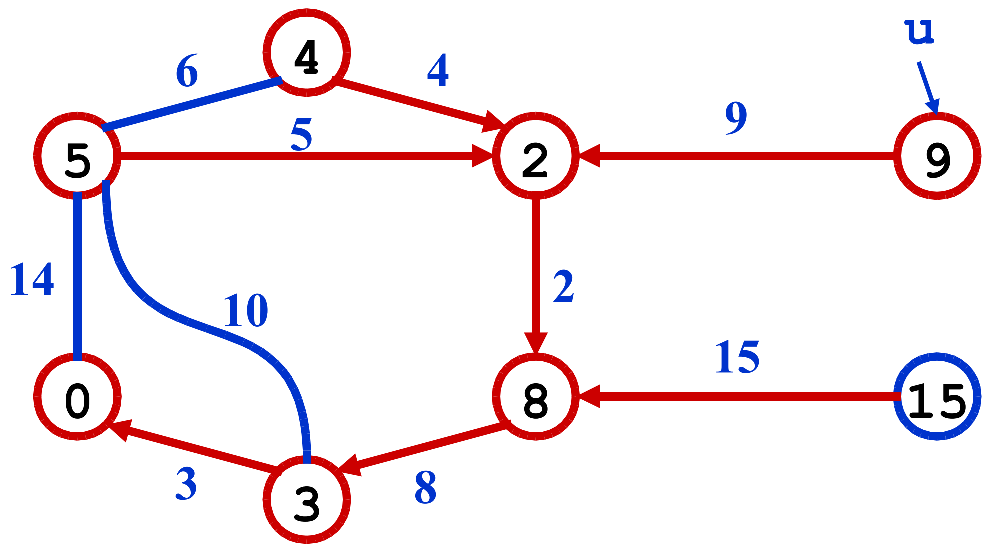

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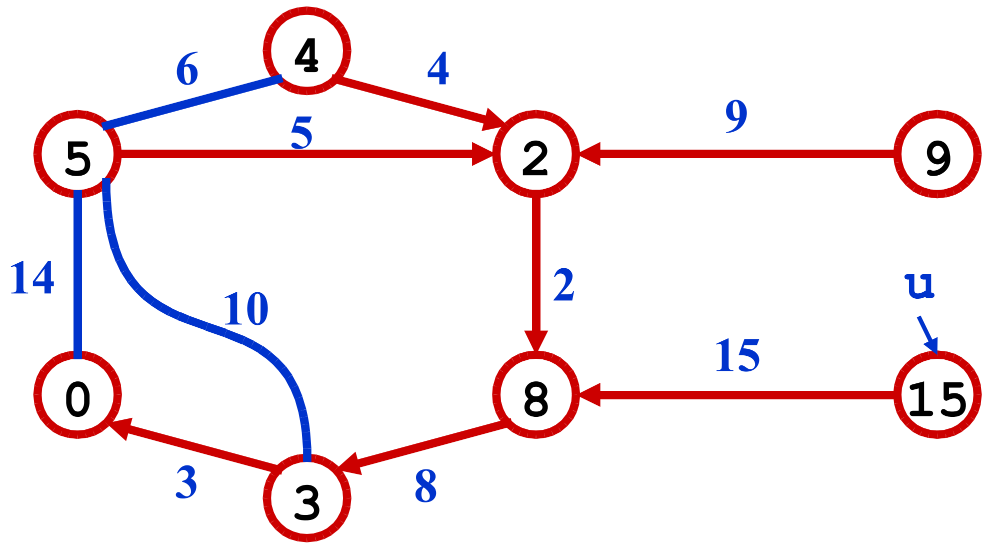

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What is the hidden cost in this code?

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How often is ExtractMin() called?

How often is DecreaseKey() called?

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What will be the running time?

A: Depends on queue

binary heap: O(E lg V)

Fibonacci heap: O(V lg V + E)

Disjoint-Set Union Problem

- Want a data structure to support disjoint sets
  - Collection of disjoint sets $\mathcal{S} = \bigcup_i \{S_i\}$, $S_i \cap S_j = \varnothing$
- Need to support following operations:
  - MakeSet(x): $\mathcal{S} = \mathcal{S} \cup \{\{x\}\}$
  - Union(S_i, S_j): $\mathcal{S} = \mathcal{S} - \{S_i, S_j\} \cup \{S_i \cup S_j\}$
  - FindSet(x): return $S_i \in \mathcal{S}$ such that $x \in S_i$
- Before discussing implementation details, we look at example application: MSTs

