(1) Given $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ &

$|\psi^*\rangle = \beta^*|0\rangle - \alpha^*|1\rangle$.

Prove that they are orthogonal & find a unitary matrix B such that it transform $|\psi\rangle$ to $|0\rangle$ & $|\psi^*\rangle$ to $|1\rangle$.

Ans: $|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ & $|\psi^*\rangle = \begin{bmatrix} \beta^* \\ -\alpha^* \end{bmatrix}$.

$$\langle\psi^*|\psi\rangle = \begin{bmatrix} (\beta^*)^* & (-\alpha^*)^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$= \begin{bmatrix} \beta & -\alpha \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \beta\alpha - \alpha\beta.$$

$$= \alpha\beta - \alpha\beta \quad [\because \alpha\beta = \beta\alpha]$$

$$= 0.$$

So, $|\psi\rangle$ & $|\psi^*\rangle$ are orthogonal.

Now, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$|\psi^*\rangle = \beta^*|0\rangle - \alpha^*|1\rangle$

Now, $A = \begin{bmatrix} \alpha & \beta^* \\ \beta & -\alpha^* \end{bmatrix}$

Now, as we have $|\alpha|^2 + |\beta|^2 = 1$, then each coloumn of A are orthonormal, then A is unitary.

Let inverse of $A = A^* = B$ (say). Then B will be also unitary.

$$B = A^* = \begin{bmatrix} \alpha^* & \beta^* \\ \beta & -\alpha \end{bmatrix}$$

Now, $B|\psi\rangle = \begin{bmatrix} \alpha^* & \beta^* \\ \beta & -\alpha \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} |\alpha|^2 + |\beta|^2 \\ \alpha\beta - \alpha\beta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$B|\psi^*\rangle = \begin{bmatrix} \alpha^* & \beta^* \\ \beta & -\alpha \end{bmatrix} \begin{bmatrix} \beta^* \\ -\alpha^* \end{bmatrix} = \begin{bmatrix} \alpha^*\beta^* - \alpha^*\beta^* \\ \beta\beta^* + \alpha\alpha^* \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle.$$

So, $B = \begin{bmatrix} \alpha^* & \beta^* \\ \beta & -\alpha \end{bmatrix}$ is our required matrix & also it is unitary ($\because$ Each coloumns of B are orthonormal).

② Prove that the quantum bits cannot be cloned.

<u>Ans</u>:

Suppose there is an unitary matrix U which does the copying procedure. In mathematical sense it can be expressed as

$U(|\psi\rangle|e\rangle) = |\psi\rangle|\psi\rangle$, $|e\rangle$ is a normalised state.

As U is unitary then we know $UU^* = I = U^*U$. where U^* is the complex conjugate transpose of U.

Suppose $|\psi\rangle$ & $|\phi\rangle$ are two pure states & the copying procedure is happening for this two things.

Then we have $U(|\psi\rangle|e\rangle) = |\psi\rangle|\psi\rangle$ — ①
$U(|\phi\rangle|e\rangle) = |\phi\rangle|\phi\rangle$. — ②

Now, we will take inner product between ① & ②.

$$\langle e|\langle\phi|U^* U|\psi\rangle|e\rangle = \langle\phi|\langle\phi||\psi\rangle|\psi\rangle.$$

$$\Rightarrow \langle e|\langle\phi|I|\psi\rangle|e\rangle = \langle\phi|\psi\rangle\langle\phi|\psi\rangle.$$

$$\Rightarrow \langle e|\langle\phi||\psi\rangle|e\rangle = [\langle\phi|\psi\rangle]^2$$

$$\Rightarrow \langle\phi|\psi\rangle\langle e|e\rangle = [\langle\phi|\psi\rangle]^2$$

$$\Rightarrow \langle\phi|\psi\rangle = [\langle\phi|\psi\rangle]^2 \quad \text{— ③} \quad \left[\because \langle e|e\rangle = 1 \text{ as } |e\rangle \text{ is normalised}\right].$$

