ISI-BUDS
Generalized Linear Models

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Linear models

- Linear models have been extensively used in practice.
- They include a large class of models such as ANOVA and linear regression.
- They owe their popularity mostly to the fact that they are **easy to fit and interpret**.
- We use these models to capture the relationship between the **response** variable, y, and a set of **explanatory** variables (predictors, covariates, ...), x.
- What does it mean for two random variables to be related?
- When we talk about relationship between y and x, we usually think about the change in the **conditional distribution** of y given x, i.e., $P(y|x)$, as x changes.

Relationship

- Regression models are based on the assumption that the only change in the conditional distribution we are interested in is the change in the **expectation** of the distribution, $E(y|x)$ (note that this by itself imposes limitations on the type of relationships we can detect).
- In general, this means $E(y|x) = g(x)$, and the relationship between x and y exists if $g(x)$ is not a constant function.
- In this setting, $g(x)$ also defines the type of relationship between x and y.

Linear regression models

- For linear regression models, $g(x)$ is a linear function in terms of model parameters, β.
- Recall that a function $f : \mathcal{R}^n \to \mathcal{R}^m$ is **linear** if
  - $f(z+t) = f(z) + f(t), \quad \forall z, t \in \mathcal{R}^n$
  - $f(az) = af(z), \quad \forall z \in \mathcal{R}^n, \forall a \in \mathcal{R}$
- The function $g(x)$ has the following general form:

$$g(x) = x\beta$$

  where x is a $n \times (p+1)$ matrix (the first column is the constant 1, and the remaining p columns are the observed value of p explanatory variables)
- β is a $(p+1)$-vector of parameters. The first element of this vector is the intercept, and the remaining parameters are called regression coefficients.

Linear regression models

- In regression terminology, $\epsilon = y - g(x)$ is called the **error**, which is a random variable assumed to be independent of x.
- We can therefore write the relationship between the response variable y and the explanatory variables x as follows:

$$y = g(x) + \epsilon$$

- For the observed data, we usually refer to the corresponding values of ϵ as **residuals**.

Least squares method

- There are many ways to estimate β, one of the most popular approach is the method of **least squares**, which is in general an optimization problem with no constraints

$$\text{minimize}||y - x\beta||_2^2$$

- Recall that ℓ_2-norm (Euclidean norm) is defined as

$$||z||_2 = (|z_1|^2 + |z_2|^2 + ... + |z_n|^2)^{1/2}$$

  In general, the ℓ_p norm ($p \geq 1$) is as follows:

$$||z||_p = (|z_1|^p + |z_2|^p + ... + |z_n|^p)^{1/p}$$

- $||y - x\beta||_2^2 = \sum_{i=1}^{n}(y_i - x_i\beta)^2$ is called **residual sum of squares**, RSS, which is a quadratic function of regression parameters, $RRS(\beta)$.

Least squares method

- To find the value of β that minimizes $RSS(\beta)$, we set the first derivative to zero,

$$\begin{aligned} \frac{\partial RRS}{\partial \beta} &= -2x'(y - x\beta) \\ \frac{\partial^2 RSS}{\partial \beta \partial \beta'} &= 2x'x \end{aligned}$$

- To have a unique solution for β, $x'x$ needs to be positive definite (x has to be full column rank).
- If this holds, the unique solution is obtained by setting the first derivative to zero

$$\begin{aligned} -2x'(y - x\beta) &= 0 \\ \hat{\beta} &= (x'x)^{-1}x'y \end{aligned}$$

Sampling distribution of parameters

- So far, we have not made any assumption regarding the distributional form of the random variables (more specifically for the response variable since x is assumed to be fixed).
- We did not need to make such assumptions if all we wanted was point estimates of regression parameters.
- Usually, we want more than point estimates.
- We, for example, want to know about variability (e.g., standard error) of the estimates.

Sampling distribution of parameters

- For this, we assume that x are fixed at the observed value and y's are uncorrelated with a constant variance; i.e., $cov(y|x) = \sigma^2 I$ (note that we have not fully specified the distribution yet).
- As the result,

$$\begin{aligned} cov(\hat{\beta}) &= (x'x)^{-1}x'[(x'x)^{-1}x']'\sigma^2 \\ &= (x'x)^{-1}x'x(x'x)^{-1}\sigma^2 \\ &= (x'x)^{-1}\sigma^2 \end{aligned}$$

- We also have

$$\begin{aligned} E(\epsilon) &= E(y) - E(E(y|x)) = E(y) - E(y) = 0 \\ var(\epsilon) &= \sigma^2 \end{aligned}$$

Estimating σ

- σ itself is almost always unknown and needs to be estimated based on the data.
- To estimate σ, we usually use the following unbiased estimator

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(y - x_i\hat{\beta})^2}{n - p - 1}$$

  We use $n - p - 1$ instead of n to make the estimate unbiased.
- The fit of the model can be measured based on $\hat{\sigma}^2$.
- For this, we use $R^2 = 1 - \frac{\hat{\sigma}^2}{S_y^2}$, which is the fraction of variance of response variable explained by the model. Here, S_y^2 is the observed variance of y.