Kruskal's Algorithm

```
Kruskal()
{
   T = ∅;
   for each v ∈ V
      MakeSet(v);
   sort E into nondecreasing order by weight w
   for each (u,v) ∈ E (in sorted order)
      if FindSet(u) ≠ FindSet(v)
         T = T U {{u,v}};
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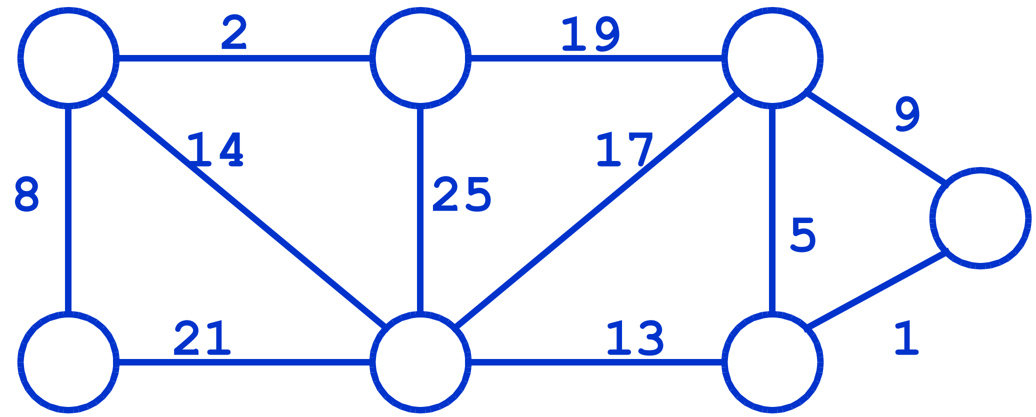
Kruskal's Algorithm

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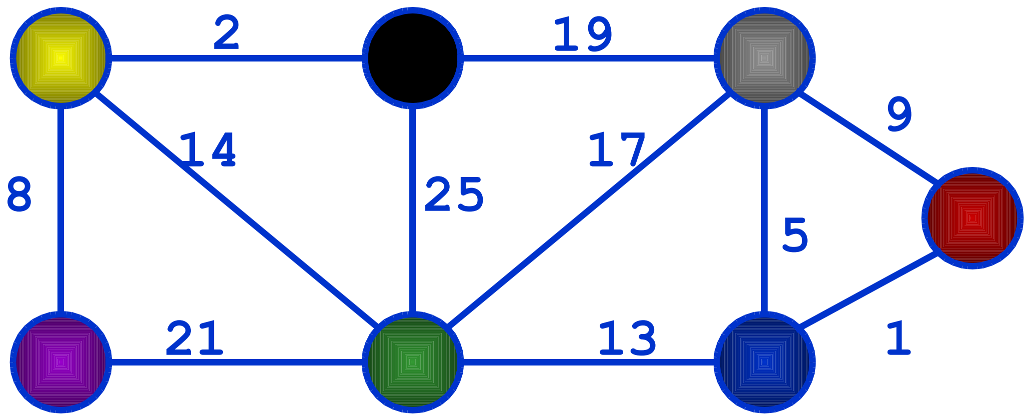


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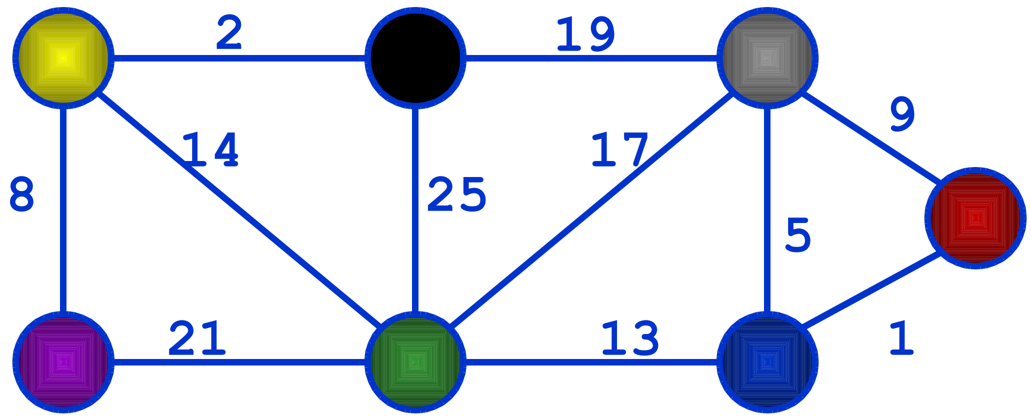


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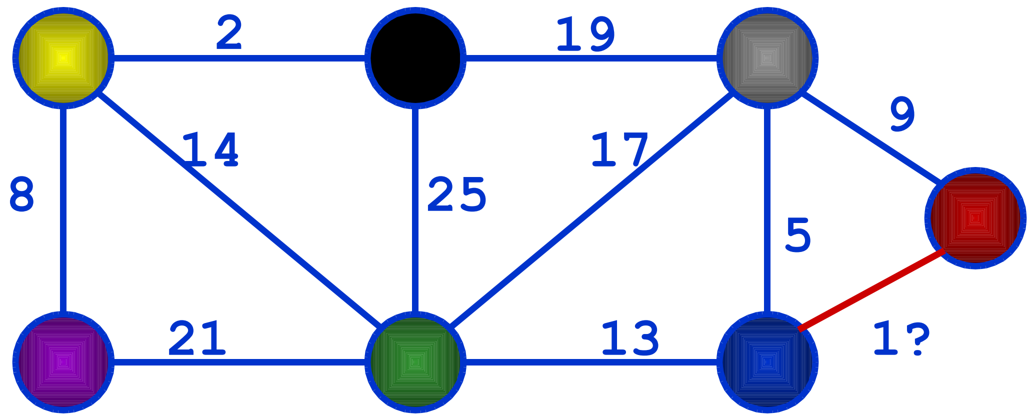
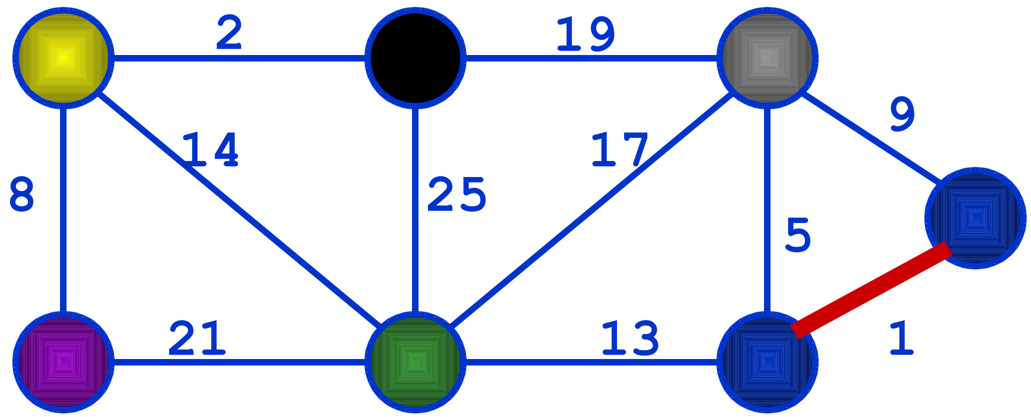
Kruskal's Algorithm




