

The background of the slide is a complex network graph. It consists of numerous small, light blue circular nodes connected by thin, grey lines. The nodes are distributed across the entire slide, with a higher density in the upper left and lower right areas. The lines connecting the nodes form a web-like structure, with some nodes having many connections and others having fewer.

ISI-BUDS PROBABILITY - PART IV

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PART V PLAN:

- Practical implications of a Monte Carlo method
- Introducing 3 algorithms:
 - Inversion method
 - Transformation method
 - Reject sampling

A KEY NEED TO IMPLEMENT MONTE CARLO METHOD

- We have seen that the Monte Carlo estimator provides us with an approximation of the expected value of a function $g(X)$ of any random variable X .
- We can calculate probabilities using expected values of appropriate functions of random variable (e.g. indicator functions).
- To implement Monte Carlo estimation, we ought to be able to simulate random values from distributions!

HOW TO SIMULATE RANDOM VARIABLES

- How do we simulate values of a random variable X that follows a certain distribution with p.d.f. $f(x)$?
- If the distribution has a name, then we can use R and the appropriate function of the form "*r+name of distribution*", such as `runif`, `rgamma`, `rnorm`, `rbinom`, `rpoiss`, `rexp`, `rbeta`, etc
- What if the random variable X follows a distribution for which we don't know the name? How do we simulate values for the random variable X ?

INVERSION METHOD

- There are different algorithms that can be used to simulate random variables.
- The inversion method is one of the most commonly used methods to simulate random variables and perhaps the simplest to implement.
- To use this method, we ought to go back to c.d.f.'s.

PROPERTIES OF CDF'S

- We have seen the definition of a c.d.f. starting from a random variable X :

$$F(x) = P(X \leq x).$$

- What guarantees that a function $F(x)$ is a c.d.f.? What are the properties of a c.d.f.?
 - $F(x)$ is increasing. [Why?]
 - $F(x)$ is right continuous, that is $F(x + \varepsilon) \rightarrow F(x)$ as $\varepsilon \rightarrow 0$
 - $F(x) \rightarrow 0$ as $x \rightarrow -\infty$ while $F(x) \rightarrow 1$ as $x \rightarrow \infty$.

THE CDF OF A RANDOM VARIABLE HAS A UNIFORM DISTRIBUTION!

- **Proposition:** If $F(x)$ is a continuous and strictly increasing function on \mathbb{R} . Let X be a random variable with c.d.f. $F(x)$. Suppose $F(x)$ has an inverse $F^{-1}(x)$.

The random variable $F(X)$ follows a uniform distribution on $[0,1]$.

GENERATING RANDOM VARIABLES FROM ANY CDF

- This result is at the basis of the Inversion method to generate random variables from a given distribution.
- **Proposition:** Let $F(x)$ be a continuous and strictly increasing c.d.f. on \mathbb{R} with inverse function $F^{-1}(x)$, which maps the interval $[0,1]$ onto \mathbb{R} .

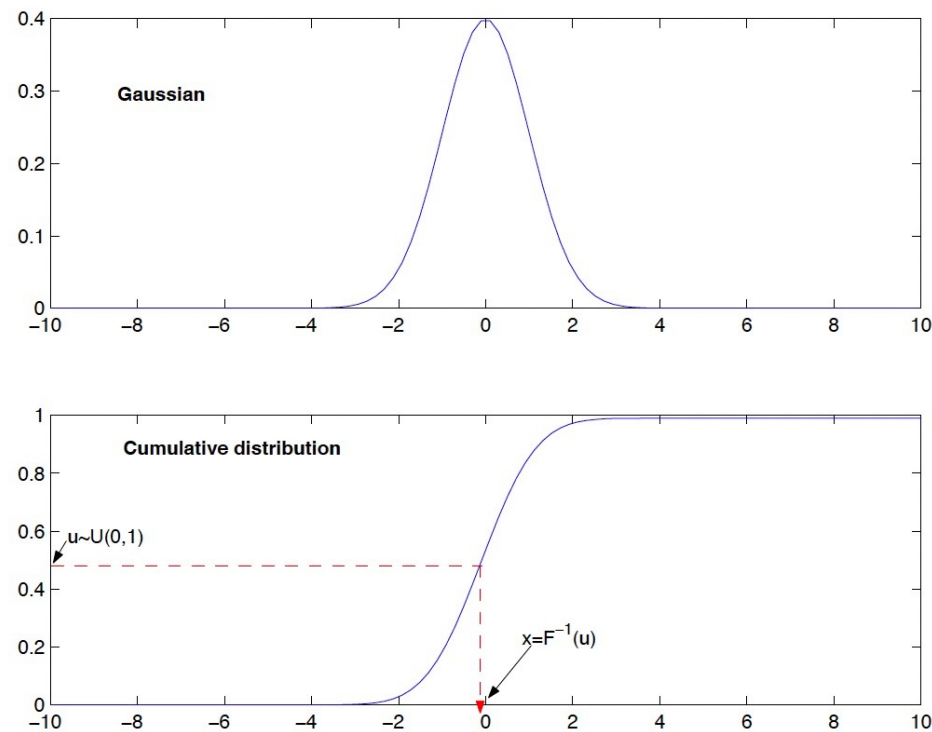
Let $U \sim \text{Uniform}([0,1])$, that is, let U be a uniform random variable on $[0,1]$, then $X = F^{-1}(U)$ is a random variable with c.d.f. $F(x)$.

