For any n-qubit state $|x\rangle$, Prove that

$$H^{\otimes n}|x\rangle = \frac{1}{2^{n/2}} \sum_{y\in\{0,1\}^n} (-1)^{x\cdot y}|y\rangle$$

(This is called Quantum Parallelism)

Proof:

Here H is the hadamard gate whose matrix is defined by $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

So, $H|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \frac{1}{\sqrt{2}}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$\therefore H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. — (*)

Again,

$H|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

$= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

$\therefore H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ — (**)

$(*)$ & $(**)$ can be written as

$$\left.\begin{aligned} H|0\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^0|1\rangle\right). \\ H|1\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^1|1\rangle\right). \end{aligned}\right\}$$

So, we can write the above two eqn. combinely as

$$H|x\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^x y\right)$$

$$\Rightarrow \boxed{H|x\rangle = \frac{1}{\sqrt{2}} \sum_{y\in\{0,1\}} (-1)^{x\cdot y} |y\rangle.}$$

Now consider, $f(n) = H^{\otimes n}|x\rangle = \frac{1}{2^{n/2}} \sum_{y\in\{0,1\}^n} (-1)^{x\cdot y} |y\rangle$

We will prove this result by the method of induction.

For $n=1$, the proof follows from the above part.

So, $f(1)$ is true i.e. $H|x\rangle = \frac{1}{\sqrt{2}} \sum_{y\in\{0,1\}} (-1)^{x\cdot y} |y\rangle.$

Hence the result is true for $n=1$.

Suppose $f(n)$ is true for $n=k$.

So, we have $H^{\otimes k}|x\rangle = \frac{1}{2^{k/2}} \sum_{y\in\{0,1\}^k} (-1)^{x\cdot y} |y\rangle$

where $|x\rangle$ is a m-qubit i.e. $|x\rangle = |x_1 x_2 \ldots x_k\rangle$.

We have to prove f(n) is true for n=k+1 assuming that f(n) is true for n=k.

Now,
$$H^{\otimes k+1}(|x\rangle) = H^{\otimes k+1} |x_1 x_2 \ldots\ldots x_{k+1}\rangle$$
$$= H^{\otimes k} |x_1 x_2 \ldots x_k\rangle \otimes H|x_{k+1}\rangle$$
$$= \frac{1}{2^{k/2}} \sum_{y \in \{0,1\}^k} (-1)^{(x_1 \ldots x_k)\cdot y} |y\rangle \otimes \frac{1}{2^{1/2}} \left(|0\rangle + (-1)^{x_{k+1}}|1\rangle\right)$$

(This we can write because f(n) is true for n=1 & n=k).

$$= \frac{1}{2^{k+1/2}} \sum_{y \in \{0,1\}^{k+1}} (-1)^{(x_1 \ldots x_{k+1})\cdot y} |y\rangle$$

So, we have $H^{\otimes k+1}(|x\rangle) = \frac{1}{2^{(k+1)/2}} \sum_{y \in \{0,1\}^{k+1}} (-1)^{(x_1 \ldots x_{k+1})\cdot y} |y\rangle$

So, f(n) is true for n=k+1.

Hence by principle of mathematical induction we can say that

$$H^{\otimes n} |x\rangle = \frac{1}{2^{n/2}} \sum_{y \in \{0,1\}^n} (-1)^{x\cdot y} |y\rangle$$