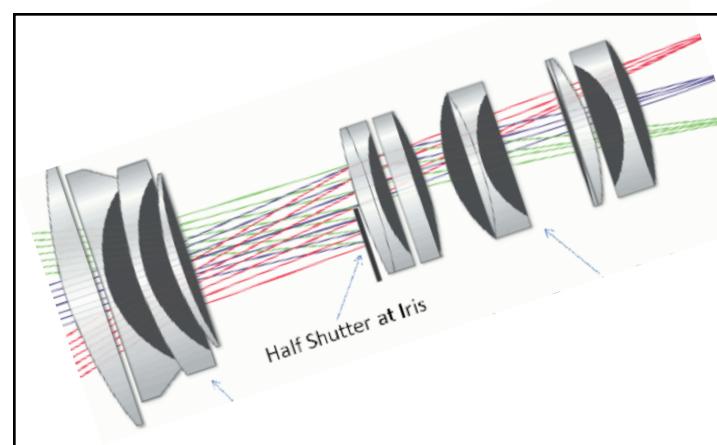
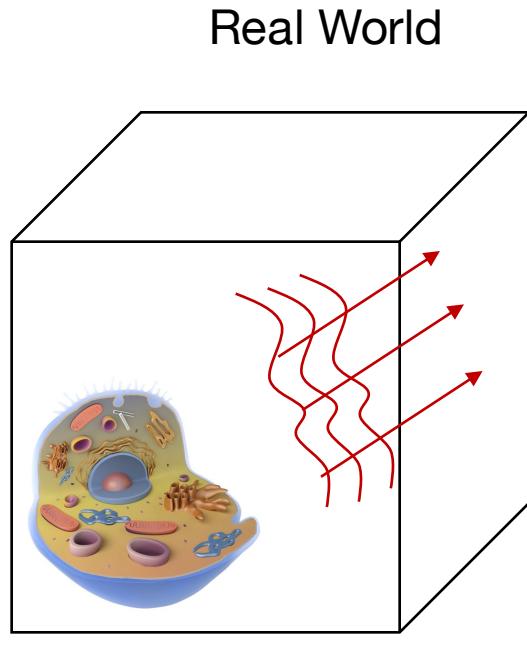


Machine Learning in Imaging

BME 590L
Roarke Horstmeyer

Lecture 3b: Discrete Mathematics, Part II

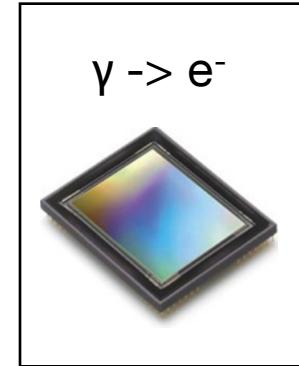
ML+Imaging pipeline introduction



Black box transformations

- Convolution
- Fourier Transform

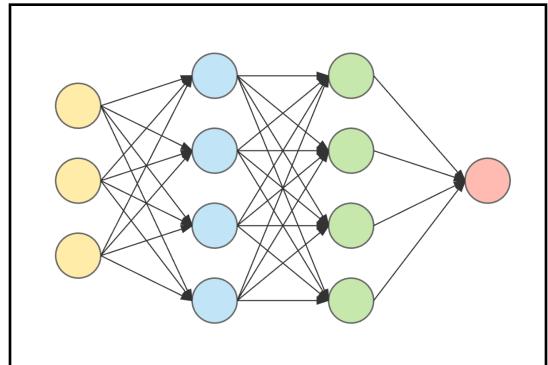
Digitization



Sampling Theorem

Discrete math & Linear algebra

Machine Learning



Optimization (this class)

Linear classification

Logistic classifier

Neural networks

Convolutional NN's

(last week)



(last week)



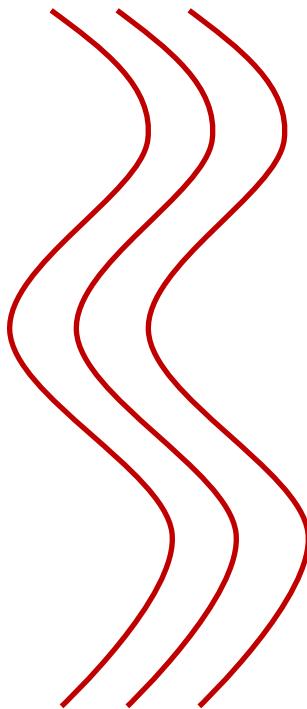
(this class)



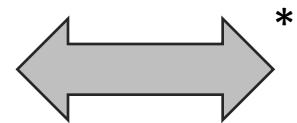
(next few weeks)

What does the Sampling Theorem mean for us?

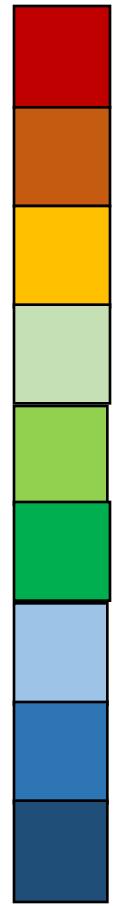
Continuous fields



Discretize vectors
(and matrices)