GENERAL INVERSION METHOD ALGORITHM

A general inversion method algorithm then would be structured as follows:

- Given a c.d.f $F(x)$, calculate $F^{-1}(x)$
- Simulate independent $U_i \sim \text{Uniform}(0,1)$
- Return $X_i = F^{-1}(U_i)$

ILLUSTRATIVE EXAMPLE



Top: pdf of a Gaussian r.v., bottom: associated cdf.

EXPONENTIAL DISTRIBUTION EXAMPLE

- Let $\lambda > 0$. The exponential CDF is given by

$$F(x) = 1 - e^{-\lambda x}$$

- Generate values of a random variable following an exponential distribution!

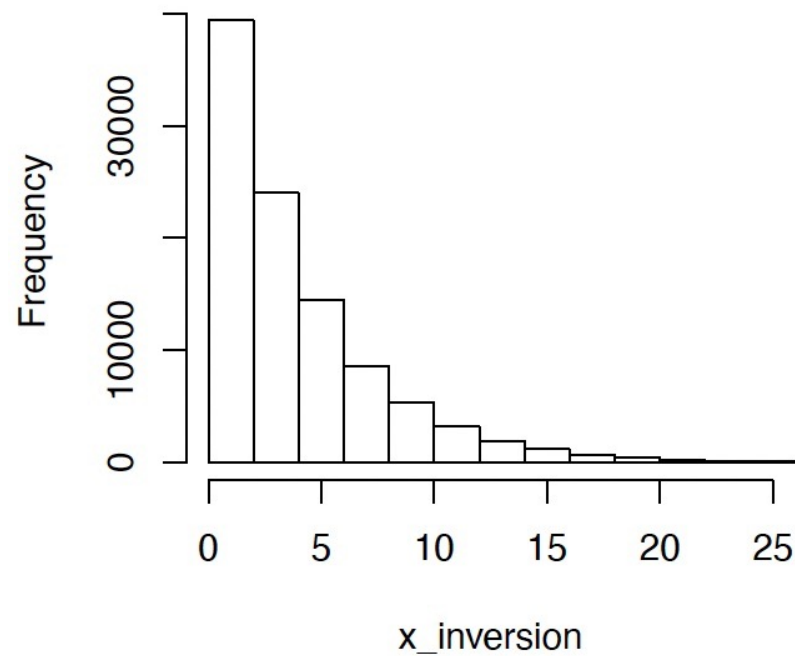
EXPONENTIAL R.V.'S USING THE INVERSION METHOD

```
# Generating values following an exponential distribution
lambda <- 0.25
n <- 10000
u <- runif(n)
x.inv <- -log(1-u)/lambda
x.rexp <- rexp(n,rate=lambda)

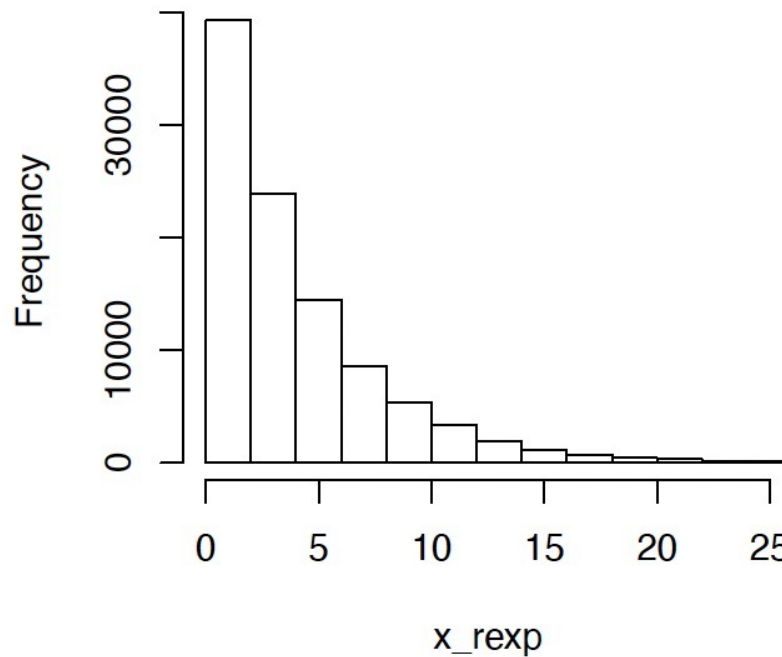
# Checking if the two samples are different
wilcox.test(x.inv,x.rexp)$p.value
```

