QUANTUM STATE AND MEASUREMENT OPERATORS

Subir Kumar Paul
Roll No.- CrS2011
MTech CrS
ISI,Kolkata

1 Introduction

Quantum computing is a rapidly emerging subject, within next decade we can expect an reasonable development in this field.This a classical combination of Quantum Mechanics and Computer science , quantum state and measurement are the heart of Quantum Mechanics and computing , which we are going to discuss.

2 Quantum State

In "layman's term", a quantum state is simply something that encodes the state of a system .The special thing about quantum state is that they allow the system to be in few state simultaneously; that called "quantum superposition".

A quantum state is a vector that contains all the information about a system . But generally we can only extract some of that information from the quantum state . This is partly due to the uncertainty principle and mostly just due to the nature of quantum mechanics itself.

3 Postulates of Quantum Theory

3.1 Postulate-1

A closed physical system is described by a state vector , which is a unit vector in a Hilbert space called the state space of the physical system.

3.2 Example

The smallest non-trivial state space is $\mathcal{H}_2$. Let, $\{|0\rangle, |1\rangle\}$ be a basis of $\mathcal{H}_2$, with vector representation

$$|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

This basis is called the computational basis. A general state (unit vector) in $\mathcal{H}_2$ can be expressed as ,

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

with $\alpha, \beta \quad in \ \mathbb{C}$, satisfying

4 Definition

A state vector in $\mathcal{H}_2$ is called a qubit.

5 Measuring a qubit

According to the laws of quantum mechanics , if we measure a qubit in computational basis we would get the result $|0\rangle$ with probability $|\alpha|^2$ and $|1\rangle$ with probability $|\beta|^2$.
Moreover after the measurement , the qubit "collapses" to the state measured .So in this case qubit acts like a regular bit,with probability distribution $\{|\alpha|^2, |\beta|^2\}$.

The measurement process changes the qubit state , measuring in different bases result in different probability distribution .Consider the following example .

Define the two orthonormal states in $\mathcal{H}_2$:

$$|+\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$|-\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

These states can be written in a vector notations as

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \qquad |1\rangle = \begin{pmatrix}0\\1\end{pmatrix} \qquad |+\rangle = \tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad |-\rangle = \tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

1. **Measurement in computational base :**
   We get one of the result $\{0, 1\}$ with equal probability .The system collapse to one of the states $\{|0\rangle, |1\rangle\}$

2. **Measurement in $\{|+\rangle, |-\rangle\}$:**
   We get result $|+\rangle$ with probability 1.The state does not change.

3. **Measurement in the basis$\{|+\rangle, |-\rangle\}$ after measuring in computational base:**
   We get $\{|+\rangle, |-\rangle\}$ with equal probability.After the first measurement(in the computational base) the qubit collapse to either $|0\rangle$ or $|1\rangle$,and after second measurement we get $\{|+\rangle, |-\rangle\}$ with equal probability.

6 Composite system

As we have seen before ,one qubit is a normalized vector in a two-dimentional space (with analogy to the classical bit).Following this logic ,two qubits are defined as a normalized vector in four-dimentional space.(two classical bits corresponding to four distinct values)

$$|00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad |01\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad |10\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad |11\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \tag{1}$$

A general state in four dimensional space is described by a linear combination of the vectors above (1) it is given by ,

$|\psi\rangle = \sum_{i,j\in 0,1} \alpha_{ij} |ij\rangle$ for some $\{\alpha_{ij}\}$ satisfying

$\sum_{i,j\in 0,1} |\alpha_{ij}|^2 = 1$

where $|\alpha_{ij}|^2$ is the probability of measuring $|ij\rangle$,when performing a measurement in computational base (1).

7 Definition

Let V and W be two Hilbert Spaces of dimensions m,n respectively.The tensor product of V and W ,denoted by $V \otimes W$, is a Hilbert Space of dimension mn, in which each state is a linear combination of states of the form $|v\rangle \otimes |w\rangle = |vw\rangle$, for every $|v\rangle \in V$, $|w\rangle \in W$

The natural inner product in the space $V \otimes W$(induced by the inner product of the spaces V and W) is:
$(\sum_i a_i |v_i w_i\rangle , \sum_j b_j |v_j^{'} w_j^{'}\rangle) = \sum_{i,j} a_i^* b_j \langle v_i|v_j^{'}\rangle \langle w_i|w_j^{'}\rangle$

8 Properties of tensor product

1) Linearity in both variables
2) Let $\{|i\rangle\}_{i=1}^n$ and $\{|j\rangle\}_{j=1}^n$ be two orthogonal bases of the spaces V and W respectively. Then $\{|ij\rangle\}$ is an orthonormal basis of $V \otimes W$.

