Problem 137
Fibonacci golden nuggets

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1 Definition

Consider the infinite polynomial series $A_F(x) = xF_1 + x^2F_2 + x^3F_3 + ...$, where F_k is the kth term in the Fibonacci sequence: 1, 1, 2, 3, 5, 8, ... ; that is, $F_k = F_{k-1} + F_{k-2}, F_1 = 1$ and $F_2 = 1$.

For this problem we shall be interested in values of x for which $A_F(x)$ is a positive integer.

Surprisingly:

$$\begin{aligned} A_F(1/2) &= (1/2) \cdot 1 + (1/2^{)}2 \cdot 1 + (1/2)^3 \cdot 2 + (1/2)^4 \cdot 3 + (1/2)^5 \cdot 5 + ... && (1) \\ &= 1/2 + 1/4 + 2/8 + 3/16 + 5/32 + ... && (2) \\ &= 2 && (3) \end{aligned}$$

The corresponding values of x for the first five natural numbers are shown below.

x	$A_F(x)$
$\sqrt{2} - 1$	1
$1/2$	2
$(\sqrt{13} - 2)/3$	3
$(\sqrt{89} - 5)/8$	4
$(\sqrt{34} - 3)/5$	5

We shall call $A_F(x)$ a golden nugget if x is rational, because they become increasingly rarer; for example, the 10th golden nugget is 74049690.

Find the 15th golden nugget.

2 Solution(s) and proof

If we check `http://en.wikipedia.org/wiki/Fibonacci_number`, it seems that the formula for $A_F(x)$ is actually $s(x)$, as given in the "Power series" section of the article, which is convergent and has a closed form for $|x| < 1/\varphi$:

$$s(x) = \sum_{k=0}^{\infty} F_k x^k \quad (4)$$

$$s(x) = \frac{x}{1 - x - x^2} \quad (5)$$

Our only task is to find values of x such that $s(x)$ is integer. If we make $s(x) = n$ in Eq. 5, and solve for x:

$$
\begin{aligned}
n &= \frac{x}{1-x-x^2} && (6)\\
n(1-x-x^2) &= x && (7)\\
nx^2+(n+1)x-n &= 0 && (8)\\
x &= \frac{-(n+1)+\sqrt{5n^2+2n+1}}{2n} && (9)
\end{aligned}
$$

In Eq. 9 se remove the negative solution, as $x > 0$.

2.1 f0

Eq. 9 already gives as a method for solving p137. We can try succesive integer values of n, and check whether the result of $5n^2+2n+1$ is a perfect square. If (and only if) it is, x will be rational. This method is too slow to go beyond the 11th golden nugget.

2.2 f1

We can take Eq. 9 and equate $5n^2+2n+1$ to some squared integer k^2, then solve for n:

$$
\begin{aligned}
5n^2+2n+1 &= k^2 && (10)\\
n &= \frac{-1+\sqrt{5k^2-4}}{5} && (11)
\end{aligned}
$$

According to Eq. 11, we can take succesive k values, square them, then check whether the equation returns an integer value for n. This method turns out to be slower than **f0**.

2.3 f2

I have realized that the values of k in Eq. 11 that return an integer n are members of the Fibonacci sequence, more precisely of the form F_{5+4m} for $m = 0, 1, 2, 3...$ It is then trivial to iterate over every 4 Fibonacci numbers from the 5th on, use them as k in Eq. 11 to get n, and return the 15th such value.

2.4 f3

Actually, n only needs to be calculated for the 15th k. *Actually*, we do know that the k value will be $k = F_{5+4\cdot 14} = F_{61}$, and thus the n value we are looking for would be directly:

$$
n = \frac{-1+\sqrt{5F_{61}^2-4}}{5} \qquad (12)
$$

In general, mth golden nugget will be:

$$
n_m = \frac{-1+\sqrt{5F_{4m+1}^2-4}}{5} \qquad (13)
$$