Assignment – 4

1. a) $v = (0, 1, 0, 1)$ $\quad w = (0, 1, 1, 1)$

$$v \cdot w = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 2 .$$

let $v = |0101\rangle$ $\quad w = |0111\rangle$

$$v = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad w = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore \langle 0111 | 0101 \rangle = 0 .$

2) $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$H|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

$$H|0\rangle \otimes H|1\rangle = \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$$

$$H \otimes H = \frac{1}{2}\begin{bmatrix} 1\cdot\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} & 1\cdot\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \\ 1\cdot\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} & -1\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$|0\rangle \otimes |1\rangle = |01\rangle$$

So, $\begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix} 1\cdot\begin{bmatrix}0\\1\end{bmatrix} \\ 0\cdot\begin{bmatrix}0\\1\end{bmatrix} \end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$

So $(H \otimes H)(|0\rangle \otimes |1\rangle)$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}$$

$$= \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$$

$$\therefore\ H|0\rangle \otimes H|1\rangle = (H \otimes H)(|0\rangle \otimes |1\rangle)$$

2.

3) To show, $HXH = Z$

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \text{bit flip}$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \text{phase flip.}$$

$$HXH = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\therefore \; HXH = Z.$

Another proof: $|\alpha|^2 + |\beta|^2 = 1.$

$$HXH(\alpha|0\rangle + \beta|1\rangle) = HX\left(\frac{\alpha}{\sqrt{2}}(|0\rangle + |1\rangle) + \frac{\beta}{\sqrt{2}}(|0\rangle - |1\rangle)\right)$$

$$= HX\left(\frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle\right)$$

$$= H\left(\frac{\alpha+\beta}{\sqrt{2}}|1\rangle + \frac{\alpha-\beta}{\sqrt{2}}|0\rangle\right)$$

$$= \frac{\alpha+\beta}{2}(|0\rangle - |1\rangle) + \frac{\alpha-\beta}{2}(|0\rangle + |1\rangle)$$

$$= \alpha|0\rangle - \beta|1\rangle = Z(\alpha|0\rangle + \beta|1\rangle), \therefore HXH = Z$$

4)

$(H \otimes H)\ CNOT\ (H \otimes H)\ \cancel{|10\rangle}$

$\cancel{H \otimes H\ |00\rangle}$

$$H \otimes H = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$(H \otimes H)\ CNOT\ (H \otimes H) = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$(H \otimes H)\, CNOT\, (H \otimes H) \cancel{\{} |01\rangle$

$$= \begin{bmatrix} 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= |11\rangle$$

$(H \otimes H)\, CNOT\, (H \otimes H)\, |11\rangle$

$$= \begin{bmatrix} 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |01\rangle$$

$(H \otimes H)\, CNOT\, (H \otimes H)\, |00\rangle = |00\rangle$

$(H \otimes H)\, CNOT\, (H \otimes H)\, |10\rangle = |10\rangle$

So $(H \otimes H)\, CNOT\, (H \otimes H)$ flips first ~~qubit~~ qubit whenever second ~~bit~~ qubit (i.e select ~~qubit~~ qubit) is 1, else it doesn't flip first ~~qubit~~ qubit.

5) $(\langle 0| \otimes I)\,(\alpha_{00}|00\rangle + \alpha_{01}|11\rangle + \alpha_{10}|10\rangle$
$+ \alpha_{11}|11\rangle).$

$$\left([1\ \ 0] \otimes \begin{bmatrix}1&0\\0&1\end{bmatrix}\right)\begin{bmatrix}\alpha_{00}\\ \alpha_{01}\\ \alpha_{10}\\ \alpha_{11}\end{bmatrix} = \begin{bmatrix} 1\cdot\begin{bmatrix}1&0\\0&1\end{bmatrix} & 0\cdot\begin{bmatrix}1&0\\0&1\end{bmatrix} \end{bmatrix}\begin{bmatrix}\alpha_{00}\\ \alpha_{01}\\ \alpha_{11}\\ \alpha_{11}\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0&0\\0&1&0&0\end{bmatrix}_{2\times 4}\begin{bmatrix}\alpha_{00}\\ \alpha_{01}\\ \alpha_{10}\\ \alpha_{11}\end{bmatrix} = \begin{bmatrix}\alpha_{00}\\ \alpha_{01}\end{bmatrix}$$

$$= \alpha_{00}|0\rangle + \alpha_{01}\cdot|1\rangle$$