Problem 141
Investigating progressive numbers, n, which are also square

Iñaki Silanes

November 2, 2014

1 Definition

A positive integer, n, is divided by d and the quotient and remainder are q and r respectively. In addition d, q, and r are consecutive positive integer terms in a geometric sequence, but not necessarily in that order.

For example, 58 divided by 6 has quotient 9 and remainder 4. It can also be seen that 4, 6, 9 are consecutive terms in a geometric sequence (common ratio 3/2).

We will call such numbers, n, progressive.

Some progressive numbers, such as 9 and $10404 = 102^2$, happen to also be perfect squares. The sum of all progressive perfect squares below one hundred thousand is 124657.

Find the sum of all progressive perfect squares below one trillion (10^12).

2 Solution(s) and proof

2.1 f0

We can take the most obvious route, and check all perfect squares below 10^{12}, to see whether they are progressive or not. For each of those 10^6 numbers m, we will check all potential divisors d, from $d = 2$ to $d = \sqrt{m}$, obtaining q and r for each d. Then we will check whether they are members of a geometrical progression.

The progression check is easy, since by construction our values will be in $q > d > r$ order. This means we will have to check $q/d = d/r$, or reordering for better computation suitability: $qr = d^2$.

This method is usable, but too slow to be acceptable (around 5 hours on my PC).

2.2 f1

We can define R as the coefficient of the geometric progression (not necessarily integer). By definition we have $r < d$, obviously, but theoretically q could be either larger than d, between d and r, or smaller than r. If the smallest of the three is r, then either $d = Rr$ and $q = Rd = R^2r$ or $q = Rr$ and $d = Rq = R^2r$. In both cases:

$$\begin{aligned} n &= dq + r && (1) \\ n &= R^3r^2 + r && (2) \end{aligned}$$

However, if the smallest one is q, then $r = Rq$ (r must be the second largest, as $r < d$ by definition), and $d = R^2$, so:

$$\begin{aligned} n &= dq + r && (3) \\ n &= R^2q^2 + Rq && (4) \end{aligned}$$

We will argue that all the numbers we seek are of the form in Eq. 2, and not Eq. 4. Indeed, in the latter Rq must be integer for n to be integer. This means $n = c^2 + c$, with n and c integer. If n is a square number, then $n = k^2$ for some integer k. So:

$$\begin{aligned}
k^2 &= c^2 + c && (5)\\
k &> c && (6)\\
k &\geq c + 1 && (7)\\
k^2 &\geq c^2 + 2c + 1 > c^2 + c = k^2 && (8)
\end{aligned}$$

From Eq. 8, it is clearly impossible that $n = c^2 + c$ be square.

So, without loss of generality, we can define $R = \alpha/\beta$, and use Eq. 2 to build up a method:

1. check all integer r from 1 upwards
2. From r, obtain all possible d and q pairs, from $\alpha r/\beta$ and $\alpha^2 r/\beta^2$
3. If d and q are integer, use Eq.2 to get n
4. If n is integer (and smaller than limit), add it up
5. If n is larger than limit, skip to next r
6. If r^2 is larger than limit, break from main loop

This method is a bit slow, but acceptable (~ 5 minutes).