Inference

- Note that while we could provide a measure of variability for the estimator of regression parameters, to perform statistical inference about these parameters, we need to make more assumptions about the distribution of y.
- We assume that

$$y|x, \beta, \sigma \sim N(x\beta, \sigma^2 I)$$

- Therefore,

$$\epsilon|\sigma \sim N(0, \sigma^2 I)$$

- As the result, we have

$$\begin{aligned} \hat{\beta}|\sigma &\sim N(\beta, (x'x)^{-1}\sigma^2) \\ \frac{n\hat{\sigma}^2}{\sigma^2} &\sim \chi^2(n-p-1) \end{aligned}$$

- Moreover, we can show that $\hat{\beta}$ and $\hat{\sigma}^2$ are independent.

Inference

- Using the sampling distribution of β, we can obtain the confidence interval for a given confidence level c.
- For each individual β_j (corresponding to x_j), the standard error is the square-root of the i^{th} element of the covariance matrix $(x'x)^{-1}\sigma^2$.
- The c level confidence interval for β_j can be obtained as

$$\hat{\beta}_j \pm t_c^* se(\hat{\beta}_j)$$

where t_c^* is the corresponding t-critical value based on $t(n-p-1)$ distribution.

Inference

- To test the null hypothesis $H_0 : \beta_j = 0$, we can use the following T-statistics:

$$T = \frac{\hat{\beta}_j}{se(\hat{\beta}_j)}$$

- Under H_0, T has a $\mathbf{t(n - p - 1)}$ distribution.
- If we want to test the null hypothesis with respect to a group of coefficients, i.e., $H_0 : \beta_1 = \beta_2 = ..., \beta_s = 0$ that reflects a reduced (simpler) model, we use the **F** statistic

$$F = \frac{(RSS_r - RSS)/s}{RSS/(n - p - 1)} \sim \mathcal{F}(s, n - p - 1)$$

where RSS_r is the residual sum of squares for the reduced model.

Likelihood function

- An alternative approach for estimating the parameters of linear regression model (and in general, most statistical models) is based on the **likelihood function**.
- To find the likelihood function, we first need to assume a probability distribution for the data, i.e., $P(y|\theta)$, where θ are unknown parameters.
- This distribution is based on our opinion regarding the mechanism that generates the data.
- The likelihood function is defined by plugging-in the observed data in the probability distribution and expressing it as a function of model parameters, i.e., $f(\theta, y)$.

Likelihood function

- For linear regression models, the data include the response variables y and the explanatory variables x. Therefore, in general we need to specify $P(x, y)$.
- However, since x are assumed to be fixed at their observed value, $P(x) = 1$, the joint distribution reduces to the conditional distribution of $y|x$.

$$P(x, y) = P(x)P(y|x) = P(y|x)$$

- Therefore, we only need to specify the conditional distribution of y given x.

Likelihood function

- We assume this $P(y|x)$ is a normal distribution.
- As we mentioned, we model the expectation of this distribution as a linear function of x, i.e., $E(y|x) = x\beta$, and we assume the variance of this distribution is σ^2 (which is independent of x and β).
- Therefore, assuming that the observations are independent, we have

$$y|x, \beta \quad \sim \quad (2\pi\sigma^2)^{-n/2} \exp(-\frac{\sum_{i=1}^{n}(y_i - x_i\beta)^2}{2\sigma^2})$$

- The likelihood function is specified by plugging-in the observed values of x and y in the probability distribution and expressing the result as a function of β (for now, we assume σ is fixed).

$$f(\beta) \quad = \quad (2\pi\sigma^2)^{-n/2} \exp(-\frac{\sum_{i=1}^{n}(y_i - x_i\beta)^2}{2\sigma^2})$$

Maximum likelihood estimation

- To estimate model parameters, we can find their values such that the probability of the observed data is maximum.
- For this, we maximize the likelihood function with respect to model parameters. Of course, it is easier to maximize the **log of likelihood function**, i.e., $L(\beta) = log(f(\beta))$.
- In general, this is a convex optimization problem.
- To maximize the likelihood function, we can focus on the part of the function that is related to the parameter (i.e., **kernel**).
- For linear regression models,

$$L(\beta) = -\sum_{i=1}^{n}(y_i - x_i\beta)^2 - log(2\sigma^2)$$

Maximum likelihood estimation

- For simplicity, we can also remove all the constant (not related to the parameters) parts;

$$L(\beta) = -\sum_{i=1}^{n}(y_i - x_i\beta)^2$$

- Now we can simply set the first derivative to zero (likelihood equation) to obtain the maximum likelihood estimate

$$\begin{aligned} \frac{\partial L(\beta)}{\partial \beta} &= 2\sum_{i=1}^{n} x_i(y_i - x_i\beta) \\ x'(y - x\beta) &= 0 \\ \hat{\beta} &= (x'x)^{-1}x'y \end{aligned}$$