9 Example

(1) Let $V = W = \mathcal{H}_2$ and consider the respective states:

$$|\psi_v\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$|\psi_w\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Therefore,

$$|\psi_v \psi_w\rangle = \frac{1}{2}(|00\rangle + |10\rangle) - |01\rangle\rangle - |11\rangle)$$

where, $|\psi_v \psi_w\rangle \in V \otimes W$.
If we had two chosen the orthonormal basis $\{|+\rangle , |-\rangle\}$ for W, we would have gotten

$$|\psi_v \psi_w\rangle = |+-\rangle$$

(2) Continuing the previous example, we define two states in the tensor product space $V \otimes W$.

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \text{ (EPR state)}$$
$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \times |0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

The inner product of these two states is: $\langle\psi_1|\psi_2\rangle = \frac{1}{2}$

10 Postulate-II

The state space of a "composite" system (a system which is composed of several sub-system) in the tensor product of the state spaces of the sub-systems. Furthermore, if the k-th ($1 \leq k \leq n$) sub-system is in the state $|\psi_k\rangle$, then the composite system is in state $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \ldots \ldots \cdots \otimes |\psi_n\rangle$

11 Remarks

The converse is not necessarily true: not every state of the composite system can be sub-systems comparing it.

This leads us to the notion of entanglement.

12 Entanglement

12.1 Definition:

A state of composite system, which cannot be written as a tensor product of states of the sub-system composing it, is said to be entangled. The two sub-systems are also called entangled.

12.2 Example:

The state $|\psi\rangle = \frac{1}{\sqrt{2}} \times (|00\rangle + |11\rangle)$ cannot be decomposed into the form $|\psi\rangle = |a\rangle \times |b\rangle$ and thus entangled.
In an entangled state, it is not possible to define the state of each of the sub-systems. Each sub-system is found in a state which is a combination of several states, with a certain probability distribution.

13 Unitary Evolution

13.1 Postulate-III

The evolution of a closed quantum system is described by a unitary operator. Namely if $|\psi_1\rangle$ and $|\psi_2\rangle$ are the states of the system in time t_1 and then time t_2, then $|\psi_2\rangle = U\,|\psi_1\rangle$
Where U in unitary and depends on t_1 and t_2

13.2 Remarks

(i) The transformation must be unitary in order to preserve the norm. The resulting state must be of unit length at all times. (ii) The evolution of a closed physical system is reversible. It is possible to go from $|\psi_2\rangle$ to state $|\psi_1\rangle$ by applying inverse operator $U^{-1} = U^{+}$

14 Definition

The tensor product between matrix $A \in \mathbb{C}^{m\times n}$ and matrix $B \in \mathbb{C}^{p\times q}$ is a matrix of size $pm \times qn$ which has form

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12} & \dots & a_{1n} \\ a_{11}B & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{11}B & a_{12} & \dots & a_{1n} \end{pmatrix}$$

After applying a tensor product of two operator $U \otimes V$ on a composite state $|\psi_1\psi_2\rangle$, the resulting state is the tensor product of the sub-states after applying the local operation.

$$|\psi_1\psi_2\rangle \to (U \otimes V)\,|\psi_1\psi_2\rangle = (U\,|\psi_1)(V\psi_2\rangle)$$

14.1 Example (Tensor product of vectors):

$$|0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |01\rangle \quad and \quad |-\rangle \otimes |+\rangle = |-+\rangle = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

15 Some simple tensor product properties:

(i) Linearity in both variables
(ii) Associativity: $(A \otimes B) \otimes C = A \otimes (B \otimes C)$
(iii) $(A \otimes C)\dot{(}B \otimes D) = (A\dot{B}) \otimes (C\dot{D})$, when ever the right hand side is defined.
(iv) The tensor product preserves Unitary, Hermiticity, Positivity, and Projection properties.
(v) If v_a, V_b are vectors consisting of all the eigen values of A, B respectively, then $V_a \otimes V_b$ is a vector consisting of all the eigen values of $A \otimes B$.
(vi)$tr(A \otimes B) = tr(A)\dot{t}r(B)$ and $rank(A \otimes B) = rank(A)\dot{r}ank(B)$

