



Vector Tools for Computer Graphics

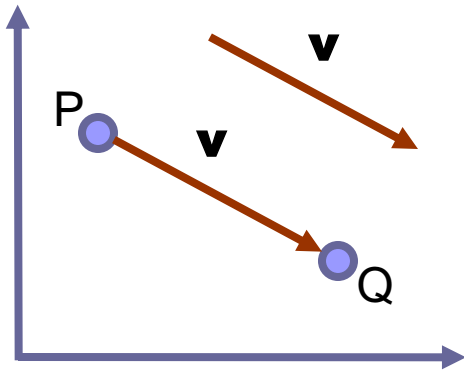
Computer Graphics

Basic Definitions

- Points specify location in space (or in the plane).
- Vectors have magnitude and direction (like velocity).

Points \neq Vectors

Basics of Vectors



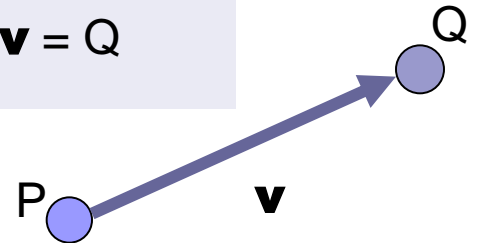
Vector as displacement:

v is a vector from point P to point Q.

The **difference** between two points is a vector: $\mathbf{v} = Q - P$

Another way:

The **sum** of a point and a vector is a point : $P + \mathbf{v} = Q$



Operations on Vectors

Two operations

Addition

$$\mathbf{a} + \mathbf{b}$$

$$\mathbf{a} = (3, 5, 8), \mathbf{b} = (-1, 2, -4)$$

$$\mathbf{a} + \mathbf{b} = (2, 7, 4)$$

Multiplication by scalars

$$s\mathbf{a}$$

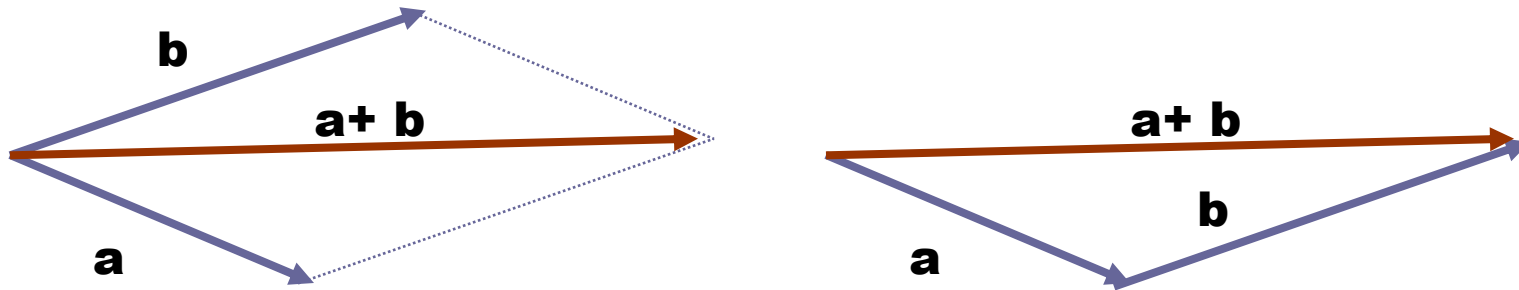
$$\mathbf{a} = (3, -5, 8), s = 5$$

$$5\mathbf{a} = (15, -25, 40)$$

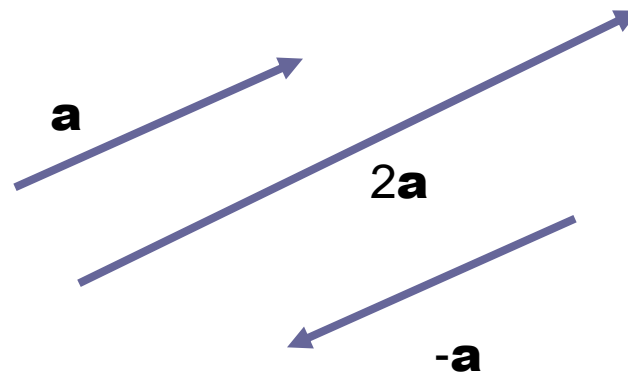
operations are done ***componentwise***

Operations on vectors

Addition

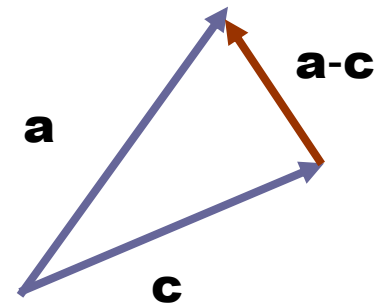
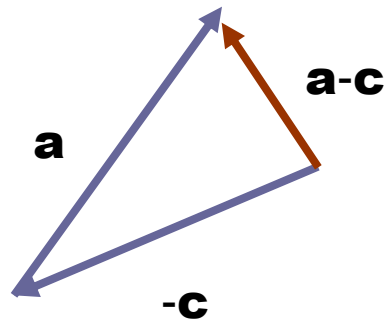
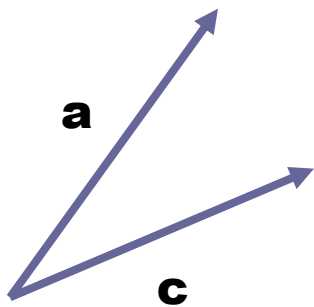


