1.

(a) The inner product between the real vectors $(0,1,0,1)$ & $(0,1,1,1)$ are

$$= [0\ 1\ 0\ 1] \cdot [0\ 1\ 1\ 1]^T$$

$$= [0\ 1\ 0\ 1] \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$= 0 \times 0 + 1 \times 1 + 0 \times 1 + 1 \times 1$$

$$= 1 + 1 = 2.$$

So, the required inner product is 2.

(b)

The matrix representation of $|0101\rangle$ is $= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Here each coloumn vector of states will be a $2^4 \times 1$ i.e. 16×1 coloumn vector.

So, for $|0111\rangle$ we have

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The conjugate transpose of $|0101\rangle$ is

$$= \langle 0101| = [0\,0\,0\,0\,0\,1\,0 \ldots 0].$$

So, the inner product between the states $|0101\rangle$ and $|0111\rangle$ is $= \langle 0101|0111\rangle$

$$= [0\,0\,0\,0\,0\,1\,0 \ldots 0] \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

(Here the coloumn vectors are all real then the conjugate transpose will be same as only transpose)

$$= 0+0+0+0+0+1\times 0+0+0\times 1+0+\cdots+0.$$
$$= 0.$$

So, in this case the required inner product is 0.

2.

Here $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

So, $H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$.

$H|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$.

$H|0\rangle \otimes H|1\rangle$

$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

$= \frac{1}{2}\left[|0\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle - |1\rangle \otimes |1\rangle\right]$.

$= \frac{1}{2}\left[|00\rangle + |10\rangle - |01\rangle - |11\rangle\right]$.

Now, $H \otimes H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

$$= \frac{1}{2}\begin{bmatrix} 1.\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & 1.\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ 1.\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & (-1)\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$|01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Now, $(H \otimes H)(|0\rangle \otimes |1\rangle)$ $\left\{ \therefore \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = |00\rangle - |01\rangle + |10\rangle - |11\rangle \right\}$

$= (H \otimes H)(|01\rangle)$

$= \frac{1}{2}\left[|00\rangle + |10\rangle - |01\rangle - |11\rangle\right]$

So, we have $H|0\rangle \otimes H|1\rangle = (H \otimes H)(|0\rangle \otimes |1\rangle)$

<u>Alternative</u>:

We know $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$H|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ & $H|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$$H|0\rangle \otimes H|1\rangle = \frac{1}{2}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 1 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ 1 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

So, $H|0\rangle \otimes H|1\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$. —(*)

$$H \otimes H = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$|0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$(H \otimes H)(|0\rangle \otimes |1\rangle) = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$= \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \quad - \quad **$$

From $*$ & $**$ we can see that

$$H|0\rangle \otimes H|1\rangle = (H \otimes H)(|0\rangle \otimes |1\rangle)$$

3. The standard basis in qubit is

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \& \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Now, Hadamard transform matrix, $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Bit flip, $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

Phase flip, $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Now, $HXH = H(XH)$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right).$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

So, $HXH(|0\rangle) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = Z(|0\rangle)$.

$HXH(|1\rangle) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = Z(|1\rangle)$.

So, HXH & Z are equal on the standard basis $|0\rangle$ & $|1\rangle$.

So, $HXH = Z$.

4. The matrix representation of CNOT

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

and $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

$$H \otimes H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

$$= \frac{1}{2}\begin{bmatrix} 1.\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & 1.\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \\ 1.\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & (-1).\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{bmatrix}.$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Now, the matrix of $(H\otimes H)\,\text{CNOT}\,(H\otimes H)$,

$$M = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

In 2-qubit the standard basis are

$$|00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad |01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad |10\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad |11\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Now, $M|00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = |00\rangle$

$M|01\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = |11\rangle$

$M|10\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = |10\rangle$

$$M|11\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |01\rangle.$$

So, we have the following transformation under matrix $M = (H \otimes H)\, CNOT\, (H \otimes H)$.

$$|00\rangle \longrightarrow |00\rangle$$
$$|01\rangle \longrightarrow |11\rangle$$
$$|10\rangle \longrightarrow |10\rangle$$
$$|11\rangle \longrightarrow |01\rangle.$$

So, we can the second bit controls the flipness of first bit if the second bit is 1, then the first bit will flip, & if second bit is 0, then there is no change with the first bit. Hence the required proof is done.