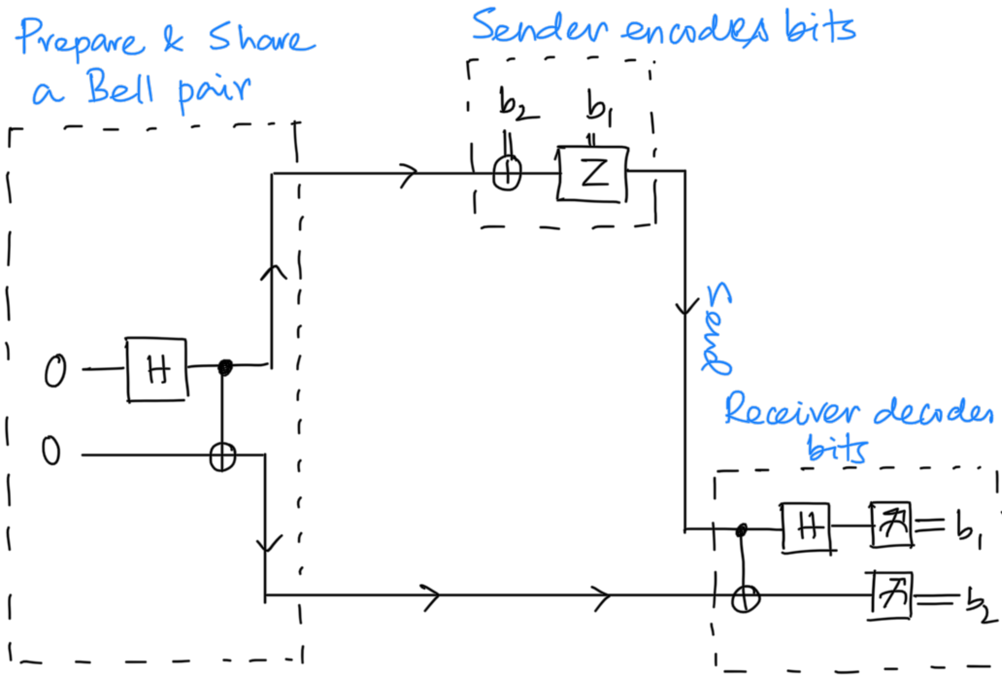
Let $x = \langle\phi|\psi\rangle$

Then ③ becomes, $x = x^2 \Rightarrow x(x-1) = 0$.
$\Rightarrow x = 0$ or $x = 1$.

So, we have $\langle\phi|\psi\rangle = 0$ or $\langle\phi|\psi\rangle = 1$.
This means that only if the pure states form an orthonormal basis then, only the cloning is possible otherwise not.
Hence in general cloning is not possible for qubits.

③ Make a presentation on dense coding.

<u>Ans</u>:



- Super dense coding, also referred to as dense coding is a quantum communication protocol to communicate a number of clasical bits of information by only transmitting a smaller number of qubits, under the assumption of sender & received pre-sharing an entangled resource.

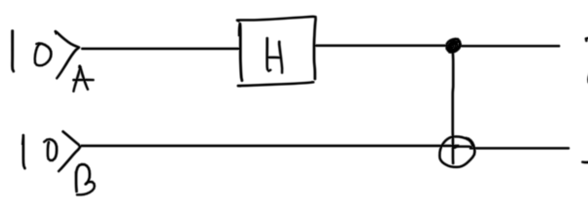
- In this protocol it involves two parties, Alice & Bob, which share a pair of maximally entangled qubits & allows Alice to transmit two bits (i.e. one of 00, 01, 10 or 11) to Bob by sending only one qubit.

Overview :

- Suppose Alice wants to send two classical bit of information (00, 01, 10 or 11) to Bob using qubits (instead of classical bits).
- To do this, an Bell state is prepared using Bell circuit & then sends one of this qubit to Alice & other to Bob.
- Once Alice obtains her qubit state she applies a certain quntum gate to her qubit depending on which two-bit messege (00, 01, 10 or 11) she wants to send to Bob.
- Her entangled qubit is then sent to Bob who, after applying the appropriate quantum gate & making a measurement, can retrieve the classical two-bit messege.

Protocol :

Preparation : The protocol starts with the preparation of an entangled state, which is later shared between Alice & Bob.



$$\frac{1}{\sqrt{2}}\left(|0_A 0_B\rangle + |1_A 1_B\rangle\right).$$

By the Bell circuit the Bell state

$|+\rangle = \frac{1}{\sqrt{2}}\left(|0_A 0_B\rangle + |1_A 1_B\rangle\right)$ is prepared

$\boxed{\text{Sharing}}$: After the preparation of $|+\rangle$, the qubit denoted by subscript A is sent to Alice & the qubit denoted by subscript B is sent to Bob.

$\boxed{\text{Encoding}}$: By applying a quntum gate Alice can transform the entangled state $|+\rangle$ into any of the four Bell states (including $|+\rangle$).