Multiplication by scalar



Operations on vectors

Subtraction



Properties of vectors

Length or size

$$\mathbf{w} = (w_1, w_2, \dots, w_n)$$

$$|\mathbf{w}| = \sqrt{w_1^2 + w_2^2 + \dots + w_n^2}$$

Unit vector

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

- The process is called **normalizing**
- Used to refer **direction**

The **standard unit vectors**: $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$

Dot Product

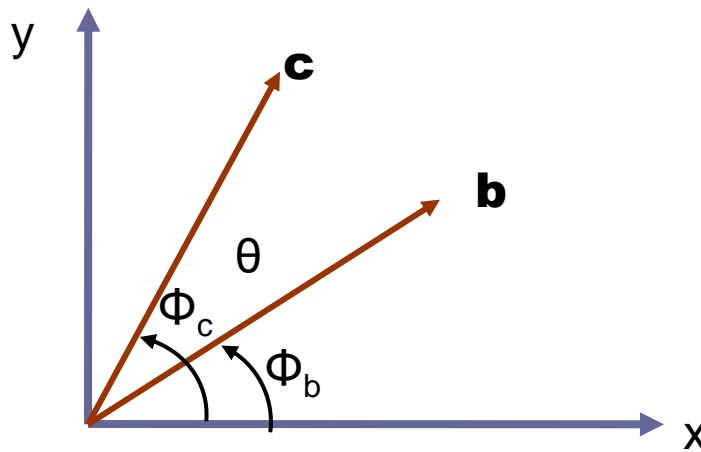
The dot product \mathbf{d} of two vectors $\mathbf{v} = (v_1, v_2, \dots, v_n)$ and $\mathbf{w} = (w_1, w_2, \dots, w_n)$:

Properties

1. Symmetry: $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
2. Linearity: $(\mathbf{a} + \mathbf{c}) \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b} + \mathbf{c} \cdot \mathbf{b}$
3. Homogeneity: $(s\mathbf{a}) \cdot \mathbf{b} = s(\mathbf{a} \cdot \mathbf{b})$
4. $|\mathbf{b}|^2 = \mathbf{b} \cdot \mathbf{b}$

Application of Dot Product

Angle between two unit vectors **b** and **c**



$$\cos \theta = \hat{\mathbf{b}} \cdot \hat{\mathbf{c}}$$

Two vectors **b** and **c** are perpendicular (orthogonal/normal) if
 $\mathbf{b} \cdot \mathbf{c} = 0$

2D perp Vector

Which vector is perpendicular to the 2D vector $\mathbf{a} = (a_x, a_y)$?

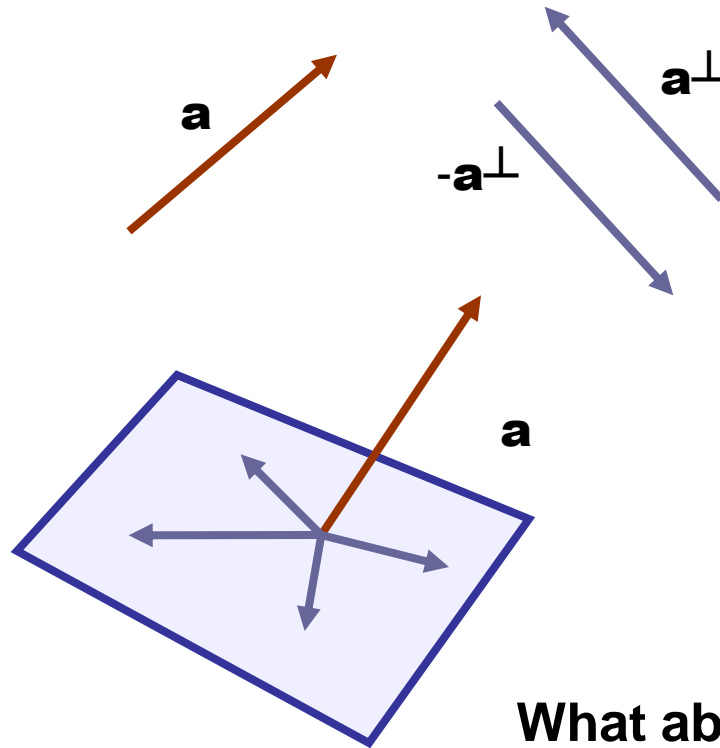
Let $\mathbf{a} = (a_x, a_y)$.

Then

$\mathbf{a}^\perp = (-a_y, a_x)$

is the

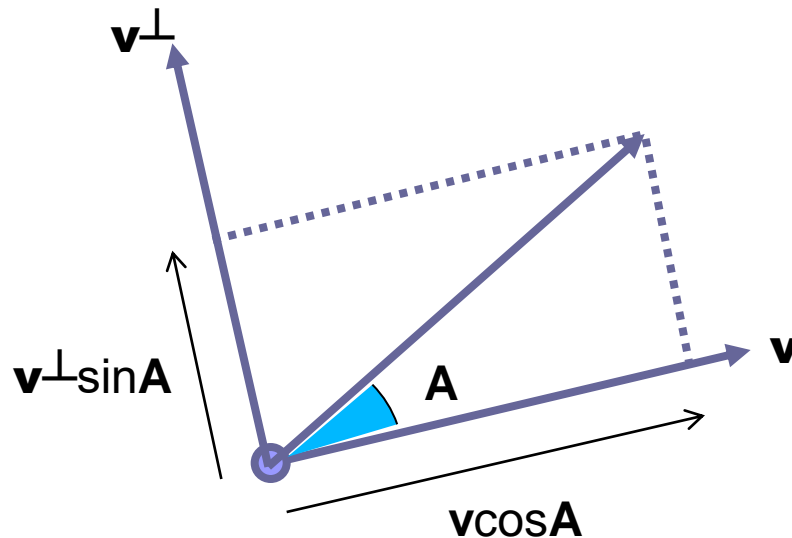
**counterclockwise
perpendicular** to \mathbf{a} .