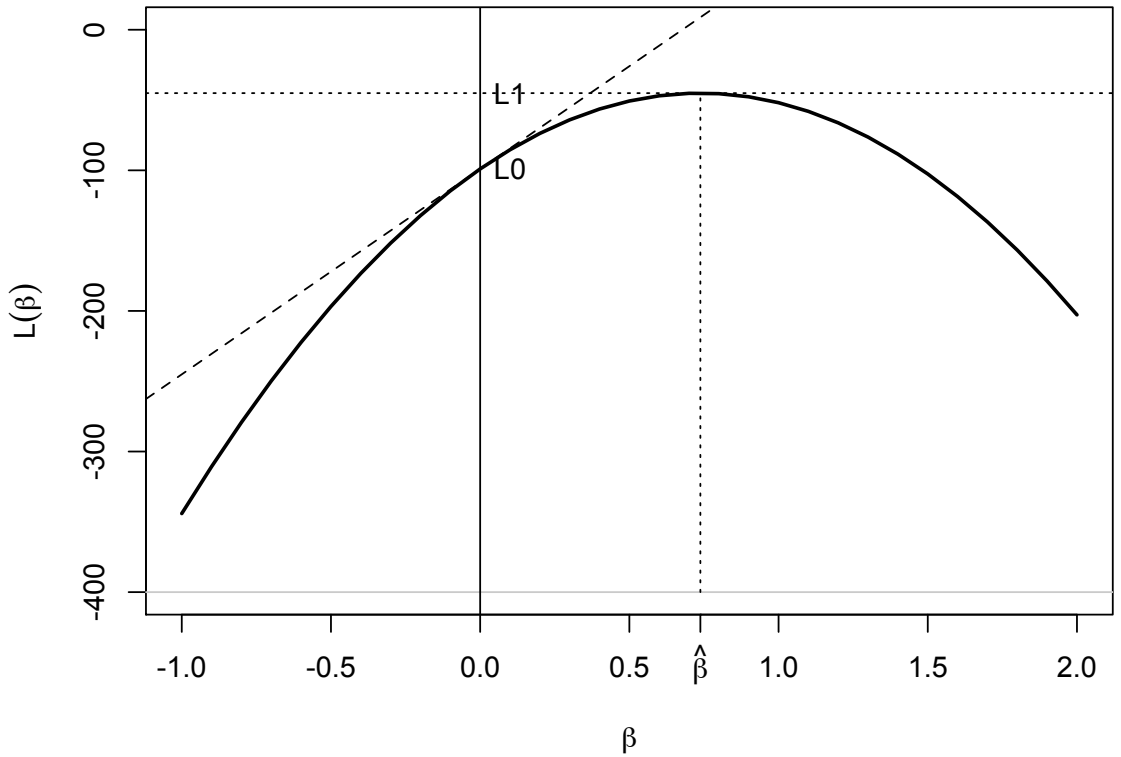
- In this case, MLE is the same as the least squares estimate.

Maximum likelihood estimation

- Under weak regularity conditions, the MLE demonstrates attractive properties as $n \to \infty$: MLE is asymptotically
  - normal
  - consistent
  - efficient

Maximum likelihood estimation

- This graphs shows the log-likelihood function and the location of MLE for randomly simulated data.



Wald, score, and likelihood ratio tests

- Using the above graph, we can create three different tests for statistical inference:
  - **Wald**: based on the distance between $\hat{\beta}$ and β_0
  - **Score**: based on difference between the slopes at $\hat{\beta}$ and β_0
  - **Likelihood ratio**: based on the distance between L_1 and L_0

Wald, score, and likelihood ratio tests

- Consider the null hypothesis $H_0 : \beta = \beta_0$, where β_0 is the value of β under the null.
- Due to large-sample normality of MLE, we have

$$w = \frac{\hat{\beta} - \beta_0}{SE(\hat{\beta})}$$

  where w has an approximately $N(0, 1)$ distribution.
- This type of statistics where we use the standard error of the estimator (as opposed to standard deviation of the null distribution) is referred to as **Wald statistic**.

