(1) Given $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ &
$|\psi^*\rangle = \beta^*|0\rangle - \alpha^*|1\rangle$.

Prove that they are orthogonal & find a unitary matrix B such that it transform $|\psi\rangle$ to $|0\rangle$ & $|\psi^*\rangle$ to $|1\rangle$.

<u>Ans:</u> $|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ & $|\psi^*\rangle = \begin{bmatrix} \beta^* \\ -\alpha^* \end{bmatrix}$.

$$\langle\psi^*|\psi\rangle = \begin{bmatrix} (\beta^*)^* & (-\alpha^*)^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$= \begin{bmatrix} \beta & -\alpha \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \beta\alpha - \alpha\beta.$$

$$= \alpha\beta - \alpha\beta \quad [\because \alpha\beta = \beta\alpha]$$

$$= 0.$$

So, $|\psi\rangle$ & $|\psi^*\rangle$ are orthogonal.

Now, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
$|\psi^*\rangle = \beta^*|0\rangle - \alpha^*|1\rangle$

Now, $A = \begin{bmatrix} \alpha & \beta^* \\ \beta & -\alpha^* \end{bmatrix}$

Now, as we have $|\alpha|^2 + |\beta|^2 = 1$, then each coloumn of A are orthonormal, then A is unitary.
Let inverse of $A = A^* = B$ (say). Then B will be also unitary.

$$B = A^* = \begin{bmatrix} \alpha^* & \beta^* \\ \beta & -\alpha \end{bmatrix}$$

Now, $$B|\psi\rangle = \begin{bmatrix} \alpha^* & \beta^* \\ \beta & -\alpha \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} |\alpha|^2 + |\beta|^2 \\ \alpha\beta - \alpha\beta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$B|\psi^*\rangle = \begin{bmatrix} \alpha^* & \beta^* \\ \beta & -\alpha \end{bmatrix} \begin{bmatrix} \beta^* \\ -\alpha^* \end{bmatrix} = \begin{bmatrix} \alpha^*\beta^* - \alpha^*\beta^* \\ \beta\beta^* + \alpha\alpha^* \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle.$$

So, $B = \begin{bmatrix} \alpha^* & \beta^* \\ \beta & -\alpha \end{bmatrix}$ is our required matrix & also it is unitary ($\because$ Each coloumns of B are orthonormal).

② Prove that the quantum bits cannot be cloned.

<u>Ans</u>:

Suppose there is an unitary matrix U which does the copying procedure. In mathematical sense it can be expressed as

$$U(|\psi\rangle|e\rangle) = |\psi\rangle|\psi\rangle$$, $|e\rangle$ is a normalised state.

As U is unitary then we know $UU^* = I = U^*U$. where U^* is the complex conjugate transpose of U.

Suppose $|\psi\rangle$ & $|\phi\rangle$ are two pure states & the copying procedure is happening for this two things.

Then we have $U(|\psi\rangle|e\rangle) = |\psi\rangle|\psi\rangle$ — ①

$U(|\phi\rangle|e\rangle) = |\phi\rangle|\phi\rangle$. — ②

Now, we will take inner product between ① & ②.

$$\langle e|\langle\phi|U^* U|\psi\rangle|e\rangle = \langle\phi|\langle\phi||\psi\rangle|\psi\rangle.$$

$$\Rightarrow \langle e|\langle\phi|I|\psi\rangle|e\rangle = \langle\phi|\psi\rangle\langle\phi|\psi\rangle.$$

$$\Rightarrow \langle e|\langle\phi||\psi\rangle|e\rangle = [\langle\phi|\psi\rangle]^2$$

$$\Rightarrow \langle\phi|\psi\rangle\langle e|e\rangle = [\langle\phi|\psi\rangle]^2$$

$$\Rightarrow \langle\phi|\psi\rangle = [\langle\phi|\psi\rangle]^2 \quad \text{—③} \quad \left[\because \langle e|e\rangle = 1 \text{ as } |e\rangle \text{ is normalised}\right].$$

let $x = \langle\phi|\psi\rangle$

Then ③ becomes, $x = x^2 \Rightarrow x(x-1) = 0$.

$\Rightarrow x = 0$ or $x = 1$.

So, we have $\langle\phi|\psi\rangle = 0$ or $\langle\phi|\psi\rangle = 1$.

This means that only if the pure states form an orthonormal basis then, only the cloning is possible otherwise not.

Hence in general cloning is not possible for qubits.