What about 3D case?

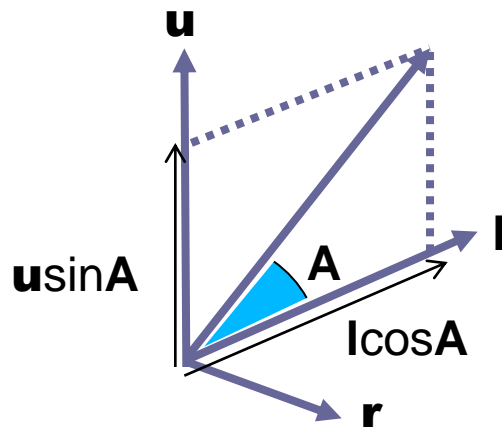
Rotation in 2d

- We want to rotate a 2d vector \mathbf{v} counterclockwise by an angle \mathbf{A}
- First we determine $\text{perp}(\mathbf{v})$, \mathbf{v}^\perp
- Then we scale \mathbf{v} by $\cos\mathbf{A}$ and scale \mathbf{v}^\perp by $\sin\mathbf{A}$ and take their sum



Rotation in 3d

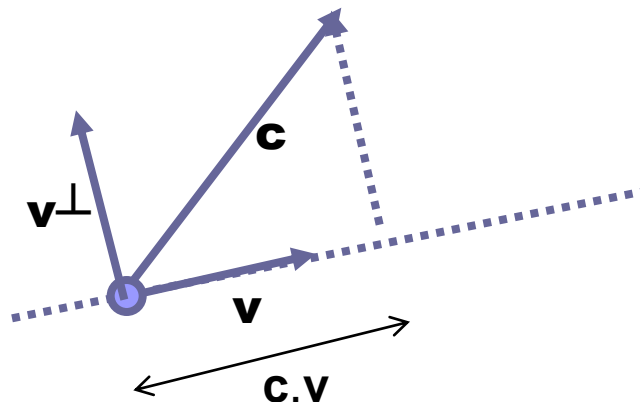
- We want to rotate a 3d vector \mathbf{l} counterclockwise with respect to a 3d unit vector \mathbf{r} by an angle \mathbf{A} , where \mathbf{l} and \mathbf{r} are perpendicular to each other
- First we determine the vector \mathbf{u} , that is perpendicular to both \mathbf{l} and \mathbf{r} and have a length equal to that of \mathbf{l}
- So, $\mathbf{u} = \mathbf{r} \times \mathbf{l}$
- Then we scale \mathbf{l} by $\cos \mathbf{A}$ and scale \mathbf{u} by $\sin \mathbf{A}$ and take their sum



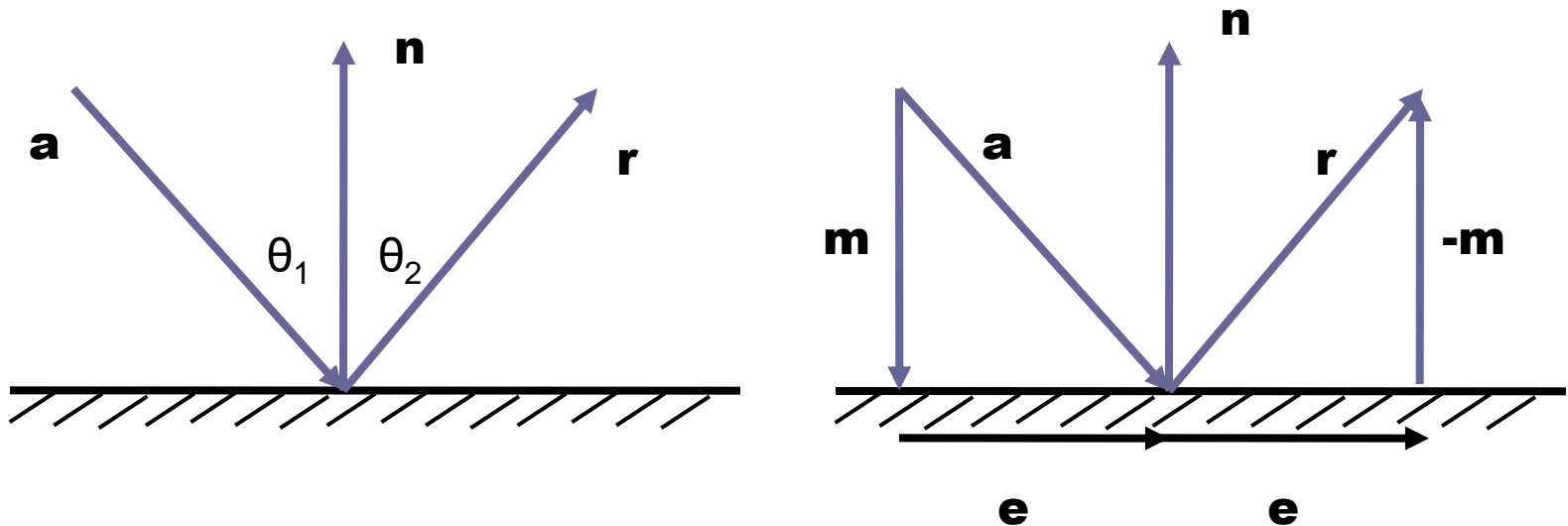
* note that, this method is applicable only in cases where the axis of rotation and the vector which is to be rotated are perpendicular to each other