```
Kruskal()
{
    T = ∅;
    for each v ∈ V
        MakeSet(v);
    sort E into nondecreasing order by weight w
    for each (u,v) ∈ E (in sorted order)
        if FindSet(u) ≠ FindSet(v)
            T = T U {{u,v}};
            Union(FindSet(u), FindSet(v));
}
```

Kruskal's Algorithm

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Kruskal()
{
   T = ∅;
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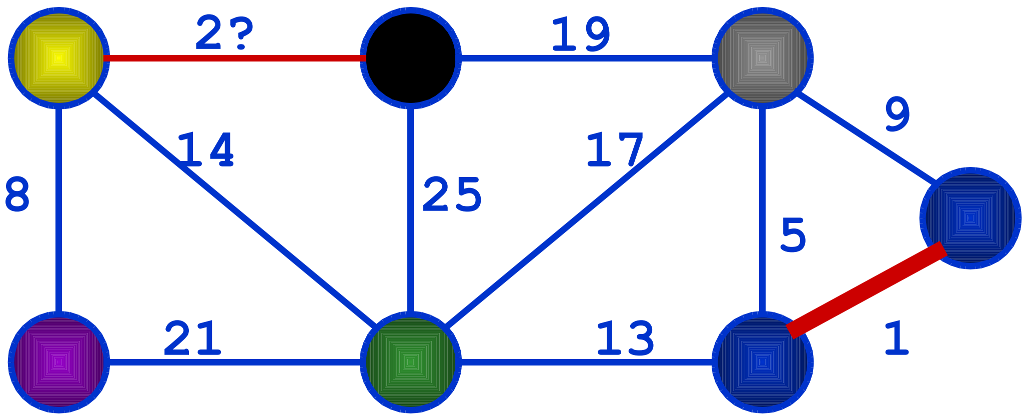
Kruskal's Algorithm

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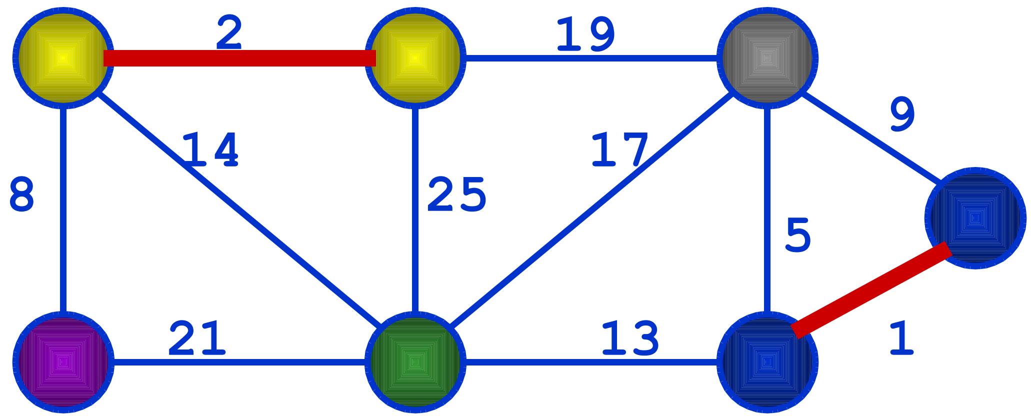
Kruskal's Algorithm

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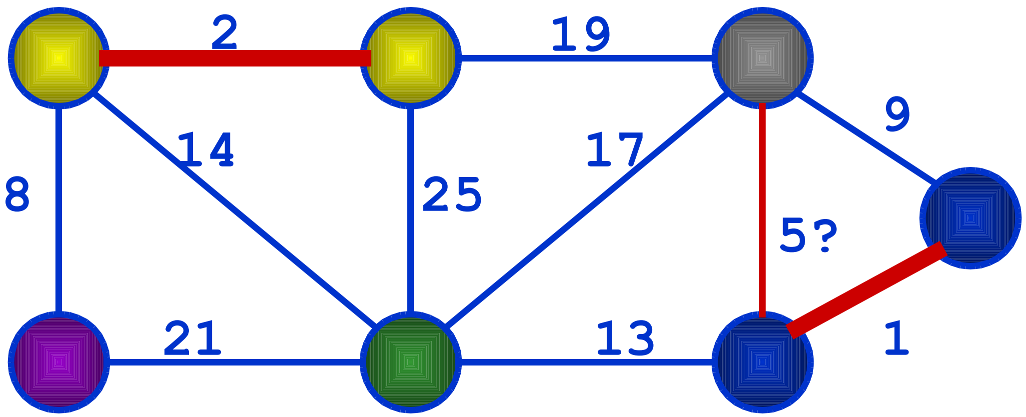
Kruskal’s Algorithm

