Problem 140
Modified Fibonacci golden nuggets

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1 Definition

Consider the infinite polynomial series $A_G(x) = xG_1 + x^2G_2 + x^3G_3 + ...$, where G_k is the kth term of the second order recurrence relation $G_k = G_{k-1} + G_{k-2}, G_1 = 1$ and $G_2 = 4$.

For this problem we shall be concerned with values of x for which $A_G(x)$ is a positive integer.

The corresponding values of x for the first five natural numbers are shown below.

x	$A_G(x)$
$(\sqrt{5}-1)/4$	1
$2/5$	2
$(\sqrt{22}-2)/6$	3
$(\sqrt{137}-5)/14$	4
$1/2$	5

We shall call $A_G(x)$ a golden nugget if x is rational, because they become increasingly rarer; for example, the 20th golden nugget is 211345365.

Find the sum of the first thirty golden nuggets.

2 Solution(s) and proof

It is evident that this is a modified version of problem 137. We will first derive a closed formula for $A_G(x)$, as the one we got from Wikipedia por $A_F(x)$.

$$s(x) = \sum_{k=1}^{\infty} x^k G_k \tag{1}$$

$$s(x) = xG_1 + x^2G_2 + \sum_{k=3}^{\infty} x^k G_k \tag{2}$$

$$s(x) = x + 4x^2 + \sum_{k=3}^{\infty} x^k (G_{k-1} + G_{k-2}) \tag{3}$$

$$s(x) = x + 4x^2 + x\sum_{k=1}^{\infty} x^k G_k - x^2G_1 + x^2 \sum_{k=1}^{\infty})x^k G_k \tag{4}$$

$$s(x) = x + 4x^2 + xs(x) - x^2 + x^2s(x) \tag{5}$$

$$s(x) = \frac{x + 3x^2}{1 - x - x^2} \tag{6}$$

Our only task is to find values of x such that $s(x)$ is integer. If we make $s(x) = n$ in Eq. 6, and solve for x:

$$
\begin{aligned}
n &= \frac{x+3x^2}{1-x-x^2} && (7)\\
n(1-x-x^2) &= x+3x^2 && (8)\\
(3+n)x^2+(n+1)x-n &= 0 && (9)\\
x &= \frac{-(n+1)+\sqrt{5n^2+14n+1}}{2(3+n)} && (10)
\end{aligned}
$$

In Eq. 10 se remove the negative solution, as $x > 0$.

2.1 f0

Eq. 10 already gives as a method for solving p140. We can try succesive integer values of n, and check whether the result of $5n^2+14n+1$ is a perfect square. If (and only if) it is, x will be rational. This method is too slow to go beyond the 21st golden nugget.

2.2 f1

We can take Eq. 10 and equate $5n^2+14n+1$ to some squared integer k^2, then solve for n:

$$
\begin{aligned}
5n^2+14n+1 &= k^2 && (11)\\
n &= \frac{-7+\sqrt{5k^2+44}}{5} && (12)
\end{aligned}
$$

According to Eq. 12, we can take succesive k values, substitute them, then check whether the equation returns an integer value for n. This method turns out to be slower than **f0**.

2.3 f2

I wondered whether the valid k values for an integer n in Eq. 12 belong to any series, as in Problem 137 they belonged to the Fibonacci series.

The first 10 values of k are: 7, 14, 50, 97, 343, 665, 2351, 4558, 16114, 31241. We can readily see that the even terms $(14, 97, 665, 4558, 31241)$, are members of the G_i, series, namely G_i for $i = 5, 9, 13, 17, 21$. In other words:

$$
k_i = G_{2i+1} \qquad (13)
$$

for even i.

However, the odd values are more puzzling: $(7, 50, 343, 2351, 16114, ...)$. Playing around, and searching a bit in internet, I found out they are members of another modified Fibonacci series, namely $H_i = H_{i-1} + H_{i-2}$, with $H_1 = 7$ and $H_2 = 12$. More precisely, they'd be elements H_i for $i = 1, 5, 9, 13, 17, ...$ In other words:

$$
k_i = H_{2i-1} \qquad (14)
$$

for odd i.

We are requested to find the value of the first 30 golden nuggets, which amounts to finding G_{2i+1} for $i = 2, 4, ..., 30$, then H_{2i-1} for $i = 1, 3, ...29$. These 30 values of k will produce 30 values of n, via Eq. 12. Add up those 30 values of n, and be done with it.

No need to say this method is amazingly fast.