Assignment 4

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Q1 a) What is the inner product between the real vectors (0, 1, 0, 1) and (0, 1, 1, 1) ?

$$x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \qquad y = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\langle x | y \rangle = x^T y = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= 0 + 1 + 0 + 1$$

$$= 2$$

b) What is the inner product between the states $|0101\rangle$ and $|0111\rangle$

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$$

$$H|1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} [|0\rangle - |1\rangle]$$

Now, $|0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |01\rangle$$

$$H \otimes H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$(H \otimes H)(|0\rangle \otimes |1\rangle) = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \quad \text{——— } ①$$

$$H|0\rangle \otimes H|1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ 1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \quad \text{——— } ②$$

From ① & ②

$$H|0\rangle + H|1\rangle = (H \otimes H)(|0\rangle \otimes |1\rangle)$$

Q3 Show that a bitflip operation, preceded and followed by Hadamard transformations, equals a phaseflip op.

$$HXH = Z$$

$$H|0\rangle = 1/\sqrt{2}\,\left[|0\rangle + |1\rangle\right]$$

$$H|1\rangle = 1/\sqrt{2}\,\left[|0\rangle - |1\rangle\right]$$

$$XH|0\rangle = 1/\sqrt{2}\,\left[|1\rangle + |0\rangle\right]$$

$$XH|1\rangle = 1/\sqrt{2}\,\left[|1\rangle - |0\rangle\right]$$

$$HXH|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$

$$= Z|0\rangle \quad \text{— ①}$$

$$HXH|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -|1\rangle = Z|1\rangle \quad \text{— ②}$$

From ① & ② $\quad HXH = Z$

Q4 Show that surrounding a CNOT gate with H switches the role of control bit and target bit of the CNOT.

$$H \otimes H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^T \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$[H \otimes H] \cdot CNOT \cdot [H \otimes H]$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Now,

$$(H\otimes H)\,CNOT\,(H\otimes H)\,|00\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = |00\rangle$$

$$(H\otimes H)\,CNOT\,(H\otimes H)\,|01\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = |01\rangle$$

$$(H\otimes H)\,CNOT\,(H\otimes H)\,|10\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = |10\rangle$$

$$(H\otimes H)\,CNOT\,(H\otimes H)\,|11\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |11\rangle$$

Q5 Simplify the following:

$$(\langle I| \otimes I)(\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle).$$

$$= \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_{00} \\ \alpha_{01} \end{bmatrix}$$

$$= \alpha_{00}|0\rangle + \alpha_{01}|1\rangle$$

Q6 Prove that an EPR pair $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is an entangled state, i.e., that it cannot be written as the tensor product of two seperate qubits.

Let us assume, for some complex α_1, β_1 and α_2, β_2 that satisfy $|\alpha_1|^2 + |\beta_1|^2 = 1$ and $|\alpha_2|^2 + |\beta_2|^2 = 1$ the following holds:

$$|\psi_1\rangle = \alpha_1 |0\rangle + \beta_1 |1\rangle$$
$$|\psi_2\rangle = \alpha_2 |0\rangle + \beta_2 |1\rangle$$
$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$

$$= \left[\alpha_1 |0\rangle + \beta_1 |1\rangle\right] \otimes \left[\alpha_2 |0\rangle + \beta_2 |1\rangle\right]$$

$$= \alpha_1\alpha_2 \left[|0\rangle \otimes |0\rangle\right] + \alpha_1\beta_2 \left[|0\rangle \otimes |1\rangle\right] + \beta_1\alpha_2 \left[|1\rangle \otimes |0\rangle\right] + \beta_1\beta_2 \left[|1\rangle \otimes |1\rangle\right]$$

$$= \alpha_1\alpha_2 |00\rangle + \alpha_1\beta_2 |01\rangle + \beta_1\alpha_2 |10\rangle + \beta_1\beta_2 |11\rangle \quad \text{—— (2)}$$

Comparing (1) & (2) we have

$$\alpha_1\alpha_2 = \frac{1}{\sqrt{2}} \quad \text{—— (3)}$$

$$\alpha_1\beta_2 = 0 \quad \text{—— (4)}$$

$$\beta_1\alpha_2 = 0 \quad \text{—— (5)}$$

$$\beta_1\beta_2 = \frac{1}{\sqrt{2}} \quad \text{—— (6)}$$

From (3) $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$

From (4) $\alpha_1 = 0$ or $\beta_2 = 0$ $\Rightarrow \beta_2 = 0$ — (7)

From (5) $\beta_1 = 0$ or $\alpha_2 = 0$ $\Rightarrow \beta_1 = 0$ — (8)

$\Rightarrow \beta_1 \beta_2 = 0$

But in eqn (6) it is $1/\sqrt{2}$.

Hence there cannot be any $\alpha_1, \beta_1, \alpha_2, \beta_2$ and hence no ψ_1 & ψ_2 that satisfy eqn (1).