Machine Learning in Imaging

BME 590L
Roarke Horstmeyer

Lecture 17: Introduction to Fourier optics

Let’s take a step back: how does light propagate?

Maxwell’s equations
without any charge

$$\nabla \times \vec{\mathcal{E}} = -\mu \frac{\partial \vec{\mathcal{H}}}{\partial t}$$

$$\nabla \times \vec{\mathcal{H}} = \epsilon \frac{\partial \vec{\mathcal{E}}}{\partial t}$$

$$\nabla \cdot \epsilon \vec{\mathcal{E}} = 0$$

$$\nabla \cdot \mu \vec{\mathcal{H}} = 0.$$

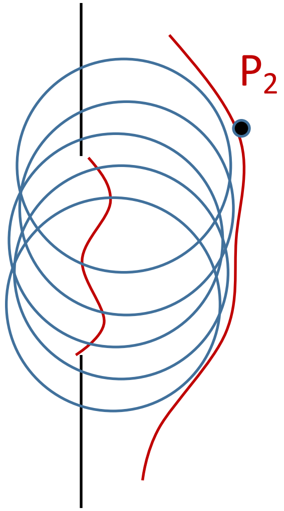
1. Take the curl of both sides of first equation
2. Substitute 2nd and 3rd equation
3. Arrive at the wave equation:

$$\nabla^2 \vec{\mathcal{E}} - \frac{n^2}{c^2} \frac{\partial^2 \vec{\mathcal{E}}}{\partial t^2} = 0 \qquad n = \left(\frac{\epsilon}{\epsilon_0}\right)^{1/2} \qquad c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}.$$

Plane-to-plane light propagation via the "paraxial approximation"

The Huygens-Fresnel Equation

$$U(P_2) = \frac{1}{j\lambda} \iint_{\Sigma} U(P_1) \frac{\exp(jkr_{21})}{r_{21}} \cos\theta \, ds$$



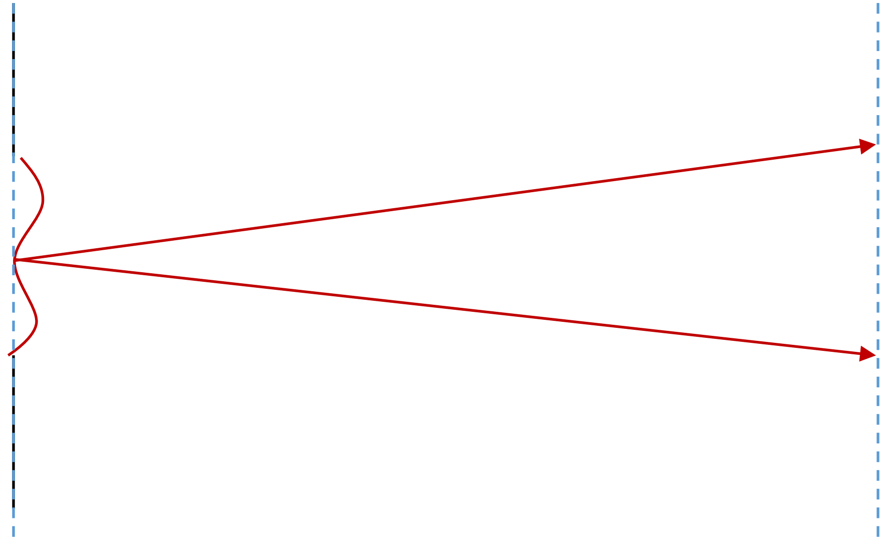
Generally connects two points in 3D:

$$U(P_1) = U(x_1, y_1, z_1)$$

$$U(P_2) = U(x_2, y_2, z_2)$$

Plane-to-plane light propagation via the "paraxial approximation"

We are usually concerned about propagation between two planes (almost always in an optical system):



$$U(P_1) = U(x_1, y_1, z_1 = z_{p1}) \qquad U(P_2) = U(x_2, y_2, z_2 = z_{p2})$$

$$U(P) = E(x, y, z)e^{ikz}$$

Plane-to-plane light propagation via the ”paraxial approximation”

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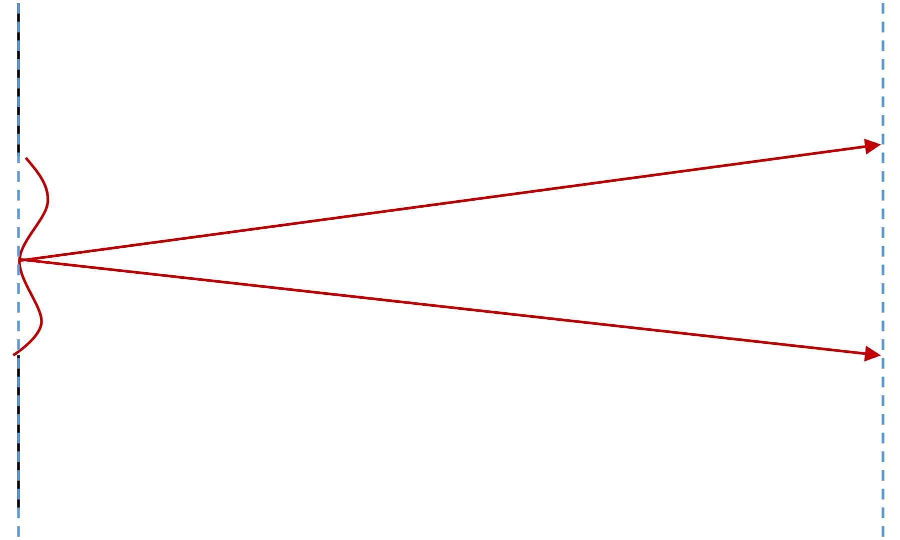

Paraxial approximation:

$$\boxed{\nabla_\perp^2 U + 2ik\frac{dU}{dz} = 0}$$

$$\nabla_\perp^2 \stackrel{\text{def}}{=} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$U(P_1) = U(x_1, y_1, z_1 = z_{p1}) \qquad U(P_2) = U(x_2, y_2, z_2 = z_{p2})$$

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Plane-to-plane light propagation via the ”paraxial approximation”

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Paraxial approximation:

$$\nabla_\perp^2 U + 2ik\frac{dU}{dz} = 0$$

Substitute in $U(P) = E(x, y, z)e^{ikz}$ and crank the wheel,

$$\nabla_\perp^2 E + 2ik\frac{dE}{dz} + 2k^2 E = 0$$

Paraxial Helmholtz Equation. This has an exact integral solution:

Plane-to-plane light propagation via the "paraxial approximation"

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Paraxial Helmholtz Equation. This has an exact integral solution:

$$\boxed{E(x, y, z) = \frac{e^{ikz}}{i\lambda z} \iint_{-\infty}^{+\infty} E(x', y', 0)e^{\frac{ik}{2z}\left[(x-x')^{2}+(y-y')^{2}\right]} dx' dy'}$$

Fresnel diffraction integral

This is how light propagates from one plane to the next. It's a convolution!

