XC9151 Mathematics II

1. Evaluate
   $$\int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx$$
   is terms of Gamma function.

2. Show that
   $$\Gamma\left(n+\frac{1}{2}\right) = \frac{1.3.5.....(2n-1)}{2n}\sqrt{\pi}.$$

3. Verify whether the improper integral
   $$\int_{-1}^{1} \frac{dx}{x^5}$$
   is convergent.

4. Evaluate
   $$\int_0^4 \int_0^{x^2} e^{y/x}\, dy\, dx.$$

5. Evaluate
   $$\int_0^1 \int_1^{\sqrt{4-y}} (x+y)\, dx\, dy$$
   by changing the order of integration.

6. Using Leibnitz's rule for differentiation under integral sign evaluate
   $\int_0^{\infty} e^{-ax} \frac{\sin x}{x} dx$, $a \geq 0$ and hence deduce that (i) $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$
   (ii)
   $$\int_0^{\infty} e^{-x} \frac{\sin x}{x} dx = \frac{\pi}{4}.$$

7. Integrating by parts show that
   $$\int erf\, x \;\; dx = x\; erfx + \frac{e^{-x^2}}{\sqrt{\pi}}.$$

8. Express

$$\int_0^1 x^m\left(1-x^n\right)^p\,dx$$

in terms of Gamma functions and hence evaluate

$$\int_0^1 x^5\left(1-x^3\right)^{10}\,dx.$$

9. Evaluate

$$\int_0^\infty x^4 e^{-x^4}\,dx.$$

10. Evaluate

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,\mathrm{dy}\,\mathrm{dx}$$

by changing into polar coordinates.

11. Find the greatest rate of increases of u=x^2 + yz^2 at the point (1,-1,3).

12. Find the area lying between y=x^2 and y=x by double integration.

13. Find by double integration, the area of the lemniscate
$r^2 = a^2\cos 2\theta$.

14. Find the volume bounded by the cylinder x^2 + y^2=4 and the planes y+z=4and z=0.

15. Verify Green's theorem for

$$\int_c (3x-8y^2)\,dx+(4y-6xy)\,dy$$

where c is the boundary of the region bounded by x=0, y=0,x+y=1.

16. Verify divergence theorem for

$$\overrightarrow{F} = 4xz\bar{i} - y^2\bar{j} + yz\bar{k},$$

taken over the cube bounded by x=0, x=1, y=0, y=1,z=0 and z=1.

17. Find the half range sine series for f(x)=1, 0<x<1.

18. The function

$$f(x) = 1+\frac{2x}{\pi} \qquad -\pi \le x \le 0$$
$$= 1-\frac{2x}{\pi} \qquad 0 \le x \le \pi .$$

has Fourier series expansion

$$f(x) = \frac{8}{\pi^2}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+-\right)$$

Deduce the values of

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+--.$$

19. Obtain the Fourier cosine transform of f(x)=e^(-ax), a >0.

20. Find the Fourier series of f(x) in (-2,2) if

$$f(x)=0 \qquad -2<x<0$$
$$=1 \qquad 0<x<2$$

Deduce the value of

i) 1 - 1/3 + 1/5 - 1/7 –

ii) 1 + 1/3^2 + 1/5^2 + 1/7^2 + …

21. Using Fourier Integral representation show that

$$\int_0^{\infty}\frac{\sin y \cos xy\,dy}{y} = \frac{\pi}{2} \text{ when } 0 \le x < 1$$

22. Find Fourier sine transform of

f(x)=1 ,0<=x<=a

=0 , x>=a

23. Solve the integral equation

$$\int_0^{\infty} f(x)\cos\theta x \; dx = e^{-\theta}$$

24. Using Parsevals Identity show that

$$\int_0^{\infty}\frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$$

25. Verify whether the improper integral

$$\int_0^{\pi/2}\tan x\,dx$$

is convergent or not.

26. Evaluate $\int_0^{\pi/2} (\tan\theta)^{1/2} d\theta$ in terms of Gamma functions.

27. Find the area enclosed between the curves y^2=x and x^2=y by double integration.

28. Evaluate the triple integral $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)\,\mathrm{dxdydz}.$

29. Find a unit vector normal to the surface x^3 y^3+3xyz=3 at the point (1,2,-1)

30.