Orthogonal Projection

- We want to decompose the vector \mathbf{c} into two vectors, one along the direction of a unit vector \mathbf{v} and another along $\text{perp}(\mathbf{v})$
- The length of the orthogonal projection of \mathbf{c} along \mathbf{v} is $\mathbf{c} \cdot \mathbf{v}$ (as \mathbf{v} is a unit vector)
- Thus the component (or orthogonal projection) of \mathbf{c} along \mathbf{v} is $(\mathbf{c} \cdot \mathbf{v})\mathbf{v}$
- So the component of \mathbf{c} along $\text{perp}(\mathbf{v})$ is $\mathbf{c} - (\mathbf{c} \cdot \mathbf{v})\mathbf{v}$



Reflection [13]



$$\mathbf{r} = \mathbf{a} - 2(\mathbf{a} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}$$

- Here \mathbf{m} is the orthogonal projection of \mathbf{a} along \mathbf{n}
- \mathbf{m} equals $(\mathbf{a} \cdot \mathbf{n})\mathbf{n}$ as \mathbf{n} is a unit vector

Cross Product

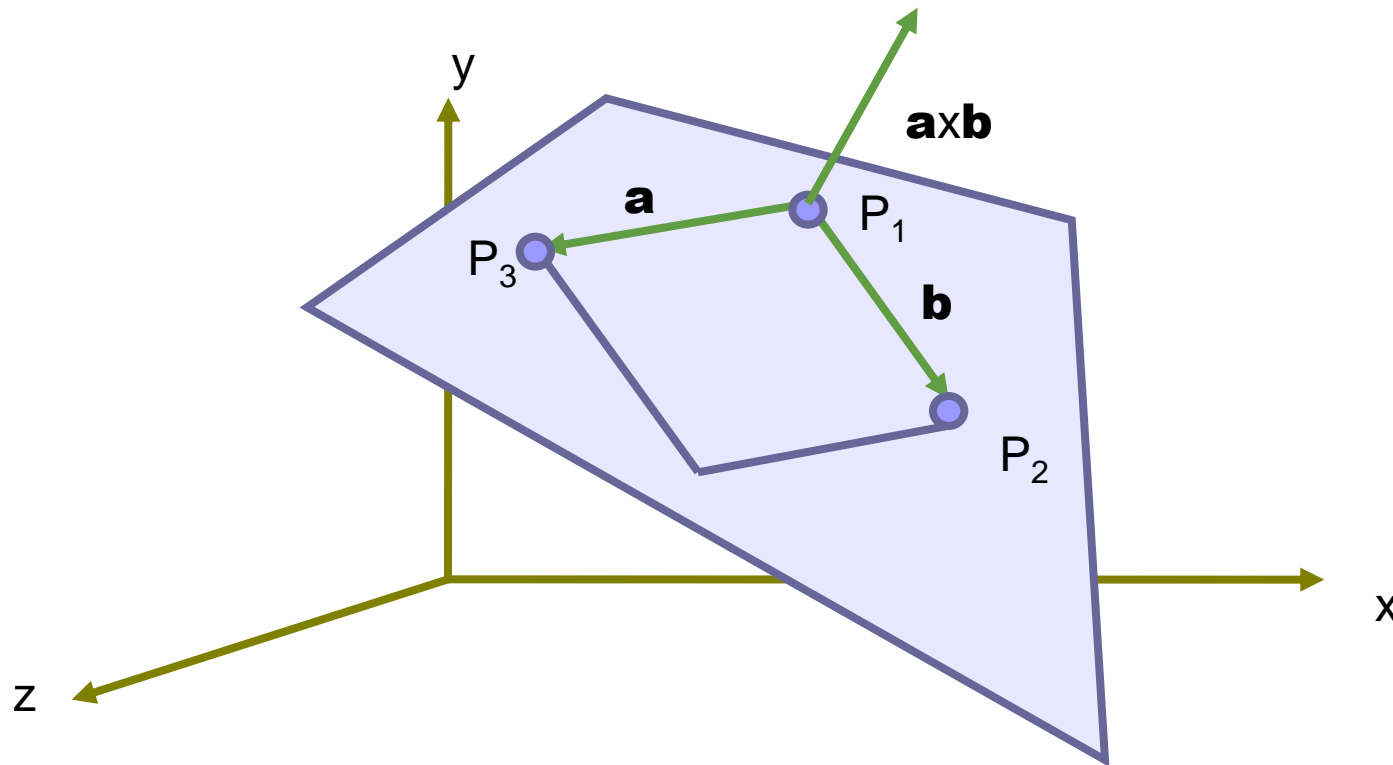
- Also called **vector product**.
- Defined for **3D** vectors only.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Properties

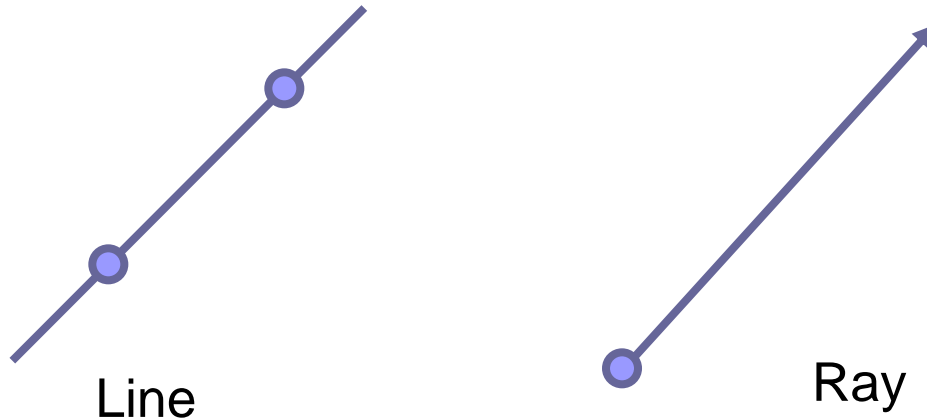
1. Antisymmetry: $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. Linearity: $(\mathbf{a} + \mathbf{c}) \times \mathbf{b} = \mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{b}$
3. Homogeneity: $(s\mathbf{a}) \times \mathbf{b} = s(\mathbf{a} \times \mathbf{b})$