1. If Alice wants to send two classical bit 00 to Bob, then she applies $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ gate to her qubit & she have

$$|B_{00}\rangle = \frac{1}{\sqrt{2}}\left(|0_A 0_B\rangle + |1_A 1_B\rangle\right)$$

2. To send classical two bit 01 she applies $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ gate to her qubit & she have

$$|B_{01}\rangle = \frac{1}{\sqrt{2}}\left(|1_A 0_B\rangle + |0_A 1_B\rangle\right).$$

3. To send the classical bit 10 she applies $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ quntum gate to her qubit & she have

$$|B_{10}\rangle = \frac{1}{\sqrt{2}}\left(|0_A 0_B\rangle - |1_A 1_B\rangle\right)$$

4. If Alice wants to send classical bit 11, she applies, $Z * X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ quntum gate to her qubit & she have

$$|B_{11}\rangle = \frac{1}{\sqrt{2}}\left(|0_A 1_B\rangle - |1_A 0_B\rangle\right).$$

$\boxed{\text{Sending}}$: After having performed one of the operation in the Encoding stage, Alice now send her entangled qubit to Bob. i.e. Alice will send one of $\{B_{00}, B_{01}, B_{10}, B_{11}\}$ as she wants to send 00, 01, 10 or 11 respectively.

$\boxed{\text{Decoding}}$: After receiving the quntum entangled state B_{00}, B_{01}, B_{10} or B_{11}, Bob will first apply CNOT gate with A as control qubit & B as target qubit & then $H \otimes I$ unitary operation on the entangled qubit A.

1. If B_{00} is received after applying above unitary operation Bob will get $|00\rangle$.

$$|B_{00}\rangle = \frac{1}{\sqrt{2}}\left(|0_A 0_B\rangle + |1_A 1_B\rangle\right).$$

After CNOT : $\frac{1}{\sqrt{2}}\left(|0_A 0_B\rangle + |1_A 0_B\rangle\right).$

After $H \otimes I$:

$$\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(|0\rangle_A+|1\rangle_A\right)|0\rangle_B+\frac{1}{\sqrt{2}}\left(|0\rangle_A-|1\rangle_A\right)|0\rangle_B\right).$$

$$=\frac{1}{2}\left[|0_A 0_B\rangle+|1_A 0_B\rangle+|0_A 0_B\rangle-|1_A 0_B\rangle\right].$$

$$=|0_A 0_B\rangle.$$

Then Bob knows Alice wants to send 00 classical bit

2. If B_{01} is received after applying above unitary operation Bob will get $|01\rangle$.

$$|B_{01}\rangle=\frac{1}{\sqrt{2}}\left(|0_A 1_B\rangle+|1_A 0_B\rangle\right).$$

After CNOT : $\frac{1}{\sqrt{2}}\left(|0_A 1_B\rangle+|1_A 1_B\rangle\right).$

After $H\otimes I$:

$$\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(|0\rangle_A+|1\rangle_A\right)|1\rangle_B+\frac{1}{\sqrt{2}}\left(|0\rangle_A-|1\rangle_A\right)|1\rangle_B\right).$$

$$=\frac{1}{2}\left[|0_A 1_B\rangle+|1_A 1_B\rangle+|0_A 1_B\rangle-|1_A 1_B\rangle\right].$$

$$=|0_A 1_B\rangle.$$

Then Bob knows Alice wants to send 01 classical bit

3. If B_{10} is received after applying above unitary operation Bob will get $|10\rangle$.

$$|B_{10}\rangle = \frac{1}{\sqrt{2}}\left(|0_A 0_B\rangle - |1_A 1_B\rangle\right).$$

After CNOT : $\frac{1}{\sqrt{2}}\left(|0_A 0_B\rangle - |1_A 0_B\rangle\right).$

After $H \otimes I$:

$$\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(|0\rangle_A + |1\rangle_A\right)|0\rangle_B - \frac{1}{\sqrt{2}}\left(|0\rangle_A - |1\rangle_A\right)|0\rangle_B\right).$$

$$= \frac{1}{2}\left[|0_A 0_B\rangle + |1_A 0_B\rangle - |0_A 0_B\rangle + |1_A 0_B\rangle\right].$$

$$= |1_A 0_B\rangle.$$

Then Bob knows Alice wants to send 10 classical bit

4. If B_{11} is received after applying above unitary operation Bob will get $|11\rangle$.

$$|B_{11}\rangle = \frac{1}{\sqrt{2}}\left(|0_A 1_B\rangle - |1_A 0_B\rangle\right).$$

After CNOT : $\frac{1}{\sqrt{2}}\left(|0_A 1_B\rangle + |1_A 1_B\rangle\right)$

After $H \otimes I$:

$$\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(|0\rangle_A + |1\rangle_A\right)|1\rangle_B + \frac{1}{\sqrt{2}}\left(|0\rangle_A - |1\rangle_A\right)|1\rangle_B\right).$$

$$= \frac{1}{2}\left[|0_A 1_B\rangle + |1_A 1_B\rangle + |0_A 1_B\rangle - |1_A 1_B\rangle\right].$$

$$= |0_A 1_B\rangle.$$

Then Bob knows Alice wants to send 01 classical bit