Problem 142
Perfect square collection

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1 Definition

Find the smallest $x+y+z$ with integers $x>y>z>0$ such that $x+y$, $x-y$, $x+z$, $x-z$, $y+z$, $y-z$ are all perfect squares.

2 Solution(s) and proof

2.1 f0

The most obvious method would be to set a maximum x to consider, then check all x from 3 to x_{max}, y from 2 to $x-1$, and z from 1 to $y-1$, to see if they fullfill the equations. This method is too slow.

2.2 f1

We can define $y = z + A$ and $x = z + B$, where $B > A > 0$ without loss of generality. From the equations given, we see that both A and B must be perfect squares themselves:

$$\begin{aligned}
x + y &= 2z + A + B = k_1^2 && (1)\\
x - y &= B - A = k_2^2 && (2)\\
x + z &= 2z + B = k_3^2 && (3)\\
x - z &= B = k_4^2 && (4)\\
y + z &= 2z + A = k_5^2 && (5)\\
y - z &= A = k_6^2 && (6)
\end{aligned}$$

Rearranging the equations we see that:

$$\begin{aligned}
k_2^2 &= B - A && (7)\\
k_3^2 &= k_1^2 - A && (8)\\
k_5^2 &= k_1^2 - B && (9)\\
z &= \frac{k_1^2 - A - B}{2} && (10)
\end{aligned}$$

In other words, we must look for integers $k_1 > k_4 > k_6$, such that defining $A = k_6^2$ and $B = k_4^2$, the above four equations hold (i.e., $B - A$ is a perfect square, $k_1^2 - A - B$ is even, etc).

This method solves the problem in about 6 seconds.