Fresnel light propagation as a convolution

$$E(x,y,z) = \frac{e^{ikz}}{i\lambda z} \iint_{-\infty}^{+\infty} E(x',y',0) e^{\frac{ik}{2z}\left[(x-x')^2+(y-y')^2\right]} dx'\,dy'$$

$$h(x,y,z) = \frac{e^{ikz}}{i\lambda z} e^{i\frac{k}{2z}(x^2+y^2)}$$

$$E(x,y,z) = E(x,y,0) * h(x,y,z)$$

Fresnel light propagation as a convolution

$$E(x,y,z) = \frac{e^{ikz}}{i\lambda z} \iint_{-\infty}^{+\infty} E(x',y',0)\, e^{\frac{ik}{2z}\left[(x-x')^2+(y-y')^2\right]} dx'\, dy'$$

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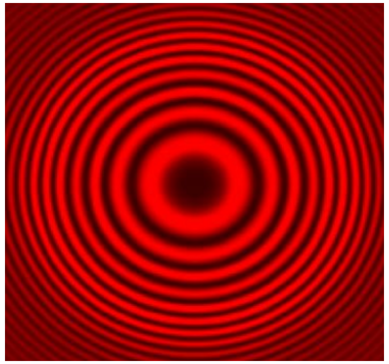

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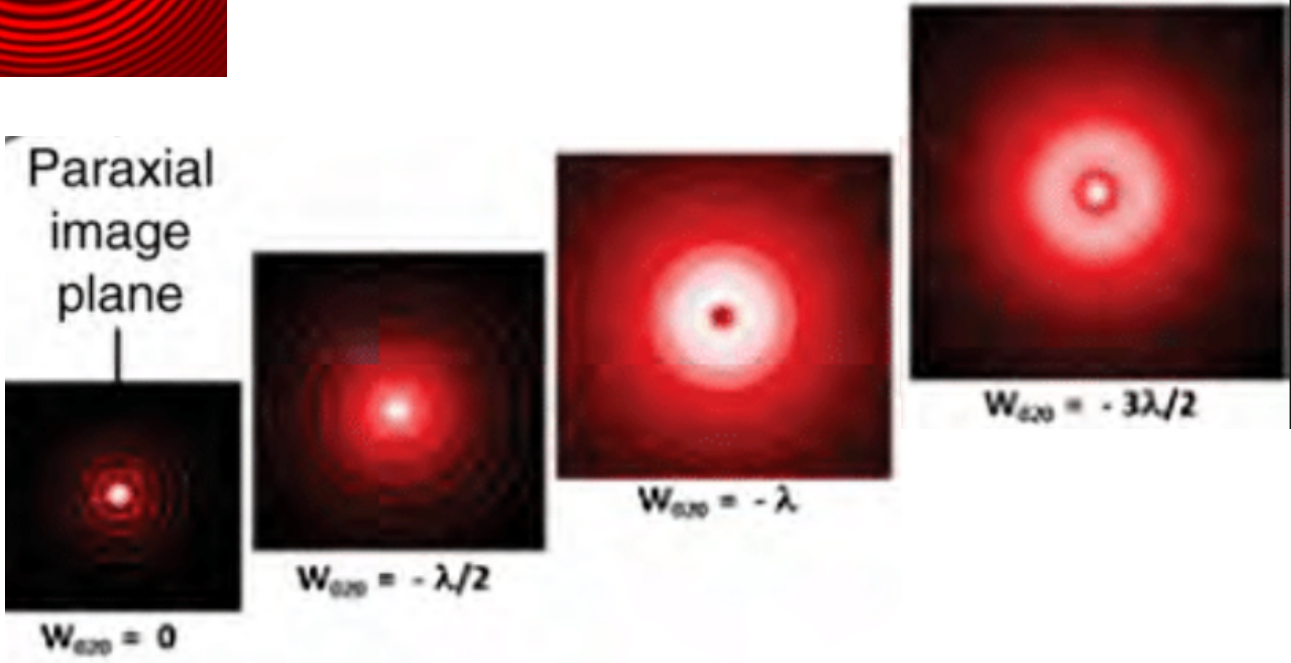

From the Fresnel approximation to the Fraunhofer approximation

Fresnel Approximation:

$$E(x,y,z) = \frac{e^{ikz}}{i\lambda z}\iint_{-\infty}^{+\infty} E(x',y',0)\, e^{\frac{ik}{2z}\left[(x-x')^2+(y-y')^2\right]}\, dx'\, dy'$$

Lets assume that the second plane is “pretty far away” from the first plane. Then, $\boxed{z > \frac{2D^2}{\lambda}}$

From the Fresnel approximation to the Fraunhofer approximation

Fresnel Approximation:

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Lets assume that the second plane is "pretty far away" from the first plane. Then,

$$\boxed{z > \frac{2D^2}{\lambda}}$$

1. Expand the squaring

$$E(x,y,z) = \frac{e^{ikz}}{i\lambda z}\iint E(x',y',0)e^{\frac{ik}{2z}(x^2+y^2)}e^{\frac{ik}{2z}(x'^2+y'^2)}e^{\frac{ik}{2z}(xx'+yy')}dx'dy'$$

From the Fresnel approximation to the Fraunhofer approximation

Fresnel Approximation:

$$E(x,y,z) = \frac{e^{ikz}}{i\lambda z} \iint_{-\infty}^{+\infty} E(x',y',0) e^{\frac{ik}{2z}\left[(x-x')^2+(y-y')^2\right]} dx'dy'$$

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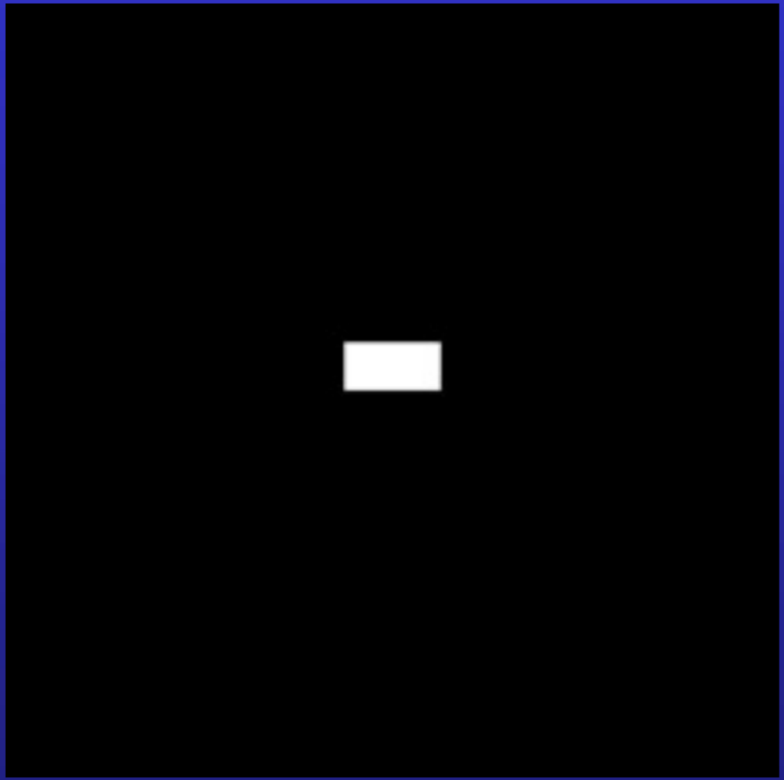
$$E(x,y,z) = \frac{e^{ikz}}{i\lambda z} \iint E(x',y',0) e^{\frac{ik}{2z}(x^2+y^2)} e^{\frac{ik}{2z}(x'^2+y'^2)} e^{\frac{ik}{2z}(xx'+yy')} dx'dy'$$