Geometric Interpretation of Cross Product



1. $\mathbf{a} \times \mathbf{b}$ is perpendicular to both \mathbf{a} and \mathbf{b}
2. $|\mathbf{a} \times \mathbf{b}| = \text{area of the parallelogram defined by } \mathbf{a} \text{ and } \mathbf{b}$

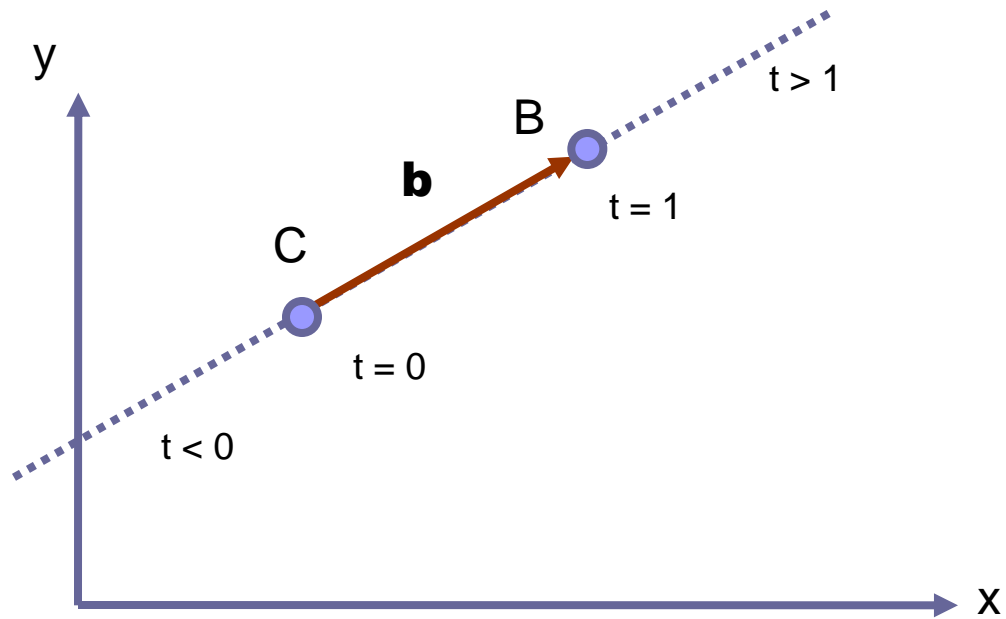
Representing Lines [11][12]



2 types of representations:

1. Two point form
2. Parametric representation

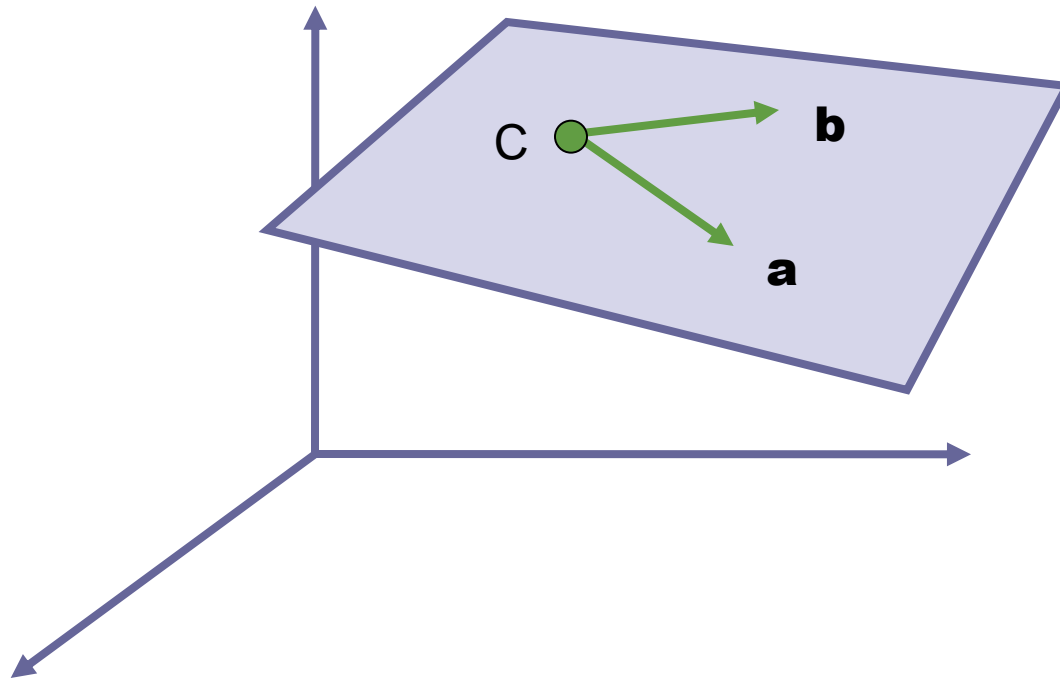
Parametric Representation of a Line



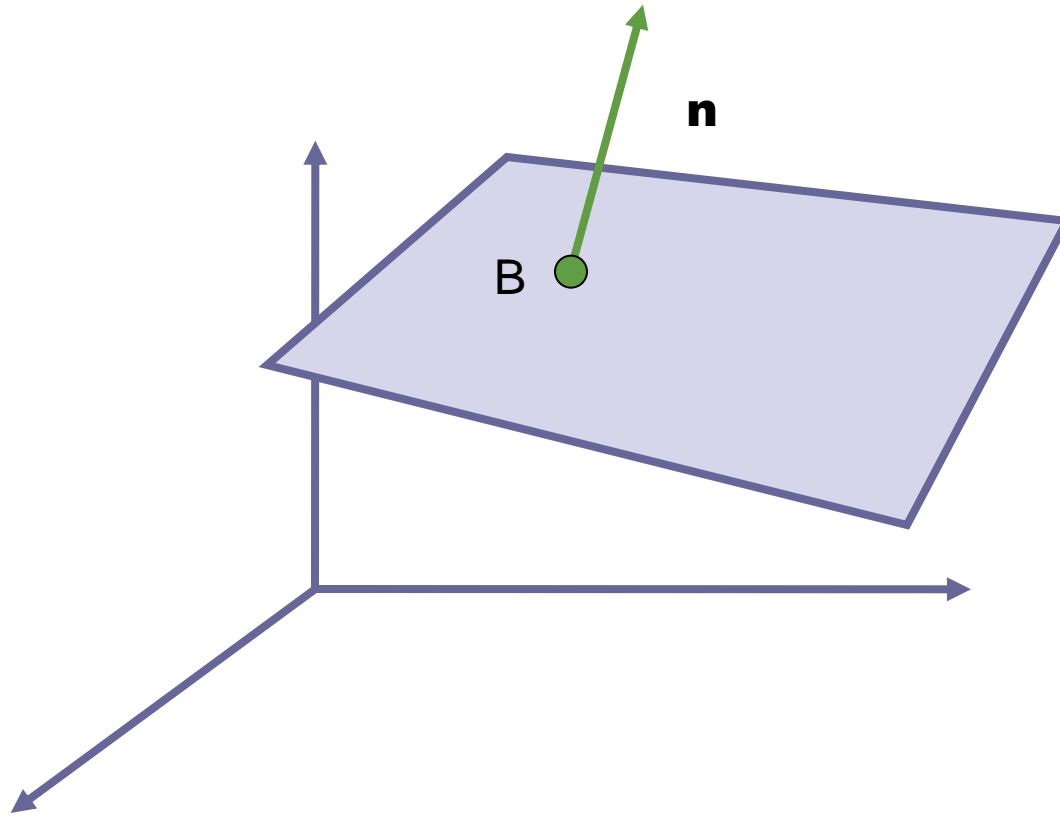
Planes in 3D

- 4 fundamental forms
 - Three-point form
 - Parametric representation
 - Point normal form
 - Equation

