FUNDAMENTAL PROBLEMS AND ALGORITHMS

Branch and Bound

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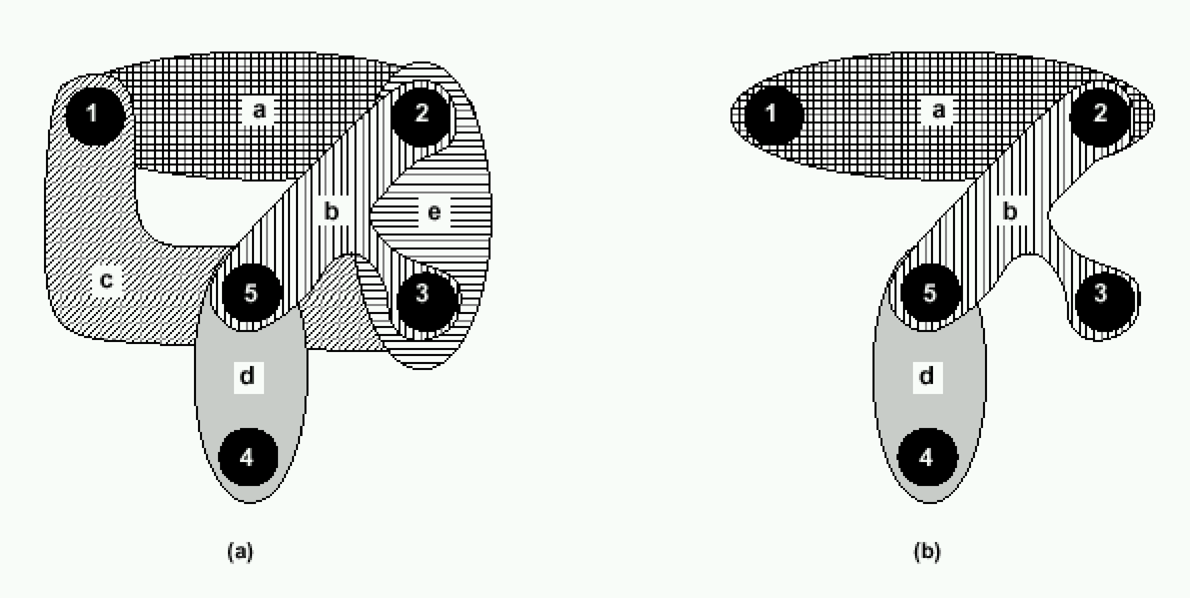
Stanford University

Branch and bound algorithm for covering
Reduction strategies

- Partitioning:
  - If **A** is block diagonal:
    - Solve covering problem for corresponding blocks.
- Essentials:
  - Column incident to one (or more) row with single 1:
    - Select column.
    - Remove covered row(s) from table.

Discuss the historic example of essential subset and function core

Example



$$
\begin{array}{c c}
 & \begin{array}{ccccc} a & b & c & d & e \end{array} \\
\mathbf{A} = & \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\end{array}
$$

I want to cover rows by columns

Explain row and column domination

Branch and bound algorithm for covering. Reduction strategies

- Column dominance:
  - If $a_{ki} \geq a_{kj}$ $\forall k$:
    - remove column j .
- Row dominance:
  - If $a_{ik} \geq a_{jk}$ $\forall k$:
    - Remove row i .

Example reduction

- Fourth column is essential.
- Fifth column is dominated.
- Fifth row is dominant.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\begin{array}{ccc} & \begin{array}{ccccc} a & b & c & d & e \end{array} & \\ \mathbf{A} = & \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \end{array}$$