2. Front term comes out, assume second term goes away, then,

$$\boxed{E(x,y,z) = C \iint E(x',y',0) e^{\frac{ik}{2z}(xx'+yy')} dx'dy'} \qquad C = \frac{e^{ikz}}{i\lambda z} e^{\frac{ik}{2z}(x^2+y^2)}$$

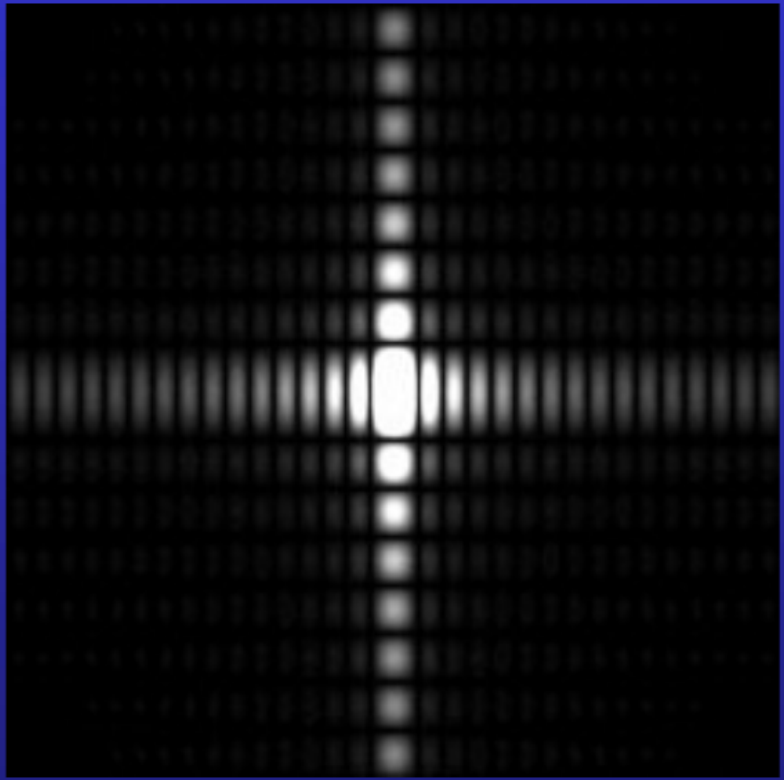
"Simple" propagation (diffraction) is a Fourier transform!!!!!!!

This is the aperture



Two-dimensional rectangle function as an image

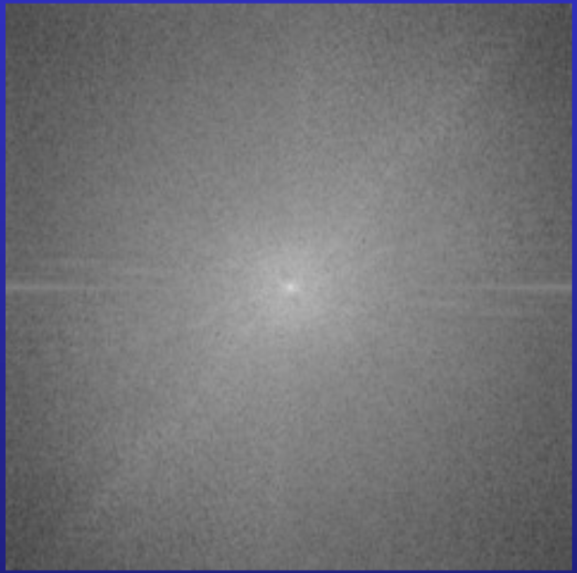
This is what you see far away



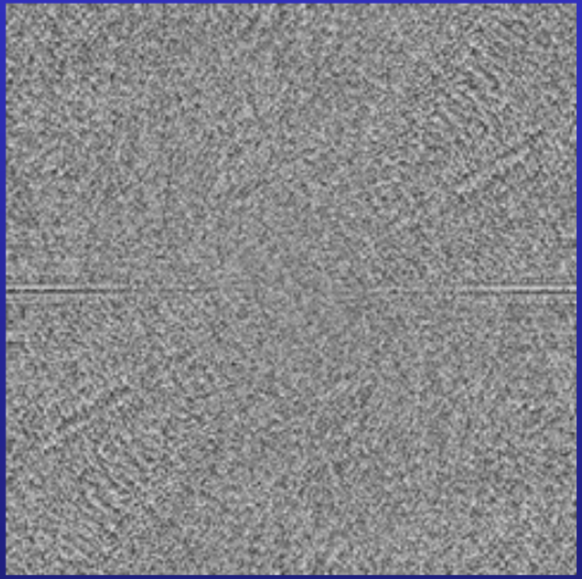
d) Magnitude of Fourier spectrum of the 2-D rectangle