Wald, score, and likelihood ratio tests

- **Score test** on the other hand is based on the slope at β_0.
- This is in fact the value of **score function**,

$$u(\beta) = \partial L(\beta)/\partial\beta$$

  evaluated at β_0.
- The dashed line in the above graph shows the slope at $\beta_0 = 0$.
- As we expect, the further β_0 is away from the MLE, the larger this slope becomes in absolute value (i.e., we can reject the null hypothesis more confidently).

Wald, score, and likelihood ratio tests

- The score statistic, s, is obtained by dividing the $u(\beta_0)$ by its corresponding standard error.
- Under the null hypothesis:

$$s \sim N(0, 1)$$

- The advantage of score test is that we do not need to estimate the maximum likelihood estimate.

Wald, score, and likelihood ratio tests

- The third test statistic is the likelihood ratio test.
- Here, we maximize the likelihood function under H_0 and under $H_0 \cup H_a$ (where H_a is the alternative hypothesis).
- The ratio of these two maximums is called the likelihood ratio test. In general,

$$LR = \frac{\sup_{\theta \in \Omega_0} f(\theta)}{\sup_{\theta \in \Omega} f(\theta)}$$

where Ω_0 is the parameter space under to H_0.

Wald, score, and likelihood ratio tests

- In general, the likelihood ratio cannot exceed 1, since the maximized value under H_0 would be less than or equal to the maximum value under $H_0 \cup H_a$.
- For hypothesis testing, we have $-2\log(LR) = -2(L_0 - L_1)$ has asymptotic χ^2 distribution with the degrees of freedom equal to the difference between the dimension of parameter space under $H_0 \cup H_a$ and under H_0.
- Here L_1 is the maximum value of log-likelihood under $H_0 \cup H_a$, and L_0 is the maximum value of log-likelihood under H_0.
- For the simple linear regression, when testing the null hypothesis, $H_0 : \beta = \beta_0$, $L_1 = L(\hat{\beta})$ and $L_0 = L(\beta_0)$.
- L_1 and L_0 (assuming $H_0 : \beta = 0$) are shown in the above figure.

What could go wrong with linear regression models

- In practice, one or more assumptions of linear regression models might be violated.
- This could result in wrong inference.
- Here, we discuss these assumptions.

Linear relationship

- Using linear models, we implicitly assume that the relationship between x and y is **linear** (note that this is general different from the linearity assumption of the function in terms of parameters; i.e., $g(x) = x\beta$).
- If the assumption of linear relationship does not hold, we might still be able to use linear regression models after some transformation of original variables.
- Typical transformations are (we could use a combination of these with the original variables)
  - $\log(x)$: For variables with positive values and heavily right-skewed distribution.
  - $\sqrt{x}$:This transformation has milder effect compared to log transformation, and it is usually recommended for counts.
  - $x^2, x^3, ...$: To create nonlinear relationships in the form of polynomial function.

Additivity of effects

- In linear regression models, the effects of explanatory variables on the response variable are assumed to be additive.
- We could of course fixe this by adding interactions terms
- We could also regress the log transformation of response variable on explanatory variables.
- This way, a model of the form,

$$\hat{y} = \beta_0 x_1^{\beta_1} x_2^{\beta_2}$$

where the effects are not additive, would become

$$\log(\hat{y}) = \log(\beta_0) + \beta_1 \log(x_1) + \beta_2 \log(x_2)$$

where the effects (on log scale) are additive

Independence and constant variance assumptions for errors

- In linear regression models, the error terms are assumed to be independent.
- In this case, the covariance matrix of error terms is not diagonal ($\sigma^2 I$) anymore, we need to use a full covariance matrix (Σ).
- Moreover, they are assumed to have equal variance. When this is not the case, we could use **weighted least squares**, where the weight of each data point is inversely proportional to its variance.
- The assumption of normality for errors is not as important as the above two.

Bounded response variable

- In linear regression analysis, we model the expected value of the response variable as a function of explanatory variables, $E(y|x) = x\beta$.
- The right had side of this equation is unbounded in general. This could cause a problem, if the left hand side, $E(y|x)$, is bounded.
- For example, if the response variable is binary, $y \in \{0, 1\}$, its expectation is between 0 and 1.
- For count variables, the expectation would be a non-negative number.

Generalized linear model

- To deal with some of these issues, we need a more flexible family of models.
- The class of generalized linear models (GLM), that includes linear models as a special case, provides such flexibility while it is still easy to use.
- Generalized linear models have three components:
  - A **random** component
  - A **systematic** component
  - A **link** function

Generalized linear model

- The random component identifies the response variable and its probability distribution.
- In most situations, we assume parametric model $P(y|\theta)$ for the distribution of y from the **exponential family**.
- Recall that the exponential family includes most of the well-known distributions such as normal, binomial, multinomial and Poisson.
- In general, if the outcome variable is continuous and real-valued, we use the **normal** distribution.
- If the outcome is binary, we use the **binomial** distribution. For outcome variables with multiple categories, we use the **multinomial** instead.
- If the outcome variable represent counts data, we use the **Poisson** distribution.