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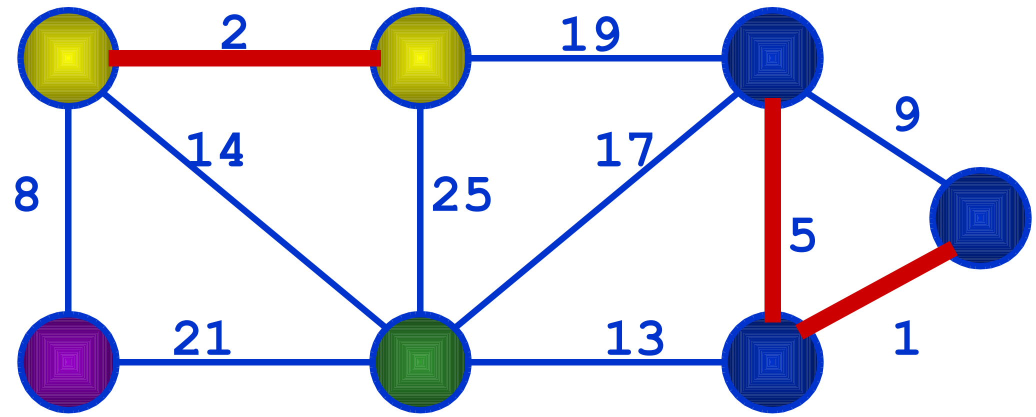
Kruskal's Algorithm

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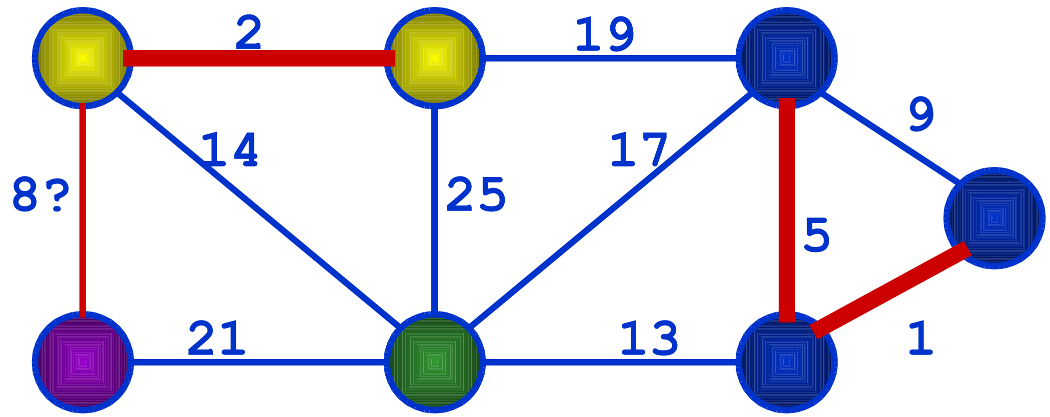
Kruskal's Algorithm

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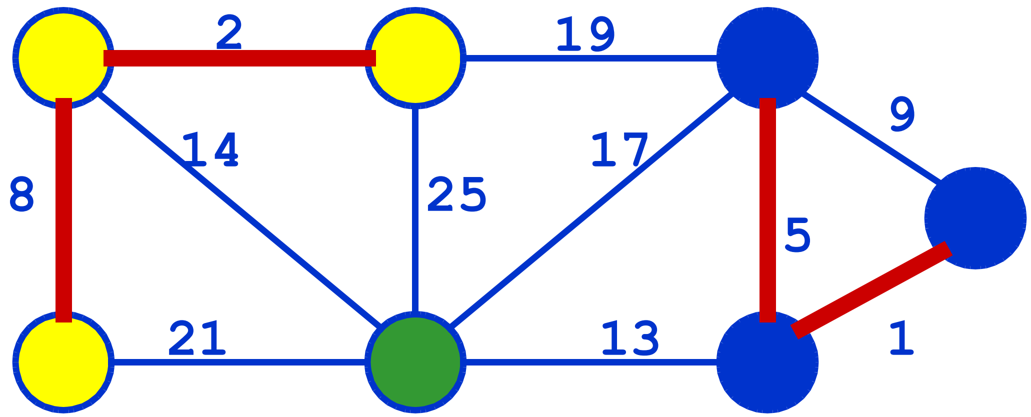
Kruskal's Algorithm

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{
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        MakeSet(v);
    sort E by increasing edge weight w
    for each (u,v) ∈ E (in sorted order)
        if FindSet(u) ≠ FindSet(v)
            T = T ∪ {{u,v}};
            Union(FindSet(u), FindSet(v));
}
```
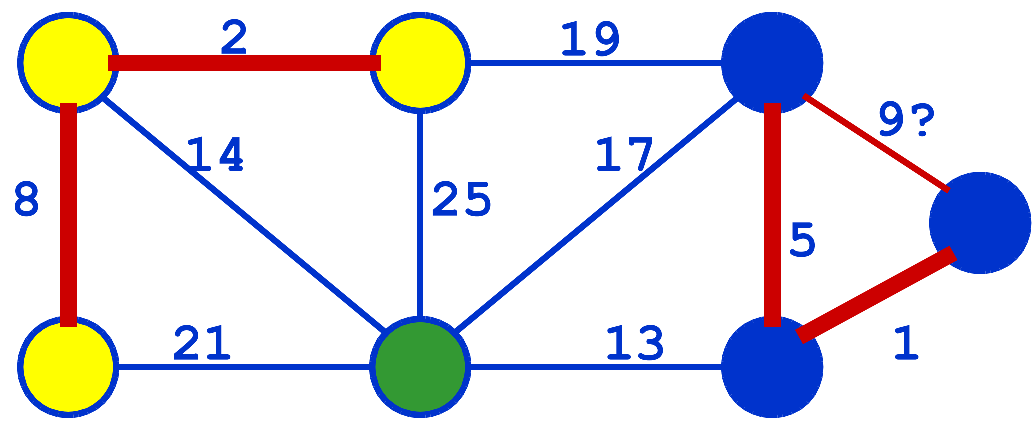