(*) Under certain
conditions



17
20
22
21
23
25
24
26
29

Discrete convolution

$$V(x_o) = \int_{-\infty}^{\infty} U(x_i)h(x_o - x_i)dx_i$$



$$v[x_0] = \sum_{x_i=-M}^{M} u[x_i]h[x_o - x_i]$$

Example 2: Find the convolution of the two sequences $x[n]$ and $h[n]$ given by,

$$x[k] = [3 \ 1 \ 2] \quad h[k] = [3 \ 2 \ 1]$$

k:	-2	-1	0	1	2	3	4	5
----	----	----	---	---	---	---	---	---

x[k]:			3	1	2			
-------	--	--	---	---	---	--	--	--

h[-k]:	1	2	3					
--------	---	---	---	--	--	--	--	--

h[1-k]:	1	2	3					
---------	---	---	---	--	--	--	--	--

h[2-k]:	1	2	3					
---------	---	---	---	--	--	--	--	--

h[3-k]:		1	2	3				
---------	--	---	---	---	--	--	--	--

h[4-k]:			1	2	3			
---------	--	--	---	---	---	--	--	--

h[5-k]:				1	2	3		
---------	--	--	--	---	---	---	--	--

y:	9	6+3	3+2+6	1+4+0				
----	---	-----	-------	-------	--	--	--	--

y:	[9	9	11	5	2	0]
----	-----	---	----	---	---	-----

Discrete convolution

$$V(x_o) = \int_{-\infty}^{\infty} U(x_i)h(x_o - x_i)dx_i$$



$$v[x_0] = \sum_{x_i=-M}^{M} u[x_i]h[x_o - x_i]$$

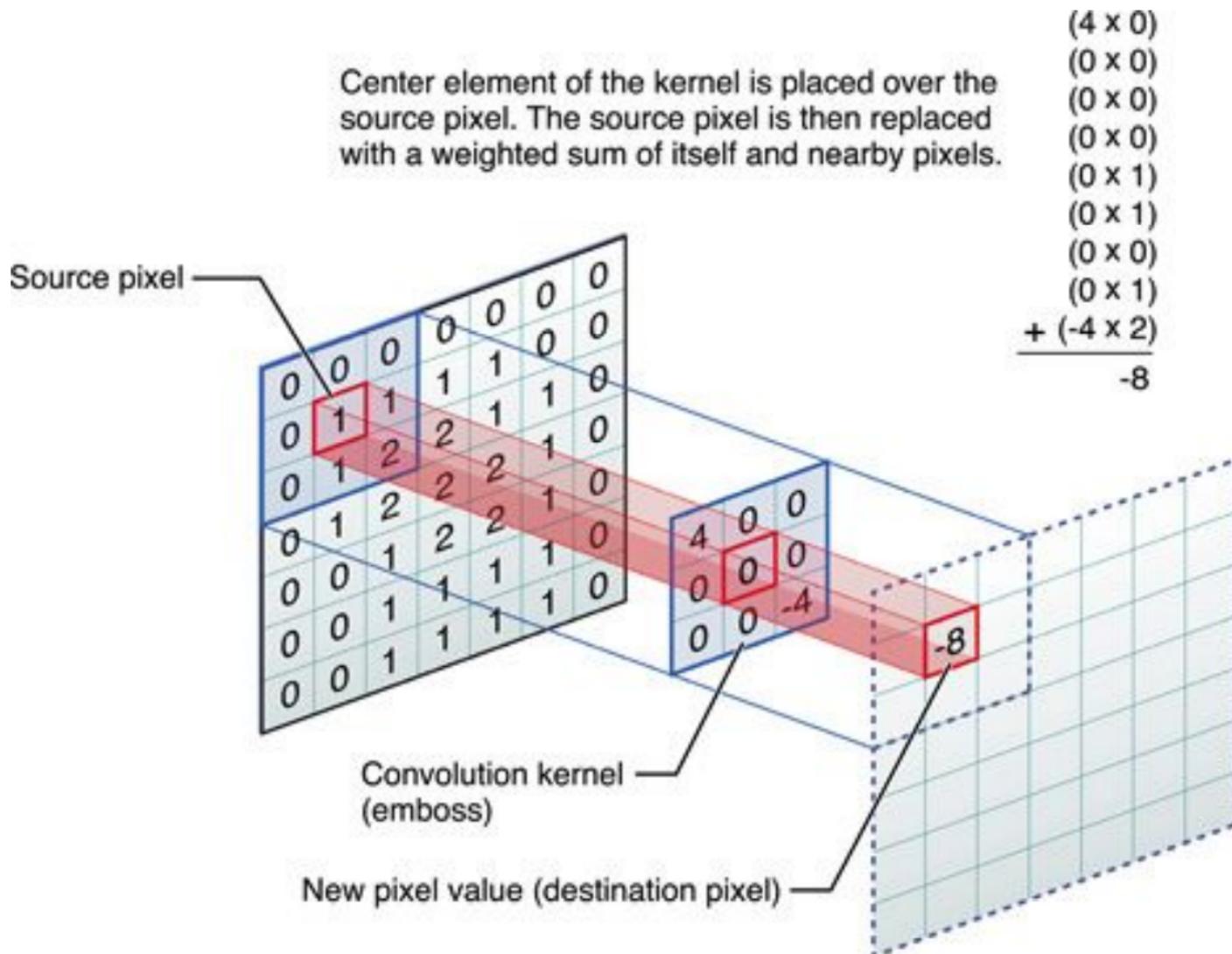
Discrete 2D convolution

$$V(x_o, y_o) = \iint_{-\infty}^{\infty} U(x_i, y_i)h(x_o - x_i, y_o - y_i)dx_idy_i$$



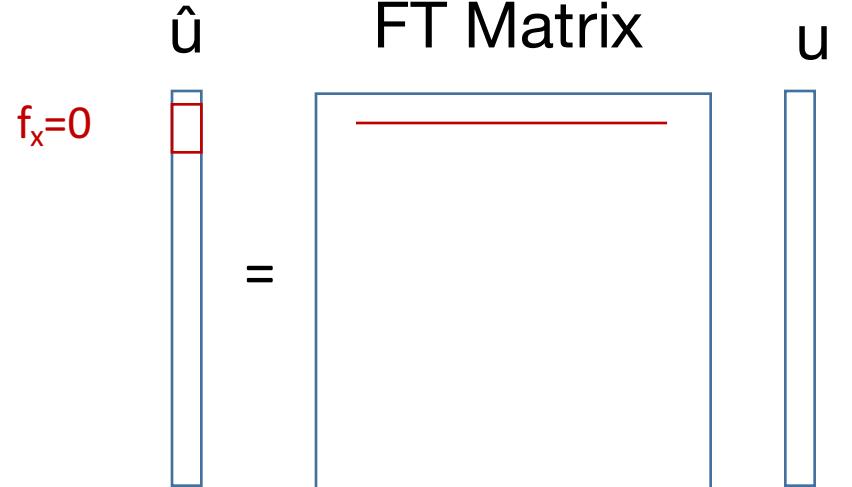
$$v[x_0, y_o] = \sum_{y_i=-L}^{L} \sum_{x_i=-M}^{M} u[x_i, y_i]h[x_o - x_i, y_o - y_i]$$

Discrete 2D convolution



Discrete Fourier Transforms

$$\hat{U}(f_x) = \int_{-\infty}^{\infty} U(x) \exp(-2\pi i(f_x x)) dx$$



$$\hat{u}[f_x] = \sum_{x=0}^{M-1} u[x] \exp(-2\pi i f_x x / M)$$

Inner product of u with different complex expon.

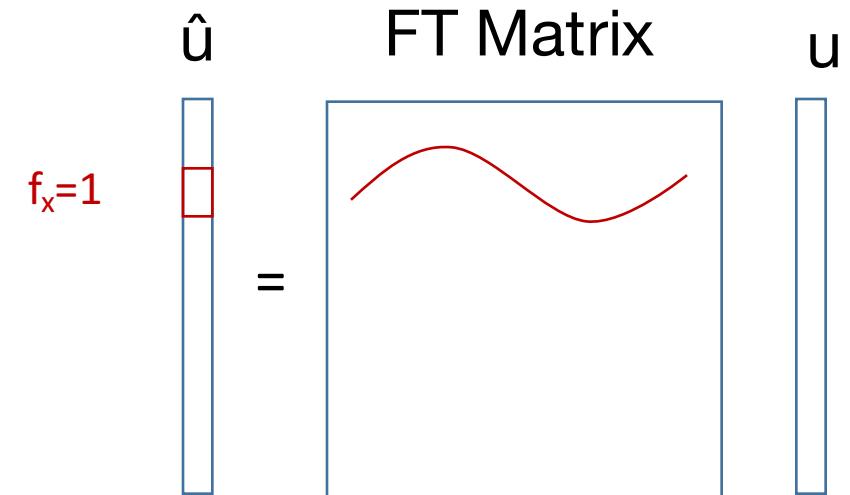
- `np.fft(u)`, `np.fftshift(np.fft(np.ifftshift(u)))`
- `fft` = fast Fourier transform, much more comp. efficient than matrix multiplication!