Generalized linear model

- The systematic component specifies the set of predictors (i.e., explanatory variables) $x = (x_1, ..., x_p)$ used in a **linear predictor** function.
- As before, we also append a vector of ones at the beginning of x.
- In the matrix form, the linear predictor function $\eta = x\beta$, where $\beta = (\beta_0, \beta_1, ..., \beta_p)$.
- Alternatively,for each observation i, where $i = 1, ..., n$, the linear predictor function is $\eta_i = \beta_0 + \sum_j^p x_{ij}\beta_j$.
- Also, as before, some of predictors could be a transformation (e.g., x^2) of original predictors.

Generalized linear model

- The link function is a monotonic differentiable function that connects the random and systematic components.
- More specifically, if $\mu = E(y|x)$, the link function g connects μ to η such that $g(\mu_i) = \eta_i = \beta_0 + \sum_{j=1}^{p} x_{ij}\beta_j$ for each observation i.
- For the ordinary linear model we discussed before, the link function is identity: $g(\mu_i) = \mu_i$. That is $\mu_i = \eta_i = x_i\beta$.

Logistic regression model

- As mentioned before, for binary outcome variable, we use the binomial distribution.

$$y_i | n_i, \mu_i \sim \text{binomial}(n_i, \mu_i)$$

  with the Bernoulli distribution as its special case when $n_i = 1$.
- As usual, we define the systematic part of the model $\eta_i = x_i\beta$ (where x_i is a row vector of all observed values for subject i, and β is a column vector of size $p + 1$).
- A common link function for this model is the **logit** function and defined as

$$g(\mu_i) = \log(\frac{\mu_i}{1 - \mu_i}) = x_i\beta$$

  where μ_i is the probability of success (i.e., $y_i = 1$).
- As the result

$$\mu_i = \frac{\exp(x_i\beta)}{1 + \exp(x_i\beta)}$$

Logistic regression model

- The likelihood is therefore defined in terms of β as follows:

$$
\begin{aligned}
p(y|\mu) &\propto \prod_{i=1}^{n} \mu_i^{y_i}(1-\mu_i)^{n_i-y_i} \\
p(y|\beta) &\propto \prod_{i=1}^{n} \left(\frac{\exp(x_i\beta)}{1+\exp(x_i\beta)}\right)^{y_i} \left(\frac{1}{1+\exp(x_i\beta)}\right)^{n_i-y_i}
\end{aligned}
$$

- Note that in this model the variance of $y|x$ depends on the mean and therefore will not be constant

$$
var(y_i|x_i) = \mu_i(1-\mu_i)
$$

Interpretation

- To interpret β, note that

$$\log[\frac{P(y=1|x,\beta)}{1-P(y=1|x,\beta)}] = \exp(\beta_0 + \beta_1 x_1 + \ldots + \beta_p x_p)$$

  is the log of odds for the outcome of interest, $y = 1$.
- The intercept β_0 is therefore the log of odds when the value of all covariates is 0.
- Or we can say, $\exp(\beta_0)$ is the odds when all covariates are 0 (sometimes called the baseline odds).

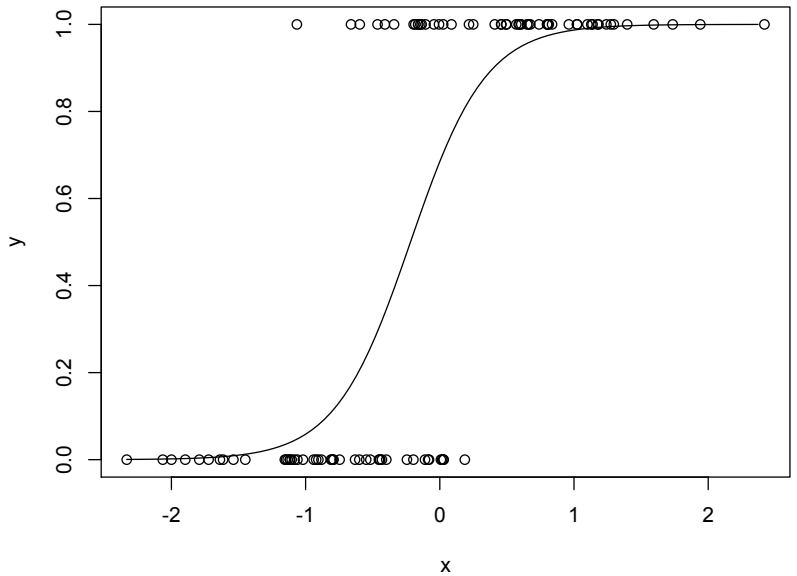
Interpretation

- $\exp(\beta_j)$ on the other hand is how much the odds multiplicatively increases for one unit increase in x_j when all other covariates are fixed.
- Or we can say, $\exp(\beta_j)$ is the odds ratio for subjects with $X_j = x_j + 1$ compared to subjects with $X_j = x_j$ when all other covariates are fixed.
- Positive β_j indicates that the odds increases as x_j increases (everything else fixed), whereas for negative estimate of β_j the odds decreases as x_j increases (everything else fixed).