Kruskal's Algorithm

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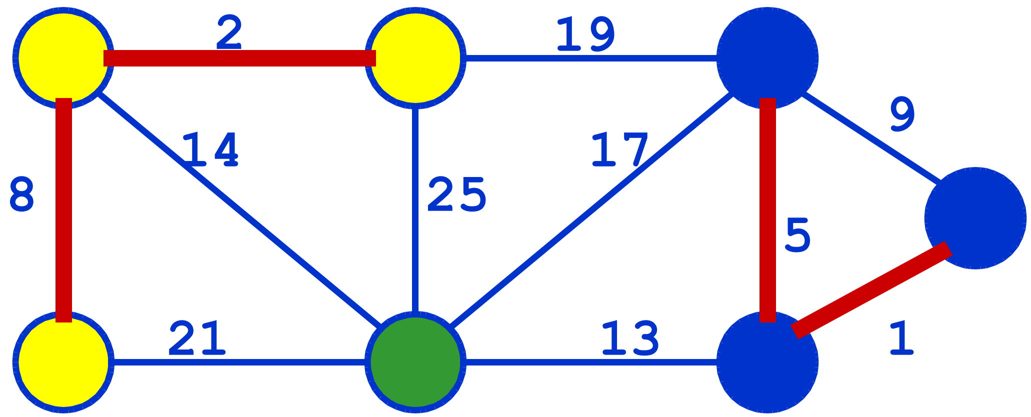


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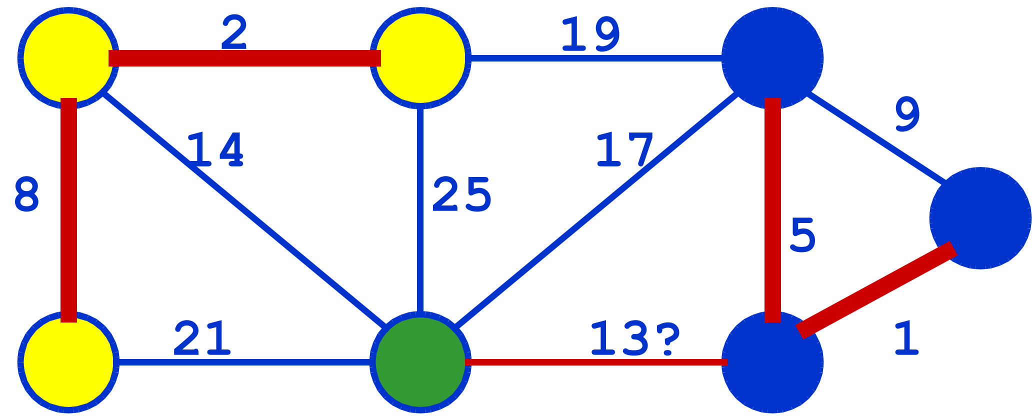
Kruskal's Algorithm

