Assignment 6.

No cloning theorem!

~~the~~ One cannot copy / clone general pure ~~qubits~~ state qubits using an Hermitian operator.

~~let~~ Suppose one can clone copy, pure state

Let $|\psi\rangle$ be a pure state qubit.

Let $|s\rangle$ be an target slot.

Let U be an Hermitian which which does this.

$$\therefore \quad |\psi\rangle \otimes |s\rangle \xrightarrow{U} |\psi\rangle \otimes |\psi\rangle.$$

$$U|\psi\rangle \otimes |s\rangle = |\psi\rangle \otimes |\psi\rangle$$

$$U(|\phi\rangle \otimes |s\rangle) = |\phi\rangle \otimes |\phi\rangle$$

for two $|\phi\rangle$ & $|\psi\rangle$.

$$\langle\psi|\phi\rangle = \langle\psi\psi|\phi\phi\rangle = \langle\psi|\phi\rangle\langle\psi|\phi\rangle$$

$$= (\langle\psi|\phi\rangle)^2$$

$$\Rightarrow \quad \langle\psi|\phi\rangle = 1 \quad \& \quad \langle\psi|\phi\rangle = 0.$$

$\langle \psi | \phi \rangle = 1 \Rightarrow |\psi\rangle = |\phi\rangle$.

$\langle \psi | \phi \rangle = 0 \Rightarrow |\psi\rangle \perp |\phi\rangle$

But if $|\psi\rangle = |0\rangle$ & $|\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |2\rangle)$

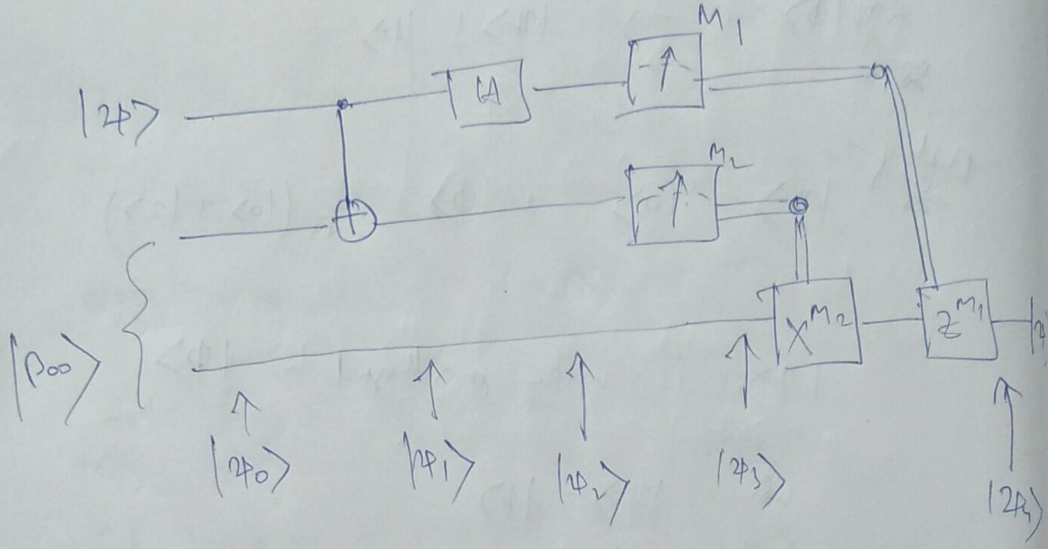
$|\psi\rangle$ is neither ~~pe~~ orthogonal to $|\phi\rangle$

nor $|\psi\rangle = |\phi\rangle$

$\therefore$ there is no ~~Hermitian operator~~ unitary operator which can copy both $|\phi\rangle$ & $|\psi\rangle$.

Thus a generic copy gate isn't possible, which ~~can copy all qubits~~. Can make a copy of all qubits.

Quantum Teleportation :



$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle.$

$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right).$

$|\psi_0\rangle = |\psi\rangle|\beta_{00}\rangle$

$= \frac{1}{\sqrt{2}}\left(\alpha|0\rangle|00\rangle + \alpha|0\rangle|11\rangle + \beta|1\rangle|00\rangle + \beta|1\rangle|11\rangle\right).$

$|\psi_1\rangle = CNOT \otimes I \; \frac{1}{\sqrt{2}}\left(\alpha|0\rangle\left(|00\rangle + |11\rangle\right) + \beta|1\rangle\left(|00\rangle + |11\rangle\right)\right).$

$|\psi_2\rangle = H \otimes I \otimes I \; \frac{1}{\sqrt{2}}\left(\alpha|0\rangle\left(|00\rangle + |11\rangle\right) + \beta|1\rangle\left(|10\rangle + |01\rangle\right)\right)$

$|\psi_2\rangle = \frac{1}{2}\left(\alpha\left(|0\rangle + |1\rangle\right)\left(|00\rangle + |11\rangle\right) + \beta\left(|0\rangle - |1\rangle\right)\left(|10\rangle + |01\rangle\right)\right.$

$$|\psi_2\rangle = \frac{1}{2}\Big[\,|00\rangle\,(\alpha|0\rangle + \beta|1\rangle) + |01\rangle\,(\alpha|1\rangle + \beta|0\rangle) + |10\rangle\,(\alpha|0\rangle - \beta|1\rangle) + |11\rangle\,(\alpha|1\rangle - \beta|0\rangle)\Big].$$

