Problem 139
Pythagorean tiles

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1 Definition

Let (a, b, c) represent the three sides of a right angle triangle with integral length sides. It is possible to place four such triangles together to form a square with length c.

For example, $(3, 4, 5)$ triangles can be placed together to form a 5 by 5 square with a 1 by 1 hole in the middle and it can be seen that the 5 by 5 square can be tiled with twenty-five 1 by 1 squares.

However, if (5, 12, 13) triangles were used then the hole would measure 7 by 7 and these could not be used to tile the 13 by 13 square.

Given that the perimeter of the right triangle is less than one-hundred million, how many Pythagorean triangles would allow such a tiling to take place?

2 Solution(s) and proof

We will call d the value of the side of the hole in the middle, and (i, j, k) the values of the sides of the triangle, in increasing order. From that is evident that $j - i = d$, and $i^2 + j^2 = k^2$. Also, as k must be divisible by d, we can define $k = dk_0$. From that:

$$
\begin{aligned}
i^2 + j^2 &= k^2 & (1) \\
i^2 + (j + d)^2 &= d^2 k_0^2 & (2) \\
2i^2 + 2id + d^2(1 - k_0^2) &= 0 & (3)
\end{aligned}
$$

Solving for i, we get:

$$
\begin{aligned}
i &= \frac{d}{2}(-1 + \sqrt{2k_0^2 - 1}) = i_0 d & (4) \\
i_0 &= \frac{-1 + \sqrt{2k_0^2 - 1}}{2} & (5)
\end{aligned}
$$

Since $j = i + d$, we can put it as a function of i_0 and d: $j = (i_0 + 1)d$. From that, we can solve for the perimeter:

$$
\begin{aligned}
P &= i + j + k & (6) \\
P &= i_0 d + (i_0 + 1)d + k_0 d & (7) \\
P &= d(2i_0 + k_0 + 1) & (8) \\
P &= P_0 d & (9)
\end{aligned}
$$

As we know that $P < P_{max}$ must hold, then $d < P_{max}/P_0$.

2.1 f0

Take i values from 1 on. Take j values from $i+1$ on. Check if that (i, j) pair corresponds to an integer k. If so, check whether $k \mod d = 0$, where $d = j - i$.

This method is too slow, and scales badly.

2.2 f1

Check from $k_0 = 1$ upwards whether Eq. 5 yields an integer i_0. If it does, $(i_0 d, (i_0 + 1)d, k_0 d)$ triplets will be valid solutions for every d, up to $d_{max} < P_{max}/P_0$, as explained above. This means d_{max} different solutions for each valid k_0.

So, the solution will then consist on finding all valid k_0 values, and corresponding d_{max} values, up to the given maximum perimeter (P_{max}). The requested result is the sum of all d_{max} values.