Cheetah

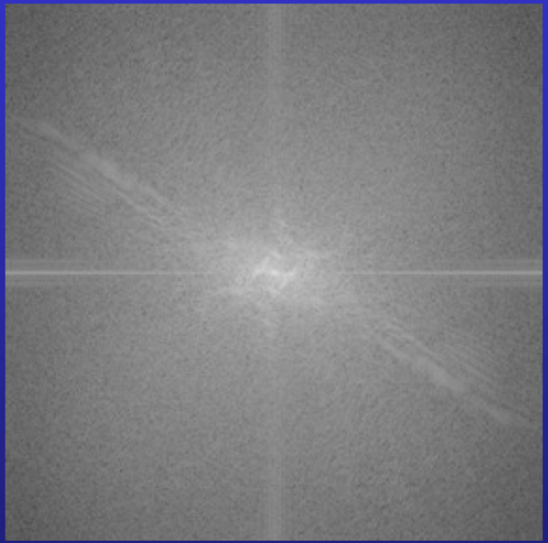
Zebra



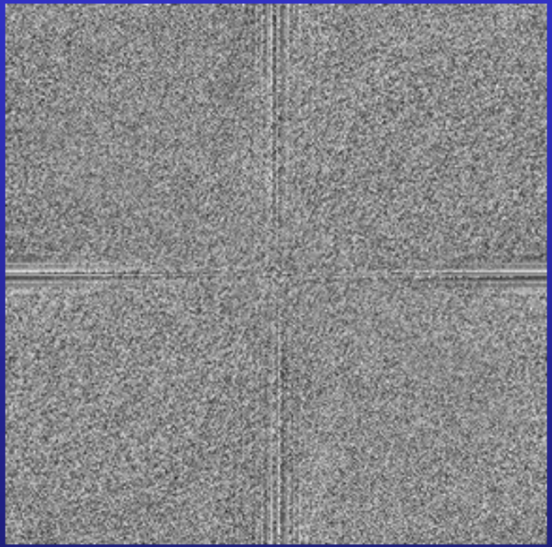
magnitude of cheetah



phase of cheetah

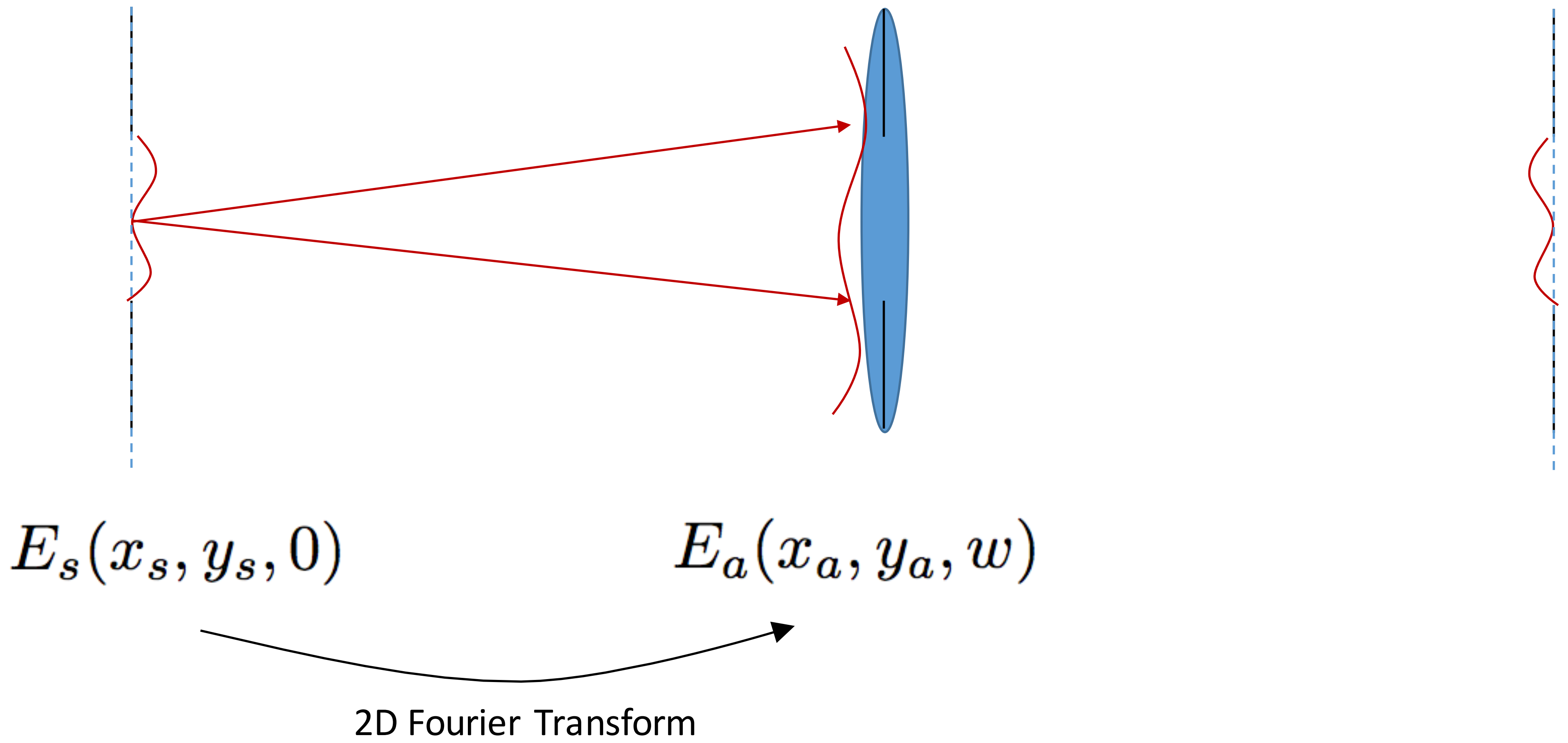


magnitude of zebra

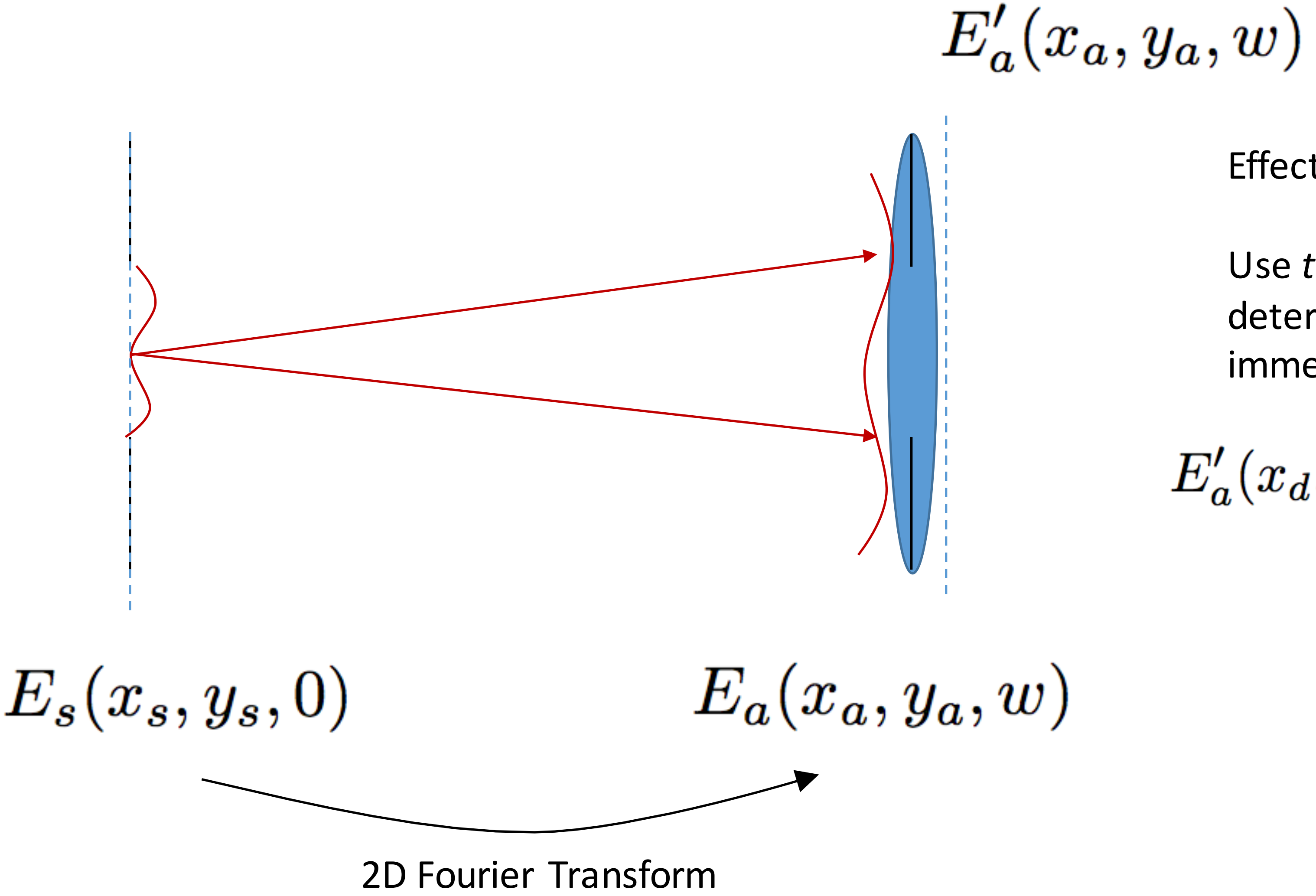


phase of zebra

Model of a microscope (or camera) using Fourier transforms:



Model of a microscope (or camera) using Fourier transforms:



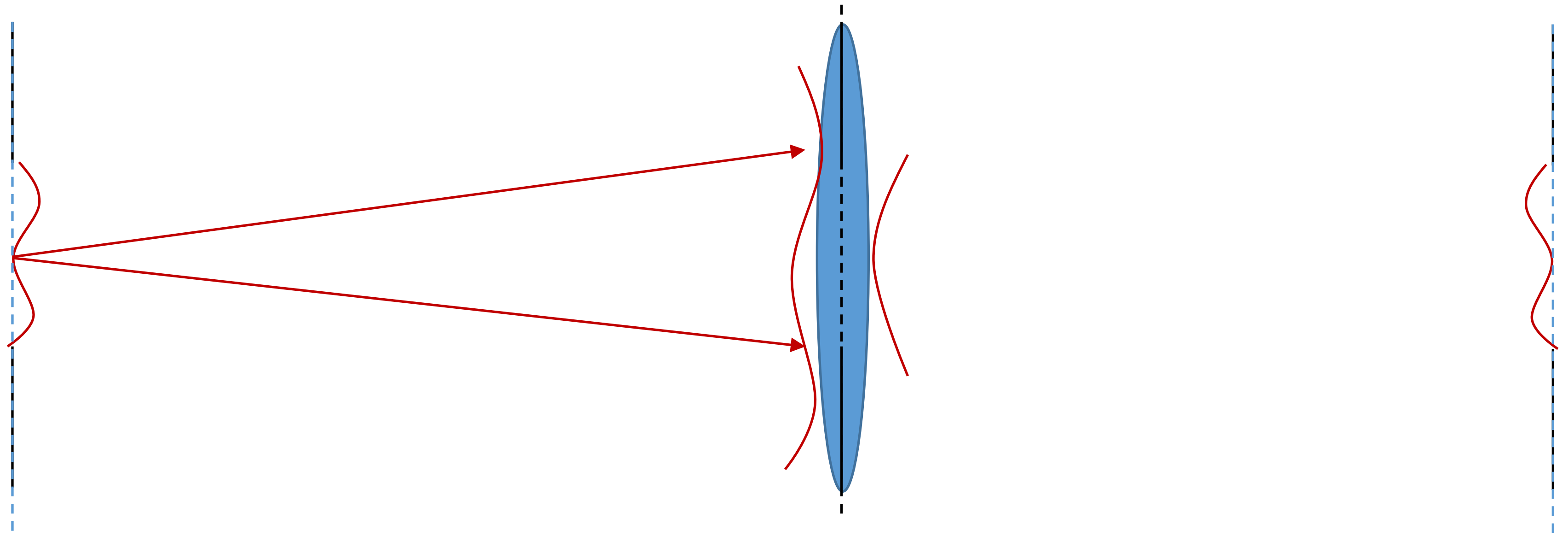
Effect of the lens is to block light.

