Problem 135

Iñaki Silanes

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1 Definition

Given the positive integers, x, y, and z, are consecutive terms of an arithmetic progression, the least value of the positive integer, n, for which the equation, $x^2 - y^2 - z^2 = n$, has exactly two solutions is $n = 27$:

$$34^2 - 27^2 - 20^2 = 12^2 - 9^2 - 6^2 = 27 \tag{1}$$

It turns out that $n = 1155$ is the least value which has exactly ten solutions.

How many values of n less than one million have exactly ten distinct solutions?

2 Solution(s) and proof

If x, y, z are members of an arithmetic progression, then we can define, without losing generality, $x = y + d$ and $z = y - d$, for some integer $d < y$ (lest $z \leq 0$). So:

$$\begin{aligned} x^2 - y^2 - z^2 &= n & (2)\\ (y+d)^2 - y^2 - (y-d)^2 &= n & (3)\\ 4dy - y^2 &= n & (4) \end{aligned}$$

Solving for y, we get:

$$y = 2d \pm \sqrt{4d^2 - n} \tag{5}$$

From Eq. 5 we can obtain the maximum d possible. We know that the content of the square root must be a perfect square, so, defining $D = 2d$:

$$\begin{aligned} 4d^2 - n &= k^2 & (6)\\ D^2 - n &= k^2 & (7) \end{aligned}$$

For Eq. 7 to hold, $D^2 - n$ must be equal to or less than $(D-1)^2$, since $D - 1$ is the largest value k could take. Developing further:

$$\begin{aligned} D^2 - n &\leq (D-1)^2 & (8)\\ D^2 - n &\leq D^2 + 1 - 2D & (9)\\ n &\geq 2D - 1 = 4d - 1 & (10)\\ d &\leq \frac{n+1}{4} & (11) \end{aligned}$$

Now, for a given d, what would the limits for y be? Since $z = y - d > 0$, then $y \geq d + 1$. Also, from Eq. 4:

$$
\begin{aligned}
4dy - y^2 &= n > 0 && (12)\\
y(4d - y) &> 0 && (13)\\
4d - y &> 0 && (14)\\
y &< 4d && (15)
\end{aligned}
$$

2.1 Solution f3

The procedure to solve this problem would then be the following:

1. Take all d from 1 to $(n_{max} + 1)/4$
2. Take all y from $d + 1$ to $4d$ (see above)
3. Calculate n from Eq. 4
4. If n is within 0 and n_{max}, add 1 to the amount of combinations that yield n
5. Once all (d, y) taken, check all n to see if its amount of combinations is 10, and print how many of them are

2.2 Solution f5

We must take into account that in the $y = d + 1$ to $y = 4d$ region there can be a sizeable region where $n > n_{max}$ for sure. See that Eq. 4 is a parabola, when plotting n vs. y for a given d. Its maximum will be at $y = 2d$, with a value of $n = 4d^2$. If this value is less than n_{max} all y values in interval will yield a valid n. However, if $n = 4d > n_{max}$, there will be a $y = 2d \pm \delta y$ region around the maximum where we do not need to check y, because we know it will yield too large an n.

The width of that region, δy can be obtained from Eq. 5, substituting n with n_{max}, and turns out to be $\delta y = \sqrt{4d^2 - n_{max}}$.

Taking advantage of this fact, we can loop over y only in the $(d+1, 2d-\delta y)$ and $(2d+\delta y, 4d)$ regions. For larger d values this saves quite a bit of time.