16 Quantum Measurement

16.1 Postulate-IV

A general quantum measurement is described by a collection M_m of "measurement operators" which act on the state space, and satisfy the "complete equation".

$$\sum_m M_m^{+} M_m = I$$

The index m refers to the measurement outcomes that may occur in the experiment .
If the state prior to the measurement is $|\psi_v\psi_w\rangle$, then the probability of measuring m, is

$$p(m) = \langle\psi|M_m^{+} M_m|\psi\rangle$$

and the state after the measurement,given that m was measured is

$$|\psi_m\rangle = \frac{M_m |\psi\rangle}{\sqrt{p(m)}}$$

Note that the "completeness quation" is equivalent to the requirement that the sum of all probabilities is equal to 1

$$\sum p(m) = \sum \langle\psi|M_m^+ M_m|\psi\rangle = \langle\psi|(\sum M_m^+ M_m)|\psi\rangle = 1$$

16.2 Example

To perform a measurement in the computational base, the projector on the axes are used:
$M_0 = |0\rangle\langle 0|, M_1 = |1\rangle\langle 1|$
Using the property of projection operator $P^2 = P$,we note that these projectors satisfy the completeness equation : $M_0^+ M_0 + M_1^+ M_1 = M_0^2 + M_1^2 = M_0 + M_1 = \langle 0| |0\rangle + \langle 1| |1\rangle = I$
Here, $M_0^2 = (\langle 0| |0\rangle)(\langle 0| |0\rangle) = \langle 0| \langle 0|0\rangle |0\rangle = \langle 0| |0\rangle = M_0$
Similarly for M_1, we have $M_1^2 = M_1$
Now measuring the qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, using the projector above , we get result 0 with probability

$$\begin{aligned} p(0) &= \langle\psi| M_0^+ M_0 |\psi\rangle \\ &= (\alpha^* \langle 0| + \beta^* \langle 1|) |0\rangle \langle 0| (\alpha|0\rangle + \beta|1\rangle) \\ &= (\alpha^* \langle 0\rangle 0 + \beta^* \langle 1\rangle 0)(\alpha \langle 0\rangle 0 + \beta \langle 0\rangle 1) \\ &= (\alpha^* + 0)(\alpha + 0) \\ &= \alpha^* \alpha \\ &= |\alpha|^2 \end{aligned}$$

Therefore, we get **p(0)=**$|\alpha|^2$ and, we get result 1, with probability, **p(1)=**$|\beta|^2$
Now, given that the result of the measurement was 0, the state collapses to the new state.

$$\begin{aligned} |\psi_0\rangle &= \frac{M_0 |\psi\rangle}{\sqrt{p(0)}} \\ &= \frac{|0\rangle\langle 0| (\alpha|0\rangle + \beta|1\rangle)}{\sqrt{|\alpha|^2}} \\ &= \frac{\alpha|0\rangle}{|\alpha|} \\ &= e^{j\theta} |0\rangle \\ &= |0\rangle \end{aligned}$$

Therefore, we have $|\psi_0\rangle = |0\rangle$

16.3 Remark

The postulates can be modified such that a state vector is defined upto a global phase, i.e., upto a multiplication by any $e^{j\theta}$ .This does not change anything , since a global phase has no influence on measurements or their outcomes/probabilities .Thus we can always disregard a global phase.

In a similar manner, to perform a measure in the basis $\{|+\rangle, |-\rangle\}$,one needs to use the measurement operator,

$$M_0 = |+\rangle\langle+| \quad and \quad M_1 = |-\rangle\langle-|$$

16.3.1 Example (Measuring one of two qubits):

Let $|\psi\rangle = \sum_{i,j\in\{0,1\}} \alpha_{ij} |ij\rangle$, be the state of some composite system AB. We want to measure only the first qubit in the computational base .The measurement operators which archive this are:

$$\begin{aligned} M_0 &= (|0\rangle\langle 0|) \otimes I_B \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{aligned}$$

Therefore, $M_1 = (|1\rangle\langle 1|) \otimes I_B = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I} \end{pmatrix}$ The probability of the possible outcomes are:

$$p(0)=\langle\psi| M_0 |\psi\rangle = |\alpha_{00}|^2 + |\alpha_{01}|^2$$

p(1)=$\langle\psi| M_1 |\psi\rangle = |\alpha_{10}|^2 + |\alpha_{11}|^2$ After measuring the result 0, the state collapses to the state:

$$|\psi_0\rangle = \frac{\alpha_{00}|0\rangle + \alpha_{01}|1\rangle}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}}$$

17 Distinguishing quantum states

An important application of measurement postulate is distinguishing quantum states.
Distinguishability,like many ideas in quantum computation and information is must easily understood using a game involving two parties , Alice and Bob.
Alice choose a state $|\psi_i\rangle$ from a fixed set of states $|\psi_1\rangle, |\psi_2\rangle, ..., |\psi_n\rangle$(Known to both Alice and Bob and gives this state to Bob , whose task is to identify i)

17.1 claim 1

There is a winning strategy for Bob if $|\psi_1\rangle, |\psi_2\rangle, ..., |\psi_n\rangle$ are orthonormal states.