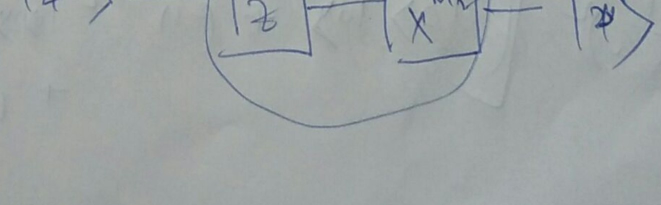
This expression naturally breaks down into 4 terms.

M_1	M_2	
0	0	$\longmapsto |\psi_3(00)\rangle = [\alpha|0\rangle + \beta|1\rangle]$
0	1	$\longmapsto |\psi_3(01)\rangle = [\alpha|1\rangle + \beta|0\rangle]$
1	0	$\longmapsto |\psi_3(10)\rangle = [\alpha|0\rangle - \beta|1\rangle]$.
1	1	$\longmapsto |\psi_3(11)\rangle = [\alpha|1\rangle - \beta|0\rangle]$.

depending on M_1 & M_2 Bob's qubit lie in one of the 4-states. Once he learns about ~~the~~ M_1 & M_2 He can fix up his qubit and recover $|\psi\rangle$.

Suppose he gets. $|\psi'\rangle$ and M_1 & M_2 are given to by ~~b~~ Alice.

Now he can apply $Z^{M_1} X^{M_2} |\psi'\rangle$ to recover $|\psi\rangle$.



Q.

$$H^{\otimes n}|a_1 a_2 \dots a_n\rangle$$

let's suppose there are k-ones in a_i's.

$|a_1\rangle$ — H — $\frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{a_1}|1\rangle\right)$.

$|a_2\rangle$ — H — $\frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{a_2}|1\rangle\right)$.

$|a_3\rangle$ — H — $\frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{a_3}|1\rangle\right)$.

$\vdots$

$|a_n\rangle$ — H — $\frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{a_n}|1\rangle\right)$.

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{a_1}|1\rangle\right) \otimes \dots \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{a_n}|1\rangle\right)$$

$$= \left(\frac{1}{\sqrt{2}}\right)^n \left(|0\rangle \left[\left(|0\rangle + (-1)^{a_2}|1\rangle\right) \otimes \dots \otimes \left(|0\rangle + (-1)^{a_n}|1\rangle\right)\right] + (-1)^{a_1}|1\rangle\right)$$

let's check for $n=2$.

$|a_1\rangle$ — $\boxed{H}$ — $\frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{a_1}|1\rangle\right)$

$|a_2\rangle$ — $\boxed{H}$ — $\frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{a_2}|1\rangle\right)$

$$\left(\frac{1}{\sqrt{2}}\right)^2 \left[|0\rangle|0\rangle + (-1)^{a_2}|0\rangle|1\rangle + (-1)^{a_1}|10\rangle + (-1)^{a_1+a_2}|11\rangle\right)$$

$n=3$.

$$\frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{a_1}|1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{a_2}|1\rangle\right)$$
$$\otimes \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{a_3}|1\rangle\right).$$

$$= \left(\frac{1}{\sqrt{2}}\right)^3 \left[(-1)^{0.a_1}(-1)^{0.a_2}|00\rangle + (-1)^{0.a_1}(-1)^{1.a_2}|01\rangle + (-1)^{1.a_1}(-1)^{0.a_2}|10\rangle + (-1)^{1.a_1}(-1)^{1.a_2}|11\rangle \right] \left[|0\rangle + (-1)^{a_3}|1\rangle\right]$$

$$= \left(\frac{1}{\sqrt{2}}\right)^3 \left[(-1)^{0.a_1}(-1)^{0.a_2}(-1)^{0.a_3}|000\rangle + (-1)^{0.a_1}(-1)^{0.a_2}(-1)^{1.a_3}|001\rangle + \cdots + (-1)^{1.a_1}(-1)^{1.a_2}(-1)^{1.a_3}|111\rangle \right].$$

Proof by induction.

Suppose for $k=n-1$ the formula is true.

