Problem 140
Modified Fibonacci golden nuggets

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1 Definition

Consider the infinite polynomial series $A_G(x) = xG_1 + x^2G_2 + x^3G_3 + ...$, where G_k is the kth term of the second order recurrence relation $G_k = G_{k-1} + G_{k-2}, G_1 = 1$ and $G_2 = 4$.

For this problem we shall be concerned with values of x for which $A_G(x)$ is a positive integer.

The corresponding values of x for the first five natural numbers are shown below.

x	$A_G(x)$
$(\sqrt{5}-1)/4$	1
$2/5$	2
$(\sqrt{22}-2)/6$	3
$(\sqrt{137}-5)/14$	4
$1/2$	5

We shall call $A_G(x)$ a golden nugget if x is rational, because they become increasingly rarer; for example, the 20th golden nugget is 211345365.

Find the sum of the first thirty golden nuggets.

2 Solution(s) and proof

It is evident that this is a modified version of problem 137. We will first derive a closed formula for $A_G(x)$, as the one we got from Wikipedia por $A_F(x)$.

$$s(x) = \sum_{k=1}^{\infty} x^k G_k \quad (1)$$

$$s(x) = xG_1 + x^2G_2 + \sum_{k=3}^{\infty} x^k G_k \quad (2)$$

$$s(x) = x + 4x^2 + \sum_{k=3}^{\infty} x^k (G_{k-1} + G_{k-2}) \quad (3)$$

$$s(x) = x + 4x^2 + x\sum_{k=1}^{\infty} x^k G_k - x^2G_1 + x^2\sum_{k=1}^{\infty})x^k G_k \quad (4)$$

$$s(x) = x + 4x^2 + xs(x) - x^2 + x^2s(x) \quad (5)$$

$$s(x) = \frac{x + 3x^2}{1 - x - x^2} \quad (6)$$

Our only task is to find values of x such that $s(x)$ is integer. If we make $s(x) = n$ in Eq. 6, and solve for x:

$$
\begin{aligned}
n &= \frac{x}{1 - x - x^2} && (7)\\
n(1 - x - x^2) &= x && (8)\\
nx^2 + (n+1)x - n &= 0 && (9)\\
x &= \frac{-(n+1) + \sqrt{5n^2 + 2n + 1}}{2n} && (10)
\end{aligned}
$$

In Eq. 10 se remove the negative solution, as $x > 0$.

2.1 f0

Eq. 10 already gives as a method for solving p137. We can try succesive integer values of n, and check whether the result of $5n^2 + 2n + 1$ is a perfect square. If (and only if) it is, x will be rational. This method is too slow to go beyond the 11th golden nugget.

2.2 f1

We can take Eq. 10 and equate $5n^2 + 2n + 1$ to some squared integer k^2, then solve for n:

$$
\begin{aligned}
5n^2 + 2n + 1 &= k^2 && (11)\\
n &= \frac{-1 + \sqrt{5k^2 - 4}}{5} && (12)
\end{aligned}
$$

According to Eq. 12, we can take succesive k values, square them, then check whether the equation returns an integer value for n. This method turns out to be slower than **f0**.

2.3 f2

I have realized that the values of k in Eq. 12 that return an integer n are members of the Fibonacci sequence, more precisely of the form F_{5+4m} for $m = 0, 1, 2, 3...$ It is then trivial to iterate over every 4 Fibonacci numbers from the 5th on, use them as k in Eq. 12 to get n, and return the 15th such value.

2.4 f3

Actually, n only needs to be calculated for the 15th k. *Actually*, we do know that the k value will be $k = F_{5+4\cdot 14} = F_{61}$, and thus the n value we are looking for would be directly:

$$
n = \frac{-1 + \sqrt{5F_{61}^2 - 4}}{5} \qquad (13)
$$

In general, mth golden nugget will be:

$$
n_m = \frac{-1 + \sqrt{5F_{4m+1}^2 - 4}}{5} \qquad (14)
$$