17.2 claim 2

There is no winning strategy for Bob if there are non-orthogonal states.

17.3 Proof of claim 1

Define measurement operator $M_i = |\psi_i\rangle\langle\psi_i|$ and also define $M_0 = \sqrt{I - \sum_{i=1}^{n} M_i}$,
where $(I - \sum_{i=1}^{n} M_i)$ is a positive operator, whose square root is well defined.
Now, $\sum_{i=1}^{n} M_i = I$ (satisfies the completeness relation) and $p(i) = \langle\psi_i| M_i |\psi_i\rangle = 1$, so the result i occurs with certainly. Thus it is possible to reliably distinguish the orthonormal states $|\psi_i\rangle$.

Proof of Claim 2:

We shall prove it by contradiction. Suppose there is a measurement possible which distinguishes two non-orthogonal states $|\psi_1\rangle, |\psi_2\rangle$.
If the state $|\psi_1\rangle$ $(|\psi_2\rangle)$ is prepared then the probability of measuring j such that $f(j) = 1(f(j) = 2)$ must be 1.
Defining $E_i = \sum_{j:f(j)=1} M_j^{+} M_j$, these observation may be written as:

$$\langle\psi_1| E_1 |\psi_1\rangle = 1; \quad \langle\psi_2| E_2 |\psi_2\rangle = 1 \tag{2}$$

Since, $\sum_i E_i = I$, it follows that, $\sum_i \langle\psi_1| E_i |\psi_1\rangle$. since, $\langle\psi_1| E_1 |\psi_2\rangle = 1$, we have,

$$\langle\psi_1| E_2 |\psi_1\rangle = 0$$
$$\Rightarrow \sqrt{E_2}\, |\psi_1\rangle = 0$$

Suppose, we decompose $|\psi_2\rangle = \alpha|\psi_1\rangle + \beta|\phi\rangle$ where $|\phi\rangle$ is orthonormal to $|\psi_1\rangle$.
$\therefore |\alpha|^2 + |\beta|^2 = 1$ and $|\beta| < 1$

$[\because |\psi_1\rangle\, and\, |\psi_2\rangle$ are not orthogonal$]$

$\therefore \sqrt{E_2}\,|\psi_2\rangle = \beta\sqrt{E_2}\,|\phi\rangle$
By (1) we get,

$$\begin{aligned}\langle\psi_2| E_2 |\psi_2\rangle &= |\beta|^2 \langle\phi| E_2 |\phi\rangle \\ &\leq |\beta|^2 \\ &< 1\end{aligned}$$

Which is a contradiction.

$$\because \langle\phi| E_2 |\phi\rangle \;\leq\; \sum_i \langle\phi| E_i |\phi\rangle \;=\; <\phi|\phi> \;=\; 1$$

So, no such measurement exist.

Projective Measurement

A projective measurment is described by an observable, M, a Hermitian operator on the state space of the system being observed has a spectral decomposition-

$$\sum_m mP_m$$

$m \equiv$ Measurement outcome and $P_m \equiv$ orthogonal projector
Where P_m is the projector onto the eigenspace of M with eigenvalue m.
Now measuring the state $|\psi\rangle$, the probability of setting result m is:

$$p(m) = \langle\psi| P_m |\psi\rangle$$

Projective measurement have many nice properties. In particular, it is very easy to calculate average value for projective measurements.
The expected value of the outcome of the measurement is:

$$\begin{aligned}
< M > &= E(M) \\
&= \sum_m mP_m \\
&= \sum_m m \langle\psi| P_m |\psi\rangle \\
&= \langle\psi| (\sum_m mP_m) |\psi\rangle \\
&= \langle\psi| M |\psi\rangle \\
&= tr(M |\psi\rangle \langle\psi|)
\end{aligned}$$

$$\therefore < M > = tr(M |\psi\rangle \langle\psi|)$$

Example:

1. The observable $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ describe a measurement in the computational basis $\{|0\rangle, |1\rangle\}$ with measurement outcomes(eigenvalues) $\{\pm 1\}$.
2. The observable $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ describes a measurement in the basis $\{|+\rangle, |-\rangle\}$ with the measurement outcomes (eigenvalues) $\{\pm 1\}$.If we calculate the expected value of the outcome when measuring the state $|0\rangle$, we have:

$$\begin{aligned}
tr(X |0\rangle \langle 0|) &= tr\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \\
&= tr \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
&= 0
\end{aligned}$$

Since, we get $|+\rangle$ and $|-\rangle$ with equal probabilities.

POVM(Positive Operator Valued Response) Measurement

The description of projective measurement turns out to be restrictive. There are measurements that can be performed on a system that cannot be described within this formalism.

These are called non-projective measurements, generalized measurements, or positive operator-valued measures (POVM).

A generalized measurement in Quantum Mechanics is described as a collection of positive operators $E_i \geq 0$, that satisfy:

$$\sum E_i = I$$

We denote such measurement as $M = \{E_i\}$. Each Ei is associated with an outcome of the measurement and since, $E_i \geq 0$, it has the decomposition
$E_i = M_i^+ M_i$. For a state ρ (pure or mixed) the probability of obtaining the result associated with E_i is:

$$Pr\{i\} = tr(E_i \rho)$$

Similarly, the post-measurement state after obtaining result I is,

$$\rho \to \frac{M_i \rho M_i^+}{tr(E_i \rho)} = \frac{M_i \rho M_i^+}{Pr\{i\}}$$

We see from these definition that the projective measurements described above are a special case where $M_i = M_i^+ = P_i$ as well since,

$$P_i^+ P_i = P_i$$

The main difference between projective measurements and POVM elements is that the POVM elements do not have to be orthogonal i.e., while $P_i P_j = P_i \delta_{ij}$ the same orthogonality relation does not exist for the M_i or E_i.

Example 1

The simplest example of the need for a POVM description of measurement is given by the following scenario:

Consider a measurement apparatus that measures a single qubit on a computational basis. However, the apparatus can fail with probability p, and when it does, it does not interact with the qubit and therefore, returns no measurement result. In this case, there is no projective description of the measurement, instead, it is described by POVM elements:

$$\begin{aligned}
E_1 &= PI \\
E_2 &= (1-p) \left|0\right\rangle \left\langle 0\right| \\
E_3 &= (i-p) \left|1\right\rangle \left\langle 1\right|
\end{aligned}$$

Example 2:

Suppose Alice gives bob a qubit prepared in one of two states

$$|\psi_1\rangle = |0\rangle \; or \; |\psi_2\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

It is impossible to distinguish between $|\psi_1\rangle$ and $|\psi_2\rangle$ (we know before) with perfect reliability.

But it is possible to perform a measurement that distinguishes the state some of the time, but never make an error of mis-identification.
Consider a POVM containing three elements,

$$\begin{aligned}
E_1 &= \frac{\sqrt{2}}{1+\sqrt{2}} |1\rangle \langle 1| \\
E_2 &= \frac{\sqrt{2}}{1+\sqrt{2}} \frac{(|0\rangle - |1\rangle)(\langle 0| - \langle 1|)}{2} \\
E_3 &= I - E_1 - E_2
\end{aligned}$$

It is easy to verify that these are positive operator, which satisfies

$$\sum_m E_m = I$$

So, it is a POVM.
Note that $E_i E_j \neq \delta_{ij}$ for any i, j and hence, this is clearly not a projective measurement.
This POVM allows one to discriminate between the state $|\psi_1\rangle = |0\rangle$ and $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.
Suppose given state is $|\psi_1\rangle = |0\rangle$. Bob performs the measurement described by the POVM $\{E_1, E_2, E_3\}$. There is a zero possibility that he will observe the result E_1 since, $\langle\psi_1| E_1 |\psi 1\rangle = 0$. If result is E_1 then he received $|\psi_2\rangle$.

A similar line of reasoning shows that if the measurement outcome be E_2 occurs, then it must be the state $|\psi_1\rangle$.
The cost for this discrimination is that if he get a result corresponding to E_3, then the measurement tells us nothing.
NOTE:
Even though POVMs strictly generalize projective measurement, one can show that every POVM can be "simulated" by a projective measurement on a slightly larger space that yeilds the exact same probability distribution over the measurement outcome.