Problem 138
Special isosceles triangles

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1 Definition

Consider the isosceles triangle with base length, $b = 16$, and legs, $L = 17$.

By using the Pythagorean theorem it can be seen that the height of the triangle, $h = \sqrt{17^2 - 8^2} = 15$, which is one less than the base length.

With $b = 272$ and $L = 305$, we get $h = 273$, which is one more than the base length, and this is the second smallest isosceles triangle with the property that $h = b \pm 1$.

Find $\sum L$ for the twelve smallest isosceles triangles for which $h = b \pm 1$ and b, L are positive integers.

2 Solution(s) and proof

Defining $B = b/2$ without losing generality, since b must be even, we have:

$$\begin{aligned} L^2 &= B^2 + h^2 && (1) \\ L^2 &= B^2 + (B \pm 1)^2 && (2) \\ 5B^2 \pm 4B + 1 - L^2 &= 0 && (3) \end{aligned}$$

Solving for B, we get:

$$B = \frac{\pm 2 \pm \sqrt{5L^2 - 1}}{5} \qquad (4)$$

We can safely remove the negative square root solution from Eq. 4, since $B > 0$.

2.1 f0

Eq. 3 already gives as a method for solving p138. We can try succesive integer values of B, and check whether the result of $5B^2 \pm 4B + 1$ is a perfect square. If it is, calculate L and add up. This method is too slow to go beyond the 7th triangle.

2.2 f1

According to Eq. 4, we can take succesive L values, square them, then check whether the equation returns an integer value for B. This method turns out to be a bit slower than **f0**.

2.3 f2

I have realized that the values of L in Eq. 4 that return an integer B are exactly half the value of members of the Fibonacci sequence, more precisely of the form $F_{9+6m}/2$ for $m = 0, 1, 2, 3...$ It is then trivial to iterate over every 6th Fibonacci number from the 9th on, adding their halves up as needed.