Exercises from QC Notes book by Ronald de Wolf
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1 (a) Inner product between two real vectors $(0,1,0,1)$ and $(0,1,1,1)$ is

$$\langle (0,1,0,1), (0,1,1,1) \rangle = (0\,1\,0\,1)(0,1,1,1)^T$$

$$= (0\,1\,0\,1)\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 2.$$

(b) We know that

$$|0101\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{16\times 1} \quad \text{and} \quad |0111\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{16\times 1}.$$

as $0101 = 5$

& $0111 = 7$

then then inner product is defined as —

$$\langle 0101 | 0111 \rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0.$$

② We know that $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$H|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Now, we know that $|0\rangle \otimes |1\rangle$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

& $H \otimes H = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$= \frac{1}{2} \begin{pmatrix} 1 \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & 1 \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ 1 \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & -1 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

So, $H|0\rangle \otimes H|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$= \frac{1}{2} \begin{pmatrix} 1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ 1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix}$$

$\frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$

& $(H \otimes H)(|0\rangle \otimes |1\rangle)$

$= \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$

$\therefore H|0\rangle \otimes H|1\rangle = (H \otimes H)(|0\rangle \otimes |1\rangle)$.

③ We know ~~that~~ $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Now, $|0\rangle$ & $|1\rangle$ form a basis. So,

$HXH(|0\rangle) = HX\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$

$= \frac{1}{\sqrt{2}}H(X|0\rangle + X|1\rangle)$

$= \frac{1}{\sqrt{2}}H(|1\rangle + |0\rangle)$

$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) + \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right)$

$= \frac{1}{\sqrt{2}}\left(\frac{2}{\sqrt{2}}|0\rangle\right) = |0\rangle$.

& $HXH(|1\rangle) = HX\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)$

~~$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) - \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)$~~

$= \frac{1}{\sqrt{2}}H(|1\rangle - |0\rangle)$

$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) - \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right)$

$= \frac{1}{2}(-2|1\rangle) = -|1\rangle$.

So, $HXH(|0\rangle) = Z(|0\rangle)$ & $HXH(|1\rangle) = Z(|1\rangle)$

As HXH & Z are equal on basis elements so, they are same linear transformation

$$\therefore \quad HXH = Z$$

(4) Here $H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ & $CNOT = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix} = \begin{pmatrix}I_2 & O\\ O & X\end{pmatrix}$

As $X = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

& $$H \otimes H = \frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}H & H\\ H & -H\end{pmatrix}$$

$\Rightarrow (H\otimes H)\, CNOT\, (H \otimes H)$

$$= \frac{1}{2}\begin{pmatrix}H & H\\ H & -H\end{pmatrix}\begin{pmatrix}I_2 & O\\ O & X\end{pmatrix}\begin{pmatrix}H & H\\ H & -H\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}H & H\\ H & -H\end{pmatrix}\begin{pmatrix}H & H\\ XH & -XH\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}H^2 + HXH & H^2 - HXH\\ H^2 - HXH & H^2 + HXH\end{pmatrix}$$

(we know that $H^2 = I_2$ & $HXH = Z$)

$$= \frac{1}{2}\begin{pmatrix}I_2 + Z & I_2 - Z\\ I_2 - Z & I_2 + Z\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2 & 0 & 0 & 0\\ 0 & 0 & 0 & 2\\ 0 & 0 & 2 & 0\\ 0 & 2 & 0 & 0\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\end{pmatrix} = A \text{ (say)}$$

$\therefore$ ~~$(H\otimes H)\, CNOT\, (H\otimes H)\,|00\rangle =$~~

$A|00\rangle = A\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} = |00\rangle$

$A|01\rangle = A\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} = |11\rangle$

$A|10\rangle = A\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} = |10\rangle$

$A|11\rangle = A\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} = |01\rangle$

The A is a 2-qubit gate where the 2nd bit controls ~~whose~~ the operation whether the first bit is negated.

i.e, when 2nd bit is 0 there is no change but ~~o~~ when 2nd bit is 1 then first bit flipped.

(5) $(\langle 0| \otimes I)\left(\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle\right)$

$$= \left(\begin{pmatrix}1\\0\end{pmatrix}^{*}\right)^{T} \otimes \begin{pmatrix}1&0\\0&1\end{pmatrix} \left(\quad " \quad \right)$$

$$= \left((1\ 0) \otimes \begin{pmatrix}1&0\\0&1\end{pmatrix}\right)\left(\quad " \quad \right)$$

$$= \begin{pmatrix}1&0&0&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}\alpha_{00}\\\alpha_{01}\\\alpha_{10}\\\alpha_{11}\end{pmatrix} = \begin{pmatrix}\alpha_{00}\\\alpha_{01}\end{pmatrix}$$

$$= \underline{\alpha_{00}|0\rangle + \alpha_{01}|1\rangle}.$$