Assignment 6.

No cloning theorem!

~~We~~ One cannot copy / clone a general pure ~~qubits~~ state qubits using an Hermitian operator.

~~Let~~ Suppose one can clone ~~copy~~, pure state

Let $|\psi\rangle$ be a pure state qubit.

Let $|s\rangle$ be an target slot.

Let U be an Hermitian which which does this.

$$\therefore \quad |\psi\rangle \otimes |s\rangle \xrightarrow{U} |\psi\rangle \otimes |\psi\rangle.$$

$$U\,|\psi\rangle \otimes |s\rangle = |\psi\rangle \otimes |\psi\rangle$$

$$U\left(|\phi\rangle \otimes |s\rangle\right) = |\phi\rangle \otimes |\phi\rangle$$

for two $|\phi\rangle$ & $|\psi\rangle$.

$$\langle \psi | \phi \rangle = \langle \psi\psi | \phi\phi \rangle = \langle \psi | \phi \rangle \langle \psi | \phi \rangle$$

$$= \left(\langle \psi | \phi \rangle\right)^2$$

$$\Rightarrow \quad \langle \psi | \phi \rangle = 1 \quad \& \quad \langle \psi | \phi \rangle = 0.$$

$\langle\psi|\phi\rangle = 1 \Rightarrow |\psi\rangle = |\phi\rangle$.

$\langle\psi|\phi\rangle = 0 \Rightarrow |\psi\rangle \perp |\phi\rangle$

But if $|\psi\rangle = |0\rangle$ & $|\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |2\rangle)$

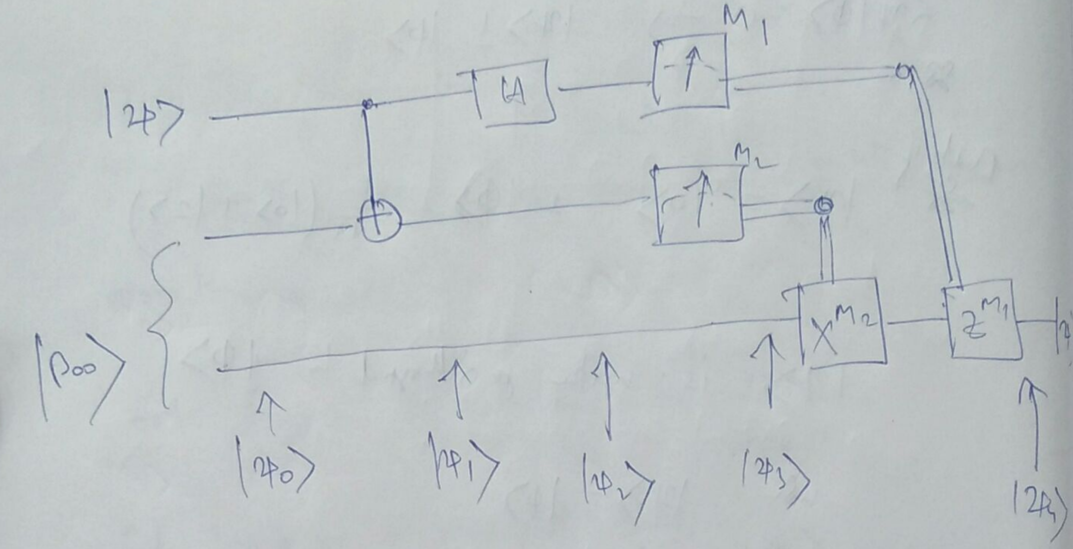
$|\psi\rangle$ is neither ~~pa~~ orthogonal to $|\phi\rangle$

nor $|\psi\rangle = |\phi\rangle$

$\therefore$ there is no ~~Hermitian operator~~ unitary operator which can copy both $|\phi\rangle$ & $|\psi\rangle$.

Thus a generic copy gate isn't possible, which ~~can copy all qubits~~ can make a copy of all qubits.