If $u = x^2+y^2+z^2, \overline{v} = x\overline{i}+y\overline{j}+z\overline{k},$ show that $\mathrm{div}(u\overline{v}) = 5u.$

31.

State Dirichlets' conditions for Fourier series expansion of a function f(x) in $a<x<b$.

32.

A function f(x) given by

$$f(x) = -x \qquad \text{when } -\pi < x \le 0$$
$$= x \qquad \text{when } 0 < x < \pi$$

has the Fourier series expansion

$$f(x) = \pi/2 - \frac{4}{\pi}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\ldots\right]$$

Deduce the value of $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{3^2}+--$

33.

Find the Fourier cosine transform of

$$f(x) = \cos x \qquad 0<x<a$$
$$= 0 \qquad x \ge a.$$

34.

Find Fourier transform of f(x-a) in terms of Fourier transform of f(x).

35.

(i)

Evaluate the integral $\int_0^{\infty}\int_x^{\infty}\frac{e^{-y}}{y}\,dydx$ by changing the order of integration.

(ii)

Evaluate $\iint r\sin\theta\,drd\theta$ over the cardioid r=a(1-cos θ) above the initial line.

(iii)

Find the volume of the tetrahedson bounded by the coordinate planes and the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

36.

(i)

Using differentiation under integral sign prove that

$$\int_0^{\infty}\frac{e^{-x}}{x}(1-e^{-ax})\,dx=\log(1+a),\quad a>-1$$

(ii)

Using Beta and Gamma functions prove that $\int_0^1\frac{x^2dx}{\sqrt{1-x^4}}\times\int_0^1\frac{dx}{\sqrt{1+x^4}}=\frac{\pi}{4\sqrt{2}}$

(iii)

Express $\int_0^1 x^m(1-x^n)^p\,dx$ in terms of gamma functions and hence evaluate

$$\int_0^1 x^5(1-x^3)^{10}\,dx.$$

(iv)

Show that $\int_0^{\infty}\frac{e^{-\sqrt{x}}}{x^{7/4}}dx=\frac{8}{3}\sqrt{\pi}$

37.

(i)

Applying Green's theorem evaluate $\int_c (x^2+xy)dx+(x^2+y^2)dy$ *where* c is the square formed by the lines $x=\pm1, y=\pm1$

(ii)

Using Divergence Theorem evaluate $\iint_S \overline{F}.\overline{n}\, ds$ where $\overline{F} = 4x\overline{i} - 2y^2\overline{j} + z^2\overline{k}$ and S is the surface bounding the region $x^2 + y^2 = 4, z = 0\ and\ z = 3$

(iii)

Verify Stoke's Theorem for $\overline{F} = (y - z + 2)\overline{i} + (yz + 4)\overline{j} - xz\ \overline{k}$ where S is the surface of the cube x = 0, y = 0, z = 0, x = 2, y = 2, z = 2 above the xy plane.

38.

(i)

If $f(x) = \pi x \qquad 0 \le x \le 1$

$\quad = \pi(2 - x) \qquad 1 \le x \le 2$

show that in the interval (0,2)

$$f(x) = \pi/2 - \frac{4}{\pi}\left[\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots\ldots\right]$$

Deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + - = \pi^2/8$$

(ii)

Express f(x) = x as a half-range sine series in 0<x<2

39.

(i)

Find the Fourier series of the periodic function f(x) = -k when $-\pi < x < 0$

= k when $0<x<\pi$

and f(x+2x) = f(x)

Deduce that $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - = \pi/4$

(ii)

Obtain the Fourier series for $y = x^2$ in $-\pi < x < \pi$.

Using Parseval's formula show that

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + - = \frac{\pi^4}{90}$$

40.

(i)

Express the function $f(x) = 1$ for $|x| < 1$

$= 0$ for $|x| > 1$

as a Fourier integral. Hence evaluate $\int_0^{\infty} \frac{\sin \lambda \cos \lambda x \, d\lambda}{\lambda}$

(ii)

Using Parseval's identities

prove that $\int_0^{\infty} \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$

(iii)

Find the Fourier sine transform of $\frac{e^{-ax}}{x}$.

(iv)

Solve the integral equation

$$\int_0^{\infty} f(\theta) \cos a\theta \, d\theta = 1 - a \qquad 0 \le a \le 1$$

$$= 0 \qquad a > 1$$

Hence evaluate $\int_0^{\infty} \frac{\sin^2 t}{t^2} dt$