Discrete Fourier Transforms

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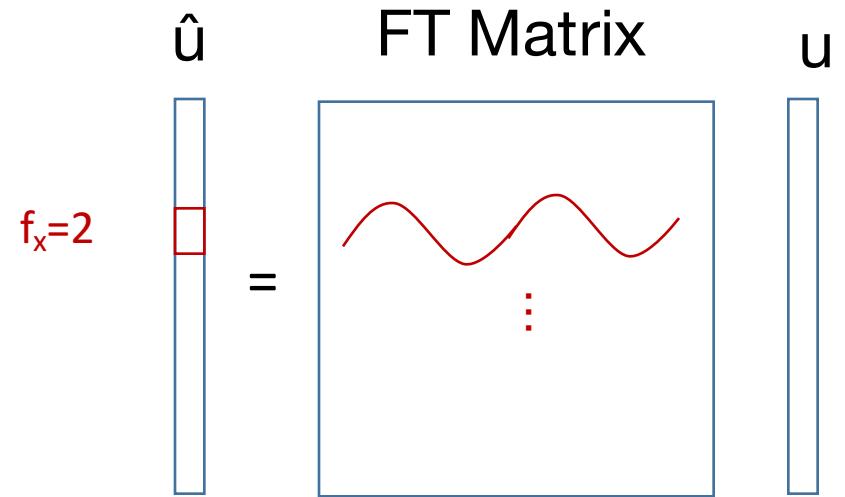
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Inner product of u with different complex expon.

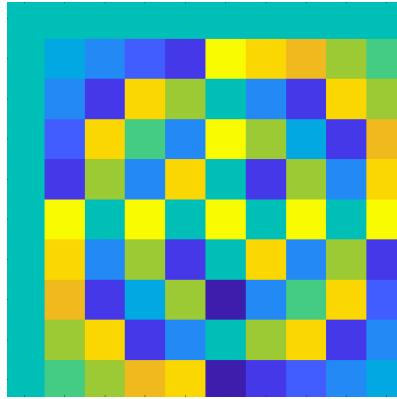


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Discrete Fourier Transforms

$$\hat{U}(f_x) = \int_{-\infty}^{\infty} U(x) \exp(-2\pi i(f_x x)) dx$$

$$\hat{u}[f_x] = \sum_{x=0}^{M-1} u[x] \exp(-2\pi i f_x x / M)$$

$$\hat{u} = \text{FT Matrix, } \theta u$$


`np.fft(np.eye(10))`

Treats 1st entry of \hat{u} as $f_x=0$

Discrete Fourier Transforms

$$\hat{U}(f_x) = \int_{-\infty}^{\infty} U(x) \exp(-2\pi i(f_x x)) dx$$

$$\hat{u}[f_x] = \sum_{x=0}^{M-1} u[x] \exp(-2\pi i f_x x / M)$$

$$\begin{array}{c|c|c} \hat{u} & \text{FT Matrix, } \theta & u \\ \hline & = & \\ \hline \end{array}$$

```
np.fftshift(np.fft(np.ifftshift(np.eye(10))))
```

Treats middle entry of \hat{u} as $f_x=0$

Discrete convolution theorem

Convolution theorem. If $\mathcal{F}\{g(x, y)\} = G(f_X, f_Y)$ and $\mathcal{F}\{h(x, y)\} = H(f_X, f_Y)$, then

$$\mathcal{F} \left\{ \iint_{-\infty}^{\infty} g(\xi, \eta) h(x - \xi, y - \eta) d\xi d\eta \right\} = G(f_X, f_Y) H(f_X, f_Y). \quad (2-15)$$

Discrete convolution theorem

Convolution theorem. If $\mathcal{F}\{g(x, y)\} = G(f_x, f_y)$ and $\mathcal{F}\{h(x, y)\} = H(f_x, f_y)$, then

Continuous

$$\mathcal{F} \left\{ \iint_{-\infty}^{\infty} g(\xi, \eta) h(x - \xi, y - \eta) d\xi d\eta \right\} = G(f_x, f_y) H(f_x, f_y). \quad (2-15)$$

If $\mathcal{F}[g[x, y]] = G[f_x, f_y]$ and $\mathcal{F}[h[x, y]] = H[f_x, f_y]$, and if we know that

Discrete

$$g[x, y] * h[x, y] = \sum_{l=-L}^{L} \sum_{m=-M}^{M} g[m, l] h[x - m, y - l],$$

then from the Convolution Theorem we have,

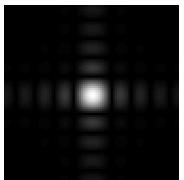
$$\mathcal{F}[g[x, y] * h[x, y]] = G[f_x, f_y] H[f_x, f_y]$$

