Quantum Parallelism

For any n-qubit state $|x\rangle$, $H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle$

Proof:- We have $H(|0\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^0 |1\rangle)$

and $H(|1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^1 |1\rangle)$

So $H(|x\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x |1\rangle)$

or, $H(|x\rangle) = \frac{1}{\sqrt{2^1}} \sum_{y \in \{0,1\}} (-1)^{x \cdot y} |y\rangle$ ——— (1)

Let $P(n)$: $H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle$

For $n = 1$

$P(1)$: $H^{\otimes 1}|x\rangle = \frac{1}{\sqrt{2^1}} \sum_{y \in \{0,1\}} (-1)^{x \cdot y} |y\rangle$ is True (from (1))

Let for $n = m$ P is true ie, $P(m)$ is true,

ie, $H^{\otimes m}|x\rangle = \frac{1}{\sqrt{2^m}} \sum_{y \in \{0,1\}^m} (-1)^{x \cdot y} |y\rangle$ $\quad (|x\rangle = |x_1 x_2 \cdots x_m\rangle)$

Now we have to show that $P(m+1)$ is true ie, we have to show

$H^{\otimes m+1}|x\rangle = \frac{1}{\sqrt{2^{m+1}}} \sum_{y \in \{0,1\}^{m+1}} (-1)^{x \cdot y} |y\rangle$ $\quad (|x\rangle = |x_1 x_2 \cdots x_m x_{m+1}\rangle)$

Now $H^{\otimes(m+1)}(|x\rangle) = H^{\otimes m+1}|x_1 x_2 \cdots x_m x_{m+1}\rangle$

$= H^{\otimes m}|x_1 x_2 \cdots x_m\rangle \otimes H|x_{m+1}\rangle$

$$= \frac{1}{\sqrt{2^m}} \sum_{y\in\{0,1\}^m} (-1)^{(x_1 x_2 \cdots x_m)\cdot y} |y\rangle \otimes \frac{1}{\sqrt{2}} \left(|0\rangle + (-1)^{x_{m+1}} |1\rangle\right)$$

$$= \frac{1}{\sqrt{2^{m+1}}} \sum_{y\in\{0,1\}^{m+1}} (-1)^{(x_1 x_2 \cdots x_m x_{m+1})\cdot y} |y\rangle$$

There fore $P(m+1)$: $H^{\otimes m+1}|x\rangle = \frac{1}{\sqrt{2^{m+1}}} \sum_{y\in\{0,1\}^{m+1}} (-1)^{x\cdot y} |x\rangle$ is true if $P(m)$ is True.

There fore by principle of mathematical induction We have $P(n)$ is True i.e

$$H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y\in\{0,1\}^n} (-1)^{x\cdot y} |y\rangle$$