1. (a) What is the inner product between the real vectors (0, 1, 0, 1) and (0, 1, 1, 1)?
   (b) What is the inner product between the states $|0101\rangle$ and $|0111\rangle$?

Ans:

ⓐ In real vector space. the inner product between two vectors $(0,1,0,1)$ and $(0,1,1,1)$ is defined by

$$\langle (0,1,0,1), (0,1,1,1) \rangle = 0\cdot 0 + 1\cdot 1 + 0\cdot 1 + 1\cdot 1$$
$$= 2.$$

ⓑ The inner product the states $|0101\rangle$ and $|0111\rangle$ is given by

$$\langle 0101 | 0111 \rangle$$
$$= (|0101\rangle^*)^T |0111\rangle$$

Now,

$$|0101\rangle = |0\rangle \otimes |1\rangle \otimes |0\rangle \otimes |1\rangle$$
$$= \underbrace{\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}} \otimes \underbrace{\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}}$$
$$= \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$
$$= (0000\ 0100\ 0000\ 0000)^T$$

So, $(|0101\rangle^*)^T = (0000\ 0100\ 0000\ 0000)$

Again.
$$|0111\rangle = |0\rangle \otimes |1\rangle \otimes |1\rangle \otimes |1\rangle$$
$$= \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}$$
$$= \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \otimes \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$
$$= (0000\ 0001\ 0000\ 0000)^T$$

Hence,
$$\langle 0101 | 0111 \rangle = (0000\ 0100\ 0000\ 0000)\,(0000\ 0001\ 0000\ 0000)^T$$
$$= 0$$

2. Compute the result of applying a Hadamard transform to both qubits of |0⟩ ⊗ |1⟩ in two ways (the first way using tensor product of vectors, the second using tensor product of matrices), and show that the two results are equal:

$$H|0\rangle \otimes H|1\rangle = (H \otimes H)(|0\rangle \otimes |1\rangle).$$

Ans: We know that, the Hadamard transformation is

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Now,

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = |+\rangle$$

$$H|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = |-\rangle$$

So,

$$H|0\rangle \otimes H|1\rangle$$

$$= |+\rangle \otimes |-\rangle$$

$$= \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

Now,

$$|0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

and

$$H \otimes H$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\left\{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right\}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

So,

$$(H \otimes H)(|0\rangle \otimes |1\rangle) = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

Hence, $H|0\rangle \otimes H|1\rangle = (H \otimes H)(|0\rangle \otimes |1\rangle)$ [Proved]

3. Show that a bitflip operation, preceded and followed by Hadamard transforms, equals a phaseflip operation: HXH = Z.

Ans: Let, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and we know that,

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}, \quad X = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad Z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

Now,

$$\begin{aligned}
&(HXH)|\psi\rangle \\
&= (HX)(H|\psi\rangle) \\
&= HX \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \\
&= \frac{1}{\sqrt{2}}(HX)\begin{pmatrix}\alpha+\beta\\ \alpha-\beta\end{pmatrix} \\
&= \frac{1}{\sqrt{2}} H \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\alpha+\beta\\ \alpha-\beta\end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\alpha-\beta\\ \alpha+\beta\end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix}\alpha-\cancel{\beta}+\alpha+\cancel{\beta}\\ \alpha-\beta-\alpha-\beta\end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix}2\alpha\\ -2\beta\end{pmatrix} \\
&= \begin{pmatrix}\alpha\\ -\beta\end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
&Z|\psi\rangle \\
&= \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \\
&= \begin{pmatrix}\alpha\\ -\beta\end{pmatrix}
\end{aligned}$$

So, $HXH|\psi\rangle = Z|\psi\rangle$

Hence, $HXH = Z$

4. Show that surrounding a CNOT gate with Hadamard gates switches the role of the control-bit and target-bit of the CNOT: (H ⊗ H)CNOT(H ⊗ H) is the 2-qubit gate where the second bit controls whether the first bit is negated (i.e., flipped).

Ans: We know that,

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \quad \text{and} \quad CNOT = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}$$

and,

$$H \otimes H = \frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix}$$

Therefore,

$$(H \otimes H)\, CNOT\, (H \otimes H)$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = U \quad \text{(say)}.$$

Then,

$$U|00\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |00\rangle.$$

$$U|01\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |11\rangle$$

$$U|10\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = |10\rangle$$

$$U|11\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |01\rangle.$$

So, by using the transformation U we have

$$|00\rangle \rightarrow |00\rangle$$
$$|01\rangle \rightarrow |11\rangle$$
$$|10\rangle \rightarrow |10\rangle$$
$$|11\rangle \rightarrow |10\rangle.$$

Clearly, we have seen that when 2nd bit is 0 there is no change but when 2nd bit is 1 then 1st bit is negeted.

5. Simplify the following: (⟨0| ⊗ I)(α00|00⟩ + α01|01⟩ + α10|10⟩ + α11|11⟩).

Ans:

$$\langle 0| \otimes I$$

$$= (|0\rangle^*)^T \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}^*\right)^T \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and,

$$\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

$$= \alpha_{00} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_{01} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_{10} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \alpha_{11} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{pmatrix}$$

Hence,

$$(\langle 0| \otimes I)(\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_{00} \\ \alpha_{01} \end{pmatrix} \quad \text{(Ans)}$$