EXPONENTIAL R.V.'S USING THE INVERSION METHOD

Histogram of x_inversion



Histogram of x_rexp



ANOTHER EXAMPLE: LOGISTIC DISTRIBUTION

- The **logistic distribution** has p.d.f. and c.d.f. given respectively by:

$$f(x) = \frac{\exp(-x)}{(1+\exp(-x))^2}$$

$$F(x) = \frac{1}{1+\exp(-x)}$$

- How would we **simulate a random sample from a logistic distribution?**

HOW TO HANDLE DISCRETE R.V.'S?

- Simulating discrete random variables with the method of inversion requires slight more care.
- Why?

GENERALIZED INVERSE

- For discrete random variables, we don't apply the inverse of the c.d.f. to a sample of values drawn from the uniform distribution. Rather, what we use is the generalized inverse to the c.d.f.
- Let $F(x)$ be a c.d.f. on \mathbb{R} . We define the generalized inverse $F^{-1}: [0, 1] \rightarrow \mathbb{R}$ as the function

$$F^{-1}(u) = \inf\{x \in \mathbb{R}: F(x) = u\}.$$

Then, if $U \sim \text{Uniform}(0,1)$, $X = F^{-1}(U)$ has $F(x)$ as c.d.f.

DISCRETE RANDOM VARIABLES

- If X is a discrete random variable with p.d.f. $P(X = k) = p(k)$, then the c.d.f. of X is defined as:

$$F(x) = \sum_{k=0}^{\lfloor x \rfloor} P(X = k) = \sum_{k=0}^{\lfloor x \rfloor} p(k)$$

and its generalized inverse is defined as follows: $F^{-1}(u)$ is $x \in \mathbb{N}$ such that

$$\sum_{k=0}^{x-1} p(k) < u \leq \sum_{k=0}^x p(k)$$

with the LHS equal to 0 if $x=0$.

EXAMPLE

```
p <- c(0.5, 0.3, 0.2) ## pmf
p_norm <- c(0, cumsum(p)) ## 0.0 0.5 0.8 1.0
m <- length(p)
n <- 100000
u <- runif(n)
x <- array(NA, n)
for(i in 1:n) {
  for(j in 1:m) {
    if ((p_norm[j] < u[i]) & (u[i] <= p_norm[j + 1])) {
      x[i] <- j
    }
  }
}
sum(is.na(x)) ## 0
table(x)
##      1      2      3
## 50227 30105 19668
```

TRANSFORMATION METHOD

- Suppose we have a random variable Y that follow some distribution with c.d.f. F_Y and defined on some space S_Y , which we know how to simulate.
- Suppose that there is a random variable X that follows some distribution with c.d.f. F_X and defined on some space S_X , that we wish to simulate.
- If we can find a function $\varphi: S_Y \rightarrow S_X$, with the property that if Y follows the distribution with c.d.f. F_Y , then $X = \varphi(Y)$ follows the distribution with c.d.f. F_X , we can simulate the random variable X , by simulating first Y and then applying the transformation φ onto the sampled values for Y .