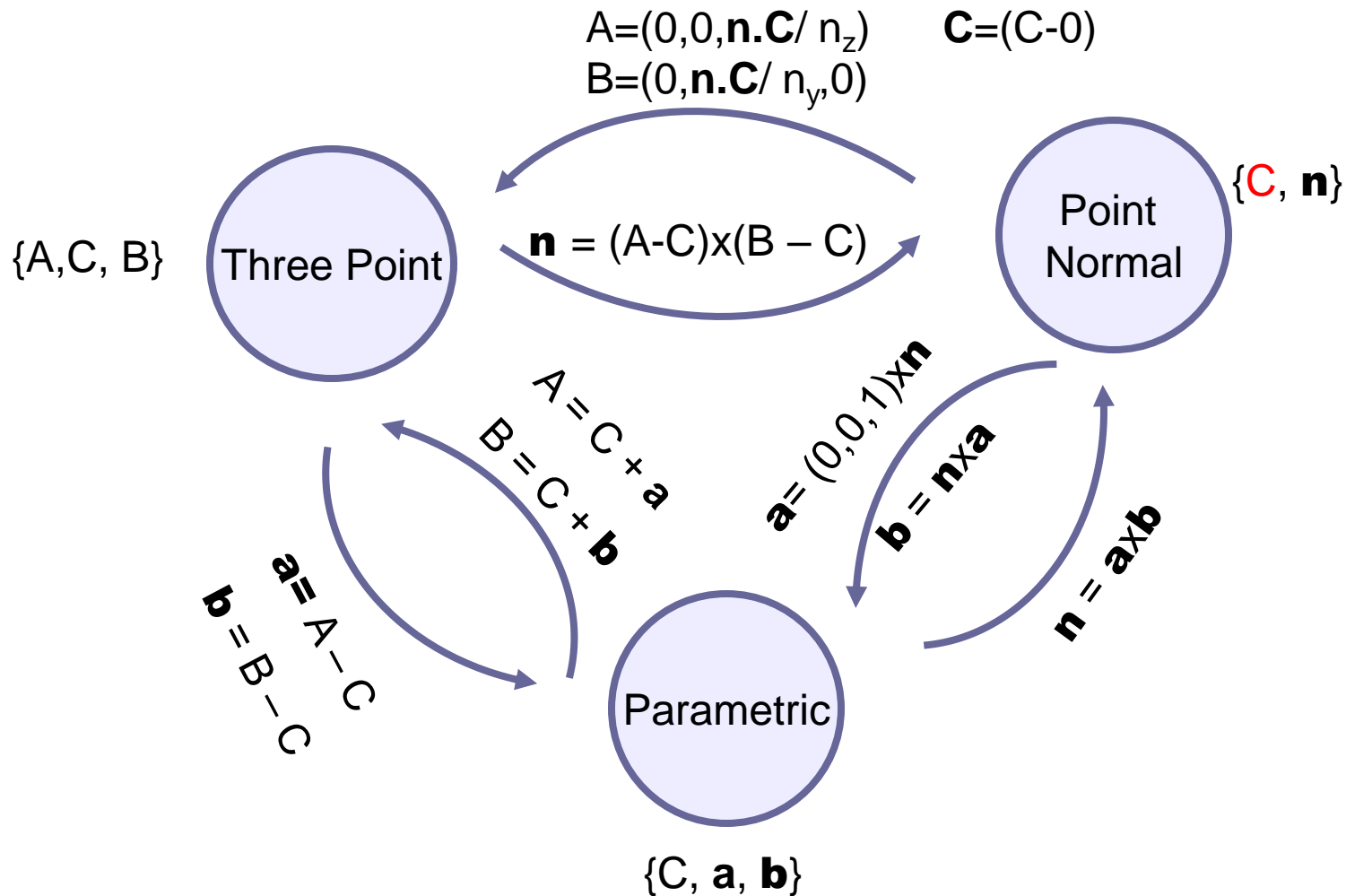
Parametric Representation of Plane



Point Normal Form of a Plane

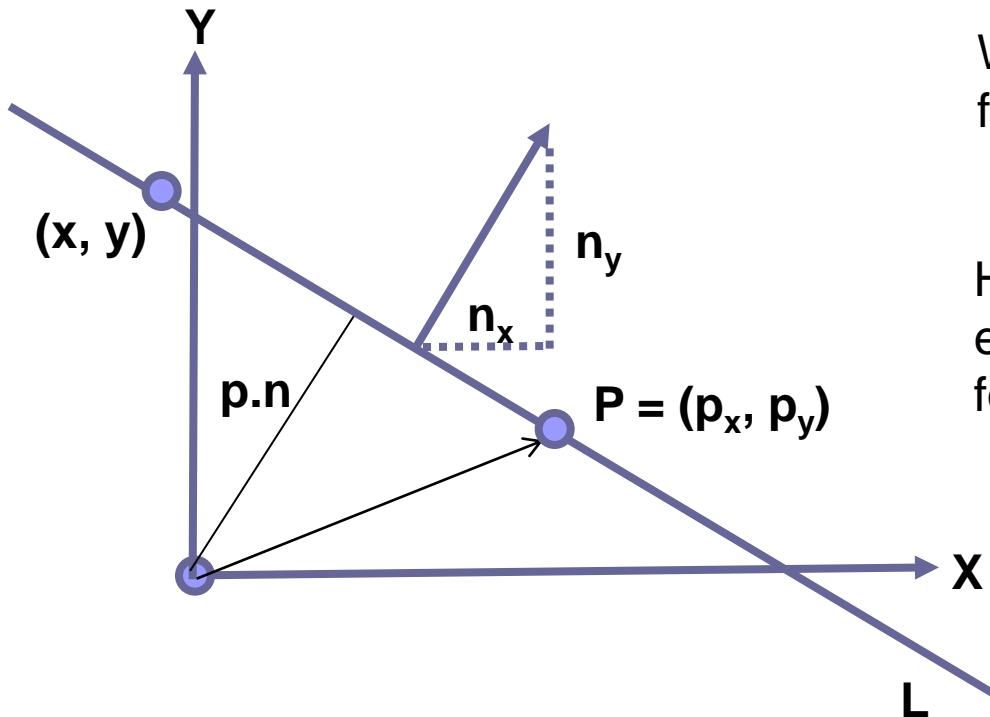


Representations of Plane [13]



Equation of a Plane [1][2][3][4]

- $ax + by + cz + d = 0$ is the standard equation of a plane in 3d
- If $\sqrt{a^2 + b^2 + c^2} = 1$, then it is called the normalized form
- In the normalized form, $|d|$ equals the distance of the plane from the origin



What about distance
from an arbitrary point?

How to convert between
equation and the other
forms?

Line-Plane Intersection [8]

- Plane: $ax + by + cz + d = 0$
- Line: $P + tV$
- Determine the specific value of t (say t') for which the equation of the plane is satisfied, i.e., the point on the line lies on the plane

Line-Line Intersection [5][6][7]

- Four possible cases:
 - Coincident
 - Parallel
 - Not parallel and do not intersect
 - Not parallel and intersect

Line-Line Intersection

- $L_1: P_1 + tV_1$
- $L_2: P_2 + sV_2$
- Parallel if V_1 and V_2 are in the same or opposite direction (i.e., the angle between them are 0 degree or 180 degree)
- Coincident if they are parallel and have at least one point in common
- If they are not parallel, how to decide whether they intersect or not?

Line-Line Intersection

- If they are not parallel, how to decide whether they intersect or not?
- One solution
 - Generate three equations for two unknowns
 - Solve the first two equations to find a solution
 - Check whether the solution satisfies the third equation
- Another solution
 - Check whether $(P_1 - P_2) \cdot (V_1 \times V_2) = 0$
 - If lines intersect this condition must hold
 - If lines do not intersect, is it sure not to hold?

Plane-Plane Intersection [9][10]

- Intersection is a line
- So we need two points on the line, or one point and the direction
- How to get a point on the intersecting line?
- How to get the direction of the intersecting line?

