ISI-BUDS
PROBABILITY - PART IV

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PART IV PLAN:

- Reviewing expectations of combinations of random variables
- Introducing the Monte Carlo idea
- Some example

EXPECTED VALUE OF A RANDOM VARIABLE

- If X is a random variable, an important quantity/statistics that we might know about it is its expected value.
- We have seen the definition in the case X is a discrete random variable:

$$\boldsymbol{E(X)} = \sum_{\textit{all values k that X can assume}} \boldsymbol{k \cdot P(X = k)}$$

- What is the definition of $E(X)$ if X is a continuous random variable?

LINEAR COMBINATIONS OF RANDOM VARIABLES

- We now know how to calculate the expected value of a random variable X, given the random variable p.d.f..
- For certain distributions (Binomial, Poisson, Normal, Gamma, etc.), we are even provided with expression for the expected value, as a function of the distribution's parameters..
- Given 2 random variables, X_1 and X_2, we can create a new variable W by taking a linear combination of X_1 and X_2:

$$W = a_1 X_1 + a_2 X_2$$

LINEAR COMBINATIONS OF RANDOM VARIABLES

- Is W a random variable? Why or why not?

- If W is a random variable:
  - Is W a discrete random variable or a continuous random variable?
  - Do we know its p.d.f.? Can we obtain it?
  - What about the expected value of W? And what about the variance of W?

EXPECTED VALUE AND VARIANCE OF THESE NEW R.V.'S

- The expected value of W can be calculated easily, as the expectation is a linear operator; hence:

$$E(W) = a_1 E(X_1) + a_2 E(X_2).$$

- The variance of W can also be derived: the general formula is

$$Var(W) = a_1^2 \cdot Var(X_1) + a_2^2 \cdot Var(X_2) + 2a_1 a_2 Cov(X_1, X_2)$$

Where $Cov(X_1, X_2)$ is the covariance between X_1 and X_2.

COVARIANCE BETWEEN TWO RANDOM VARIABLES

- If X_1 and X_2 are two random variable, the covariance of X_1 and X_2, denoted by $Cov(X_1, X_2)$, is a summary statistics that provides a measure of the strength of the relationship between X_1 and X_2 and is defined as:

$$Cov(X_1, X_2) := E(X_1 \cdot X_2) - E(X_1) \cdot E(X_2)$$

- If X_1 and X_2 are independent, then $Cov(X_1, X_2) = 0$.

COVARIANCE PROPERTIES

- Since it is defined using the expected value, it follows that:
  - If X_1, X_2, X_3 and X_4 are four random variables and a_1, a_2, a_3, a_4 are four real numbers, then:

$$Cov(a_1X_1 + a_2X_2, a_3X_3 + a_4X_4) = a_1a_3 \cdot Cov(X_1, X_3) + a_1a_4Cov(X_1, X_4) + a_2a_3 \cdot Cov(X_2, X_3) + a_2a_4 \cdot Cov(X_2, X_4)$$

EXAMPLE: TIME AT ATM MACHINE

Suppose that the length of time a person takes to use an ATM machine is normally distributed with mean $\mu = 100\ sec$. and standard deviation $\sigma = 4\ sec$.

There are n=4 people in front of Jackson in a line of people waiting to use the machine. Jackson is concerned about T="total time the 4 people will take to use the machine".

What is the mean value of T="total time for the 4 people ahead of Jackson"?

Assuming that the times for the four people are independent of each other, what is the standard deviation of T?

WHAT ABOUT FUNCTIONS OF RANDOM VARIABLES?

- We know now how to calculate the expected value of linear combinations of random variables.
- What about functions of random variables?
- For example, suppose X is random variable describing the velocity of a particle of mass m. Suppose also that X has p.d.f. given by $f(x)$. What is the expected kinetic energy of the particle?
- How do we calculate this expected value?

EXPECTED VALUE OF FUNCTIONS OF A RANDOM VARIABLE

- More in general, suppose X is a random variable with p.d.f. $f(x)$. Suppose also that $g(x)$ is a function defined on the real line.
- How do we calculate $E(\, g(X)\,)$?