Logistic regression model

- Positive β

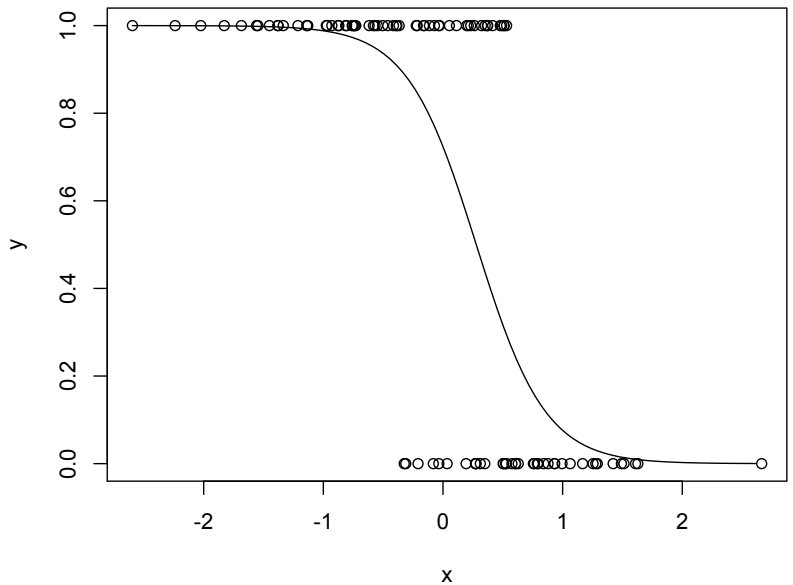
$$
\begin{aligned}
x_i &\sim N(0,1) \\
y_i &\sim \text{Bernoulli}\left(\frac{\exp(1+3x)}{1+\exp(1+3x)}\right)
\end{aligned}
$$



Logistic regression model

- Negative β

$$
\begin{aligned}
x_i &\sim N(0,1) \\
y_i &\sim \text{Bernoulli}\left(\frac{\exp(1-3x)}{1+\exp(1-3x)}\right)
\end{aligned}
$$



Logistic regression for retrospective studies

- Assume that we want to investigate the effect of smoking x_1, along with some other covariates $(x_2, ..., x_p)$ on lung cancer so we can build the following model

$$\log(\frac{\theta}{1-\theta}) = \beta_0 + \beta_1 x_1 + ... + \beta_p x_p \quad (1)$$

  where $\theta = P(y = 1|x)$ is the probability of developing lung cancer.

- Obviously, we cannot assign subjects to "smoking" and "non-smoking" groups (clinical trials) to see who develops lung cancer, or let them decide their cohort and wait for a long time to see the results (cohort study).

Logistic regression for retrospective studies

- Instead, we randomly select (with probability p_1) some subjects from the population of patients with lung cancer, and randomly select some subjects (with probability p_0) from the population of people without lung cancer. Then, we ask each subject whether he or she has been a smoker. That is, we design a **retrospective case-control** study.
- If we denote the decision to sample a subject as $z = 1$ our sampling mechanism is according to the following rules:

$$\begin{aligned} P(z = 1|y = 0) &= p_0 \\ P(z = 1|y = 1) &= p_1 \end{aligned}$$

- If we use this data in our logistic regression model, we are in fact modeling $\phi = P(y = 1|z = 1, x)$.

Logistic regression for retrospective studies

- However, using Bayes' theorem and assuming that sampling procedure does not depend on x, we have

$$\begin{aligned}\phi &= P(y=1|z=1,x) = \\ &= \frac{P(z=1|y=1,x)P(y=1|x)}{P(z=1|y=0,x)P(y=0|x)+P(z=1|y=1,x)P(y=1|x)} \\ &= \frac{p_1\theta}{p_0(\theta-1)+p_1\theta}\end{aligned}$$

- Using Model (1), and plugging it θ in the logistic regression model based on ϕ, we have

$$\log(\frac{\phi}{1-\phi}) = \beta_0^* + \beta_1 x_1 + ... + \beta_p x_p$$

where $\beta_0^* = \beta_0 + \log(p_1/p_0)$

- Therefore, the only difference between the two models would be the intercept; our inference about the effect of smoking on lung cancer, β_1, is valid even though it is based on a retrospective study.