Use *thin object approximation* to determine distribution of light on the immediate other side of the lens stop:

$$E'_a(x_d, y_d) = E_a(x_d, y_d)A(x_d, y_d)$$

Model of a microscope (or camera) using Fourier transforms:

Last piece of the puzzle: what happens from lens to sensor?



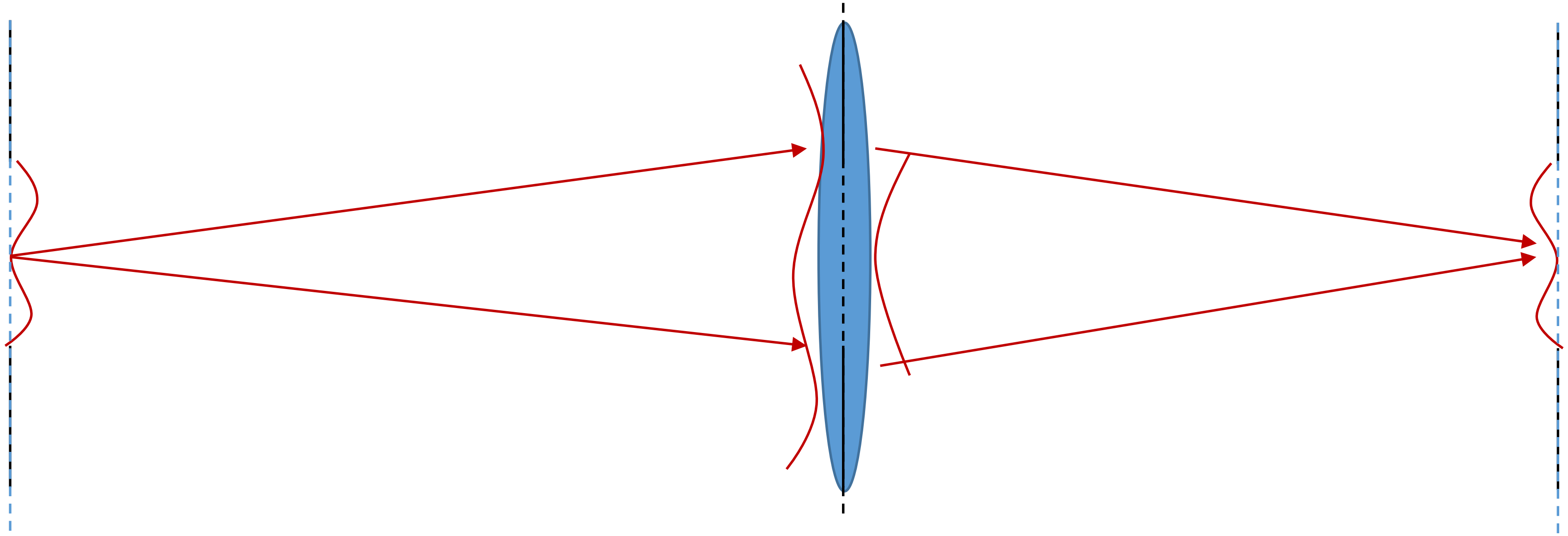
$$E_s(x_s, y_s, 0) \qquad E_a'(x_d, y_d) = E_a(x_d, y_d)A(x_d, y_d)$$

2D Fourier Transform

Model of a microscope (or camera) using Fourier transforms:

Last piece of the puzzle: what happens from lens to sensor?

inverse Fourier transform!



