(1) Given $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ & $|\psi^*\rangle = \beta^*|0\rangle - \alpha^*|1\rangle$
Prove that they are orthogonal & find a unitary matrix $B$ such that it transform $|\psi\rangle$ to $|0\rangle$ & $|\psi^*\rangle$ to $|1\rangle$.

Ans: $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ & $|\psi^*\rangle = \begin{pmatrix} \beta^* \\ -\alpha^* \end{pmatrix}$

$$\langle \psi^* | \psi \rangle = \begin{bmatrix} (\beta^*)^* & (-\alpha^*)^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$
$$= \begin{bmatrix} \beta & -\alpha \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$
$$= \beta\alpha - \alpha\beta$$
$$= 0$$

Let $A = \begin{pmatrix} \alpha & \beta^* \\ \beta & -\alpha^* \end{pmatrix}$. Now notice that $A$ is unitary as $|\alpha|^2 + |\beta|^2 = 1$. Let $B$ be the inverse of $A$. Then $B$ will also be unitary and $B = A^* = \begin{pmatrix} \alpha^* & \beta^* \\ \beta & -\alpha \end{pmatrix}$

Now $B|\psi\rangle = \begin{pmatrix} \alpha^* & \beta^* \\ \beta & -\alpha \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$

& $B|\psi^*\rangle = \begin{pmatrix} \alpha^* & \beta^* \\ \beta & -\alpha \end{pmatrix} \begin{pmatrix} \beta^* \\ -\alpha^* \end{pmatrix} = \cancel{\alpha^*\beta^* - \alpha^*} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$

(2) Prove that the quantum bits cannot be cloned.

Ans: Let if possible quantum bits can be cloned. Suppose $U$ be an unitary matrix which does the cloning. So

$$U(|\psi\rangle|e\rangle) = |\psi\rangle|\psi\rangle$$ , where $|e\rangle$ is a normalise state

As $U$ is unitary, $UU^* = U^*U = I$

Suppose $|\psi\rangle$ & $|\varphi\rangle$ are two pure states & the copying procedure is happening for this two things.

So $U(|\psi\rangle|e\rangle) = |\psi\rangle|\psi\rangle$ — (1)

& $U(|\varphi\rangle|e\rangle) = |\varphi\rangle|\varphi\rangle$ — (2)

Now $\langle e|\langle\varphi|U^*U|\psi\rangle|e\rangle = \langle\varphi|\langle\varphi||\psi\rangle|\psi\rangle$

$\Rightarrow \langle e|\langle\varphi|I|\psi\rangle|e\rangle = \langle\varphi|\psi\rangle\langle\varphi|\psi\rangle$

$\Rightarrow \langle e|\langle\varphi|1|\psi\rangle|e\rangle = [\langle\varphi|\psi\rangle]^2$

$\Rightarrow \langle\varphi|\psi\rangle\langle e|e\rangle = [\langle\varphi|\psi\rangle]^2$ — (3)

Let $x = \langle\varphi|\psi\rangle$

Then (3) becomes $x = x^2$

$\Rightarrow x(x-1) = 0$

$\Rightarrow x = 0$ or $x = 1$

Thus $\langle\varphi|\psi\rangle = 0$ or $\langle\varphi|\psi\rangle = 1$

i.e. cloning is possible only if the pure states form an orthonormal basis.

Hence cloning is not possible for qubits.