Problem 139
Pythagorean tiles

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1 Definition

Let (a, b, c) represent the three sides of a right angle triangle with integral length sides. It is possible to place four such triangles together to form a square with length c.

For example, $(3, 4, 5)$ triangles can be placed together to form a 5 by 5 square with a 1 by 1 hole in the middle and it can be seen that the 5 by 5 square can be tiled with twenty-five 1 by 1 squares.

However, if (5, 12, 13) triangles were used then the hole would measure 7 by 7 and these could not be used to tile the 13 by 13 square.

Given that the perimeter of the right triangle is less than one-hundred million, how many Pythagorean triangles would allow such a tiling to take place?

2 Solution(s) and proof

We will call d the value of the side of the hole in the middle, and (i, j, k) the values of the sides of the triangle, in increasing order. From that is evident that $j - i = d$, and $i^2 + j^2 = k^2$. Also, as k must be divisible by d, we can define $k = dk_0$. From that:

$$\begin{aligned} i^2 + j^2 &= k^2 && (1) \\ i^2 + (j+d)^2 &= d^2 k_0^2 && (2) \\ 2i^2 + 2id + d^2(1 - k_0^2) &= 0 && (3) \\ & && (4) \end{aligned}$$

Solving for i, we get:

$$\begin{aligned} i &= \frac{d}{2}(-1 + \sqrt{2k_0^2 - 1}) = i_0 d && (5) \\ i_0 &= \frac{-1 + \sqrt{2k_0^2 - 1}}{2} && (6) \end{aligned}$$

Since $j = i + d$, we can put it as a function of i_0 and d: $j = (i_0 + 1)d$. From that, we can solve for the perimeter:

$$\begin{aligned} P &= i + j + k && (7) \\ P &= i_0 d + (i_0 + 1)d + k_0 d && (8) \\ P &= d(2i_0 + k_0 + 1) && (9) \\ P &= P_0 d && (10) \end{aligned}$$

2.1 f0

Eq. **??** already gives as a method for solving p138. We can try succesive integer values of B, and check whether the result of $5B^2 \pm 4B + 1$ is a perfect square. If it is, calculate L and add up. This method is too slow to go beyond the 7th triangle.

2.2 f1

According to Eq. **??**, we can take succesive L values, square them, then check whether the equation returns an integer value for B. This method turns out to be a bit slower than **f0**.

2.3 f2

I have realized that the values of L in Eq. **??** that return an integer B are exactly half the value of members of the Fibonacci sequence, more precisely of the form $F_{9+6m}/2$ for $m = 0, 1, 2, 3...$ It is then trivial to iterate over every 6th Fibonacci number from the 9th on, adding their halves up as needed.