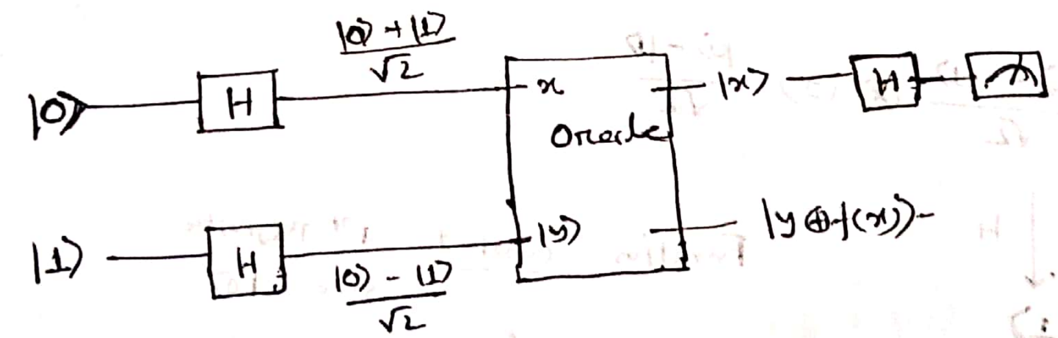
Algorithms - Basics

- State of the quantum register : linear combination of states.
- Quantum parallelism : Computation of function for each of the states in the input register.
- Oracle : A black box computation, analogous to a classical subroutine
- Measurement to extract required result.

Destuch- Alogorithm

$f : \{0,1\} \to \{0,1\}$.

- input : 1 qubit 0/1 ; output : 1 qubit 0/1.
- Function is either a constant $f(0) = f(1) = 0$ or $f(0) = f(1) = 1$.
- a balanced function either $f(0) = 0, f(1) = 1$ or $f(0) = 1, f(1) = 0$.
- Classical computation required two queries. Quantum computer can achieve it in a single query.



$$U_f\left(|x\rangle \otimes |y\rangle\right) \longrightarrow |x\rangle \otimes |y \oplus f(x)\rangle$$

$$\longrightarrow \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \left|\frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes f(x)\right\rangle$$

Input : $\frac{1}{2}\left[|00\rangle + |10\rangle - |01\rangle - |11\rangle\right]$.

Oracle : $\frac{1}{2}\Big[|0, 0\oplus f(0)\rangle + |1, 0\oplus f(1)\rangle$

$- |0, 1\oplus f(0)\rangle - |1, 1\oplus f(1)\rangle\Big]$.

$= \frac{1}{2}\left[|0, f(0)\rangle + |1, f(1)\rangle - |0, \overline{f(0)}\rangle - |1, \overline{f(1)}\rangle\right]$ — ⊛

General case

If $f(0)=f(1)$ constant f $\Big[= \frac{1}{2}\left[\left(|0\rangle + |1\rangle\right)\left(f(0) - \overline{f(0)}\right)\right]$

$\cong \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

$\xrightarrow[\text{1st register}]{H} |0\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

$\boxed{\text{Function is balanced}}$

$f(1) = \overline{f(0)}$ and $f(0) = \overline{f(1)}$

So, we can write from ⊛

$\frac{1}{2}\left[|0, f(0)\rangle + |1, f(1)\rangle - |0, f(1)\rangle - |1, f(0)\rangle\right]$

$= \cancel{\frac{1}{2}} \frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes (\pm) \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

1st register $\downarrow H$

$|1\rangle$

Function constant 1st register give $|0\rangle$

Function Balanced 1st register give $|1\rangle$

So, we can find the function is Balanced or constant by only ~~see~~ observing ~~1st qubit~~ 1st Register ~~&~~ after ~~sending~~ moving throw H gate.

Deutsch- Josze Problem

- An extension of Deutsch algorithm to the case of an n qubit input : $f: \{0,1\}^{\otimes n} \to \{0,1\}$

- The function is either a constant or balanced, i.e. exactly half of them gives 0 while other half of them give to 1.

- Input is a uniform linear combination of the n qubit computational basis states

$$\{x_{n-1}, x_{n-2}, \ldots, x_0\} \qquad x_i = 0, 1$$

In worst case we need $\left(\frac{2^n}{2}+1\right)$ trails.

