1. Given a 2-qubit state

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

it is not possible to find $|\psi_1\rangle$ & $|\psi_2\rangle$ such that

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle.$$

Ans: Suppose if possible $|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$ &

$|\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$

such that $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ for some

$\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$ & $|\alpha_1|^2 + |\beta_1|^2 = 1 = |\alpha_2|^2 + |\beta_2|^2$

Now $|\psi_1\rangle \otimes |\psi_2\rangle$

$$= (\alpha_1|0\rangle + \beta_1|1\rangle) \otimes (\alpha_2|0\rangle + \beta_2|1\rangle).$$

$$= \alpha_1\alpha_2(|0\rangle \otimes |0\rangle) + \alpha_1\beta_2(|0\rangle \otimes |1\rangle) + \beta_1\alpha_2(|1\rangle \otimes |0\rangle) + \beta_1\beta_2(|1\rangle \otimes |1\rangle).$$

$$= \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle.$$

So, in matrix notation we have.

$$|\psi\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = |\psi_1\rangle \otimes |\psi_2\rangle = \begin{bmatrix} \alpha_1\alpha_2 \\ \alpha_1\beta_2 \\ \beta_1\alpha_2 \\ \beta_1\beta_2 \end{bmatrix}.$$

Then from equality of matrix we have.

$$\alpha_1\alpha_2 = \frac{1}{\sqrt{2}}, \quad \alpha_1\beta_2 = 0, \quad \beta_1\alpha_2 = 0, \quad \beta_1\beta_2 = \frac{1}{\sqrt{2}}.$$

Now, $|\alpha_1 \alpha_2|^2 + |\alpha_1 \beta_2|^2 = \frac{1}{2} + 0.$

$\Rightarrow |\alpha_1|^2 |\alpha_2|^2 + |\alpha_1|^2 |\beta_2|^2 = \frac{1}{2}.$

$\Rightarrow |\alpha_1|^2 \left(|\alpha_2|^2 + |\beta_2|^2 \right) = \frac{1}{2}.$

$\Rightarrow |\alpha_1|^2 = \frac{1}{2} \quad [\because |\alpha_2|^2 + |\beta_2|^2 = 1].$

Again,

$|\beta_1 \alpha_2|^2 + |\beta_1 \beta_2|^2 = \frac{1}{2}.$

$\Rightarrow |\beta_1|^2 = \frac{1}{2} \quad [\because |\alpha_2|^2 + |\beta_2|^2 = 1].$

We also know, $\alpha_1 \beta_2 = 0 \Rightarrow |\alpha_1 \beta_2|^2 = 0$

$\Rightarrow |\alpha_1|^2 |\beta_2|^2 = 0.$

$\Rightarrow |\beta_2|^2 = 0 \quad [\because |\alpha_1|^2 = \frac{1}{2} \neq 0].$

Again $\beta_1 \alpha_2 = 0 \Rightarrow |\beta_1 \alpha_2|^2 = 0$

$\Rightarrow |\beta_1|^2 |\alpha_2|^2 = 0$

$\Rightarrow |\alpha_2|^2 = 0 \quad [\because |\beta_1|^2 = \frac{1}{2} \neq 0].$

So, $|\alpha_2|^2 + |\beta_2|^2 = 0 + 0 = 0.$

But in our assumption we have $|\alpha_2|^2 + |\beta_2|^2 = 1.$

So, this is a contradiction & hence our assumption is wrong.

Hence, it is not possible that

$$\frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle = (\alpha_1 |0\rangle + \beta_1 |1\rangle) \otimes (\alpha_2 |0\rangle + \beta_2 |1\rangle)$$

for any $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{C}.$

2. Find the matrix representation of CCNOT gate & CSWAP gate.

Ans:

- CCNOT gate:

In the states if the 1st & 2nd bits are 1 then only the 3rd bit will be flipped under CCNOT gate.

Here we will write all the state transformation under the CCNOT gate.

$$|000\rangle \longrightarrow |000\rangle$$
$$|001\rangle \longrightarrow |001\rangle$$
$$|010\rangle \longrightarrow |010\rangle$$
$$|011\rangle \longrightarrow |011\rangle$$
$$|100\rangle \longrightarrow |100\rangle$$
$$|101\rangle \longrightarrow |101\rangle$$
$$|110\rangle \longrightarrow |111\rangle$$
$$|111\rangle \longrightarrow |110\rangle$$

So, the matrix representation for CCNOT gate is

$$M_{CCNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

- **CSWAP gate:**

Here we will write all the state transformation under the CSWAP gate.

$$|000\rangle \longrightarrow |000\rangle$$
$$|001\rangle \longrightarrow |001\rangle$$
$$|010\rangle \longrightarrow |010\rangle$$
$$|011\rangle \longrightarrow |011\rangle$$
$$|100\rangle \longrightarrow |100\rangle$$
$$|101\rangle \longrightarrow |110\rangle$$
$$|110\rangle \longrightarrow |101\rangle$$
$$|111\rangle \longrightarrow |111\rangle$$

Hence swap will happen in the rest of the two bits of each state iff the first bit of that state i.e. the controlled bit is 1.

So, the matrix representation for CSWAP gate is

$$M_{CSWAP} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$