Induction hypothesis:

$$H^{\otimes n-1}|a_1 \ldots a_{n-1}\rangle = \sum_{(x_1 \ldots x_{n-1}) \in \{0,1\}^{n-1}} (-1)^{\sum_{i=1}^{n-1} x_i a_i} |x_1, \ldots x_{n-1}\rangle \checkmark$$

$$H^{\otimes n}|a_1 \ldots a_n\rangle = \left(H^{\otimes n-1}|a_1 \ldots a_{n-1}\rangle\right) \otimes H|a_n\rangle$$

$$= \left(\sum_{(x_1, x_2, \dots x_{n-1}) \in \{0,1\}^{n-1}} (-1)^{\sum_{i=1}^{n-1} x_i a_i} |x_1 \dots x_{n-1}\rangle \right) \otimes \left(|0\rangle + (-1)^{a_n} |1\rangle \right)$$

$$= \sum_{(x_1, x_2, \dots x_{n-1})} (-1)^{\sum_{i=1}^{n-1} x_i a_i} |x_1 \dots x_{n-1} 0\rangle + \sum_{(x_1, x_2, \dots x_{n-1})} (-1)^{a_n} (-1)^{\sum_{i=1}^{n-1} x_i a_i} |x_1 x_2 \dots x_{n-1} 1\rangle$$

$$= \sum_{(x_1 x_2 \dots x_n) \in \{0,1\}^n} (-1)^{\sum_{i=1}^{n} x_i a_i} |x_1 x_2 \dots x_n\rangle$$

$$\therefore \quad H^{\otimes n} |a_1 a_2 \dots a_n\rangle = \sum_{(x_1, x_2, x_3 \dots x_n) \in \{0,1\}^n} (-1)^{\sum_{i=1}^{n} x_i a_i} |x_1 x_2 \dots x_n\rangle$$

Given a $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ &

$$|\psi^{\perp}\rangle = \beta^{*}|0\rangle - \alpha^{*}|1\rangle$$

Prove that they are orthogonal and find an unitary matrix B such that it transforms $|\psi\rangle$ to $|0\rangle$ & $|\psi^{\perp}\rangle$ to $|1\rangle$

$$\langle\psi|\psi\rangle = 1 = \langle\psi^{\perp}|\psi^{\perp}\rangle$$

$$\langle\psi|\psi^{\perp}\rangle = \begin{bmatrix}\alpha^{*} & \beta^{*}\end{bmatrix}\begin{bmatrix}\beta^{*} \\ -\alpha^{*}\end{bmatrix} = \alpha^{*}\beta^{*} - \alpha^{*}\beta^{*}$$

$$= 0.$$

$$|0\rangle \xmapsto{A} |\psi\rangle$$

$$|1\rangle \xmapsto{A} |\psi^{\perp}\rangle$$

$$A = \begin{bmatrix}\alpha & \beta^{*} \\ \beta & -\alpha^{*}\end{bmatrix}$$

$\because$ A is unitary.

$$\therefore A^{-1} = A^{*} = \begin{bmatrix}\alpha^{*} & \beta^{*} \\ \beta & -\alpha\end{bmatrix}$$

$$\therefore \quad |\psi\rangle \xmapsto{A^{*}} |0\rangle$$

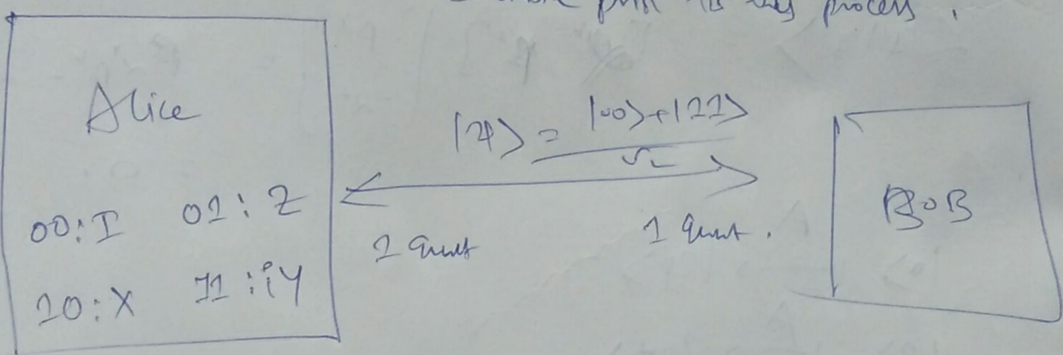
$$|\psi^{\perp}\rangle \xmapsto{A^{*}} |1\rangle.$$

Dense coding

Suppose BoB and Alice share a pair of qubits in the entangled state $|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$

& Alice gets 1 qubit & BoB gets second qubit.

This preparing & sharing can be done by some 3rd party Charlie prior to this process.



Now Alice if alice want to send $a_1 a_2$ to Bob,

If Alice want to send 00 to BoB, then she does nothing

if she wants to send 01 to BoB, then she applies phase flip Z to her qubit.

if she wishes to send 10, then then she ~~flip~~ applies X to her qubit.

if she wishes to send 11, then she applies iY gate to her qubit.

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \qquad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad iY = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$00 : |\psi\rangle \xrightarrow{I} \frac{|00\rangle + |11\rangle}{\sqrt{2}}$

phase of this bit changes

$01 : |\psi\rangle \xrightarrow{Z} \frac{|00\rangle - |11\rangle}{\sqrt{2}}$

$10 : |\psi\rangle \xrightarrow{X} \frac{|10\rangle + |01\rangle}{\sqrt{2}}$ — bit flips occur on Alice side ~~since~~

$11 : |\psi\rangle \xrightarrow{iY} \frac{|01\rangle - |10\rangle}{\sqrt{2}}$

Now BoB can measure the change of his part of qubit using Bell basis, and thus can find out the bits which Alice wants to send to him.