1 (a) What is the inner product between the real vectors $(0,1,0,1)$ and $(0,1,1,1)$?

(b) What is the inner product between the states $|0101\rangle$ and $|0111\rangle$?

Solution

(a) Here the vectors are $(0,1,0,1)$ and $(0,1,1,1)$

Now inner product of $(0,1,0,1)$ and $(0,1,1,1)$

$$= \langle (0,1,0,1), (0,1,1,1) \rangle$$

$$= \begin{pmatrix}0\\1\\0\\1\end{pmatrix}^T \begin{pmatrix}0\\1\\1\\1\end{pmatrix}$$

$$= (0\ 1\ 0\ 1)\begin{pmatrix}0\\1\\1\\1\end{pmatrix} = 1+1 = 2.$$

$\therefore$ $\boxed{\langle (0,1,0,1), (0,1,1,1) \rangle = 2}$

(b)

$$|0101\rangle = |0\rangle \otimes |1\rangle \otimes |0\rangle \otimes |1\rangle$$

$$= \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}_{16\times 1}$$

$$|0111\rangle = |0\rangle \otimes |1\rangle \otimes |1\rangle \otimes |1\rangle$$

$$= \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}$$

$$= \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \otimes \begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}_{16\times 1}$$

$\therefore$ $\langle |0101\rangle, |0111\rangle \rangle$

$$= (0000010000000000)\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} = 0$$

$\therefore$ $\boxed{\langle |0101\rangle, |0111\rangle \rangle = 0}$

(2) Compute the result of applying a Hadamard transform to both qubits of $|0\rangle \otimes |1\rangle$ in two ways. (the first ways using tensor product of vectors, the second using tensor product of matrices), and show that two result are equal.

$$\boxed{H|0\rangle \otimes H|1\rangle = (H \otimes H)(|0\rangle \otimes |1\rangle)}$$

Solution

We have $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$\cancel{H|0\rangle}$ $\quad |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$H|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$(H \otimes H) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$|0\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Now $$H|0\rangle \otimes H|1\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1/\sqrt{2} \end{pmatrix} \otimes \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \cdot -1/\sqrt{2} \\ \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \cdot -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$$

And again

$$(H \otimes H)(|0\rangle \otimes |1\rangle) = \cancel{(H \otimes H)}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$$

$$\boxed{\left(H|0\rangle \otimes H|1\rangle\right) = (H \otimes H)(|0\rangle \otimes |1\rangle)}$$

(Proved)

Q (3) Show that a bitflip operation, preceded and followed by Hadamard transformations, equals to a phaseflip operation : $HXH = Z$.

Soln :

We can write, $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}{\sqrt{2}}$

$$\therefore \boxed{H = \frac{(X+Z)}{\sqrt{2}}}$$

Also $X^2 = Z^2 = I$.

$XZ = -ZX$ (we have)

Now

$$HXH = \frac{(X+Z)}{\sqrt{2}} \cdot X \cdot \frac{(X+Z)}{\sqrt{2}}$$

$$= \frac{(X^2+ZX)(X+Z)}{2}$$

$$= \frac{(X^3+X^2Z+ZX^2+ZXZ)}{2}$$

$$= \frac{X^3+\cancel{X}Z+Z+Z(-ZX)}{2} \quad [X^2=I,\ XZ=-ZX]$$

$$= \frac{X+2Z-Z^2X}{2}$$

$$= \frac{X+2Z-X}{2} \quad [\because Z^2=I]$$

$$= Z$$

So, $\boxed{HXH = Z}$ (Proved)

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(4) Show that ~~a~~ surrounding a CNOT gate with Hadamard gates ~~with~~ switches the role of the control-bit and target bit of the CNOT: $(H \otimes H)\ CNOT\ (H \otimes H)$ is the 2-qubit gate where the ~~1st~~ second bit controls whether the first bit is ~~g~~ negated (i.e. flipped).

(Solution)

We have $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$H \otimes H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

matrix of

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\therefore (H \otimes H)\ CNOT\ (H \otimes H) = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

So the matrix of

$$(H \otimes H)\ CNOT\ (H \otimes H) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

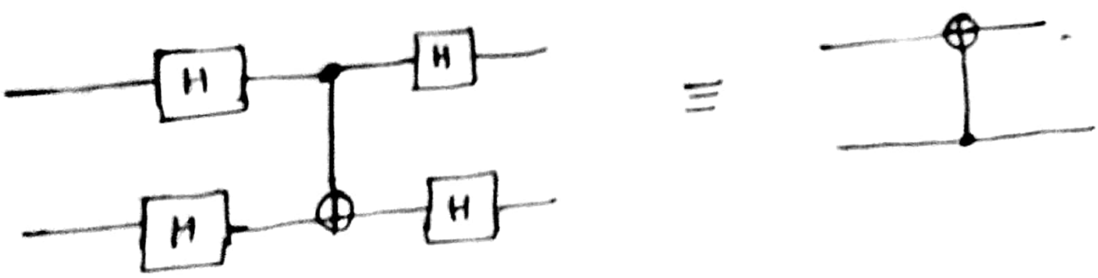
Now by the matrix representation of

$(H \otimes H)$ CNOT $(H \otimes H)$ we have the 2-qubit transformation as

$$|00\rangle \longrightarrow |00\rangle$$
$$|01\rangle \longrightarrow |11\rangle$$
$$|10\rangle \longrightarrow |10\rangle$$
$$|11\rangle \longrightarrow |01\rangle$$

From this input and output we see that 2nd bit becomes controll bit and 1st bit becomes target bit so it interchanged with CNOT Gate.

In circuit it happen.-



(5) Simplify the following:

$$\left(\langle 0| \otimes I\right) \left(\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle\right)$$

Sol^n

$$\left(\langle 0| \otimes I\right)\left(\alpha_{00}|0\rangle \otimes |0\rangle + \alpha_{01}|0\rangle \otimes |1\rangle + \alpha_{10}|1\rangle \otimes |0\rangle + \alpha_{11}|1\rangle \otimes |1\rangle\right)$$

$$= \alpha_{00}\langle 0|0\rangle \otimes I|0\rangle + \alpha_{01}\langle 0|0\rangle \otimes I|1\rangle + \alpha_{10}\langle 0|1\rangle \otimes I|0\rangle + \alpha_{11}\langle 0|1\rangle \otimes I|1\rangle.$$

$$= \alpha_{00} I|0\rangle + \alpha_{01} I|1\rangle + 0 + 0$$

$$= \alpha_{00}|0\rangle + \alpha_{01}|1\rangle$$

$[\because \langle 0|0\rangle = \langle 1|1\rangle = 1$

$\langle 0|1\rangle = 0.$

$I|0\rangle = |0\rangle$

$I|1\rangle = |1\rangle.$

$$\therefore \boxed{\left(\langle 0| \otimes I\right)\left(\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle\right) = \alpha_{00}|0\rangle + \alpha_{01}|1\rangle}$$