① ⓐ Inner product $= (0101)(0111)^T$

$$= (0101)\begin{pmatrix}0\\1\\1\\1\end{pmatrix}$$

$$= 0\cdot 0 + 1\cdot 1 + 0\cdot 1 + 1\cdot 1$$

$$= 2$$

ⓑ $|0101\rangle = \begin{pmatrix}0\\0\\0\\0\\0\\1\\0\\\vdots\\0\end{pmatrix}_{16\times 1}$ & $|0111\rangle = \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\1\\0\\\vdots\\0\end{pmatrix}_{16\times 1}$

$\therefore$ Inner product $= \langle 0101 | 0111 \rangle$

$$= \begin{pmatrix}0\\0\\0\\0\\0\\1\\0\\\vdots\\0\end{pmatrix}^{\dagger}\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\1\\0\\\vdots\\0\end{pmatrix}$$

$$= (0000010\ldots 0)\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\1\\0\\\vdots\\0\end{pmatrix}$$

$$= 0$$

② $H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

$H|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

$H|0\rangle \otimes H|1\rangle$

$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$= \frac{1}{2}\left[|0\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle\right]$

$= \frac{1}{2}\left[|00\rangle - |01\rangle + |10\rangle - |11\rangle\right]$

$H \otimes H$

$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 1 \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & 1 \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ 1 \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & -1 \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$

$|0\rangle \otimes |1\rangle$

$= |01\rangle$

$= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$\therefore\ (H \otimes H)(|0\rangle \otimes |1\rangle)$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$= \frac{1}{2}\left[|00\rangle - |01\rangle + |10\rangle - |11\rangle\right] \longrightarrow (**)$$

from $(*)$ & $(**)$, $H|0\rangle \otimes H|1\rangle = (H \otimes H)(|0\rangle \otimes |1\rangle)$

(3) HXH

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= Z$$

(4) $(H \otimes H)\, CNOT\, (H \otimes H)$

$$= \left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right]\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right]$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\therefore (H \otimes H)\,CNOT\,(H \otimes H)\,|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |00\rangle$$

$$(H \otimes H)\,CNOT\,(H \otimes H)\,|01\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |11\rangle$$

$$(H \otimes H)\,CNOT\,(H \otimes H)\,|10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = |10\rangle$$

$$(H \otimes H)\,CNOT\,(H \otimes H)\,|11\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |01\rangle$$

Thus $(H \otimes H)\,CNOT\,(H \otimes H)$ is the 2-qubit gate where the second bit controls whether the first bit is neglected.

5. $(\langle 0| \otimes I)(\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle)$

$$= \left[\begin{pmatrix}1\\0\end{pmatrix}^{\dagger} \otimes \begin{pmatrix}1&0\\0&1\end{pmatrix}\right]\left[\alpha_{00}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \alpha_{01}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \alpha_{10}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + \alpha_{11}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right]$$

$$= \left[(1\ 0) \otimes \begin{pmatrix}1&0\\0&1\end{pmatrix}\right] \begin{pmatrix}\alpha_{00}\\\alpha_{01}\\\alpha_{10}\\\alpha_{11}\end{pmatrix}$$

$$= \left(1\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix} \quad 0\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix}\right) \cdot \begin{pmatrix}\alpha_{00}\\\alpha_{01}\\\alpha_{10}\\\alpha_{11}\end{pmatrix}$$

$$= \begin{pmatrix}1&0&0&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}\alpha_{00}\\\alpha_{01}\\\alpha_{10}\\\alpha_{11}\end{pmatrix}$$

$$= \begin{pmatrix}\alpha_{00}\\\alpha_{01}\end{pmatrix}$$

$$= \alpha_{00}\begin{pmatrix}1\\0\end{pmatrix} + \alpha_{01}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \alpha_{00}|0\rangle + \alpha_{01}|1\rangle.$$

6. Done in the previous assignment