(1) Given a 2-qubit state $|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$, show that it is not possible to find $|\psi_1\rangle$ & $|\psi_2\rangle$ such that $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$

Ans: Let if possible $|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$ &
$|\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$ such that $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$
for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$ & $|\alpha_1|^2 + |\beta_1|^2 = |\alpha_2|^2 + |\beta_2|^2 = 1$

$$|\psi_1\rangle \otimes |\psi_2\rangle$$
$$= [\alpha_1|0\rangle + \beta_1|1\rangle] \otimes [\alpha_2|0\rangle + \beta_2|1\rangle]$$
$$= \alpha_1\alpha_2(|0\rangle \otimes |0\rangle) + \alpha_1\beta_2(|0\rangle \otimes |1\rangle) + \beta_1\alpha_2(|1\rangle \otimes |0\rangle) + \beta_1\beta_2(|1\rangle \otimes |1\rangle)$$

$$= \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle$$

$$\therefore |\psi\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix} = |\psi_1\rangle \otimes |\psi_2\rangle = \begin{pmatrix} \alpha_1\alpha_2 \\ \alpha_1\beta_2 \\ \beta_1\alpha_2 \\ \beta_1\beta_2 \end{pmatrix}$$

$$\Rightarrow \alpha_1\alpha_2 = \frac{1}{\sqrt{2}}, \quad \alpha_1\beta_2 = 0, \quad \beta_1\alpha_2 = 0, \quad \beta_1\beta_2 = \frac{1}{\sqrt{2}}$$

Now, $|\alpha_1\alpha_2|^2 + |\alpha_1\beta_2|^2 = \frac{1}{2} + 0$
$$\Rightarrow |\alpha_1|^2|\alpha_2|^2 + |\alpha_1|^2|\beta_2|^2 = \frac{1}{2}$$
$$\Rightarrow |\alpha_1|^2(|\alpha_2|^2 + |\beta_2|^2) = \frac{1}{2}$$
$$\Rightarrow |\alpha_1|^2 = \frac{1}{2} \quad [\because |\alpha_2|^2 + |\beta_2|^2 = 1]$$

Similarly, $|\beta_1\alpha_2|^2 + |\beta_1\beta_2|^2 = \frac{1}{2}$
$$\Rightarrow |\beta_1|^2 = \frac{1}{2} \quad [\because |\alpha_2|^2 + |\beta_2|^2 = 1]$$

Now, $\alpha_1 \beta_2 = 0$

$\Rightarrow |\alpha_1 \beta_2|^2 = 0$

$\Rightarrow |\alpha_1|^2 \, |\beta_2|^2 = 0$

$\Rightarrow \frac{1}{2} \, |\beta_2|^2 = 0 \quad \left[\because |\alpha_1|^2 = \frac{1}{2}\right]$

$\Rightarrow |\beta_2|^2 = 0$

Now, $\beta_1 \alpha_2 = 0$

$\Rightarrow |\beta_1 \alpha_2|^2 = 0$

$\Rightarrow |\beta_1|^2 \, |\alpha_2|^2 = 0$

$\Rightarrow \frac{1}{2} |\alpha_2|^2 = 0 \quad \left[\because |\beta_1|^2 = \frac{1}{2}\right]$

$\Rightarrow |\alpha_2|^2 = 0$

Thus $|\alpha_2|^2 + |\beta_2|^2 = 0 + 0 = 0$

which is a contradiction to our assumption

$|\alpha_2|^2 + |\beta_2|^2 = 1$.

Hence not possible.

(2) Find The matrix representation of CCNOT gate & CSWAP gate.

CCNOT Gate:

Ans: If the 1st & 2nd bits are 1, then only the 3rd bit will be flipped

So, we have

$|000\rangle \rightarrow |000\rangle$

$|001\rangle \rightarrow |001\rangle$

$|010\rangle \rightarrow |010\rangle$

$|011\rangle \rightarrow |011\rangle$

$|100\rangle \rightarrow |100\rangle$

$|101\rangle \rightarrow |101\rangle$

$|110\rangle \rightarrow |111\rangle$

$|111\rangle \rightarrow |110\rangle$

$$\therefore M_{CCNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

CSWAP Gate:

$$|000\rangle \rightarrow |000\rangle$$
$$|001\rangle \rightarrow |001\rangle$$
$$|010\rangle \rightarrow |010\rangle$$
$$|011\rangle \rightarrow |011\rangle$$
$$|100\rangle \rightarrow |100\rangle$$
$$|101\rangle \rightarrow |110\rangle$$
$$|110\rangle \rightarrow |101\rangle$$
$$|111\rangle \rightarrow |111\rangle$$

$$\therefore \; M_{CSWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$