Input : a linear combination of n-qubit computational basis state

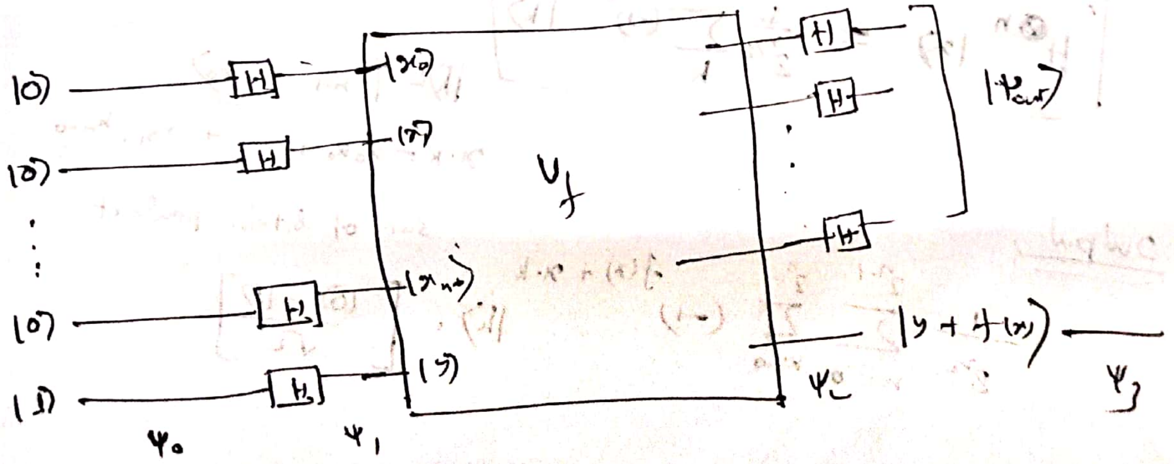
$$\left.\begin{aligned} |0\rangle &\xrightarrow{H} \frac{|0\rangle + |1\rangle}{\sqrt{2}} \\ |1\rangle &\xrightarrow{H} \frac{|0\rangle - |1\rangle}{\sqrt{2}} \end{aligned}\right\} \quad |x\rangle \xrightarrow{H} \frac{|0\rangle + (-1)^x |1\rangle}{\sqrt{2}}$$

$$|0\rangle^{\otimes n} \xrightarrow{H^{\otimes n}} \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \cdots\cdots \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$= \frac{1}{2^{n/2}} \sum_x |x\rangle$$

($|x\rangle$ - n-qubit basis state, uniform linear combination)

$$|0\rangle^{\otimes 2} \xrightarrow{H^{\otimes 2}} \frac{1}{2}\left[|00\rangle + |01\rangle + |10\rangle + |11\rangle\right].$$



Input : $\frac{1}{2^{n/2}} \sum_{x=0}^{2^n-1} |x\rangle$

Target : $\frac{|0\rangle - |1\rangle}{\sqrt{2}}$

Output : Entangled

$$\frac{1}{2^{(n+1)/2}} \sum_{x=0}^{2^n-1} |x\rangle \otimes \left[|f(x)\rangle - |\bar{f}(x)\rangle\right]$$

$f(x) = 0$ or 1

$$\frac{1}{2^{\left(\frac{n+1}{2}\right)}} \sum_{x=0}^{2^n-1} |x\rangle \otimes (-1)^{f(x)} \left[|0\rangle - |1\rangle\right]$$

$$= \left[\underbrace{\frac{1}{\sqrt{2^n}} \sum_x (-1)^{f(x)} |x\rangle} \right] \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$\Big\downarrow H^{\otimes n}$$

$$|x_i\rangle \xrightarrow{H} \frac{|0\rangle + (-1)^{x_i} |1\rangle}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}} \sum_{k_i=0}^{1} (-1)^{x_i k_i} |k_i\rangle$$

$|x\rangle := |x_{n-1}, x_{n-2} \ldots x_1 x_0\rangle$

$$|x\rangle \xrightarrow{H^{\otimes n}} \frac{1}{2^{n/2}} \sum_{k_{n-1}=0}^{1} \cdots \sum_{k_0=0}^{1} (-1)^{\sum x_i k_i} |k_{n-1} \ldots k_0\rangle$$

$$\boxed{H^{\otimes n} |x\rangle = \frac{1}{2^{n/2}} \sum_k (-1)^{x \cdot k} |k\rangle}$$

$|k\rangle = |k_{n-1} \ldots k_0\rangle$

$x \cdot k = x_0 k_0 + \ldots + x_{n-1} k_{n-1}$

sum of bitwise product

$\therefore$ Output,

$$\frac{1}{2^n} \sum_{x=0}^{2^n-1} \sum_{k=0}^{2^n-1} (-1)^{f(x) + x \cdot k} |k\rangle \cdot \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

Let $f(x)$ is constant

First Register will be in state $|0\rangle^{\otimes n}$.

For balanced function, the co-efficient of $k=0$ is

$$\sum_{x=0}^{2^{n}-1} (-1)^{f(x)} = 0.$$

because there are as many as $f(x)=1$ as there are $f(x)=0$.

* First register $= 0$: - constant function

* First register anything except 0, $\rightarrow$ balanced function

So single query determine whether may function was balanced or not.