5. $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, So, conjugate transpose of $|0\rangle$
$= \langle 0| = [1 \quad 0]$.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Also, $|00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $|01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $|10\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$,

$|11\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ are the standard basis for 2-qubit.

Now, $\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$

$$= \alpha_{00}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_{01}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_{10}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_{11}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$= \begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{bmatrix}.$$

Now, $\langle 0| \otimes I = [1 \quad 0] \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$

$$= \left[1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad 0 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Now, $(\langle 0| \otimes I)(\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle)$.

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{bmatrix}.$$

$$= \begin{bmatrix} \alpha_{00} \\ \alpha_{01} \end{bmatrix}. = \alpha_{00} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_{01} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \alpha_{00}|0\rangle + \alpha_{01}|1\rangle.$$

So, the simplified result is $\alpha_{00}|0\rangle + \alpha_{01}|1\rangle$.

6. let $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

Suppose if possible $|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$ &

$|\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$

such that $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ for some

$\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$ & $|\alpha_1|^2 + |\beta_1|^2 = 1 = |\alpha_2|^2 + |\beta_2|^2$

Now $|\psi_1\rangle \otimes |\psi_2\rangle$

$= (\alpha_1|0\rangle + \beta_1|1\rangle) \otimes (\alpha_2|0\rangle + \beta_2|1\rangle)$.

$= \alpha_1\alpha_2(|0\rangle \otimes |0\rangle) + \alpha_1\beta_2(|0\rangle \otimes |1\rangle) + \beta_1\alpha_2(|1\rangle \otimes |0\rangle)$
$+ \beta_1\beta_2(|1\rangle \otimes |1\rangle)$.

$= \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle$.

So, in matrix notation we have.

$$|\psi\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = |\psi_1\rangle \otimes |\psi_2\rangle = \begin{bmatrix} \alpha_1\alpha_2 \\ \alpha_1\beta_2 \\ \beta_1\alpha_2 \\ \beta_1\beta_2 \end{bmatrix}.$$

Then from equality of matrix we have.

$\alpha_1\alpha_2 = \frac{1}{\sqrt{2}}$, $\alpha_1\beta_2 = 0$, $\beta_1\alpha_2 = 0$, $\beta_1\beta_2 = \frac{1}{\sqrt{2}}$.

Now, $|\alpha_1\alpha_2|^2 + |\alpha_1\beta_2|^2 = \frac{1}{2} + 0$.

$\Rightarrow |\alpha_1|^2|\alpha_2|^2 + |\alpha_1|^2|\beta_2|^2 = \frac{1}{2}$.

$\Rightarrow |\alpha_1|^2(|\alpha_2|^2 + |\beta_2|^2) = \frac{1}{2}$.

$$\Rightarrow \quad |\alpha_1|^2 = \frac{1}{2} \quad [\because |\alpha_2|^2 + |\beta_2|^2 = 1].$$

Again

$$|\beta_1\alpha_2|^2 + |\beta_1\beta_2|^2 = \frac{1}{2}.$$

$$\Rightarrow \quad |\beta_1|^2 = \frac{1}{2} \quad [\because |\alpha_2|^2 + |\beta_2|^2 = 1].$$

We also know, $\alpha_1\beta_2 = 0 \Rightarrow |\alpha_1\beta_2|^2 = 0$

$$\Rightarrow |\alpha_1|^2 |\beta_2|^2 = 0.$$

$$\Rightarrow |\beta_2|^2 = 0 \quad [\because |\alpha_1|^2 = \tfrac{1}{2} \neq 0]$$

Again $\beta_1\alpha_2 = 0 \Rightarrow |\beta_1\alpha_2|^2 = 0$

$$\Rightarrow |\beta_1|^2 |\alpha_2|^2 = 0$$

$$\Rightarrow |\alpha_2|^2 = 0 \quad [\because |\beta_1|^2 = \tfrac{1}{2} \neq 0].$$

So, $|\alpha_2|^2 + |\beta_2|^2 = 0 + 0 = 0.$

But in our assumption we have $|\alpha_2|^2 + |\beta_2|^2 = 1.$

So, this is a contradiction & hence our assumption is wrong.

Hence, it is not possible that

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = (\alpha_1|0\rangle + \beta_1|1\rangle) \otimes (\alpha_2|0\rangle + \beta_2|1\rangle)$$

for any $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{C}$.

Hence an EPR-pair $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is an entangled state.