TRANSFORMATION METHOD GENERAL ALGORITHM

- The transformation method general algorithm to generate samples from a given distribution will be as follows:
 1. Find $Y \sim F_Y$ that you can simulate from, and a function φ such that $X = \varphi(Y) \sim F_X$.
 2. Simulate independently $Y_i \sim F_Y$.
 3. Return $X_i = \varphi(Y_i) \sim F_X$.

EXAMPLE: EXPONENTIAL TO GAMMA

- **Note:** Let $Y_i, i=1,2,\dots,\alpha$ be independent random variables that follow an Exponential distribution with parameter $\lambda = 1$, that is: $Y_i \sim \text{Exp}(1)$.
- Denote by $X = \frac{1}{\beta} \sum_{i=1}^{\alpha} Y_i$, then $X \sim \text{Gamma}(\alpha, \beta)$.
- Use this result to devise an algorithm to sample values from a $\text{Gamma}(7,10)$ distribution.

REJECTION SAMPLING: MAIN IDEA

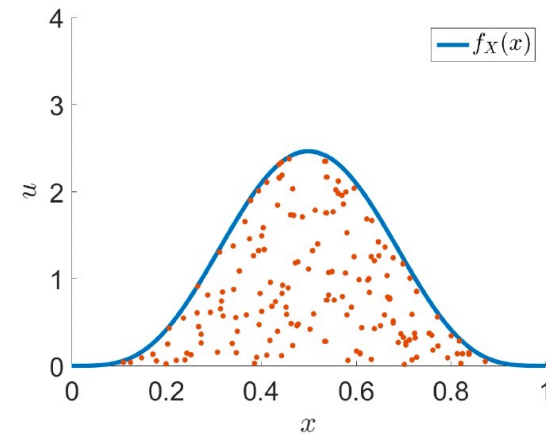
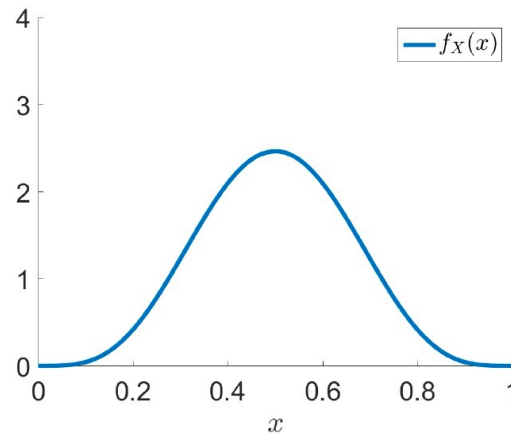
- Another algorithm to simulate random variables is based on a rejection sampling idea.
- The method consists into drawing random variables from a larger space defined by a proposal p.d.f., and "reject" those values that are not in the region defined by the target p.d.f.

FUNDAMENTAL THEOREM OF SAMPLING

Theorem (Fundamental Theorem of simulation)

Let X be a rv on Ω with pdf or pmf f_X . Simulating X is equivalent to simulating

$$(X, U) \sim \text{Unif}(\{(x, u) | x \in \Omega, 0 < u < f_X(x)\})$$



REJECTION SAMPLING IDEA

- ▶ Direct sampling of (X, U) uniformly over the set \mathcal{A} is in general challenging
- ▶ Consider some superset \mathcal{S} such that $\mathcal{A} \subseteq \mathcal{S}$, such that simulating uniform rv on \mathcal{S} is easy
- ▶ Therefore, a uniform distribution on \mathcal{A} can be obtained by drawing from a uniform distribution on \mathcal{S} , and *rejecting* samples in \mathcal{S} not in \mathcal{A}
- ▶ Rejection sampling technique:
 1. Simulate $(Y, V) \sim \text{Unif}(\mathcal{S})$, with simulated values y and v
 2. if $(y, v) \in \mathcal{A}$ then stop and return $X = y, U = v$,
 3. otherwise go back to 1.
- ▶ The resulting rv (X, U) is uniformly distributed on \mathcal{A}
- ▶ X is marginally distributed from f_X