MORE EXAMPLES: QUEUING SYSTEM

Customers arrive at a shop and queue to be served. Their requests require varying amount of time. The manager cares about customer satisfaction and does not want to exceed the 9AM-5PM work schedule of the employees.

If the shop assistants continue to deal with all customers in the shop at 5PM, what is the probability the they will have served all the customers by 5:30PM?

MORE ON QUEIENG SYSTEMS.

- Let's call by X the random variable "number of customers in the shop at 5:30PM".
- We want to calculate $P(X=0)$!
- How can we also express $P(X=0)$?

MONTE CARLO INTEGRATION OR MONTE CARLO ESTIMATOR

- In many situations we do not know the distribution of X or we do not know how to calculate analytically E(g(X)).
- What to do then? Monte Carlo integration!
- Monte Carlo integration can be thought of a stochastic way to approximate integrals.

MONTE CARLO INTEGRATION OR MONTE CARLO ESTIMATOR

- Let's denote with $\theta = E(g(X))$. Let's assume that $X_1, X_2, .., X_n$ are random variables that are independent of each other and have all the same distributions as X, then:

$$\widehat{\theta_n} = \frac{1}{n} \cdot \sum_{i=1}^{n} g(X_i)$$

Is called a Monte Carlo estimator of θ.

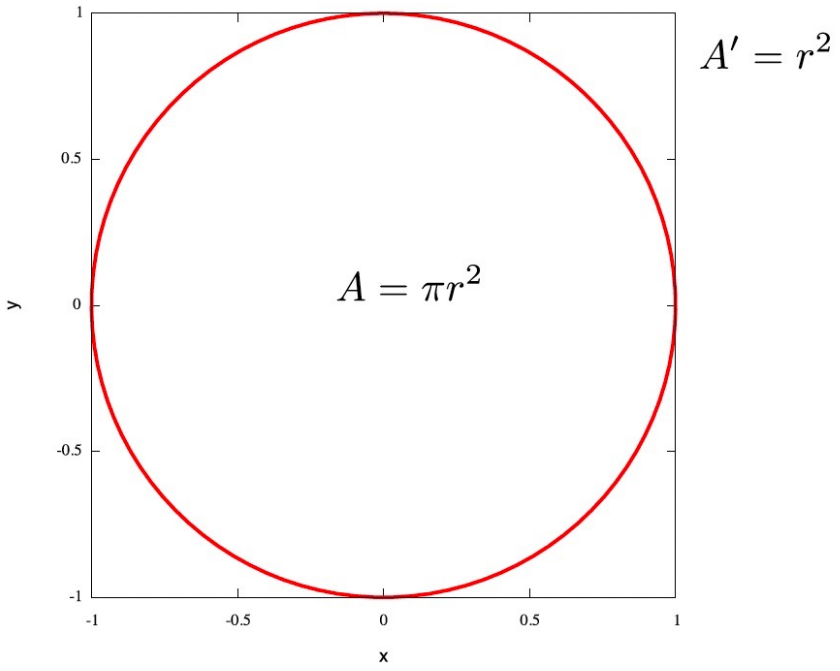
GENERIC ALGORITHM

- A generic Monte Carlo algorithm would be the following

1. Simulate independent random variables $X_1, X_2, .., X_n$ with p.d.f. $f(x)$
2. Return $\hat{\theta}_n = \frac{1}{n} \sum_{i=1}^{n} g(X_i)$

