1. Given $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|\psi^*\rangle = \beta^*|0\rangle - \alpha^*|1\rangle$
Then prove that they are orthogonal and find a unitrary matrix B such that it transform $|\psi\rangle$ to $|0\rangle$ and $|\psi^+\rangle$ to $|1\rangle$.

Sol.

Here it is given $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$

and $|\psi^*\rangle = \beta^*|0\rangle - \alpha^*|1\rangle = \begin{pmatrix}\beta^*\\ -\alpha^*\end{pmatrix}$

Now $\langle\psi|\psi^*\rangle = \begin{pmatrix}\alpha^* & \beta^*\end{pmatrix}\begin{pmatrix}\beta^*\\ -\alpha^*\end{pmatrix} = \alpha^*\beta^* - \alpha^*\beta^*$
$= 0$

$\therefore \boxed{\langle\psi|\psi^*\rangle = 0}$, $\left[\begin{array}{l}|\psi\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}\\ \langle\psi| = \begin{pmatrix}\alpha^* & \beta^*\end{pmatrix}\end{array}\right]$

So $\boxed{|\psi\rangle \text{ and } |\psi^*\rangle \text{ are orthogonal}}$

Here we went to find a matrix, A, which satisfy

$A(|0\rangle) = |\psi\rangle$ and $A|1\rangle = |\psi^+\rangle$

ie $A|0\rangle = \alpha|0\rangle + \beta|1\rangle$

$A|1\rangle = \beta^*|0\rangle + (-\alpha^*)|1\rangle$

$\therefore A = \begin{pmatrix}\alpha & \beta^*\\ \beta & -\alpha^*\end{pmatrix}$.

Now, we have to find $A^{-1} = (A^*)^T$ [A is unitrary]

$= \begin{pmatrix}\alpha^* & \beta\\ \beta^* & -\alpha\end{pmatrix}^T$

$\boxed{A^{-1} = \begin{pmatrix}\alpha^* & \beta^*\\ \beta & -\alpha\end{pmatrix}}$

$\boxed{A^{-1}(|\psi\rangle) = |0\rangle}$ and $\boxed{A^{-1}(|\psi^+\rangle) = |1\rangle}$