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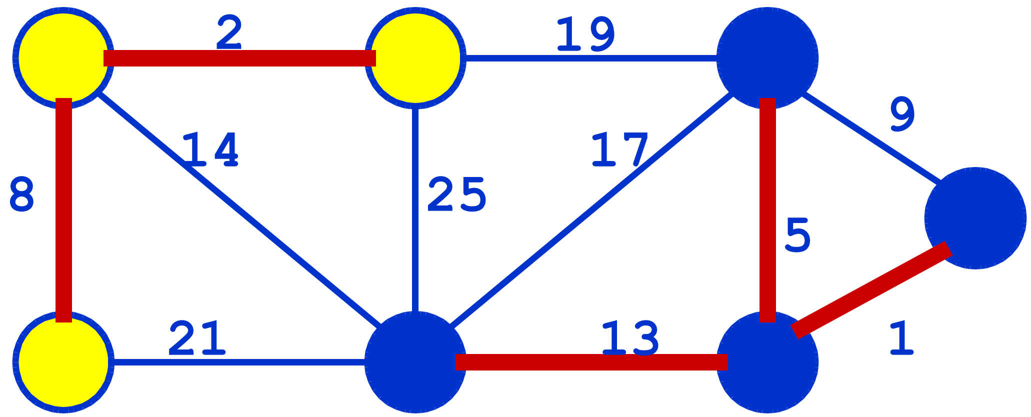


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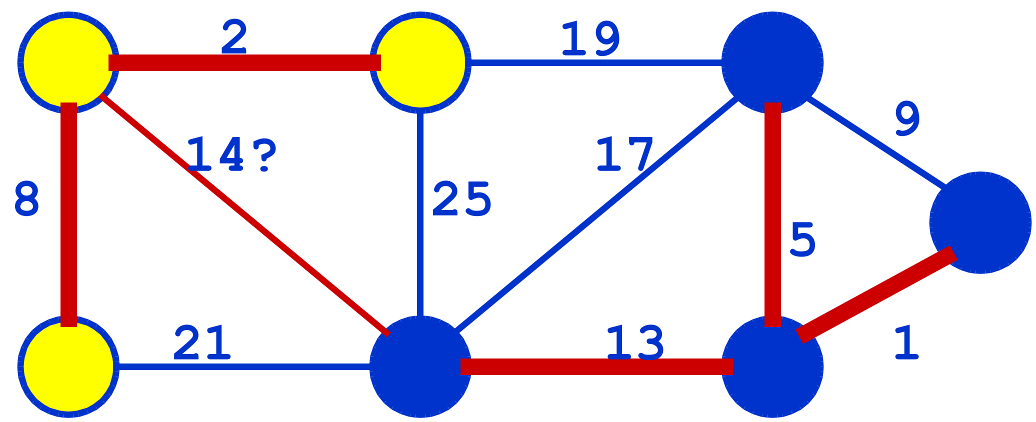


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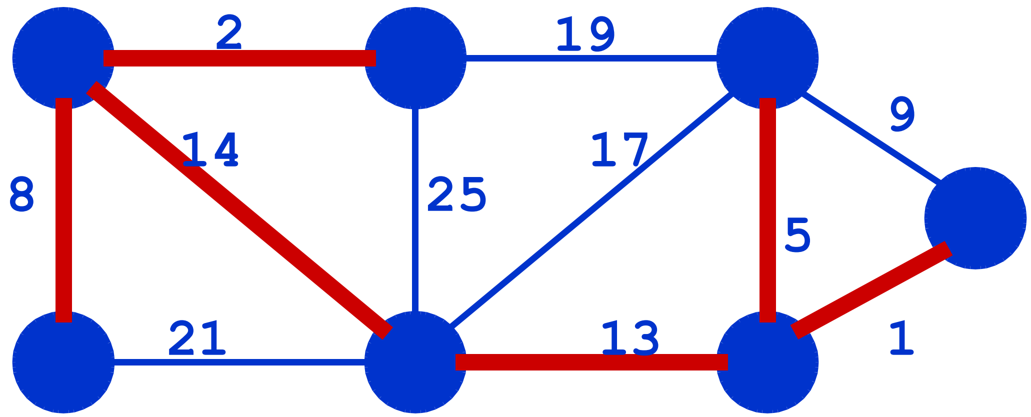


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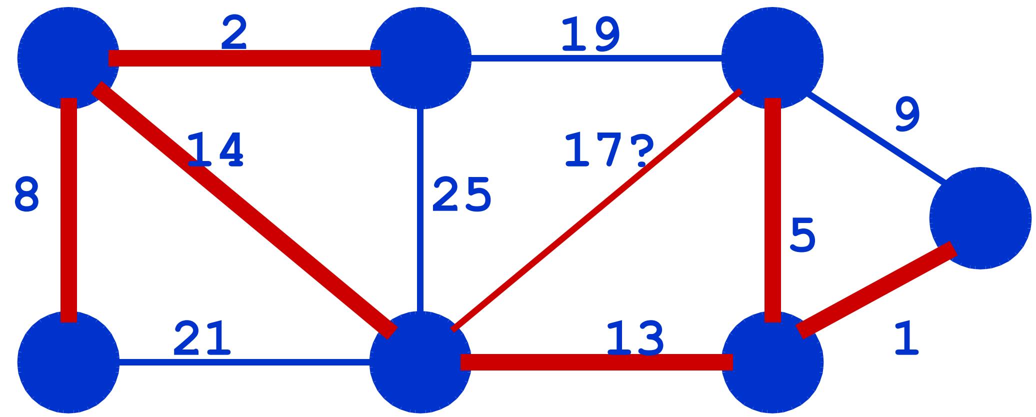
Kruskal’s Algorithm