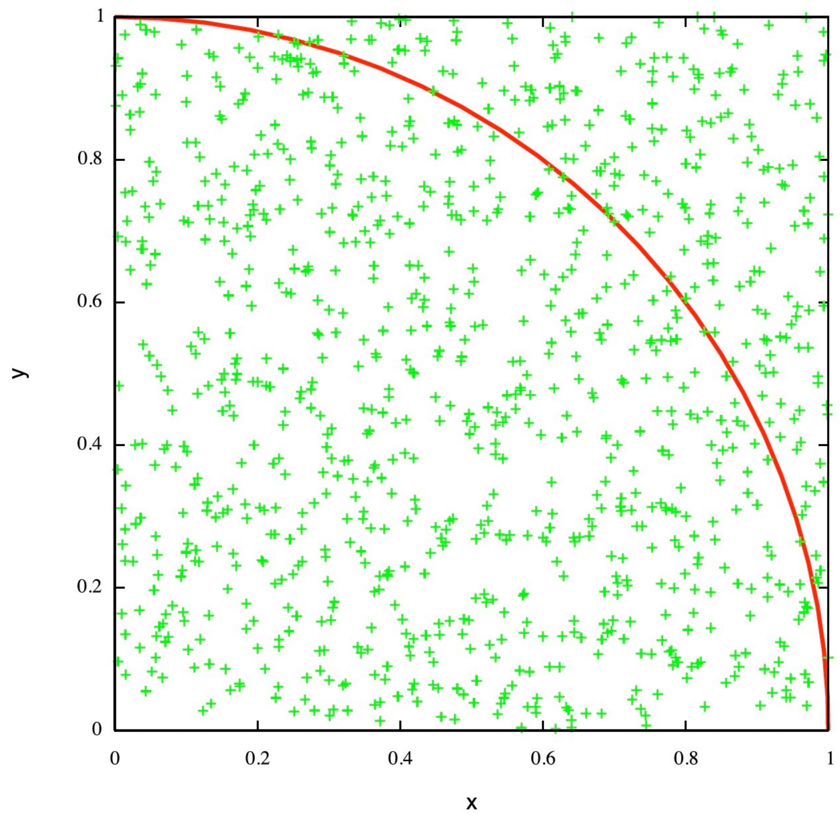
COMPUTING π VIA MONTE CARLO

- A typical example of the use of Monte Carlo estimator is the calculation of the number π.
- How can we use Monte Carlo for this computation?
- Note: the area A of a circle D of radius r=1 is given by $A = \pi r^2 = \pi$.
- On the other hand, the area of the square S of side 2 that encloses the circle is equal to 4.
- Can we use this information?



COMPUTING π VIA MONTE CARLO

- Idea: Randomly fill the square S. Then the ratio of points that falls inside a circle must be proportional to π.



STRATEGY FOR MONTE CARLO ESTIMATION

- In other words:

$$\text{Area of circle} = \iint_D dx\,dy = \pi$$

$$\text{Area of square} = \iint_S dx\,dy = 4$$

$$\frac{\text{Area circle}}{\text{Area square}} = \frac{\pi}{4} = \frac{\iint_D dx\,dy}{\iint_S dx\,dy} = \iint I\big((x,y) \in D\big) \cdot \frac{1}{4} = E\big(g(x,y)\big)$$

STRATEGY FOR MONTE CARLO ESTIMATION

- We have that $\frac{\pi}{4}$ is the expected value of the function $g(x, y) = \mathrm{I}\big((x, y) \in S\big)$.
- Since $S = (-1,1)\times(-1,1)$, we need to sample x and y uniformly on the interval $(-1,1)$.
- How can we do that? Set $\mathrm{X} = 2U - 1$ and $\mathrm{Y} = 2V - 1$ where $U \sim Unif((0,1))$ and similarly for V.

EXAMPLE CODE

```r
n <- 1000
x <- array(0,c(2,1000))
t <- array(0,c(1,1000))

for(i in 1:1000){
    # generate point in square
    # x coord
    x[1,i] <- 2*runif(1)-1
    # y coord
    x[2,i] <- 2*runif(1)-1

    # compute g(x,y); test if it is inside of circle of radius 1
    if(x[1,i]*x[1,i]+x[2,i]*x[2,i] <= 1){
        t[i] <- 1
    }
    else{
        t[i] <- 0
    }
}
print(sum(t)/n*4)
```