Quantum Teleportation:



$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle.$

$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right).$

$|\psi_0\rangle = |\psi\rangle|\beta_{00}\rangle$

$= \frac{1}{\sqrt{2}}\left(\alpha|0\rangle|00\rangle + \alpha|0\rangle|11\rangle + \beta|1\rangle|00\rangle + \beta|1\rangle|11\rangle\right).$

$|\psi_1\rangle = CNOT \otimes I \; \frac{1}{\sqrt{2}}\left(\alpha|0\rangle\left(|00\rangle + |11\rangle\right) + \beta|1\rangle\left(|00\rangle + |11\rangle\right)\right).$

$= \frac{1}{\sqrt{2}}\left(\alpha|0\rangle\left(|00\rangle + |11\rangle\right) + \beta|1\rangle\left(|10\rangle + |01\rangle\right)\right)$

$|\psi_2\rangle = H \otimes I \otimes I \; \frac{1}{\sqrt{2}}\left(\alpha|0\rangle\left(|00\rangle + |11\rangle\right) + \beta|1\rangle\left(|10\rangle + |01\rangle\right)\right)$

$|\psi_2\rangle = \frac{1}{2}\left(\alpha\left(|0\rangle + |1\rangle\right)\left(|00\rangle + |11\rangle\right) + \beta\left(|0\rangle - |1\rangle\right)\left(|10\rangle + |01\rangle\right)\right.$

$$|\psi_2\rangle = \frac{1}{2}\Big[|00\rangle(\alpha|0\rangle+\beta|1\rangle) + |01\rangle(\alpha|1\rangle+\beta|0\rangle) + |10\rangle(\alpha|0\rangle-\beta|1\rangle) + |11\rangle(\alpha|1\rangle-\beta|0\rangle)\Big]$$

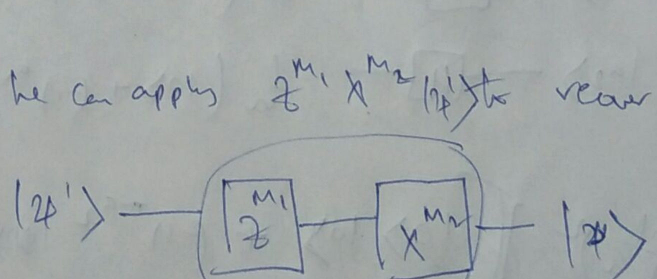
This expression naturally breaks down into 4 terms.

M_1	M_2	
0	0	$\longmapsto |\psi_3(00)\rangle = [\alpha|0\rangle+\beta|1\rangle]$
0	1	$\longmapsto |\psi_3(01)\rangle = [\alpha|1\rangle+\beta|0\rangle]$
1	0	$\longmapsto |\psi_3(10)\rangle = [\alpha|0\rangle-\beta|1\rangle]$
1	1	$\longmapsto |\psi_3(11)\rangle = [\alpha|1\rangle-\beta|0\rangle]$

depending on M_1 & M_2 Bob's qubit lie in one of the 4-states. Once he learns about M_1 & M_2 He can fix up his qubit and recover $|\psi\rangle$.

Suppose he gets $|\psi'\rangle$ and M_1 & M_2 are given to by Alice.

Now he can apply $Z^{M_1}X^{M_2}|\psi'\rangle$ to recover $|\psi\rangle$.



