① Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. Then find an unitrary matrix M such that $M|\psi\rangle = |0\rangle$ and $M|\psi^{\perp}\rangle = |1\rangle$

Where $|\psi^{\perp}\rangle = \beta^*|0\rangle - \alpha^*|1\rangle$

Ans:- Let $M = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ be an unitray matrix such that $M|\psi\rangle = |0\rangle$ and $M|\psi^{\perp}\rangle = |1\rangle$, $MM^{\dagger} = M^{\dagger}M = \mathbb{I}$

Now $M|\psi\rangle = |0\rangle$

$\Rightarrow |\psi\rangle = M^{\dagger}|0\rangle$

$\Rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} x^* & z^* \\ y^* & w^* \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x^* \\ y^* \end{pmatrix}$ where x^* is the complex conjugate of x

$\Rightarrow x^* = \alpha$, $y^* = \beta$

$\Rightarrow x = \alpha^*$ $\quad y = \beta^*$ $\quad$ as $(x^*)^* = x$

and $M|\psi^{\perp}\rangle = |1\rangle$

$\Rightarrow |\psi^{\perp}\rangle = M^{\dagger}|1\rangle$

$\Rightarrow \begin{pmatrix} \beta^* \\ -\alpha^* \end{pmatrix} = \begin{pmatrix} x^* & z^* \\ y^* & w^* \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} z^* \\ w^* \end{pmatrix}$

$\Rightarrow z^* = \beta^*$ $\quad w^* = -\alpha^*$

$\Rightarrow z = \beta$ $\quad w = -\alpha$

$\therefore M = \begin{pmatrix} \alpha^* & \beta^* \\ \beta & -\alpha \end{pmatrix}$