⊗ Prove that,

For any n-qubit state, $H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle$.

Sol^n:- We know that,

$$H(|0\rangle) = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^0 |1\rangle)$$

$$H(|1\rangle) = |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^1 |1\rangle)$$

Therefore,

$$H|x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x |1\rangle)$$

$$= \frac{1}{\sqrt{2}} \sum_{y \in \{0,1\}} (-1)^{x \cdot y} |y\rangle. \quad \text{--- (1)}$$

So, above statement is true for a 1-qubit state.

Let, above statement is true for $n = m$. i.e.

$$H^{\otimes m}|x\rangle = \frac{1}{\sqrt{2^m}} \sum_{y \in \{0,1\}^m} (-1)^{x \cdot y} |y\rangle. \quad \text{--- (2)}$$

where, $|x\rangle = |x_1 x_2 \cdots x_n\rangle$.

Now, for $n = m+1$. let,

$$|z\rangle = |z_1 z_2 \cdots z_m z_{m+1}\rangle$$

$$= |z_1 z_2 \cdots z_m\rangle \otimes |z_{m+1}\rangle.$$

So, $H^{\otimes m+1}|z\rangle$

$$= \left(H^{\otimes m}|z_1 \cdots z_m\rangle\right) \otimes \left(H|z_{m+1}\rangle\right)$$

$$= \left(\frac{1}{\sqrt{2^m}} \sum_{y \in \{0,1\}^m} (-1)^{(z_1 z_2 \cdots z_m)\cdot y} |y\rangle\right) \otimes \left(\frac{1}{\sqrt{2}} \sum_{y_{m+1} \in \{0,1\}} (-1)^{z_{m+1} \cdot y_{m+1}} |y_{m+1}\rangle\right)$$

$$= \frac{1}{\sqrt{2^{m+1}}} \sum_{y \in \{0,1\}^m} (-1)^{(z_1 z_2 \cdots z_m)\cdot y} \left(|y\rangle \otimes \sum_{y_{m+1} \in \{0,1\}} (-1)^{z_{m+1} \cdot y_{m+1}} |y_{m+1}\rangle\right)$$

$$= \frac{1}{\sqrt{2^{m+1}}} \sum_{y \in \{0,1\}^{m+1}} (-1)^{(z_1 \cdots z_{m+1}) \cdot y} |y\rangle$$

$$= \frac{1}{\sqrt{2^{m+1}}} \sum_{y \in \{0,1\}^{m+1}} (-1)^{z \cdot y} |y\rangle.$$

So, this is true for $n = m+1$.

Hence, by induction this is true for all $n \in \mathbb{N}$, i.e.

for any n-qubit state $|x\rangle$, $n \in \mathbb{N}$.

$$H^{\otimes n} |x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle. \quad \text{[Proved]}.$$