ISI-BUDS
PROBABILITY - PART I

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July 11, 2022

PART I PLAN:

- Reviewing probability laws
- Defining Random variables
- Looking at some discrete distributions

PROBABILITY IS EVERYWHERE!

- Many aspects of our life are surrounded by random circumstances, that is situations where *the outcome is not determined until we observe it*.
- Example: you are taking a medical test for a disease (e.g. COVID).
  - What are the random circumstances here? And what are the possible outcomes?

ASSIGNING PROBABILITIES TO OUTCOMES

- When dealing with random circumstances, we often use the word probability.
- What is a probability? A number between 0 and 1!
- What does it indicate? ....
- How do we assign probabilities?
  - Relative frequency or the *proportion of times an event/outcome will occur in the long run*
  - Subjective/personal interpretation of probability: the *degree to which an individual believes that an event will occur.*

EXAMPLE: BOY OR A GIRL?

- What is the probability that the next baby born at UCI Health Hospital is a boy?

EXAMPLE: BOY OR A GIRL?

What if I tell you that you have access to data on the number of boys that are born in the same hospital over the previous 52 weeks, and the data look as follows

Week no.	Total no. births	Total no. of boys
1	30	19
4	116	68
13	317	172
26	623	383
39	919	483
52	1237	639

EXAMPLE: BOY OR A GIRL?

Based on this information, would you change what you assign as the probability of a boy?

PROBABILITY DEFINITIONS AND RELATIONSHIPS (I)

- A random circumstance can result in many different outcomes. The set of all possible outcomes is called the sample space.
- A simple event indicates one of the possible outcomes.
- A compound event indicates at least two simple events.
- We denote with $P(A)$ the probability that event A occurs.

PROBABILITY DEFINITIONS AND RELATIONSHIPS (II)

- Probabilities must satisfy certain rules:
  - The probability of each event must be between 0 and 1;
  - The sum of the probabilities of all possible simple events is 1.

- If a random circumstance results in k possible outcomes, all equally likely simple events, *what is the probability of each event*?

EXAMPLE: PLAYING THE LOTTERY

In a lottery game, a player chooses a 3-digit number between 000 and 999. If the number matches the number of the day, the player wins.

What is the sample space? How many simple events are there?

What is the probability that any specific 3-digit number is a winner?

What is the probability that the winning number has all 3 same digits (e.g. 111, 222, etc.)?

PROBABILITY DEFINITIONS AND RELATIONSHIPS (III)

- One event is called the complement of another event if the two events do not contain any of the same simple events and together they cover the entire sample space.
- For an event A, the complement is denoted by A^C.
- A and A^C are called complementary events.
- If we know the probability of event A, P(A), what is the probability of its complement A^C, $P(A^C)$?

PROBABILITY DEFINITIONS AND RELATIONSHIPS (IV)

- Two events are called mutually exclusive or disjoint if they do not contain any of the same simple events.
- If two events A and B are mutually exclusive, the probability that either event A or event B occurs is the sum of the probabilities of the two events, that is:

$$\boxed{P(A \text{ or } B) = P(A) + P(B)}$$

- Note that this is not true in general.

EXAMPLE: PLAYING THE LOTTERY

Consider again the lottery game we described before. Consider these two events:

A = the three digits are the same
B = the first and third digit are different

Are A and B mutually exclusive?

Are A and B complementary events?

PROBABILITY DEFINITIONS AND RELATIONSHIPS (V)

- Two events are independent of each other if knowing that one event will occur (or has occurred) does not change the probability that the other event occurs.
- Two events are dependent if knowing that one event will occur (or has occurred) changes the probability that the other occurs.

EXAMPLE: WINNING A FREE LUNCH

Some restaurants display a glass bowl in which customers deposit their business cards, and a drawing is held once a week for a free lunch. The bowl is emptied after each week's drawing. Suppose that you and your friend each deposit a card in two consecutive days. We call

Event A= You win in week 1

Event B = Your friend wins in week 1

Event C = Your friend wins in week 2

Are events A and B independent? Are A and C independent? Are B and C independent?

