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23.5 Ionisation of Interstellar Gas near a Star

Radiation from a star

Question 1

We can determine the total energy output radiated from the star by integrating L_ν over all frequencies.

$$L = L_\nu \int_0^\infty L_\nu d\nu \tag{1}$$

$$= 4\pi R^2 \frac{2\pi h}{c^2} \int_0^\infty \frac{\nu^3}{\exp\left(\frac{h\nu}{kT_*}\right) - 1} d\nu \tag{2}$$

Using the substitution $x = \frac{h\nu}{kT_*}$, $dx = \frac{h}{kT_*} d\nu$ gives a simpler form of the integral.

$$L = 4\pi R^2 \frac{2\pi h}{c^2} \int_0^\infty \left(\frac{kT_*}{h}\right)^3 \frac{x^3}{e^x - 1} \left(\frac{kT_*}{h}\right) dx \tag{3}$$

$$= \frac{8\pi^2 k^4 T_*^4 R^2}{c^2 h^3} \int_0^\infty \frac{x^3}{e^x - 1} dx \tag{4}$$

$$= \frac{8\pi^2 k^4 T_*^4 R^2}{c^2 h^3} \underbrace{\zeta(4)}_{\frac{\pi^4}{90}} \underbrace{\Gamma(4)}_{6} \tag{5}$$

$$= \frac{8}{15} \frac{\pi^6 k^4 T_*^4 R^2}{c^2 h^3} \tag{6}$$

Where the integral has been rewritten using the Riemann zeta and gamma functions. Equation (6) can be rearranged for T_* and R as below.

$$T_* = \left(\frac{15}{8}\frac{Lc^2h^3}{\pi^6k^4R^2}\right)^{\frac{1}{4}} \tag{7}$$

$$R = \left(\frac{15}{8}\frac{Lc^2h^3}{\pi^6k^4T_*^4}\right)^{\frac{1}{2}} \tag{8}$$

We consider calculations of T_* or R for 3 stars of different solar masses with some given properties shown below in Table 1 starting with the sun.

$M\ (M_\odot)$	$L\ (W)$	$T_*\ (K)$	$R\ (m)$
1	3.9×10^{26}	$\mathbf{5,796.26 \simeq 5800}$	6.96×10^8
7	4.0×10^{29}	$20,000$	$\mathbf{1.87 \times 10^9}$
12	4.0×10^{30}	$25,000$	$\mathbf{3.79 \times 10^9}$

Table 1: Results of the *radiation_from_a_star.py* program. **Bold** values have been computed using the program. The rest of the values are given. ($M_\odot$ denotes solar mass.)

Ionisation near Stars

Question 2

Here we solve the ionisation equation balance equation.

$$n_{H^0} \int_{\nu_0}^{\infty} \frac{L_\nu}{4\pi r^2 h\nu} a_\nu e^{-\tau_\nu} d\nu = n_p n_e \alpha_B(T) \tag{9}$$

By substituting in the subsidiary equations we can express this equation as below and denoting the integral expression as I for convenience.

$$\underbrace{\frac{2\pi a_0 \nu_0^3 R^2}{\alpha_B c^2} \int_{\nu_0}^{\infty} \frac{1}{r^2} \frac{\nu^{-1}}{\exp\left(\frac{h\nu}{kT_*}\right) - 1} \exp\left(-\frac{a_0 \nu_0^3}{\nu^3} \int_0^r n_{H^0}(r) dr\right) d\nu}_{I} = \frac{n_p^2}{n_{H^0}} \tag{10}$$

$$I = \frac{n_p^2}{n_H - n_p} \tag{11}$$

$$n_p(r) = \frac{-I + \sqrt{I^2 + 4In_H}}{2} \tag{12}$$

I can now be numerically integrated over an array of r and then substituted into equation 12 where n_p and therefore n_{H^0} can be determined as functions of r and plotted.

The *photoionisation.py* program on page ?? contains the code that tries to implement the above.

I could not get the code to program to work when including the optical depth term due to not knowing the form of $n_{H^0}(r)$ in the optical depth integral.

Question 3

Below are plots where I have set the optical depth exponential to 1 to give some sort of result.

