1. Given a 2-qubit state $|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

Show that it is not possible to find $|\varphi_1\rangle$ and $|\varphi_2\rangle$ such that

$$|\psi\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle$$

Ans:- If possible let there exist $|\varphi_1\rangle$ and $|\varphi_2\rangle$ such that

$|\psi\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle$ where $|\varphi_1\rangle = \alpha|0\rangle + \beta|1\rangle$, $|\alpha|^2 + |\beta|^2 = 1$

$|\varphi_2\rangle = \gamma|0\rangle + \delta|1\rangle$, $|\gamma|^2 + |\delta|^2 = 1$

or, $|\psi\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$

or, $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle$

Comparing we get

$\alpha\gamma = 1/\sqrt{2}$ $\quad \alpha\delta = 0, \quad \beta\gamma = 0, \quad \beta\delta = \frac{1}{\sqrt{2}}$

Now $|\alpha\gamma|^2 + |\alpha\delta|^2 = \left(\frac{1}{\sqrt{2}}\right)^2 + 0^2$

or, $|\alpha|^2(|\gamma|^2 + |\delta|^2) = \frac{1}{2}$

or, $|\alpha|^2 = \frac{1}{2}$ $\quad$ since $|\gamma|^2 + |\delta|^2 = 1$

and $\alpha\delta = 0 \Rightarrow |\alpha\delta|^2 = 0$

$\Rightarrow |\alpha|^2|\delta|^2 = 0$ $\quad$ Since $|\alpha|^2 = \frac{1}{2} \Rightarrow \alpha \neq 0$

$\Rightarrow |\delta|^2 = 0$ ——— (1)

and $|\beta\gamma|^2 + |\beta\delta|^2 = \left(\frac{1}{\sqrt{2}}\right)^2$

$\Rightarrow |\beta|^2(|\gamma|^2 + |\delta|^2) = \frac{1}{2}$

$\Rightarrow |\beta|^2 = \frac{1}{2}$ $\quad$ as $|\gamma|^2 + |\delta|^2 = 1$

$\therefore \beta\gamma = 0 \Rightarrow |\beta\gamma|^2 = 0$

$\Rightarrow |\beta|^2|\gamma|^2 = 0$

$\Rightarrow |\gamma|^2 = 0$ $\quad$ Since $|\beta|^2 = \frac{1}{2}$ ——— (2)

From ① and ② we get

$|\gamma|^2 + |\delta|^2 = 0$ which contradict the fact that $|\gamma|^2 + |\delta|^2 = 1$

Therefore there does not exist any $|\psi_1\rangle$ and $|\psi_2\rangle$ such that

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$

2. Find the matrix representation of CCNOT gate and CCSWAP Gate.

Ans :- Table for CCNOT gate is

CCNOT

Input	Output
$\|000\rangle$	$\|000\rangle$
$\|001\rangle$	$\|001\rangle$
$\|010\rangle$	$\|010\rangle$
$\|011\rangle$	$\|011\rangle$
$\|100\rangle$	$\|100\rangle$
$\|101\rangle$	$\|101\rangle$
$\|110\rangle$	$\|111\rangle$
$\|111\rangle$	$\|110\rangle$

CCNOT is a 3 qubit gate. If in the input state 1st and 2nd qubit is 1 the then output state will be first two are ~~...~~ 1 and the 3rd bit is flipped.

So the matrix representation of CCNOT gate is

$$M_{CCNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Transformation table for CSWAP gate is

CSWAP

Input	Output
$\|000\rangle$	$\|000\rangle$
$\|001\rangle$	$\|001\rangle$
$\|010\rangle$	$\|010\rangle$
$\|011\rangle$	$\|011\rangle$
$\|100\rangle$	$\|100\rangle$
$\|101\rangle$	$\|110\rangle$
$\|110\rangle$	$\|101\rangle$
$\|111\rangle$	$\|111\rangle$

CSWAP is a 3 qubit gate. If in the input state the Control bit ie 1st bit is 0 then no chenge in the output state. and if the control bit is 1 Then other two bit (2nd and 3rd) are swap.

The matrix representation of CSWAP gate is

$$M_{CSWAP} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$