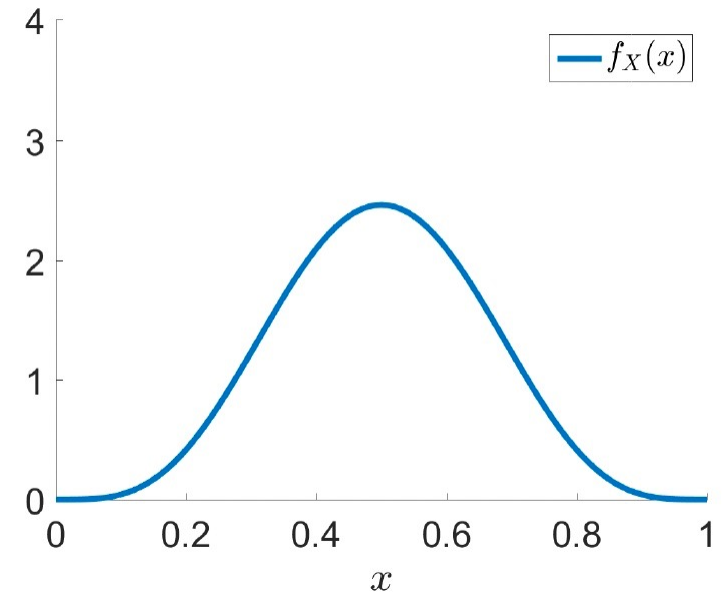
EXAMPLE: BETA DENSITY

- Let $X \sim \text{Beta}(\alpha = 5, \beta = 5)$, then the p.d.f. is given by:

$$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} \cdot (1-x)^{\beta-1}$$

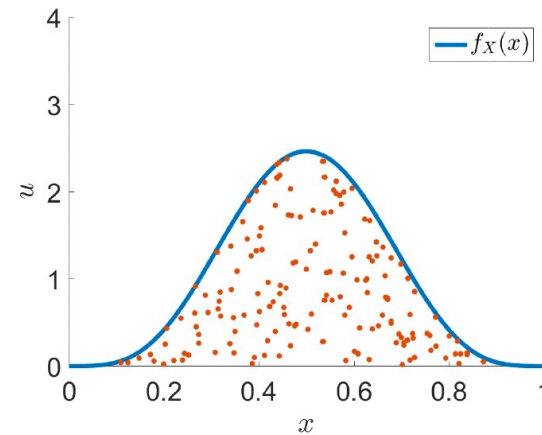
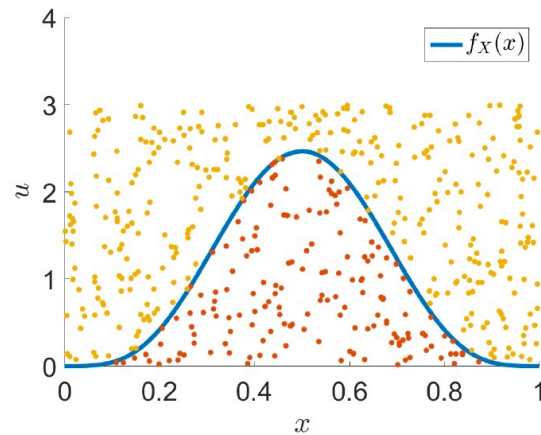
for $0 < x < 1$ and with $\alpha = 5, \beta = 5$.

Then, $f(x)$ is bounded above by 3 on $[0, 1]$.



EXAMPLE: BETA DENSITY

- ▶ Let $\mathcal{S} = \{(y, v) | y \in [0, 1], v \in [0, 3]\}$
 1. Simulate $Y \sim \mathcal{U}([0, 1])$ and $V \sim \mathcal{U}([0, 3])$, with simulated values y and v
 2. If $v < f_X(x)$, return $X = x$
 3. Otherwise go back to Step 1.
- ▶ Only requires simulating uniform random variables and evaluating the pdf pointwise



REJECTION SAMPLING MORE PRECISELY

- In general: consider a random variable X defined on a space S_X with p.d.f. $f(x)$, the **target distribution**.
- We want to **sample from $f(x)$** using a **proposal p.d.f. $q(x)$** from which can sample easily.
- We find a constant M such that $\frac{f(x)}{q(x)} \leq M$ for all $x \in S$.
- The following rejection algorithm, returns samples for a random variable X with p.d.f. $f(x)$.

REJECTION SAMPLING GENERAL ALGORITHM

1. Identify a **proposal c.d.f.** $Q(x)$ from which it is **easy to sample** with p.d.f. $q(x)$. Find M such that $\frac{f(x)}{q(x)} \leq M$ for all $x \in S$.
2. Simulate $Y_i \sim Q$ and $U_i \sim \text{Unif}((0,1))$.
3. If $U_i \leq \frac{f(Y_i)/q(Y_i)}{M}$, then return $X_i = Y_i$, otherwise do not return anything.