Discrete convolution theorem example

Input image

$$U_1(x,y)$$


Convolution filter h

$$*$$

$$=$$


Output image

$$U_2(x,y)$$

Discrete convolution theorem example

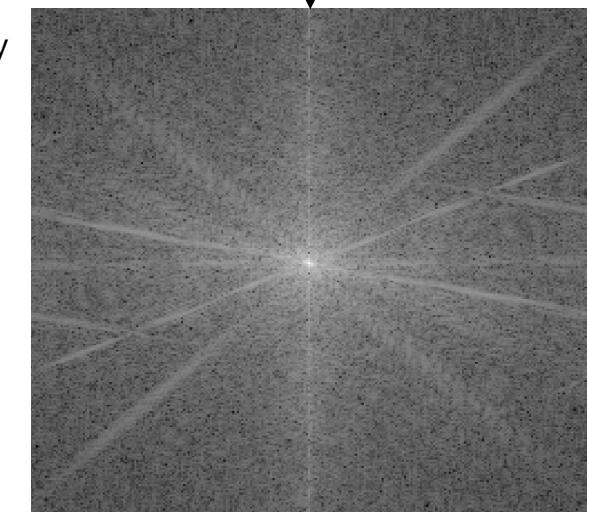
Input image

$$U_1(x,y)$$



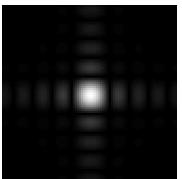
$$\mathcal{F}[U_1]$$

Input spectrum
 $\hat{U}_1(f_x, f_y)$



Convolution filter h

*



$$\mathcal{F}[h]$$

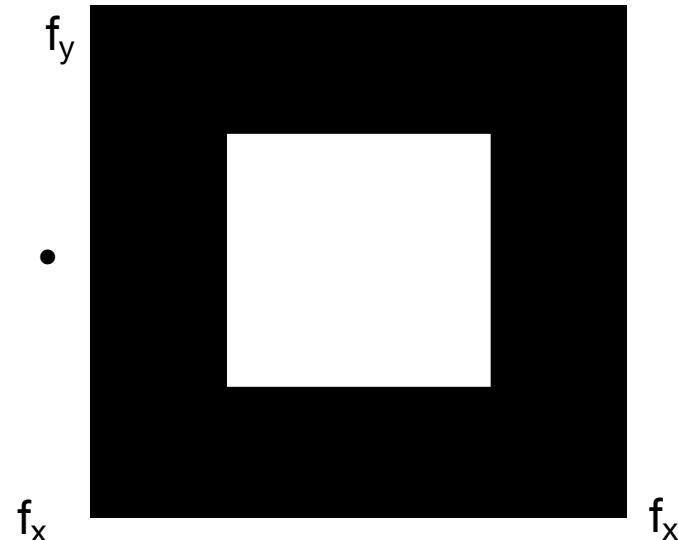
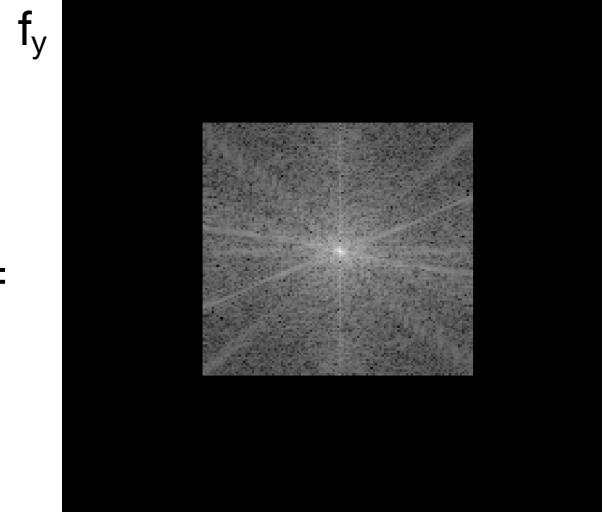
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Output image

$$U_2(x,y)$$

$$\mathcal{F}^{-1}[H\hat{U}_1]$$

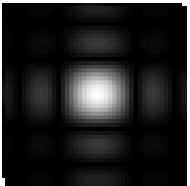


Discrete convolution theorem example

Input image

$$U_1(x,y)$$


Convolution filter h

$$*$$

$$=$$


Output image

$$U_2(x,y)$$

Discrete convolution theorem example

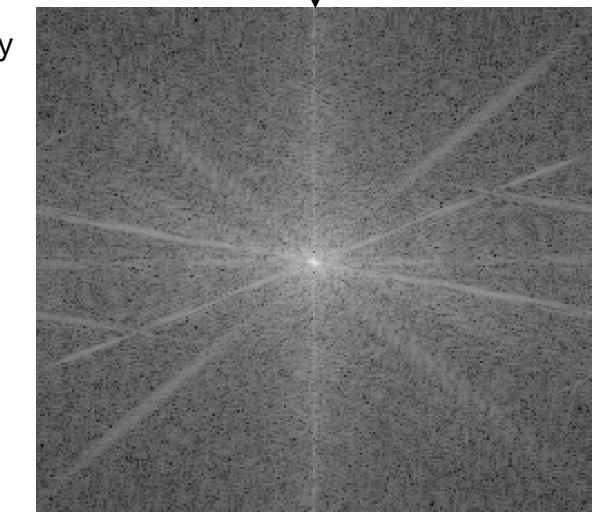
Input image

$$U_1(x,y)$$



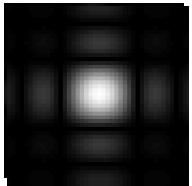
$$\mathcal{F}[U_1]$$

Input spectrum
 $\hat{U}_1(f_x, f_y)$



Convolution filter h

*



$$\downarrow \quad \mathcal{F}[h]$$

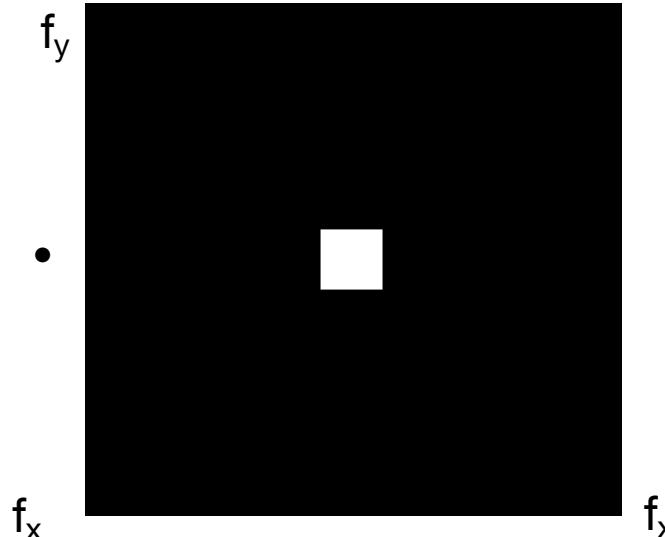
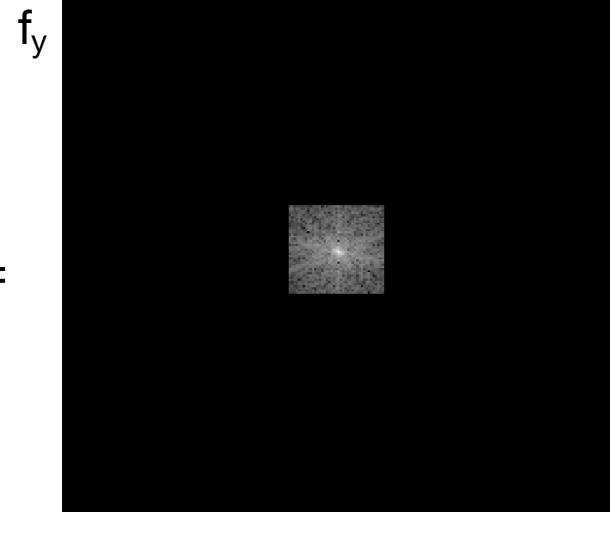
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Output image