PROBABILITY DEFINITIONS AND RELATIONSHIPS (VI)

- The conditional probability of event B, given that event A has occurred or will occur, is the long-term relative frequency with which event B occurs when circumstances are such that A has occurred or will occur.
- The conditional probability of B given A is denoted by P(B | A), and it is calculated as:

$$\boxed{P(B\,|A) = \frac{P(A \text{ and } B)}{P(A)}}$$

EXAMPLE: DIFFERENCES IN GAMBLING

Based on a survey of all ninth- and twelfth-graders in Minnesota public schools, a researcher reported proportions admitting that they had gambled at least once a week during the previous year. He found that these proportions differed considerably for males and females and for ninth- vs twelfth-graders.

One results the researcher found was that 22.9% of the ninth-grader boys and 4.5% of the ninth-grade girls admitted gambling at least weekly.

What do those two probabilities represent?

RULE 1: COMPLEMENT RULE

- To find the probability of A^C, the complement of A, we use:

$$\boxed{P(A^C) = 1 - P(A)}$$

- This rule is used often to calculate probabilities of events involving the word "at least".

Specifically: $\boxed{P(\text{at least once}) = 1 - P(\text{none})}$

RULE 2: ADDITION RULE

- To find the probability that either event A or event B or both happen, we use:

$$\boxed{P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)}$$

This is the general formula.

- If events A and B are mutually exclusive:

$$\boxed{P(A \text{ or } B) = P(A) + P(B)}$$

RULE 3: MULTIPLICATION RULE

- To find the probability that two events A and B both occur simultaneously, we use:

$$\boxed{P(A \text{ and } B) = P(A) \times P(B \mid A) = P(B) \times P(A \mid B)}$$

This is the general formula.

- If events A and B are independent events, then: $\boxed{P(A \text{ and } B) = P(A) \times P(B)}$
- If $A_1, A_2, \ldots, A_n$ are n independent events, then: $\boxed{P(A_1 \text{ and } A_2 \text{ and} .. A_n) = P(A_1) \times P(A_2) \times .. \times P(A_n)}$

BAYES' RULE

- Bayes rule is a famous theorem at the basis of Bayesian statistics.
- Mathematically very simple, but with very deep ramifications.
- If we have events A and B, and we know the conditional probability P(B | A) of event B given A, and we know the probabilities P(A) and P(B), we could calculate the conditional probability P(A | B) of event A given B as:

$$\boxed{P(A\,|B) = \frac{P(B\,|A) * P(A)}{P(B)}}$$

SOME PROBLEMS

1. Aparna drives to school for her class on M-W-F. She prefers to park in Parking Lot A because it is next to her classroom building, but she is not always able to find a spot there. If it's sunny, the probability that she finds a spot is 0.7, but if it is raining more people drive, so the probability that she finds a spot is only .4. Whether she finds a spot in Lot A is independent from one day to the next.

a) On a sunny day, what is the probability that Aparna does not find a spot in Lot A?

b) In a week that is sunny on Monday and Friday and raining on Wednesday, what is the probability that Aparna does not find a spot in Lot A on any of the 3 days?

c) In a week that is sunny on Monday and Friday and raining on Wednesday, what is the probability that Aparna finds a spot in Lot A on at least one of the 3 days?

SOME PROBLEMS

2. Suppose that there are 30 people in your statistics class and you are divided into 15 teams of 2 students each. You happen to mention that your birthday was last week, upon which you discover that your teammate's mother has the same birthday as yours (month and day, not necessarily the same year!). Assume that the probability is 1/365 for any given day.

a) What is the probability that your teammate's mother would have the same birthday as yours?

b) Suppose that your teammate has two siblings and two parents, for a family of size 5. What is the probability that at least one of your teammate's family has the same birthday as yours, assuming their birth dates are all independent?

A GAME SHOW PROBLEM

Suppose you are on a game show, and you are given the choice of 3 doors: behind one door is a car, and behind the other two doors are two goats. You pick a door, say door No. 1, and the host, who knows what's behind the doors, open another door, say door No. 3, which has a goat. He then says to you: "Do you want to pick door no. 2?".

Is it your advantage to switch your choice? Why?

HOW DO THEY KNOW IT?

A prize is place under one of 5 objects. Alexis is privately told the shape of the object and Jose' is privately told the color of the object. Alexis and Jose' are mathematicians who use perfect logical reasoning, and the initial setup is common knowledge. The host asks Alexis and Jose': "Do either of you know where the prize is?". Both of them are silent. The host then asks: "Do you know now?". Again, they are both silent. Finally, the host asks: "Do you know now?". Alexis and Jose' simultaneously say "YES!".

How do they know?