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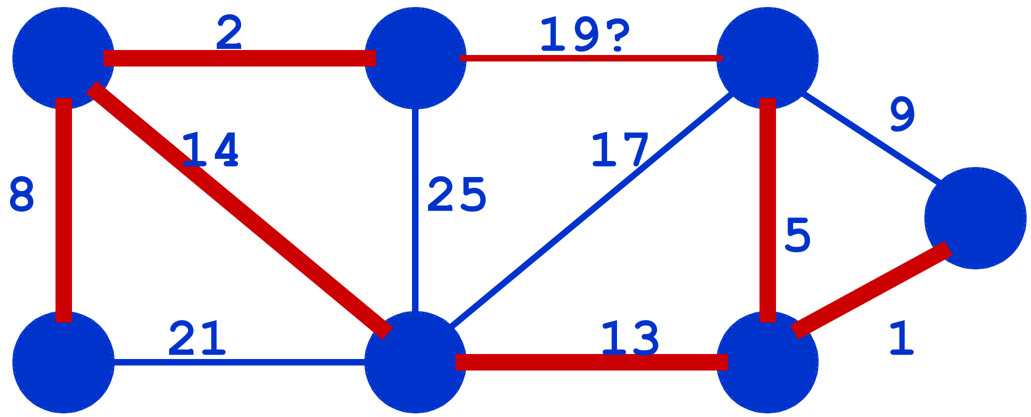


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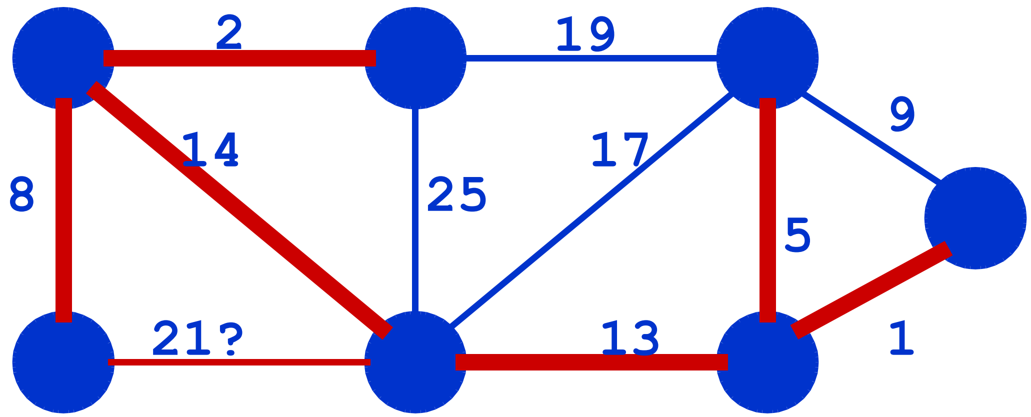


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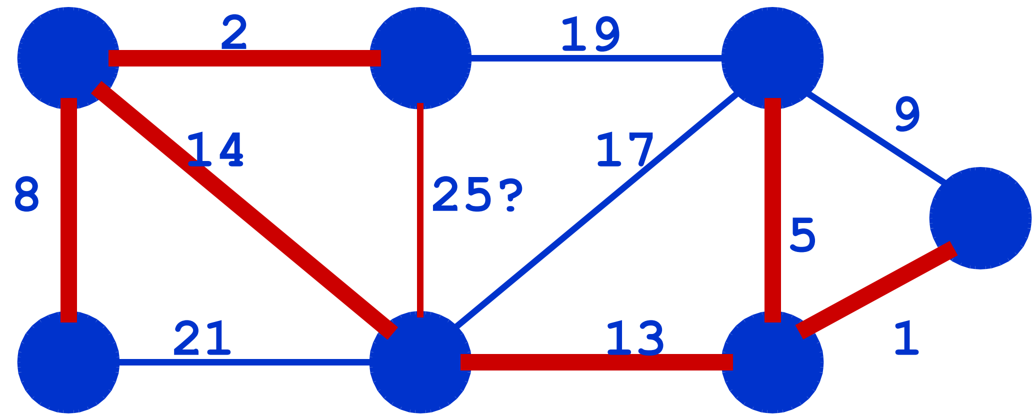


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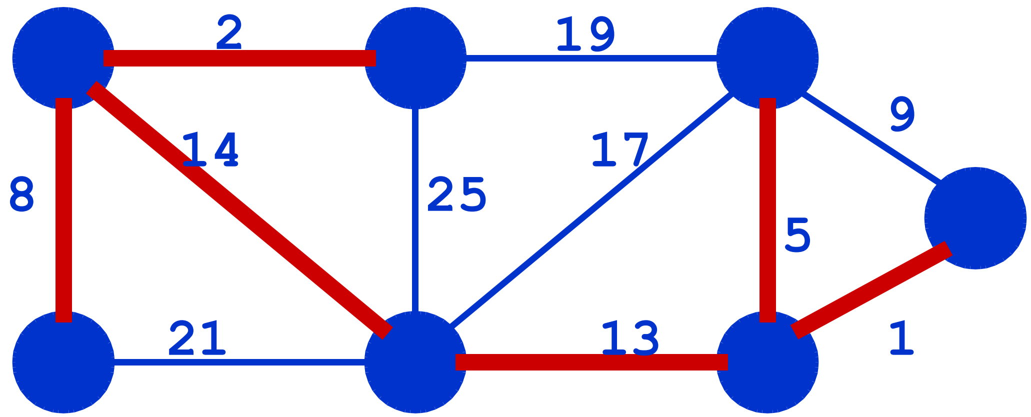