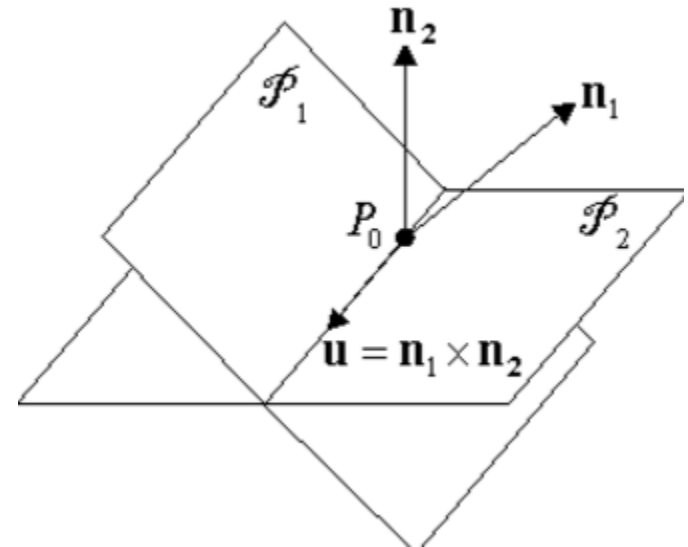


Plane-Plane Intersection

- How to get a point on the intersecting line?
 - Imagine another plane not parallel to any of the given planes, for example the plane $z = 0$ (the XY plane)
 - Now solve three plane equations to find their common intersection point

Plane-Plane Intersection

- How to get the direction of the intersecting line?
- Consider the planes in point-normal form
 - Plane 1: P_1, n_1
 - Plane 2: P_2, n_2
- n_1 and n_2 are both perpendicular to the intersecting line
- So the direction of the line of intersection is along $n_1 \times n_2$



Linear Combination of Vectors

Definition

Linear combination of m vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$:

$$\mathbf{w} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \dots + a_m \mathbf{v}_m$$

where a_1, a_2, \dots, a_m are scalars.

Types

Affine Combination

$$a_1 + a_2 + \dots + a_m = 1$$

Examples:

$$3\mathbf{a} + 2\mathbf{b} - 4\mathbf{c}$$

$$(1-t)\mathbf{a} + (t)\mathbf{b}$$

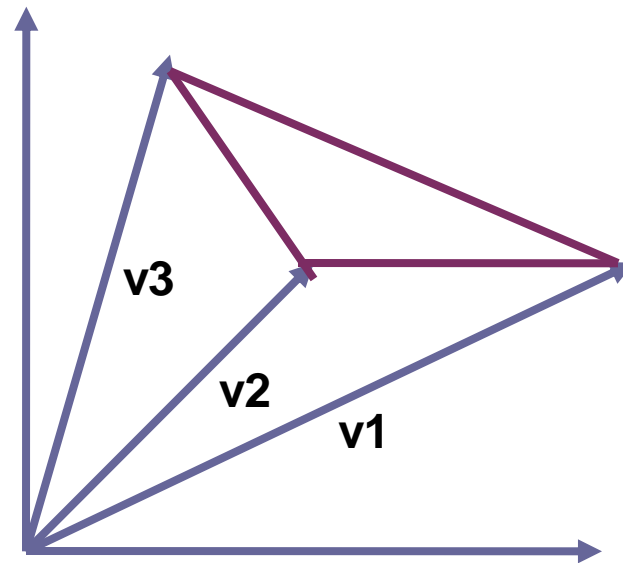
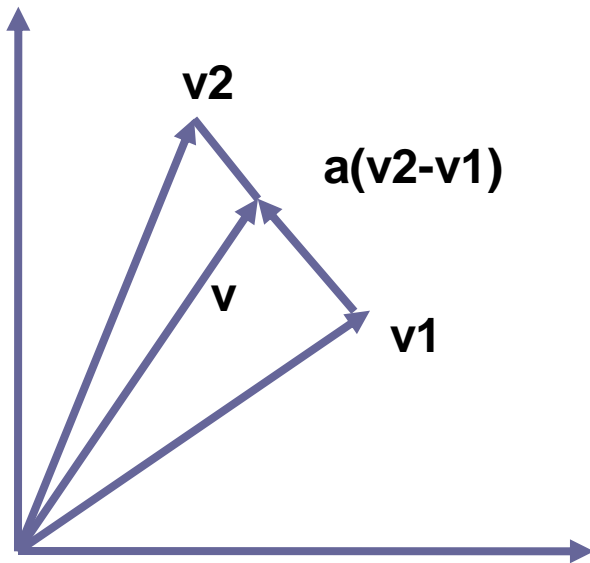
Convex Combination

$$a_1 + a_2 + \dots + a_m = 1$$

and $a_i \geq 0$, for $i = 1, \dots, m$.

$$\text{Example: } .3\mathbf{a} + .2\mathbf{b} + .5\mathbf{c}$$

Convex Combination of Vectors



Links



- [1] <http://www.songho.ca/math/plane/plane.html>
- [2] http://mathinsight.org/distance_point_plane
- [3] <https://www.youtube.com/watch?v=gw-4wltP5tY>
- [4] <https://www.youtube.com/watch?v=7rIFO8hct9g>
- [5] <https://www.youtube.com/watch?v=nKVCvY-FW5Q>
- [6] <https://www.youtube.com/watch?v=bJ56Xr9081k>
- [7] <https://www.youtube.com/watch?v=r5DwyBFxD7Q>
- [8] <https://www.youtube.com/watch?v=Td9CZGkqrSg>
- [9] <https://www.youtube.com/watch?v=SoSTdgqknvY>
- [10] <https://www.youtube.com/watch?v=LpardiBTAvU>

Links



[11] <https://www.youtube.com/watch?v=FILbI7DB0SM>

[12] <https://www.youtube.com/watch?v=nZ2mS5M4fcQ>

[13] (textbook) Chapter 4, Computer Graphics using OpenGL (2nd edition)
by Francis S Hill, Jr.