Problem 134

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1 Definition

Given the positive integers, x, y, and z, are consecutive terms of an arithmetic progression, the least value of the positive integer, n, for which the equation, $x^2 - y^2 - z^2 = n$, has exactly two solutions is $n = 27$:

$$34^2 - 27^2 - 20^2 = 12^2 - 9^2 - 6^2 = 27 \tag{1}$$

It turns out that $n = 1155$ is the least value which has exactly ten solutions.

How many values of n less than one million have exactly ten distinct solutions?

2 Solution(s) and proof

If x, y, z are members of an arithmetic progression, then we can define, without losing generality, $x = y+d$ and $z = y - d$, for some integer $d < y$ (lest $z \leq 0$). So:

$$\begin{aligned} x^2 - y^2 - z^2 &= n & (2)\\ (y+d)^2 - y*^2 - (y-d)^2 &= n & (3)\\ 4dy - y^2 &= n & (4) \end{aligned}$$

Solving for d, we get:

$$d = \frac{n + y^2}{4y} \tag{5}$$

From Eq. 5 we can obtain the maximum d possible