$$U_2(x,y)$$

$$\uparrow \quad \mathcal{F}^{-1}[H\hat{U}_1]$$



Convolutions as a big matrix multiplication

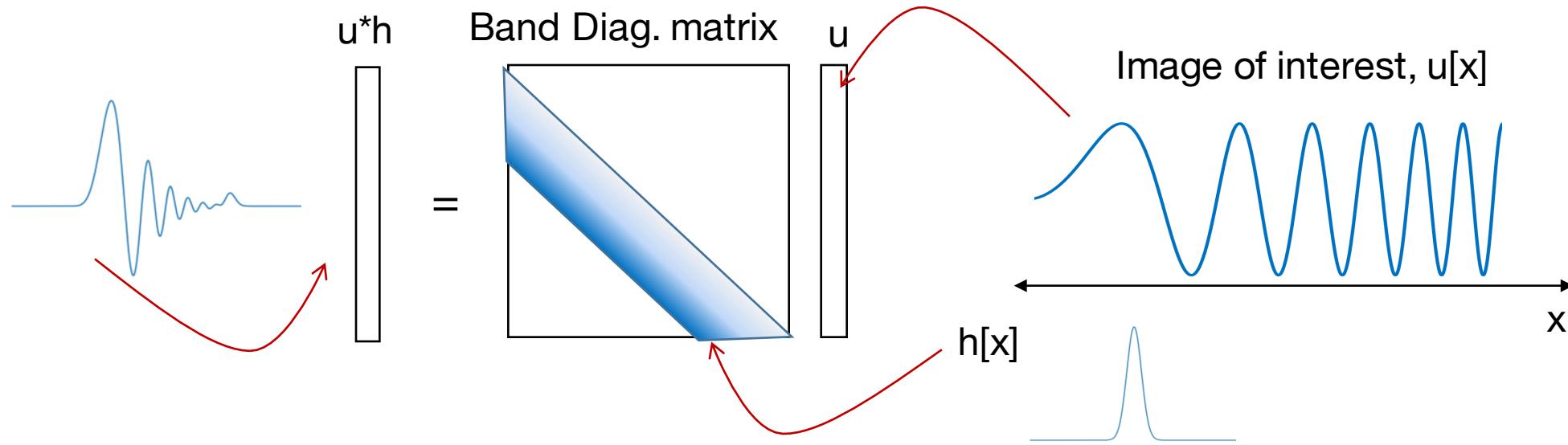
$$(u * h)[x] = \sum_{m=0}^{N+M-2} u[m]h[x-m] \longrightarrow y = u * h = \begin{bmatrix} h_1 & 0 & \dots & 0 & 0 \\ h_2 & h_1 & \dots & \vdots & \vdots \\ h_3 & h_2 & \dots & 0 & 0 \\ \vdots & h_3 & \dots & h_1 & 0 \\ h_{m-1} & \vdots & \dots & h_2 & h_1 \\ h_m & h_{m-1} & \vdots & \vdots & h_2 \\ 0 & h_m & \dots & h_{m-2} & \vdots \\ 0 & 0 & \dots & h_{m-1} & h_{m-2} \\ \vdots & \vdots & \vdots & h_m & h_{m-1} \\ 0 & 0 & 0 & \dots & h_m \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix}$$

Convolutions as a big matrix multiplication

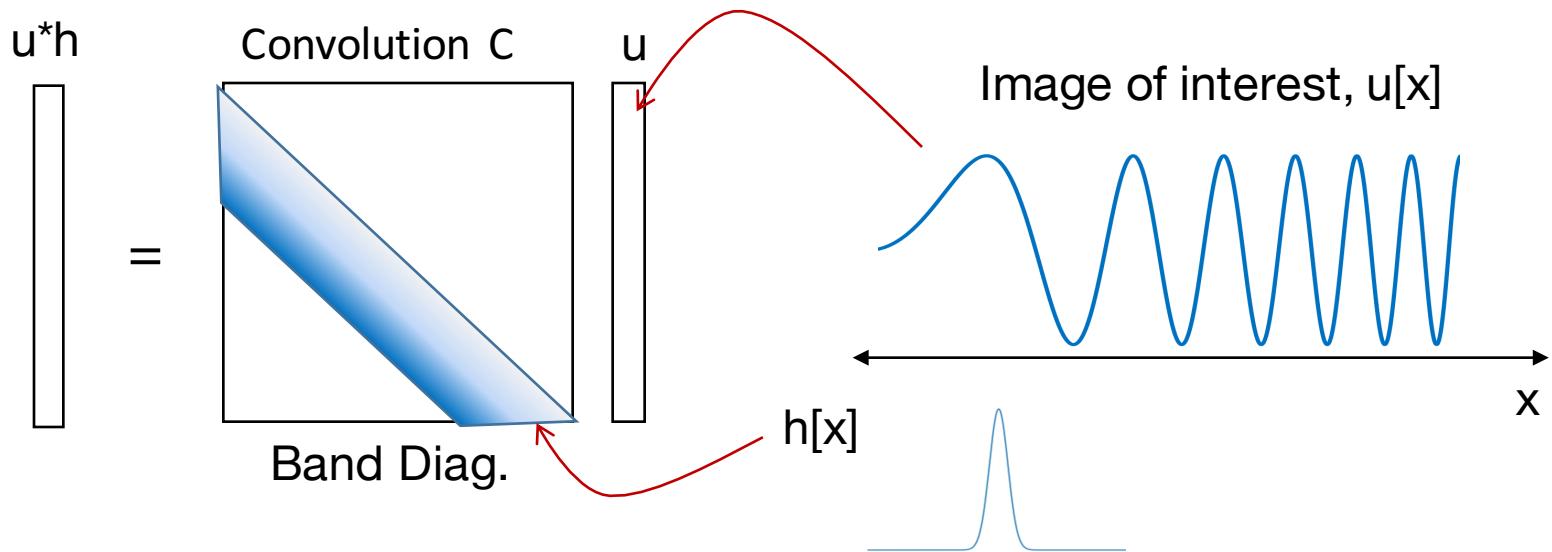
$$\begin{matrix} u^*h & \text{Band Diag. matrix} & u \\ \left| \right. & = & \left| \right. \end{matrix}$$

The diagram illustrates the multiplication of a vector u^*h by a band diagonal matrix. On the left, there is a vertical bar with a double vertical line inside, representing the vector u^*h . In the center, there is an equals sign (=). To the right of the equals sign is a large square box labeled "Band Diag. matrix". This box contains a blue shaded triangular band that is diagonally oriented, representing the non-zero elements of a band diagonal matrix. To the right of the matrix box is another vertical bar with a double vertical line inside, representing the vector u .

Convolutions as a big matrix multiplication

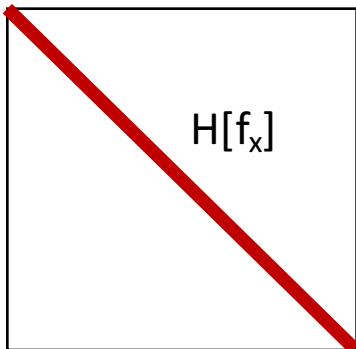


Convolutions as a big matrix multiplication

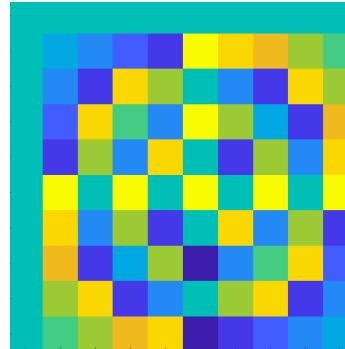


Discrete Fourier transform diagonalizes convolution matrix with $H[f_x]$ along diag.

