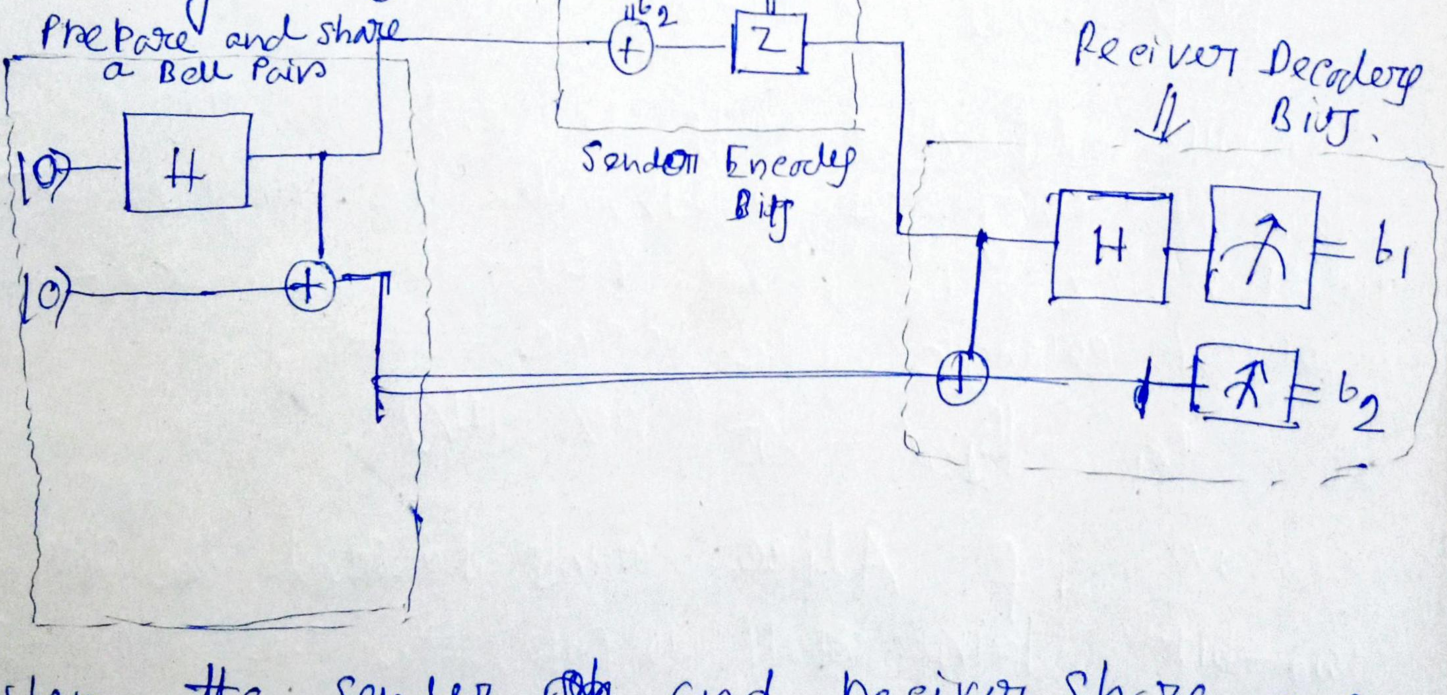
② Dense coding

Dense coding is a quantum communication protocol to communicate a ~~[illegible]~~ fixed number of classical bits of information by only transmitting a smaller number of qubits, under the assumption of sender and receiver pre-sharing an entangled state.

In this procedure, Alice & Bob shares a common entangled states & Alice ~~[illegible]~~ wants to transmit two bit of information to Bob by sending only one qubit.



When the sender ~~[illegible]~~ and receiver share a Bell state, ~~[illegible]~~ and share two classical bits through one qubit. In this diagram line carries qubits & double lines are classical bits.

Alice needs to perform on her entangled qubit, depending on which classical two-bit message she wants to send to Bob.

There are four possible cases corresponds to the four possible two-bit string.

Case-(i) If Alice wants to send the classical two-bit string 00 to Bob, then she applies the identity quantum gate, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ to here qubit, so, that it remains unchanged. The resultant entangled state is then

$$|B_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$|B_{00}\rangle$ is also used to remind us of the fact that Alice wants to send the two-bit string 00.

Case-(ii) If Alice wants to send the classical two-bit string 01 to Bob, then she applies the quantum NOT gate $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to here qubit, so, that the resultant entangle state becomes -

$$|B_{01}\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$$

Case-(iii) If Alice wants to send the classical two bit string 01 to Bob, then she applies the quantum Z gate $= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ to here qubit so, that the resultant entangle state become

$$|B_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle).$$

Case-(iv) If Alice wants to send the classical two-bit string 11 to Bob, then she applies the quantum gate $ZX = iY = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

So, that the resultant entangle state become

$$|B_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

The matrices X, Z and Y are Pauli Matrices.

In order for Bob to find out which classical bits Alice send he will perform the CNOT unitary operation, with A as control qubit and B as target qubit (A for Alice's bit and B for bob's bit). Then, he will perform $H \otimes I$ unitary operation on the entangled qubit A.

1. If the resultant entangled state was B_{00} then after the application of the above unitary operations the entangled state will become $|00\rangle$.
2. If the resultant entangled state was B_{01} then after the application of above unitary operating the entangled state will become $|01\rangle$.
3. If the resultant entangled state was B_{11} then after the application of the above unitary operations the entangled state will become $|11\rangle$.

These operation performed by Bob can be seen as a measurement which project the entangled state onto one of the four two-qubit basis vectors $|00\rangle, |01\rangle, |10\rangle$ or $|11\rangle$.

After the operations performed by Alice if the resultant entangled state was $B_{01} = 1/\sqrt{2}\,(|10\rangle + |01\rangle)$. Then a CNOT with A as control bit and B as target bit will change B_{01} to $B'_{01} = 1/\sqrt{2}\,(|01\rangle + |01\rangle)$

Now the Hadamard gate is applied only to A to obtain

$$B''_{01} = 1/\sqrt{2}\left(\left(1/\sqrt{2}\,(|0\rangle - |1\rangle)\right) \otimes |1\rangle + \left(1/\sqrt{2}\,(|0\rangle + |1\rangle)\right) \otimes |1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(|01\rangle - |11\rangle\right) + \frac{1}{\sqrt{2}}\left(|01\rangle + |11\rangle\right)\right]$$

$$= \frac{1}{2}\left(|01\rangle + |01\rangle\right) = |01\rangle .$$