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If the code were able to include the optical depth, I would expect $n_p/n_H \approx 1$ for a larger distance from the star before reaching the sharp transition. τ_ν will increase with r as photons are absorbed and scattered more.

$r = 9.20920920920921e + 17$

An approximation to r_1

Question 4

To evaluate the Strömgren sphere we integrate the ionisation balance equation over volume using the fact that for a spherically symmetric system, the volume element can be simplified to $dV = 4\pi r^2 dr$.

$$\int_V n_{H^0} \int_{\nu_0}^{\infty} \frac{L_\nu}{4\pi r^2 h\nu} a_\nu e^{-\tau_\nu} d\nu \, dV = \int_V n_p n_e \alpha_B \, dV \tag{13}$$

$$\int_0^{r_1} n_{H^0} \int_{\nu_0}^{\infty} \frac{L_\nu}{h\nu} a_\nu e^{-\tau_\nu} \, d\nu \, dr = \int_0^{r_1} n_p n_e \alpha_B \, 4\pi r^2 dr \tag{14}$$

As $n_p = n_e = n_H$ for $r \leq r_1$ and $n_p = n_e = 0$ for $r > r_1$, the limits of the integral over r are from 0 to r_1. By using the below definition of the optical depth τ_ν we can substitute for dr in the left-hand-side integral. Note that at r_1, there is no more ionisation so the optical depth τ_ν must be infinite here.

$$\frac{d\tau_\nu}{dr} = n_{H^0}(r) a_{\nu_0} \tag{15}$$

$$\int_0^{r_1} n_{H^0} \int_{\nu_0}^{\infty} \frac{L_\nu}{h\nu} a_\nu e^{-\tau_\nu} \frac{1}{n_{H^0} a_\nu} \, d\nu \, d\tau_\nu = \int_0^{r_1} n_H^2 \alpha_B \, 4\pi r^2 dr \tag{16}$$

$$\underbrace{\int_{\nu_0}^{\infty} \frac{L_\nu}{h\nu} d\nu}_{Q(H)} \underbrace{\int_0^{\infty} e^{-\tau_\nu} \, d\tau_\nu}_{1} = 4\pi \, n_H^2 \alpha_B \int_0^{r_1} r^2 dr \tag{17}$$

$$Q(H) = \frac{4\pi}{3} r_1^3 n_H^2 \alpha_B \tag{18}$$

$Q(H)$ is the total number of ionizing photons emitted by the star per second and r_1 is now the Strömgren radius.

Table 2 below gives the calculations of $Q(H)$ and r_1 for the 3 different mass stars for $T = 20,000$ K, using the program *stromgren.py* on page ??. This program contains functions to perform the numerical integration of $Q(H)$ by the midpoint rule and then using these values to compute r_1 via equation 18.

I have evaluated the integral to an upper limit of $\nu = 0.5 \times 10^{16}$ Hz because a plot of $\frac{L_\nu}{h\nu}$ against ν shows the function having negligible value from this point. This was done to reduce the computational time as otherwise evaluating to infinity would take too long.

M ($M_\odot$)	Q (s^{-1})	r_1 (m)
1	8.91×10^{35}	9.36×10^{13}
7	6.44×10^{45}	1.81×10^{17}
12	1.62×10^{47}	5.31×10^{17}

Table 2: Results of the *stromgren.py* program. The values of r_1 have been computed for a gas temperature of $T = 10,000$ K.

Comparing to previous values of r_1......

The effect of a quasar on the host galaxy

Question 5

To evaluate whether or not any of the interstellar gas in the galaxy has neutral fraction > 0.5, I use the previous method of the Strömgren radius.

The interstellar gas switches from more ionised to more neutral at $n_p/n_H = 0.5$. The Strömgren radius r_1 is approximated at this point so this is essentially a step function at this point. Therefore determining the value of r_1 and comparing this to the radius of the galaxy, which I denote as r_g, will answer the question.