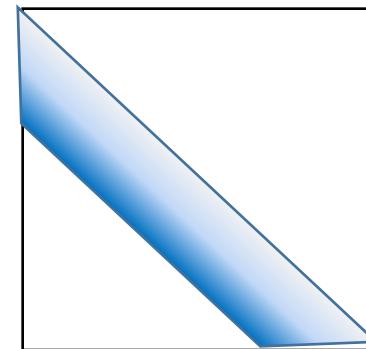
Diagonal matrix



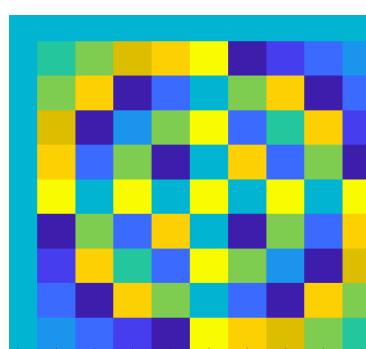
F



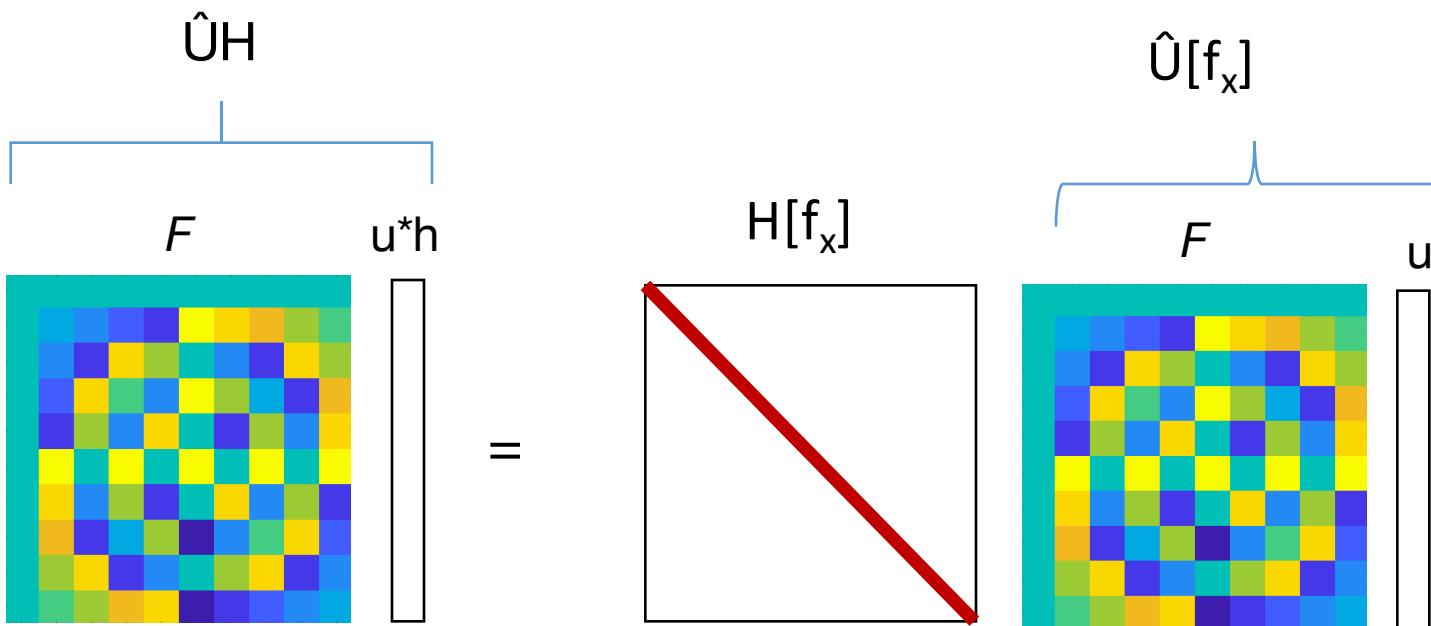
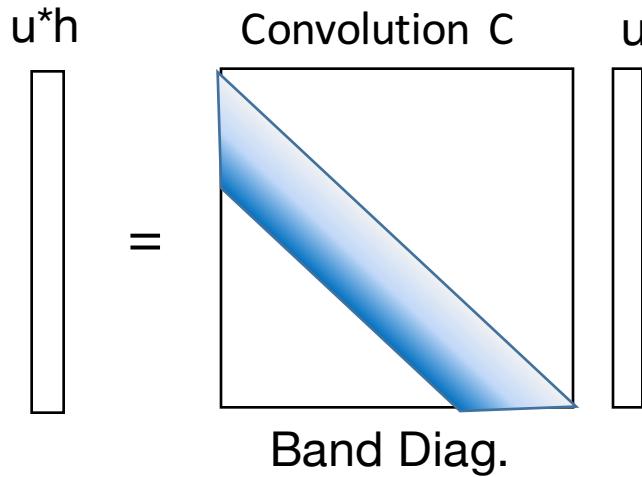
Circulant C