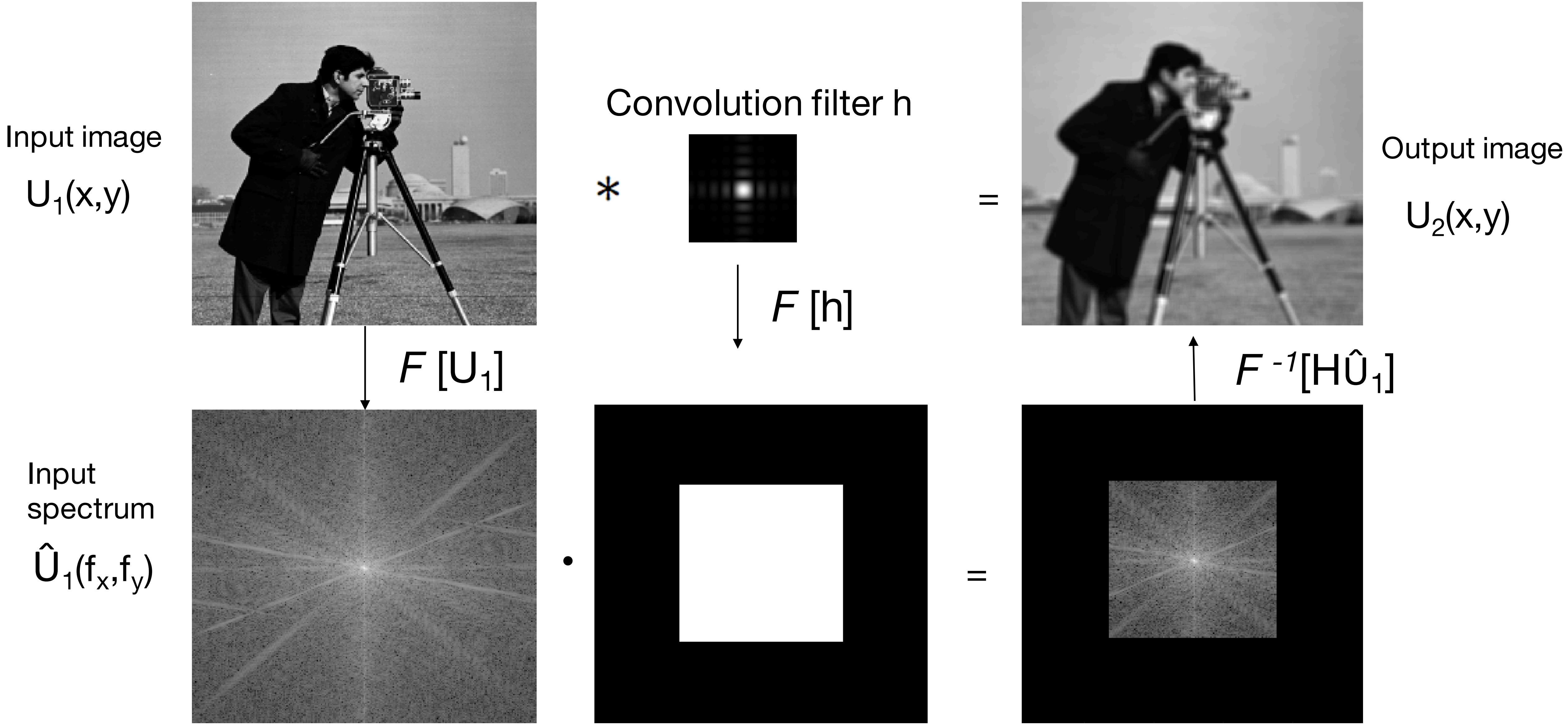
$$E_s(x_s, y_s, 0)$$

$$E_a'(x_d, y_d) = E_a(x_d, y_d)A(x_d, y_d)$$

2D Fourier Transform

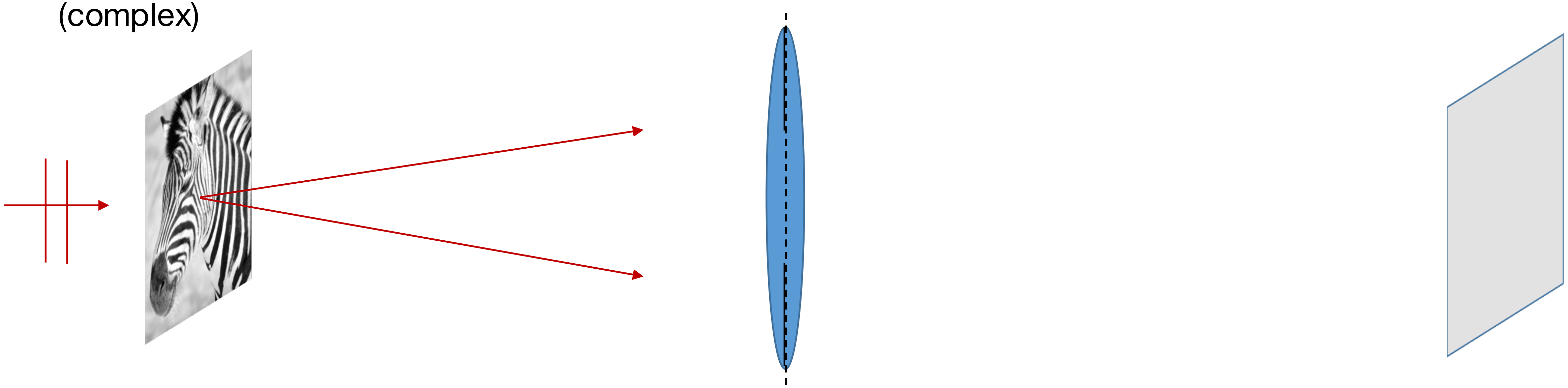
2D inverse Fourier Transform

This process should sound familiar….



