Exercise :-

1. (a) What is the inner product between the real vectors $(0,1,0,1)$ and $(0,1,1,1)$?

(b) What is the inner product between the state $|0101\rangle$ and $|0111\rangle$?

Ans:- (a) We know that inner product of two real vector $x=(x_1, x_2, \dots, x_n)$ and $y=(y_1, y_2, \dots, y_n)$ is given by

$$\langle x, y\rangle = \langle (x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)\rangle$$

$$= x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

So, inner product of $(0,1,0,1)$ and $(0,1,1,1)$ is

$$\langle (0,1,0,1), (0,1,1,1)\rangle = 0.0 + 1.1 + 0.1 + 1.1$$

$$= 2$$

(b) and inner product between two state $|x\rangle$ and $|y\rangle$ is $\langle x|y\rangle$ where $\langle x|$ is the bra notation which is conjugate transpose of $|x\rangle$

Now $|0101\rangle = |0\rangle \otimes |1\rangle \otimes |0\rangle \otimes |1\rangle$

$$= \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\0\\0\\0\\0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}$$

So $\langle 0101| = (0000010000000000)$

and $|0111\rangle = |0\rangle \otimes |1\rangle \otimes |1\rangle \otimes |1\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}$

$$= \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}$$

Therefore inner product of $|0101\rangle$ and $|0111\rangle$ is

$$\langle 0101|0111\rangle = (0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0)\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} = 0$$

② Compute the result of applying a Hadamard transformation to both qubits of $|0\rangle\otimes|1\rangle$ in two ways (the first way using tensor product of vectors, the second using tensor product of matrices), and show that the two results are equal.

$$H|0\rangle\otimes H|1\rangle = (H\otimes H)(|0\rangle\otimes|1\rangle)$$

Ans:- We know $H = \frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$

Then $H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

and $H|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

So $H|0\rangle\otimes H|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$

and $H\otimes H = \frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{pmatrix}$

and $|0\rangle\otimes|1\rangle = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$

So $(H\otimes H)(|0\rangle\otimes|1\rangle) = \frac{1}{2}\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$

Therefore $H|0\rangle\otimes H|1\rangle = (H\otimes H)(|0\rangle\otimes|1\rangle)$

③ Show that a bit flip operation, preceded and followed by Hadamard transforms, equals a phase flip operation:

$$HXH = Z$$

⇒ To show the equality we show that HXH and Z sends the basis vector to the same qubit.

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Now $HXH|0\rangle = HX\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right)$

$= H\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right)$

$= |0\rangle$

and $HXH|1\rangle = HX\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)$

$= H\left(\frac{1}{\sqrt{2}}(-|0\rangle + |1\rangle)\right)$

$= -|1\rangle$

Again we know that $Z|0\rangle = |0\rangle$

$Z|1\rangle = -|1\rangle$

Therefore $HXZ = Z$

<u>Another process</u>

to show equality we will show that matrix representation of HXH and Z are equal.

Now $HXH = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$

(4) Show that the surrounding a CNOT gate with Hadamard gates switches the role of the control-bit and target-bit of the CNOT: $(H\otimes H)\,CNOT\,(H\otimes H)$ is the 2-qubit gate where the second bit controls whether the first bit is negated (i.e. flipped)

Ans:. ~~Now let $A = (H\otimes H)\,CNOT\,(H\otimes H)$~~

, We know $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Then $H\otimes H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$

Then $A = (H\otimes H)\,CNOT\,(H\otimes H)$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

So $A|00\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |00\rangle$

$A|01\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |11\rangle$

$A|10\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = |10\rangle$

$A|11\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |01\rangle$

Thus $A|00\rangle = |00\rangle$
$A|01\rangle = |11\rangle$
$A|10\rangle = |10\rangle$
$A|11\rangle = |01\rangle$

Thus we show that second bit controls whether the first bit is negated.

⑤ Simplify the following

$$(\langle 0| \otimes I)(\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle)$$

Ans we have $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ so $\langle 0| = \begin{pmatrix} 1 & 0 \end{pmatrix}$

and $\langle 0| \otimes I = \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

So $(\langle 0| \otimes I)(\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle)$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{pmatrix} = \begin{pmatrix} \alpha_{00} \\ \alpha_{01} \end{pmatrix}$$

$$= \alpha_{00}|0\rangle + \alpha_{01}|1\rangle$$