F^{-1}

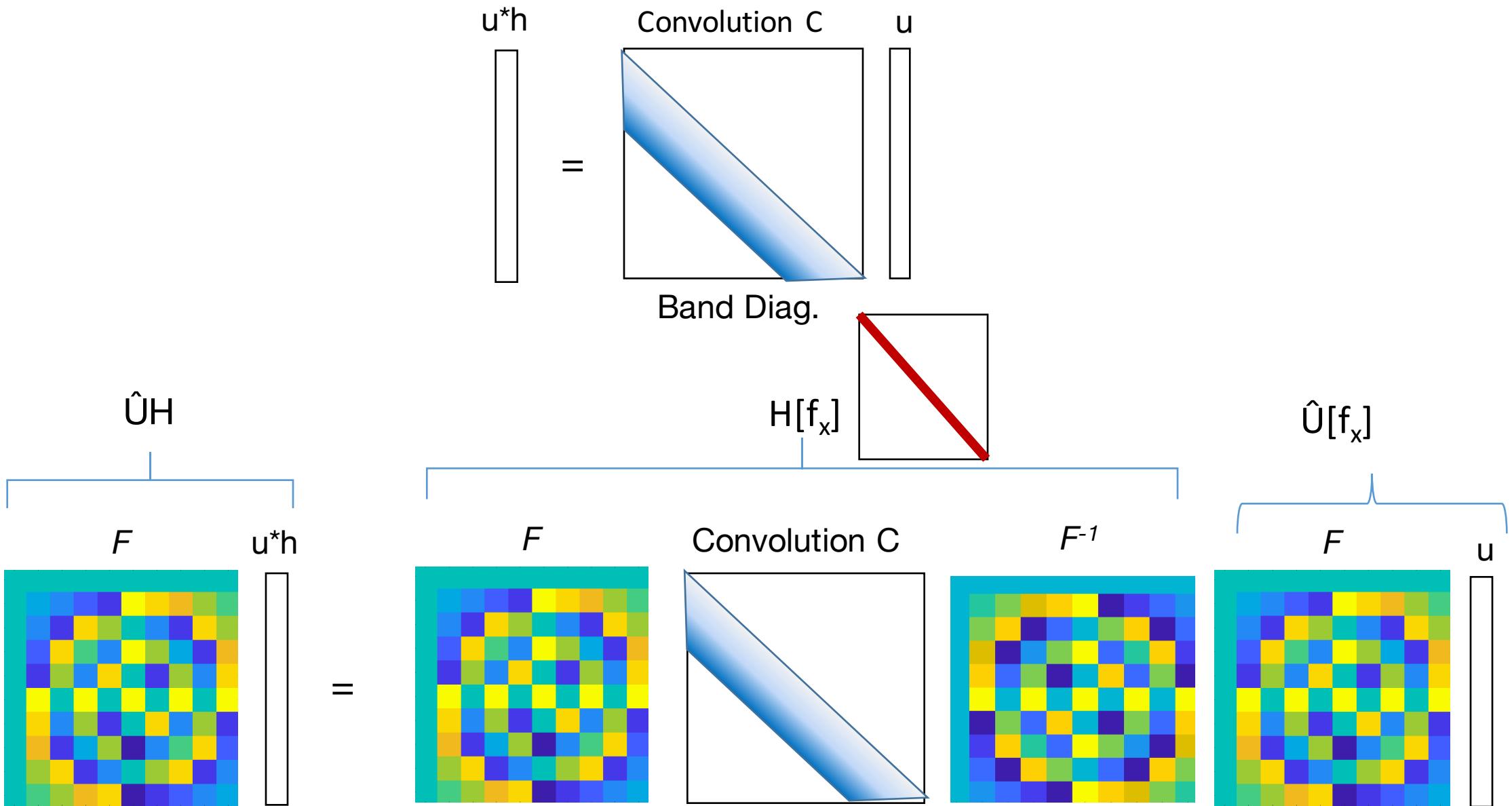


Convolutions as a big matrix multiplication

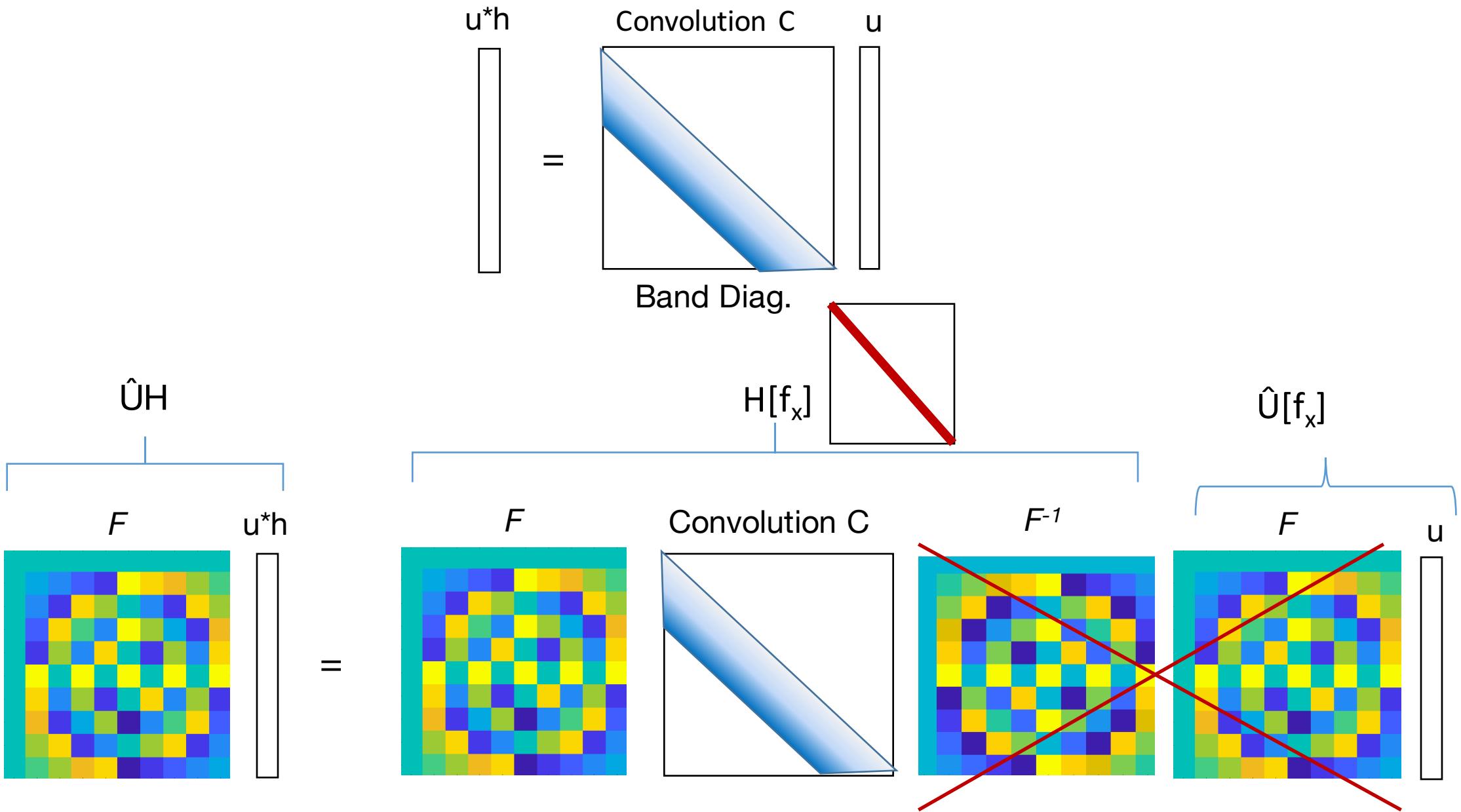


- Convolution achieved by multiplication in Fourier domain

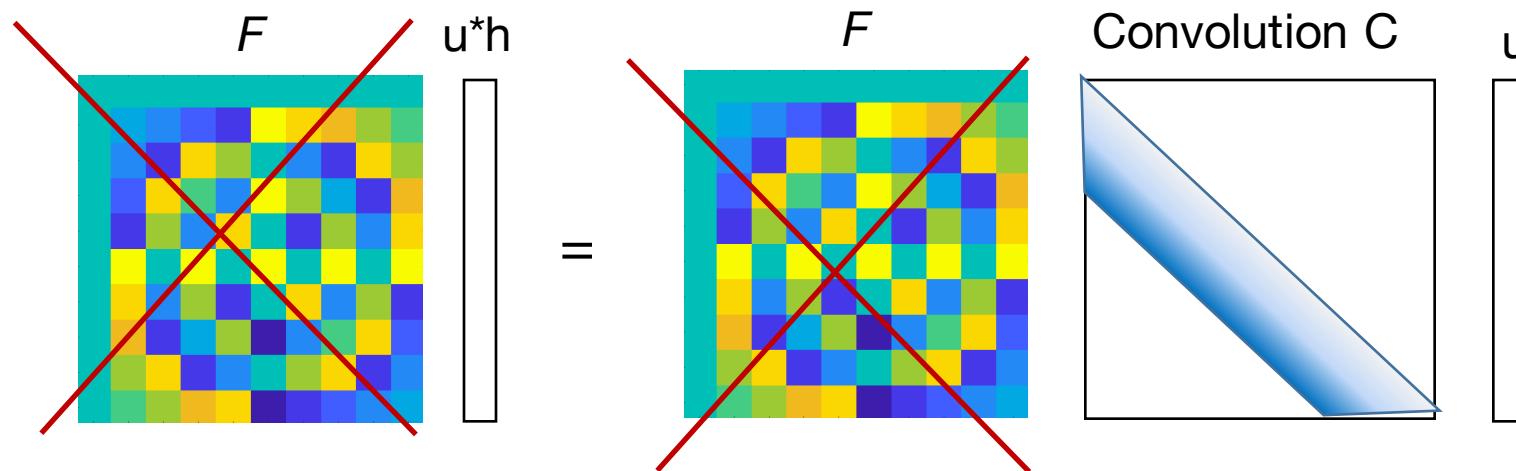
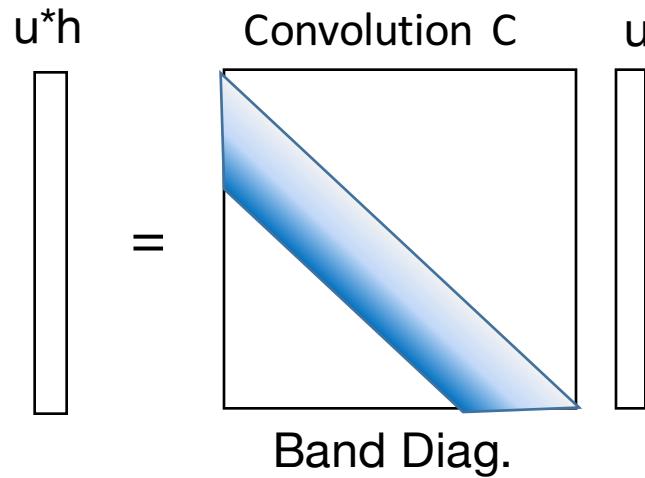
Convolutions as a big matrix multiplication



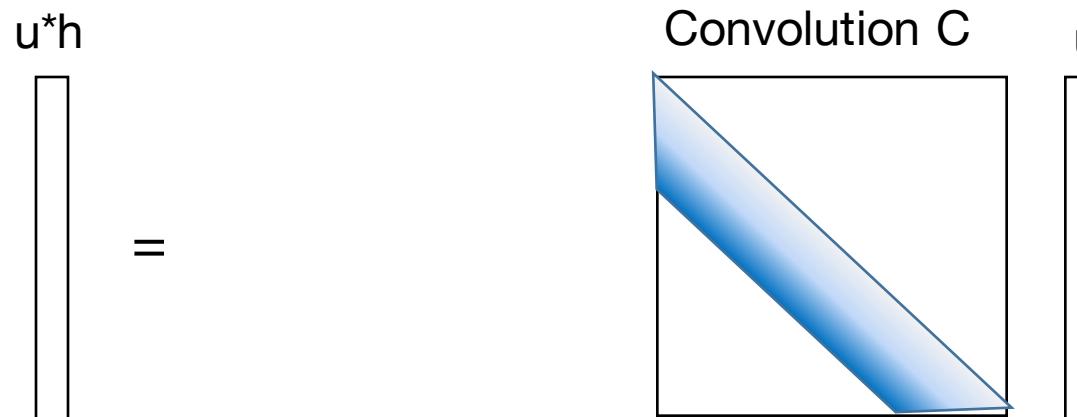
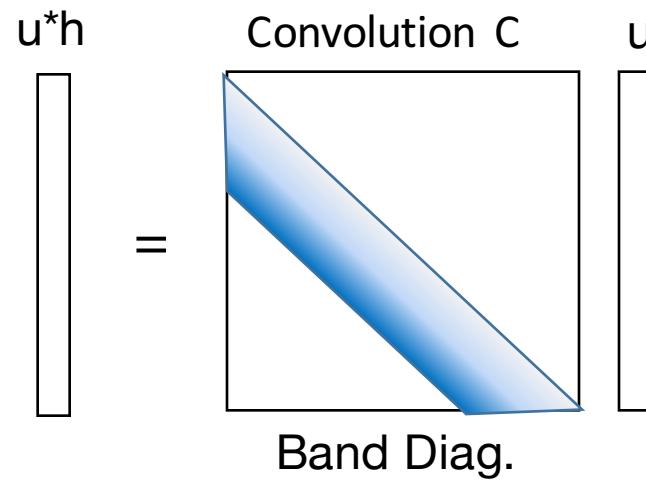
Convolutions as a big matrix multiplication



Convolutions as a big matrix multiplication



Convolutions as a big matrix multiplication



Last thing – matrix and vector derivatives

$$\mathbf{u} = \mathbf{W}\mathbf{v}$$

$$\frac{d\mathbf{u}}{d\mathbf{v}} =$$

Last thing – matrix and vector derivatives

$$\mathbf{u} = \mathbf{W}\mathbf{v}$$

$$\frac{d\mathbf{u}}{d\mathbf{v}} =$$

$$\mathbf{u}_3 = W_{3,1}v_1 + W_{3,2}v_2 + \dots + W_{3,M}v_M$$