Multinomial logistic model

- This is a generalization of logistic regression when the outcome could have multiple values (i.e., could belong to one of K classes).

$$y_i|n_i, \mu_{i1}, ..., \mu_{iK} \sim \text{multinomial}(n_i, \mu_{i1}, ..., \mu_{iK})$$

  where μ_{ik} is the probability of class k for observation i such that $\sum_{k=1}^{K} \mu_{ik} = 1$.
- y_i is also a vector of K elements with $\sum_{k=1}^{K} y_{ik} = n_i$.
- The systematic part is now a vector $\eta_{ik} = x_i\boldsymbol{\beta}$, where $\boldsymbol{\beta}$ is a matrix of size $(p+1) \times K$.

Multinomial logistic model

- Each column k (where $k = 1, ..., K$) corresponds to a set of $p + 1$ parameters associated with class k.
- This representation is redundant and results in nonidentifiability, since one of the β_k's (where $k = 1, ..., J$) can be set to zero without changing the set of relationships expressible with the model.
- Usually, either the parameters for $k = 1$ (the first column) or for $k = K$ (the last column) would be set to zero.

Multinomial logistic model

- For the multinomial logistic model, we use a generalization of the link function we used for the binary logistic regression

$$\mu_{ik} = \frac{\exp(x_i \boldsymbol{\beta}_k)}{\sum_{k'=1}^{K} \exp(x_i \boldsymbol{\beta}_{k'})}$$

- The likelihood in terms of β is as follows:

$$p(y|\mu) \propto \prod_{i=1}^{n} \prod_{k=2}^{K} \mu_{ik}^{y_{ik}}$$

$$P(y|x, \boldsymbol{\beta}) \propto \prod_{i=1}^{n} \prod_{k=1}^{K} \left(\frac{\exp(x \boldsymbol{\beta}_k)}{\sum_{k'=1}^{K} \exp(x \boldsymbol{\beta}_{k'})} \right)^{y_{ik}}$$

- Here $\boldsymbol{\beta}_k$ is a column vector of $p+1$ parameters corresponding to class k.

Multinomial logistic model

- $\boldsymbol{\beta}$ in general is a $(p+1) \times K$ matrix. The first row, $(\beta_{01}, ..., \beta_{0K})$ are intercepts, and $(\beta_{j1}, ..., \beta_{jK})$ in row $j+1$ are regression parameters associated with the j^{th} predictor.
- x_i is the row vector of predictors value for observation i (including the constant 1 at the beginning).
- y_{ik} is the number of cases in observation i that are in class k.

Poisson model

- When the outcome variable, y, represents counts, we use the Poisson model.

$$y_i|\mu_i \sim \text{Poisson}(\mu_i)$$

- The systematic components are defined as before: $\eta_i = x_i\beta$.
- The usual link function for this model is the log link:

$$g(\mu_i) = \log(\mu_i) = \eta_i$$

- We therefore have

$$\mu_i = \exp(\eta_i) = \exp(x_i\beta)$$

Poisson model

- The likelihood in terms of β can obtained as follows:

$$p(y_i|\mu_i) \propto \prod_i^n \exp(-\mu_i)\mu_i^{y_i}$$

$$p(y_i|\beta) \propto \prod_i^n \exp[-\exp(x_i\beta)][\exp(x_i\beta)]^{y_i}$$

- Similar to logistic and multinomial models, the variance of $y|x$ in Poisson model depends on the mean and therefore will not be constant

$$var(y_i|x_i) = \mu_i$$

Numerical methods for finding MLE

- As before, we maximize the likelihood function to estimate the regression parameters.
- We saw previously that the likelihood equation for linear regression models takes a simple form that can be solved analytically.
- However, the likelihood equations are in general nonlinear in β, and as the result, numerical methods are needed to find $\hat{\beta}$.
- A common approach is to use the Newton-Raphson method to maximize the likelihood function, or equivalently, maximize the log-likelihood function, or minimize the negative log-likelihood function (called the **energy function** in machine learning).

Newton-Raphson method

- The Newton-Raphson method is a general purpose iterative algorithm for solving nonlinear equations.
- In statistics, we use this method to the solve likelihood equation.
- Denote the log-likelihood as $L(\beta)$. Our objective is to find the value of β for which $L(\beta)$ is maximized.
- We start with the single parameter case.

