Q) Show $\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right)$ ~~this~~ ~~is~~ can not be written as $\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix} \otimes \begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix}$.

Ans

We assume that, $\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix} \otimes \begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

We can write

$$\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix} = \left(\alpha_1|0\rangle + \beta_1|1\rangle\right), \qquad \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$$

$$\begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix} = \left(\alpha_2|0\rangle + \beta_2|1\rangle\right). \qquad \text{where}$$

$$|\alpha_1|^2 + |\beta_1|^2 = 1$$

$$|\alpha_2|^2 + |\beta_2|^2 = 1$$

So

$$\left(\alpha_1|0\rangle + \beta_1|1\rangle\right) \otimes \left(\alpha_2|0\rangle + \beta_2|1\rangle\right) = \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |1\rangle + |1\rangle \otimes |1\rangle\right),$$

Then we obtain the system of four equation.

$$\left.\begin{aligned} \alpha_1\alpha_2 &= \frac{1}{\sqrt{2}}, \quad \alpha_1\beta_2 = 0 \\ \beta_1\alpha_2 &= 0 \\ \beta_1\beta_2 &= \frac{1}{\sqrt{2}} \end{aligned}\right\} \longrightarrow (1)$$

Since $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$ (field), [Product of non-zero elements never 0, in a field]

and $\alpha_1\alpha_2 \neq 0$ and $\beta_1\beta_2 \neq 0$

$\Rightarrow \alpha_1 \neq 0, \alpha_2 \neq 0, \beta_1 \neq 0, \beta_2 \neq 0.$

So $\boxed{\alpha_1\beta_2 = 0 \quad \text{or} \quad \beta_1\alpha_2 = 0}$ [$\because \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{C}$]

— contradiction.

So the system of equation (1) admits no solution.

So the given state can not be written as a product state.

<Q>. Give matrix representation of $\boxed{\text{CSWAP}}$ gate ?

(Solution)

We know swap gate takes two inputs and gives two output. The truth table for swap gate is

Input		Output	
0	0	0	0
0	1	1	0
1	0	0	1
1	1	1	1

. Its matrix representation is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{2^2\times 2^2}$$

Now C.SWAP gate, i.e. controlled SWAP gate. The first input is a control bit. If it is 0, the second and third inputs are unchanged. If the control bit is 1, it swaps the 2nd and 3rd inputs to the 2nd output to 3rd and 3rd to 2nd. Its truth table is given by,

Input			Output		
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	1	0
1	1	0	1	0	1
1	1	1	1	1	1

So matrix representation of CSWAP-GATE is

$I_{4\times 4}$

$$\left(\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)_{2^3\times 2^3}$$

SWAP

Q (Write matrix ~~p~~ representation of **CCNOT** Gate.)

Sol^n For CNOT Gate, takes two inputs and gives two out puts. The first bit is called controlled bit. If 1st bit is 0, then it has no effect on the 2nd bit. If it is 1, it acts like not gate on the 2nd bit.

The function is $f(x,y) = (x, x \oplus y)$.

The matrix representation of CNOT gate is

Truth table			
INPUT		OUTPUT	
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0

matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}_{2^2 \times 2^2}$$

Similarly CCNOT is controlled CNOT gate. Here first two bit are controlled bit. It is 3 input 3 out put gate. Here the 3rd bit change when first two bit are 1. This gate is reversible

$$f(x,y,z) = (x, y, (x \wedge y) \oplus z).$$

$$T(x, y, (x \wedge y) \oplus z) = (x, y, (x \wedge y) \oplus (x \wedge y) \oplus z) = (x, y, z).$$

It matrix is

Identity

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}_{2^3 \times 2^3}$$

CNOT