Model of image formation for wave optics (coherent light):

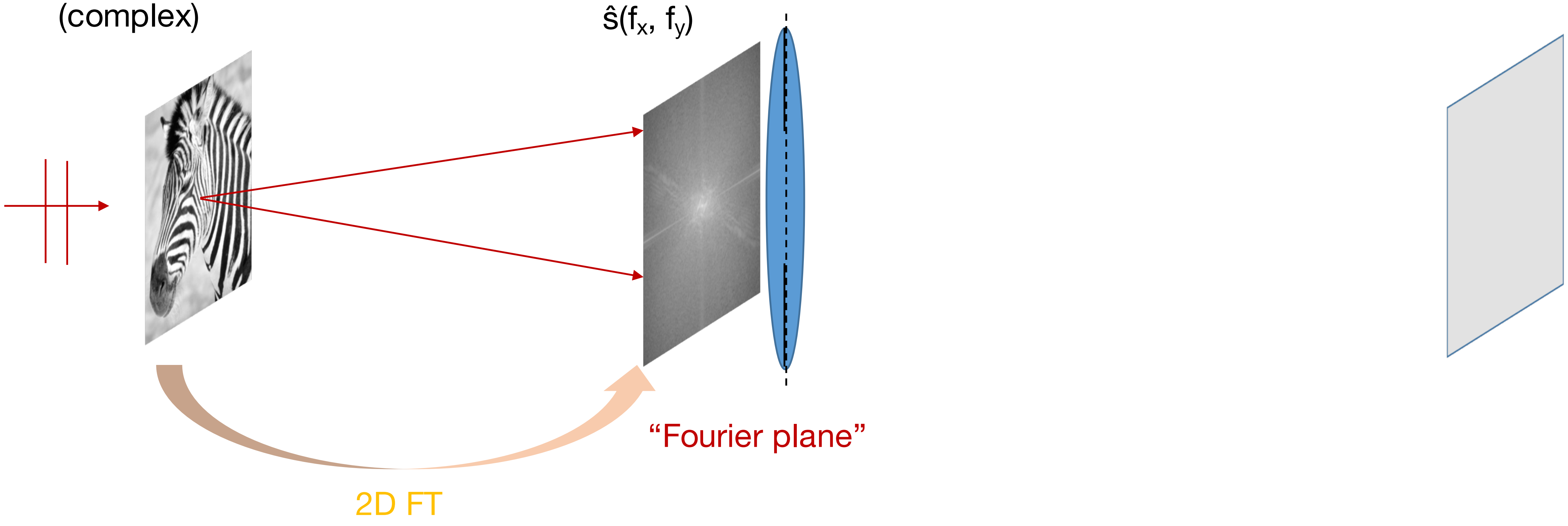
1. Discrete sample function s(x,y) (complex)



Model of image formation for wave optics (coherent light):

1. Discrete sample function s(x,y) (complex)

2. Compute its 2D Fourier transform



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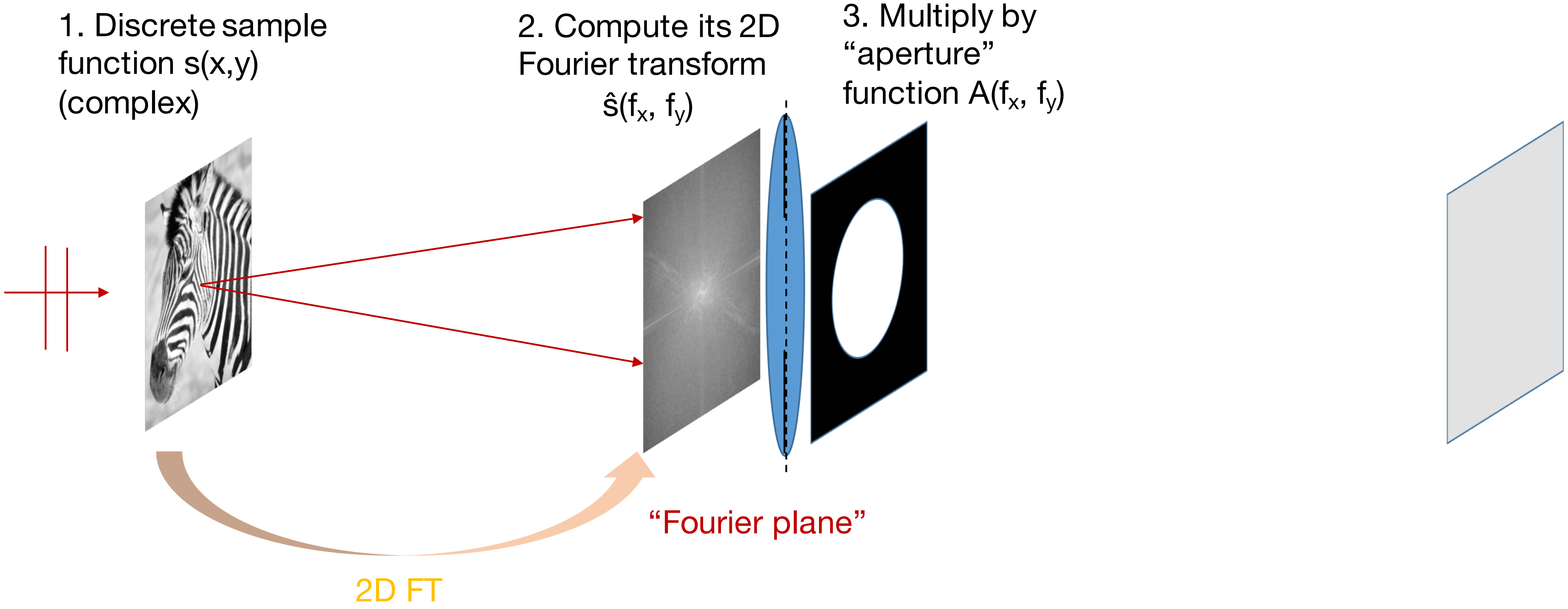

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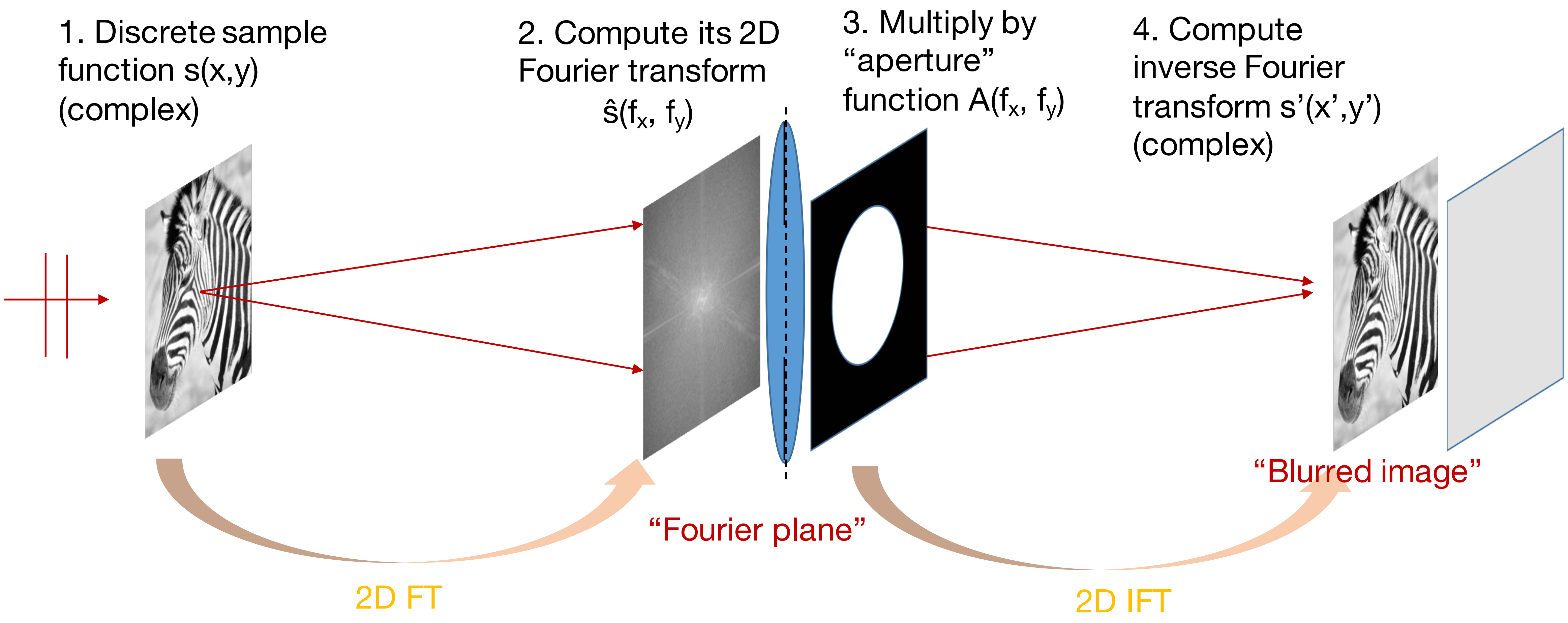