Newton-Raphson method

- Start with an initial guess $\beta^{(0)}$.
- Iteratively update your guess as follows:
  - At each iteration n, use the Taylor series expansion (up to the quadratic term) around the current value of $\beta^{(n)}$

$$L(\beta) \simeq L(\beta^{(n)}) + L'(\beta^{(n)})(\beta - \beta^{(n)}) + \frac{1}{2}L''(\beta^{(n)})(\beta - \beta^{(n)})^2$$

  - Now take the derivative of $L(\beta)$, set it to zero and solve for β.
  - Regard the answer as your next guess $\beta^{(n+1)}$:

$$\beta^{(n+1)} = \beta^{(n)} - \frac{L'(\beta^{(n)})}{L''(\beta^{(n)})}$$

- Continue the above process until the algorithm converges.

Newton-Raphson method

- We can rewrite the equation for our next guess as

$$\beta^{(n+1)} = \beta^{(n)} + \frac{u(\beta^{(n)})}{o(\beta^{(n)})}$$

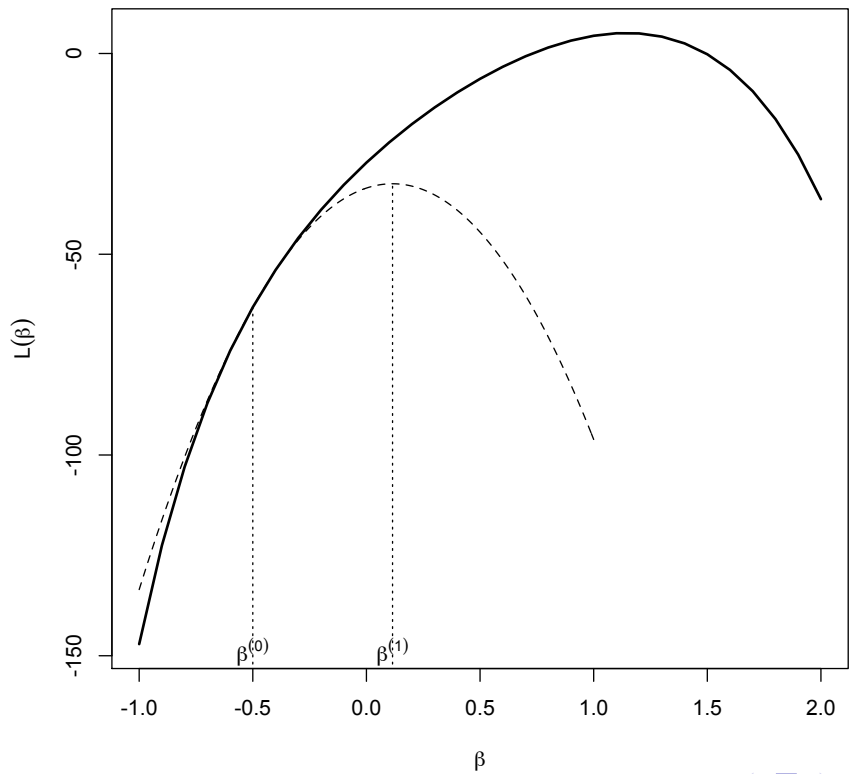
  where $u(\beta)$ is the score function, and

$$o(\beta^{(n)}) = -L''(\beta^{(n)})$$

- $o(\beta)$ is called the **observed information**, and its expectation $i(\beta) = E[-L''(\beta)]$ is called the **Fisher information**.

Newton-Raphson method

- The following graph illustrates how this method works.
- The sold line is the log-likelihood function, $\beta^{(0)}$ is our initial guess, the dashed line is the approximate quadratic function around $\beta^{(0)}$, and $\beta^{(1)}$ is our next guess.



Multiple parameter

- For multiple parameter models (where β is a vector), we have

$$\beta^{(n+1)} = \beta^{(n)} + [o(\beta^{(n)})]^{-1} u(\beta^{(n)})$$

- where $o(\beta)$ is a matrix whose (j, k) element is

$$o_{jk}(\beta) = -\frac{\partial^2 L(\beta)}{\partial \beta_j \partial \beta_k}$$

- Therefore, the observed information matrix $o(\beta)$ is the negative of the **Hessian matrix**.
- As before, the expected value of the $o(\beta)$ is the Fisher information matrix.

Fitting GLMs in R

- In practice, we use R and mainly the function `glm()` to fit generalized linear models.
- The function has the following format:

```
glm( formula, family = gaussian, data )
```

- `formula` specifies the systematic component, for example:

$$y \sim x_1 + x_2$$
$$y \sim .$$

Fitting GLMs in R

- `family` This specifies the stochastic part of the model, i.e., probability distribution for the response variable.
- It could also specify the link function:

  `family = binomial(link = "logit'').`
- Some of the default links are

```
gaussian(link = "identity")
binomial(link = "logit")
poisson(link = "log")
Gamma(link = "inverse")
```

- For multinomial logistic model, we can use the function `multinom` in the `nnet` package.
- If the categories are ordered, we can fit an ordinal logistic model using the function `polr()` in the `MASS` package.