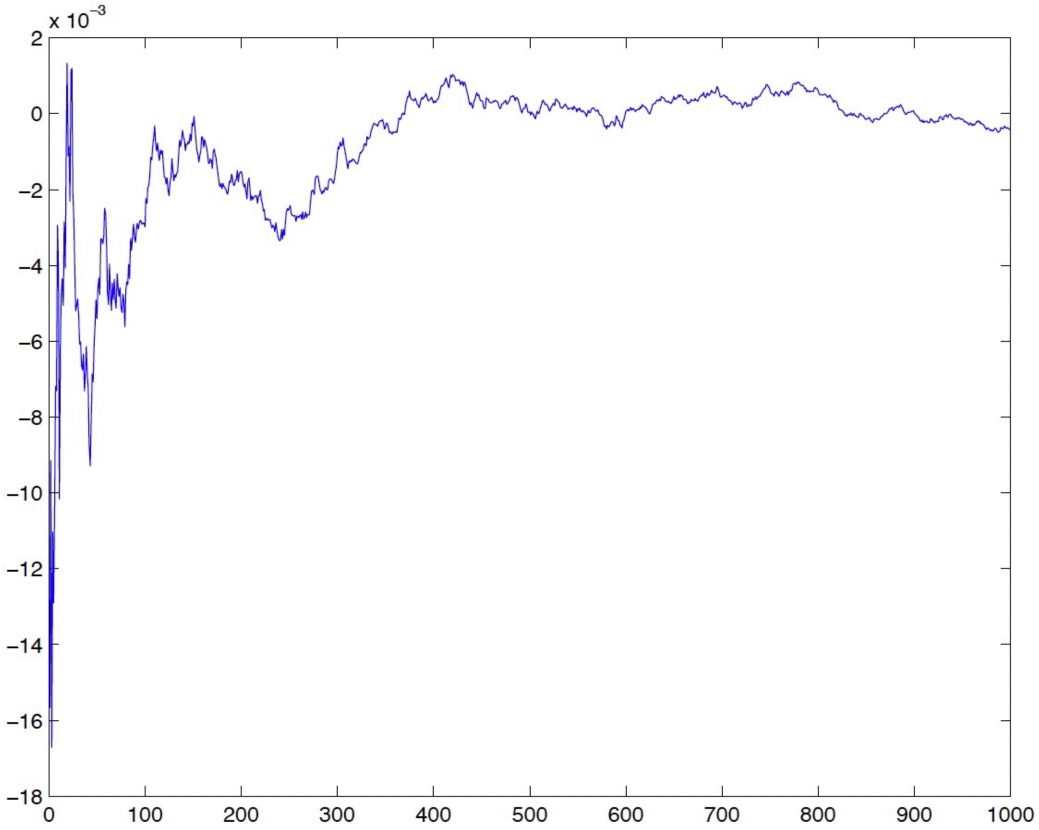
MONTE CARLO ERROR

- The Monte Carlo estimator $\hat{\boldsymbol{\theta}}_{\boldsymbol{n}}$ will have an error in estimating $\boldsymbol{\theta}$. We call this error the Monte Carlo error. And it is defined as the difference