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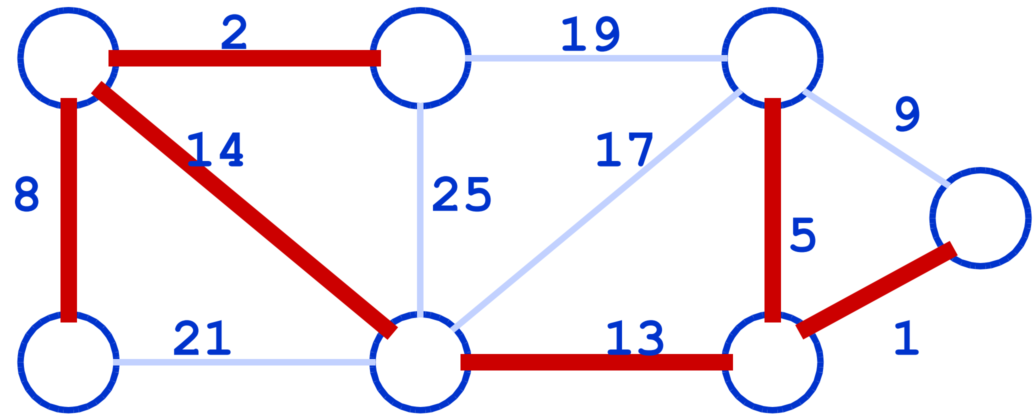


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}
```

Correctness of Kruskal's Algorithm

Theorem: In a connected weighted graph G, Kruskal's Algorithm constructs a minimum-weight spanning tree.

Proof: 1/3

- We show first that the algorithm produces a tree

  We never choose an edge that completes a cycle

  If the final graph has more than one component, then there is no edge joining two of them and G is not connected

  - Since G is connected, some such edge exists and we considered it.

  Thus the final graph is connected and acyclic, which makes it a tree.

Correctness of Kruskal's Algorithm

Proof: continue

- Let T be the resulting tree, and let T^* be a spanning tree of minimum weight.
- If $T=T^*$, we are done.
- If $T \neq T^*$, let e be the first edge chosen for T that is not in T^*. Adding e to T^* creates one cycle C. Since T has no cycle, C has an edge $e^* \notin E(T)$. Consider the spanning tree T^*+e-e^*

The first edge chosen for T that is not in T^*

$$T: \quad \ldots\ldots\ldots\ldots\; e,\; \ldots\ldots\ldots\ldots$$

$$T^*: \underbrace{\ldots\ldots\ldots\ldots}_{\text{Identical}}\; e^*, \ldots\ldots\ldots\ldots$$

Correctness of Kruskal's Algorithm

Proof: continue

- Since T^* contains e^* and all the edges of T chosen before e, both e^* and e are available when the algorithm chooses e, and hence $w(e) \le w(e^*)$
- Thus T^*+e-e^* is a spanning tree with weight at most T^* that agrees with T for a longer initial list of edges than T^* does.

$$T: \; \ldots\ldots\ldots\ldots\ldots \; e, \; \ldots\ldots\ldots\ldots$$

$$T^*: \; \underbrace{\ldots\ldots\ldots\ldots\ldots}_{\text{Identical}} \; e^*, \ldots\ldots\ldots\ldots$$

$$T: \; \ldots\ldots\ldots\ldots\ldots\ldots \; e, \; \ldots\ldots\ldots\ldots$$

$$T^*+e-e^*: \; \underbrace{\ldots\ldots\ldots\ldots\ldots\ldots \; e}_{\text{Identical}}, \; \ldots\ldots\ldots\ldots$$

- ❑ Repeating this argument eventually yields a minimum-weight spanning tree that agrees completely with T.

Kruskal's Algorithm: Running Time

What will affect the running time?

```
Kruskal()
{
   T = ∅;
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         T = T U {{u,v}};
         Union(FindSet(u), FindSet(v));
}
```

Kruskal's Algorithm: Running Time

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}
```

What will affect the running time?

1 Sort

O(V) MakeSet() calls

O(E) FindSet() calls

O(V) Union() calls

(*Exactly how many Union()s?*)

Kruskal's Algorithm: Running Time

- To summarize:
  - Sort edges: O(E lg E)

  - O(V) MakeSet()'s
  - O(E) FindSet()'s
  - O(V) Union()'s

- Upshot:
  - Best disjoint-set operation algorithm makes above three operations to take O(E lg E) time.
  - Thus overall time is O(E lg E) = O(E lg V), since $|E| < |V|^2$