The *quasar.py* on page ?? computes the numerical integration and calculation of r_1 as in question 4 but this time using the quasar's luminosity. The result is:

$$r_1 = 9.21 \times 10^{20}\,\mathrm{m} \;>\; 3 \times 10^{20}\,\mathrm{m}$$

$$\therefore\; r_1 > r_g$$

Therefore, none of the interstellar gas in the galaxy has a neutral fraction greater than 0.5.

Table 3 below contains the results of *quasar.py* for the the hydrogen neutral fraction as a function of distance from the quasar.

r (10^{18} m)	n_{H^0}/n_H
1	?
7	?
12	?

Table 3: Results of the *stromgren.py* program. The values of r_1 have been computed for a gas temperature of $T = 10,000$ K.

The graph is approximately zero from $\nu =$??????? so the tabulated values end here.

Programs

radiation_from_a_star.py - used in Question 1

```
import numpy as np

# Constants
c = 2.998e8
h = 6.626e-34
k = 1.381e-23

# Sun(M = M_solar) Properties
L_s = 3.90e26
R_s = 6.96e8

# M = 7*M_solar
T_7 = 20000
L_7 = 4.0e29

# M = 12*M_solar
T_12 = 25000
L_12 = 4.0e30

def surface_temp(L,R):
    return f'T = {((15 * L * c**2 * h**3)/(8 * np.pi**6 * k**4 * R**2)
       )**0.25} K'

def radius(L,T):
    return f'R = {((15 * L * c**2 * h**3)/(8 * np.pi**6 * k**4 * T**4)
       )**0.5:.4e} m'

print(surface_temp(L_s,R_s))
print(radius(L_7,T_7))
print(radius(L_12,T_12))
```

stromgren.py - used in Question 4

```
import numpy as np
import matplotlib.pyplot as plt

# Constants
c = 2.998e8
h = 6.626e-34
k = 1.381e-23

v_0 = 3.29e15
alpha = {"5000": 4.54e-19, "10000": 2.59e-19, "20000": 2.52e-19}
n_H = 1e6

radii = {"1": 6.96e8, "7": 1.8722e9, "12": 3.789e9}
luminosities = {"1": 3.9e26, "7": 4.0e29, "12": 4.0e30}
temperatures = {"1": 5796.25855, "7": 20000, "12": 25000}
```

```
def function(v,R, T_star):
    numerator = (8 * np.pi**2 * R**2 * v**2) / (c**2)
    denominator = np.exp((h * v) / (k * T_star)) - 1
    y = numerator / denominator
    return y

def integral(v):
    Q_values = []
    for r_key, r in radii.items():
        T_s = temperatures[r_key]
        integral_sum = 0
        for i in range(1, len(v)):
            integral_sum += function(v[i], r, T_s) * (v[i] - v[i-1])
        Q_values.append(integral_sum)
    return Q_values

def stromgren(Q_array):
    radius = []
    for Q_value in Q:
        r_1 = ((3 * Q_value) / (4 * np.pi * n_H**2 * alpha['10000']))
           **(1/3)
        radius.append(f'{r_1:4e}')
    return radius

v = np.linspace(v_0, 0.5e16, 100000)

Q = integral(v)
r_stromgren = stromgren(Q)

print(Q)
print(r_stromgren)
```

quasar.py **- used in Question 5**

```
import numpy as np
import matplotlib.pyplot as plt

# Constants
h = 6.626e-34
v_0 = 3.29e15
n_H = 1e6
alpha = {"5000": 4.54e-19, "10000": 2.59e-19, "20000": 2.52e-19}

def quasar_function(v):
    I = ((1e24 * v_0**1.4) / (h)) * v**(-2.4) * np.exp(-(v) / (10 *
       v_0))
    return I

def quasar_integral(v):
    integral_sum = 0
    for i in range(1, len(v)):
        integral_sum += quasar_function(v[i]) * (v[i] - v[i-1])
```

```
    return integral_sum

def quasar_stromgren(Q_val):
    radius = ((3 * Q_val) / (4 * np.pi * n_H**2 * alpha['10000']))
       **(1/3)
    return radius

v = np.linspace(v_0, 1e18, 1000000)
Q = quasar_integral(v)

print(f'radius = {quasar_stromgren(Q)}')
```