$$\frac{\partial u_3}{\partial v_2} = \frac{\partial}{\partial v_2}(W_{3,1}v_1 + W_{3,2}v_2 + \dots + W_{3,M}v_M) = \frac{\partial}{\partial v_2}W_{3,2}v_2 = W_{3,2}$$

Last thing – matrix and vector derivatives

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$$\frac{\partial u_i}{\partial v_j} = W_{i,j}$$

$$\frac{d\mathbf{u}}{d\mathbf{v}} = \mathbf{W}$$

Last thing – matrix and vector derivatives

$$\mathbf{u} = \mathbf{W}\mathbf{v}$$

$$\frac{d\mathbf{u}}{d\mathbf{v}} =$$

$$\mathbf{u}_3 = W_{3,1}v_1 + W_{3,2}v_2 + \dots + W_{3,M}v_M$$

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$$\frac{\partial u_i}{\partial v_j} = W_{i,j}$$

$$\frac{d\mathbf{u}}{d\mathbf{v}} = \mathbf{W}$$

- When confused, write out one entry, solve derivative and generalize
- Use dimensionality to help (if \mathbf{x} has N elements, and \mathbf{y} has M, then $d\mathbf{y}/d\mathbf{x}$ must be NxM)
- Take advantage of *The Matrix Cookbook*:
 - <https://www.math.uwaterloo.ca/~hwolkowi/matrixcookbook.pdf>