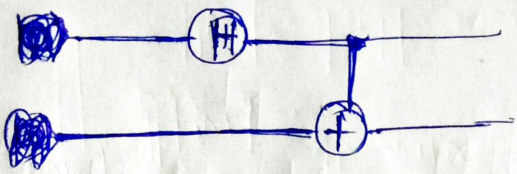
Assignment - 5

① **What is the matrix for this circuit?**



$\Rightarrow$ here $H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ let $I = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

Matrix rep for CNOT gate is $\begin{pmatrix} I & 0 \\ 0 & X\end{pmatrix}$.

where $X = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$.

So, matrix rep for the given circuit is

$$= M_{CNOT}(H \otimes I)$$

$$= \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)$$

$$= \begin{pmatrix} I & 0\\ 0 & X\end{pmatrix}\left(\frac{1}{\sqrt{2}}\begin{pmatrix} I & I\\ I & -I\end{pmatrix}\right)$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} I & 0\\ 0 & X\end{pmatrix}\begin{pmatrix} I & I\\ I & -I\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} I & I\\ X & -X\end{pmatrix}$$

& also, $\frac{1}{\sqrt{2}}\begin{pmatrix} I & I\\ X & -X\end{pmatrix}|00\rangle$

$= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

② **Generate the qubit $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$**

$\Rightarrow$ ~~If~~ we can generate this qubit from the matrix $\frac{1}{\sqrt{2}}\begin{pmatrix} I & I\\ X & -X\end{pmatrix}$ then let $|\psi\rangle$ be such a qubit for which

$$\frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ X & -X \end{pmatrix}(|\psi\rangle) = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$\Rightarrow \begin{pmatrix} I & I \\ X & -X \end{pmatrix}(|\psi\rangle) = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

as $\frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ X & -X \end{pmatrix}$ is an unitary matrix so,

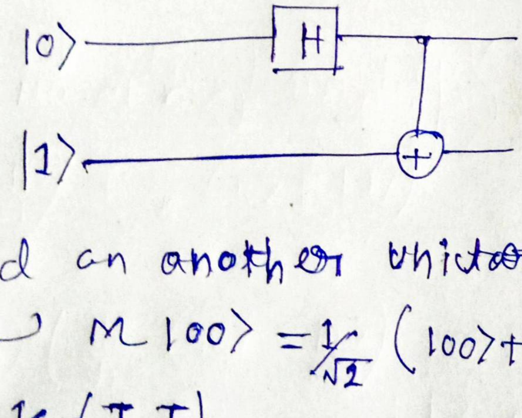
$$\begin{pmatrix} I & I \\ X & -X \end{pmatrix}^{-1} = \begin{pmatrix} I & I \\ X & -X \end{pmatrix}^{T} = \frac{1}{2}\begin{pmatrix} I & X \\ I & -X \end{pmatrix}.$$

$$\Rightarrow |\psi\rangle = \begin{pmatrix} I & I \\ X & -X \end{pmatrix}^{T}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} I & X \\ I & -X \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |01\rangle$$

So, $\frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ X & -X \end{pmatrix}(|01\rangle) = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

$\Rightarrow$



$\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$

(3) find an another unitary matrix M which $M|00\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ s.t.

$M \neq \frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ X & -X \end{pmatrix}$.

Consider, a matrix $M' = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & e^{i\theta} & 0 & 0 \\ 0 & 0 & e^{i\theta} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$

Now $M^{\dagger} = (\overline{M})^{T} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \bar{e}^{i\theta} & 0 & 0 \\ 0 & 0 & \bar{e}^{i\theta} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$

& $MM^{\dagger} = \begin{pmatrix} 1/\sqrt{2} & 0 & 0 & -1/\sqrt{2} \\ 0 & e^{i\theta} & 0 & 0 \\ 0 & 0 & e^{i\theta} & 0 \\ 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & e^{-i\theta} & 0 & 0 \\ 0 & 0 & e^{-i\theta} & 0 \\ -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{pmatrix}$

$= \quad I_4 .$

Similarly, $M^{\dagger} M = I_4 \Rightarrow M$ is an unitary matrix.

& $M(|00\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$