$$\mathit{Monte\ Carlo\ Error} = \hat{\theta}_n - \theta$$

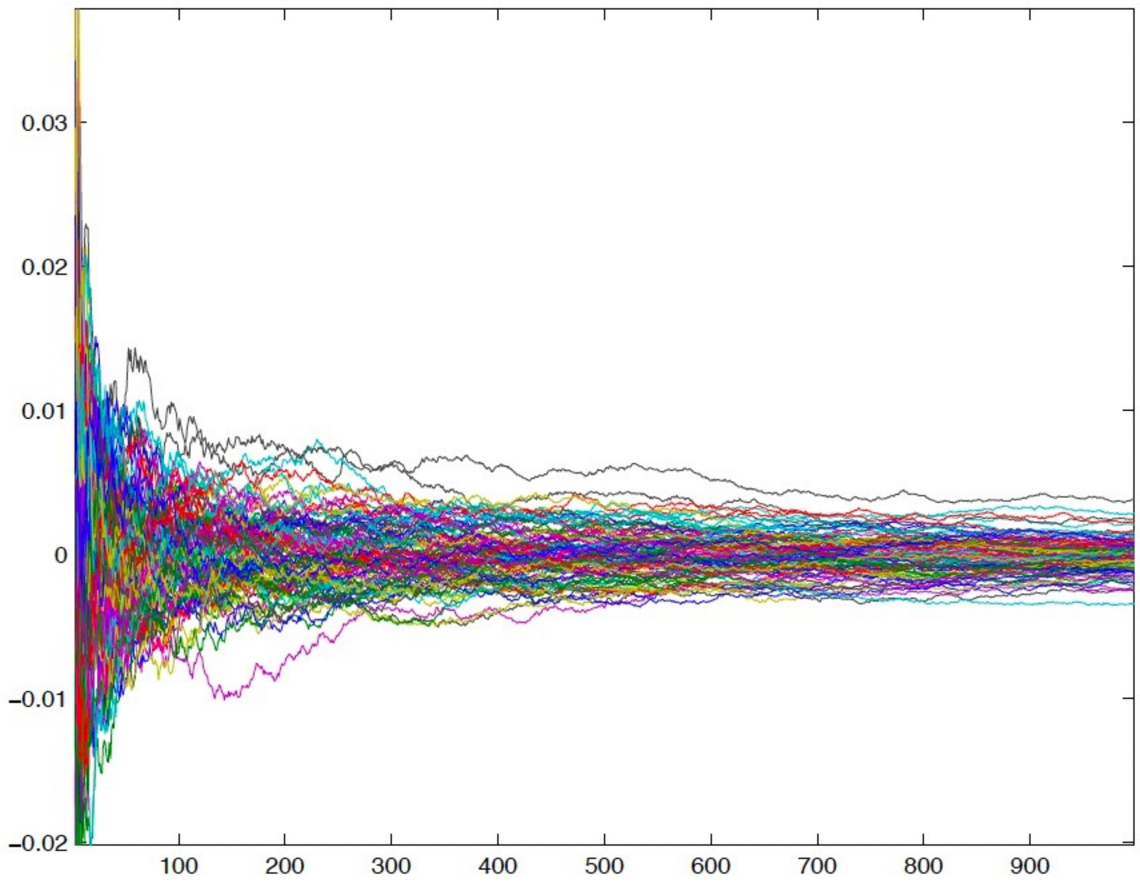
MONTE CARLO ERROR

- Monte Carlo Error in the estimation of π as a function of n, the number of simulations.



MONTE CARLO ERROR

- Monte Carlo Error in estimating π as a function of n, the number of simulations, for 100 independent realizations.



USING MONTE CARLO FOR THE QUEUING SYSTEM (PART I)

- How to solve the problem of the queueing system, and calculate the probability that at 5:30pm there are no unattended customers in the store?

USING MONTE CARLO FOR THE QUEUING SYSTEM (PART II)

- Idea: simulate a large number of n of days using the model for the arrival of customers and for the service time and compute

$$\hat{\theta}_n = \frac{1}{n}\sum_{i=1}^{n} I(X_i = 0)$$

where X_i is the number of customers in the store at 5:30pm in the i-th day.

HOW GOOD IS A MONTE CARLO ESTIMATOR?

- If X is a random variable, g(x) is a function and $\boldsymbol{\theta = E(g(X))}$ exists, then, if we denote with $\boldsymbol{\hat{\theta}_n}$ the Monte Carlo estimator, $\boldsymbol{\hat{\theta}_n}$ has the following properties:

1. Unbiasedness: $\boldsymbol{E(\hat{\theta}_n) = \theta}$
2. <u>Strong consistency</u>: As $\boldsymbol{n \to \infty, \hat{\theta}_n \to \theta}$ <u>almost surely</u>.

HOW GOOD IS A MONTE CARLO ESTIMATOR?

- We can also prove a Central Limit Theorem for the Monte Carlo estimator $\hat{\theta}_n$, and show that

$$\frac{\sqrt{n}}{\sigma} \cdot \left(\hat{\theta}_n - \theta\right) \xrightarrow{\mathrm{d}} N(0,1)$$

Where $\sigma^2 = Var(\theta) = Var(g(X))$.