Q

$$H^{\otimes n}|a_1 a_2 \dots a_n\rangle$$

let's suppose there are k-ones in a_i's.

$$|a_1\rangle - H - \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{a_1}|1\rangle\right).$$

$$|a_2\rangle - H - \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{a_2}|1\rangle\right).$$

$$|a_3\rangle - H - \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{a_3}|1\rangle\right).$$

$$\vdots$$

$$|a_n\rangle - H - \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{a_n}|1\rangle\right).$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{a_1}|1\rangle\right) \otimes \cdots \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{a_n}|1\rangle\right)$$

$$= \left(\frac{1}{\sqrt{2}}\right)^n \left(|0\rangle \left[\left(|0\rangle + (-1)^{a_2}|1\rangle\right) \otimes \cdots \otimes \left(|0\rangle + (-1)^{a_n}|1\rangle\right) \right] + (-1)^{a_1}|1\rangle \right.$$

let's check for n=2

$$|a_1\rangle - \boxed{H} - \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{a_1}|1\rangle\right)$$

$$|a_2\rangle - \boxed{H} - \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{a_2}|1\rangle\right)$$

$$\left(\frac{1}{\sqrt{2}}\right)^2 \left[|0\rangle|0\rangle + (-1)^{a_2}|0\rangle|1\rangle + (-1)^{a_1}|10\rangle + (-1)^{a_1+a_2}|11\rangle \right)$$

$\underline{n=3}$.

$$\frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{a_1}|1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{a_2}|1\rangle\right)$$

$$\otimes \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{a_3}|1\rangle\right).$$

$$= \left(\frac{1}{\sqrt{2}}\right)^3 \left[(-1)^{0.a_1}(-1)^{0.a_2}|00\rangle + (-1)^{0.a_1}(-1)^{1.a_1}|01\rangle + (-1)^{1.a_1}(-1)^{0.a_2}|10\rangle + (-1)^{1.a_1}(-1)^{1.a_2}|11\rangle \right] \left[|0\rangle + (-1)^{a_3}|1\rangle\right]$$

$$= \left(\frac{1}{\sqrt{2}}\right)^3 \left[(-1)^{0.a_1}(-1)^{0.a_2}(-1)^{0.a_3}|000\rangle + (-1)^{0.a_1}(-1)^{0.a_2}(-1)^{1.a_3}|001\rangle + \cdots + (-1)^{1.a_1}(-1)^{1.a_2}(-1)^{1.a_3}|111\rangle \right].$$

Proof by induction.

Suppose for $k = n-1$ the formula is true.

Induction hypothesis:

$$H^{\otimes n-1}|a_1 \ldots a_{n-1}\rangle = \sum_{(x_1 \ldots x_{n-1}) \in \{0,1\}^{n-1}} (-1)^{\sum_{i=1}^{n-1} x_i a_i} |x_1, \ldots x_{n-1}\rangle \checkmark$$

$$H^{\otimes n}|a_1, \ldots, a_n\rangle = \left(H^{\otimes n-1}|a_1 \ldots a_{n-1}\rangle\right) \otimes H|a_n\rangle$$

$$= \left(\sum_{(x_1, x_2 \ldots x_{n-1}) \in \{0,1\}^{n-1}} (-1)^{\sum_{i=1}^{n-1} x_i a_i} |x_1 \ldots x_{n-1}\rangle \right) \otimes \left(|0\rangle + (-1)^{a_n} |1\rangle \right)$$

$$= \sum_{(x_1, x_2 \ldots x_{n-1})} (-1)^{\sum_{i=1}^{n-1} x_i a_i} |x_1 \ldots x_{n-1} 0\rangle + \sum_{(x_1, x_2 \ldots x_{n-1})} (-1)^{a_n} (-1)^{\sum_{i=1}^{n-1} x_i a_i} |x_1 x_2 \ldots x_{n-1} 1\rangle$$

$$= \sum_{(x_1 x_2 \ldots x_n) \in \{0,1\}^n} (-1)^{\sum_{i=1}^{n} x_i a_i} |x_1 x_2 \ldots x_n\rangle$$

$$\therefore \quad H^{\otimes n} |a_1 a_2 \ldots a_n\rangle = \sum_{(x_1, x_2, x_3 \ldots x_n) \in \{0,1\}^n} (-1)^{\sum_{i=1}^{n} x_i a_i} |x_1 x_2 \ldots x_n\rangle$$