Branch and bound covering algorithm

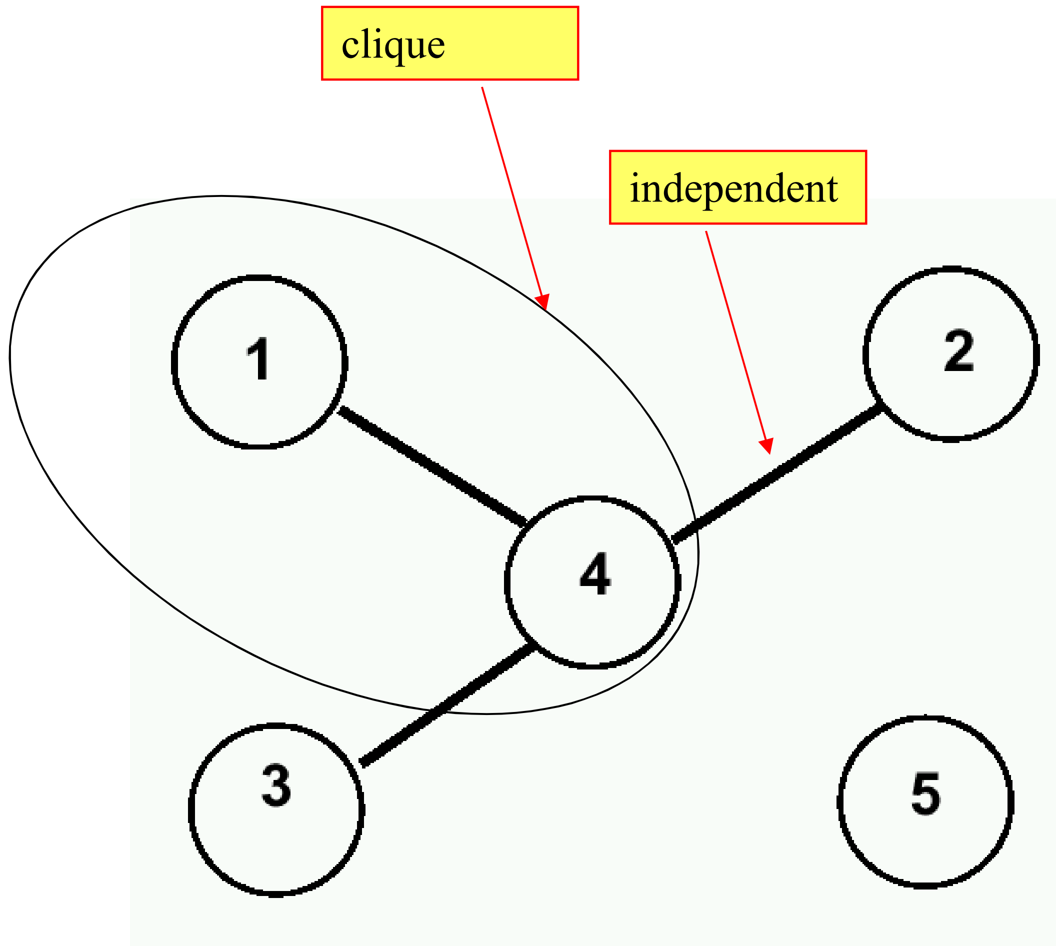
```
EXACT COVER( A; x; b) {
Reduce matrix A and update corresponding x;
if (Current est i mate j bj ) return(b);
if ( A has no rows ) return (x);
    Select a branching column c;
     xc =1 ;
     A = A after deleting c and rows incident to it;
     x =EXACT COVER(A; x; b);
if ( j xj < j bj )
     b = x ;
      xc =0 ;
     A = A after deleting c ;
      x =EXACT COVER(A; x; b);
if ( j) xj <j bj )
     b = x ;
      return (b);
}
```

Bounding function

- Estimate lower bound on the covers derived from the current x.
- The sum of the ones in x, plus bound on cover for local A:
  - Independent set of rows:
    - No 1 in same column.
  - Build graph denoting pair-wise independence.
  - Find clique number.
  - Approximation by defect is acceptable.

Example

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

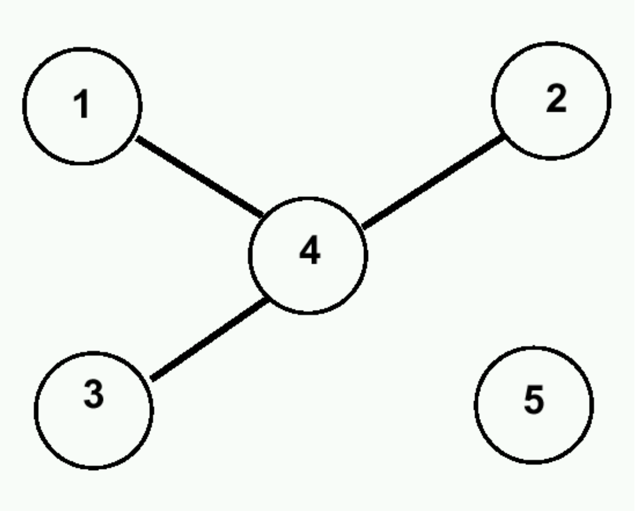


- Row 4 independent from 1,2,3.
- Clique number is 2.
- Bound is 2.

Example

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

- There are no independent rows.
- Clique number is 1 (one vertex).
- Bound is 1 + 1 (already selected essential).



Example

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

- Choose first column:
  - Recur with $\bar{\mathbf{A}} = [11]$.
    - Delete one dominated column.
    - Take other column (essential).
  - New cost is 3.
- Exclude first column:
  - Find another solution with cost 3 (discarded).

Unate and binate cover

- Set covering problem:
  - Involves a ***unate*** clause.
- Covering with implications:
  - Involves a ***binate*** clause.
- Example:
  - The choice of an element implies the choice of another element.

Unate and binate covering problems

- **Unate cover:**
  - Exact minimization of Boolean functions.
- **Binate cover:**
  - Exact minimization of Boolean relations.
  - Exact library binding.
  - Exact state minimization.

Unate and binate covering problems

- Unate cover:
  - It always has a solution.
  - Adding and element to a feasible solution preserves feasibility.
- Binate cover:
  - It may not have a solution.
  - *Adding and element to a feasible solution may make it unfeasible.*
  - Minimum-cost satisfiability problem.
  - Intrinsically more difficult.

Algorithms for unate and binate covering

- Branch and bound algorithm:
  - Extended to weighted covers.
- More complex in the binate case:
  - Dominant clauses can be discarded <u>only if weight dominates.</u>
  - Harder to bound.
- Only problems of smaller size are solvable, comparing to unate.
- Heuristic for binate cover are also more difficult to develop.

Discuss unate functions and they role

If time allows discuss symmetric functions and they role