Method and proof for Problem 133

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1 Problem definition

A number consisting entirely of ones is called a repunit. We shall define $R(k)$ to be a repunit of length k; for example, $R(6) = 111111$.

Let us consider repunits of the form $R(10^n)$.

Although $R(10)$, $R(100)$, or $R(1000)$ are not divisible by 17, $R(10000)$ is divisible by 17. Yet there is no value of n for which $R(10^n)$ will divide by 19. In fact, it is remarkable that 11, 17, 41, and 73 are the only four primes below one-hundred that can be a factor of $R(10^n)$.

Find the sum of all the primes below one-hundred thousand that will never be a factor of $R(10^n)$.

2 Method and proof

Let's take prime p, and define $A(p) = k$ as the smallest k, such that $R(k) \propto p$, where $R(n)$ is the base-10 repunit of n digits. Let's assume $R(q) \propto p$, where q is the next smallest n for which $R(n)$ is divisible by p. Then:

$$\begin{aligned}
R(q) &= R(k) + 10^k R(q-k) \propto p && (1)\\
R(q) \bmod p &= (R(k) \bmod p + (10^k R(q-k)) \bmod p) \bmod p = 0 && (2)\\
R(q) \bmod p &= (0 + (10^k R(q-k)) \bmod p) \bmod p = 0 && (3)\\
R(q) \bmod p &= (10^k R(q-k)) \bmod p = 0 && (4)
\end{aligned}$$

From Eq. 4 we conclude that either $10^k \bmod p = 0$ or $R(q-k) \bmod p = 0$. We will discard the former, as follows:

$$\begin{aligned}
R(k) = \frac{10^k - 1}{9} &\propto p && (5)\\
10^k - 1 &= n \cdot p && (6)\\
10^k &= n \cdot p + 1 && (7)\\
10^k \bmod p &= 1 && (8)
\end{aligned}$$

So, from Eqs. 4 and 8 we conclude that $R(q-k) \bmod p = 0$. Recall that q is the second-smallest n for $R(n) \bmod p = 0$, with k being the smallest. Clearly $q - k$ is a valid n for $R(n)\text{mod} = 0$, and is smaller than q. The only value smaller than q with that property is k, so $q - k = k$, or $q = 2k$.

Repeating the same argument for the third and following smallest n for $R(n) \bmod p = 0$, we conclude that if $R(q) \bmod p = 0$, then $q \propto k = A(p)$.

$$R(q) \bmod p = 0 \implies q \bmod A(p) = 0 \tag{9}$$

Going back to the definition of the problem, and applying Eq. 9, if we assume that $R(10^n) \bmod p = 0$ for some n, then it follows that $10^n \bmod A(p) = 0$. In other words, $A(p)$ must be of the form $2^i \cdot 5^j$. Any p for which $A(p)$ is not of that form will never be a divisor of $R(10^n)$, for **any** n.

2.1 f0

The method for solving the problem, then, will consist on looping over all primes p below 10^5, finding $A(p)$, and finding out whether $A(p) = 2^i \cdot 5^j$ for integer i and j.

2.2 f1

Same as **f0**, but calculate $A(p)$ using $10^k \bmod p = 1$, instead of $R(k) \bmod p = 0$, which is slightly faster.

2.3 f2

Instead of finding $A(p)$ for each p, and *then* checking whether $A(p) \bmod 2^i \cdot 5^j = 0$, only check all $k = 2^i \cdot 5^j < p$, for $R(k) \bmod p = 0$ (that is $10^k \bmod p = 1$). If no such k exists, then we don't know what $A(p)$ is for p, but we do know that its prime factors won't be just 2 and 5.