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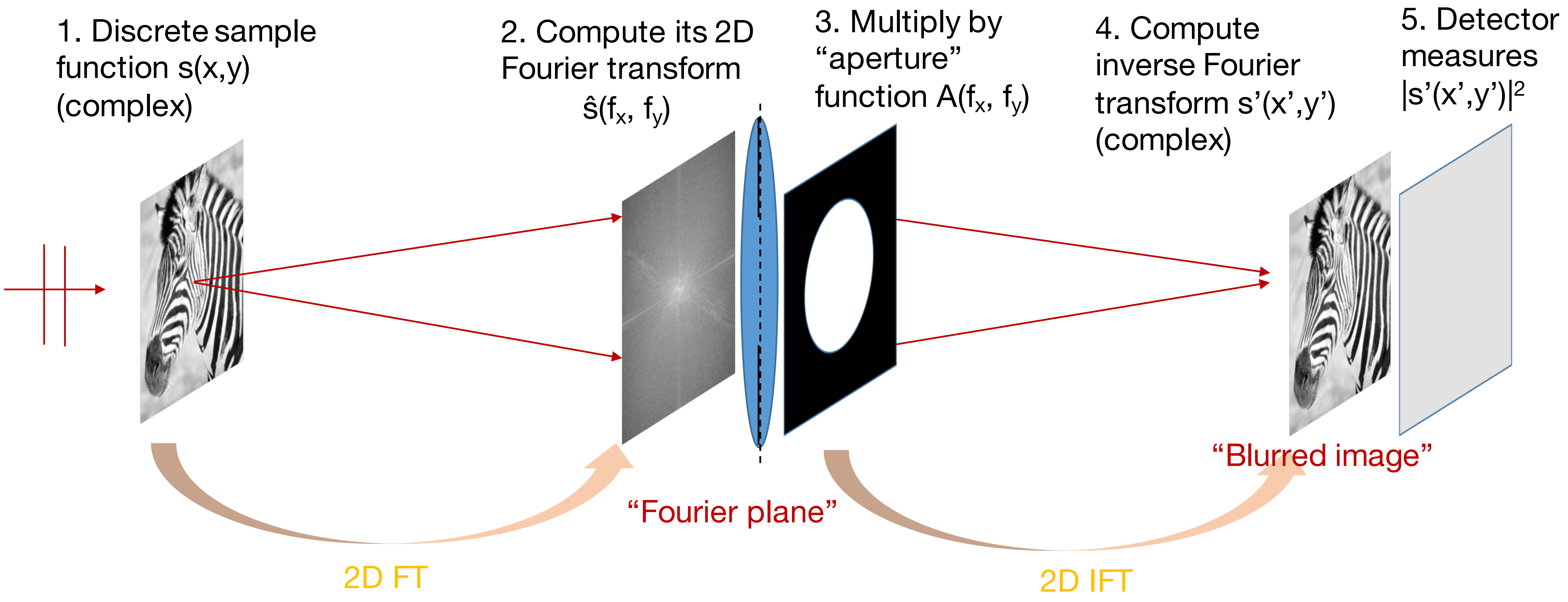


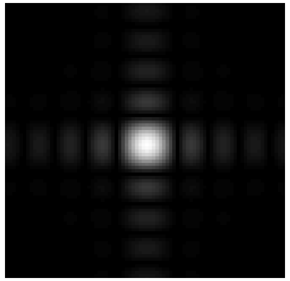
Can also model this using the Convolution Theorem

Aperture function (lens shape)

$A(f_x, f_y)$

Camera blur function (IFT of lens shape)

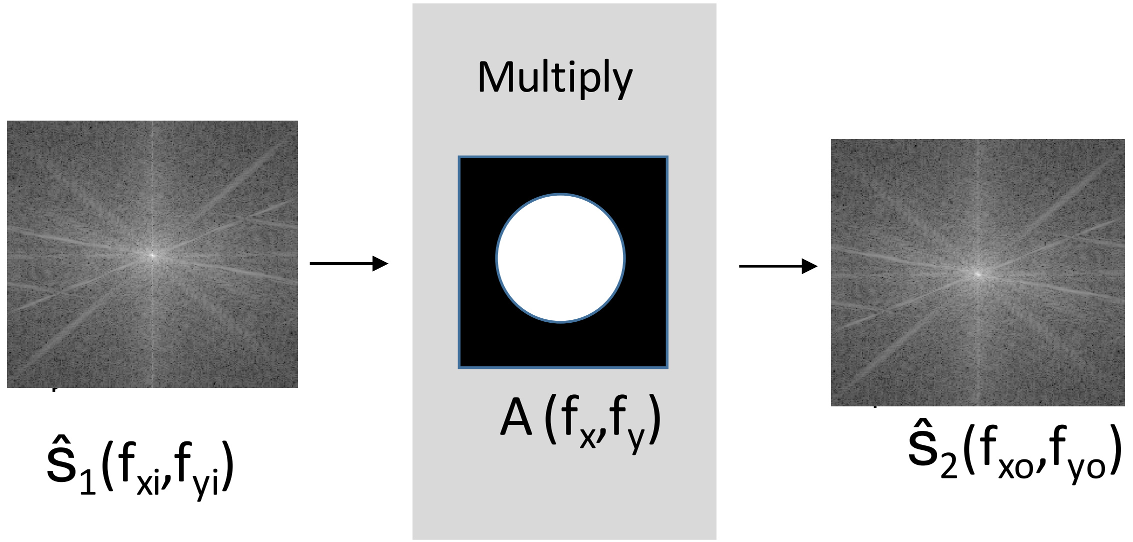
$h(x, y)$



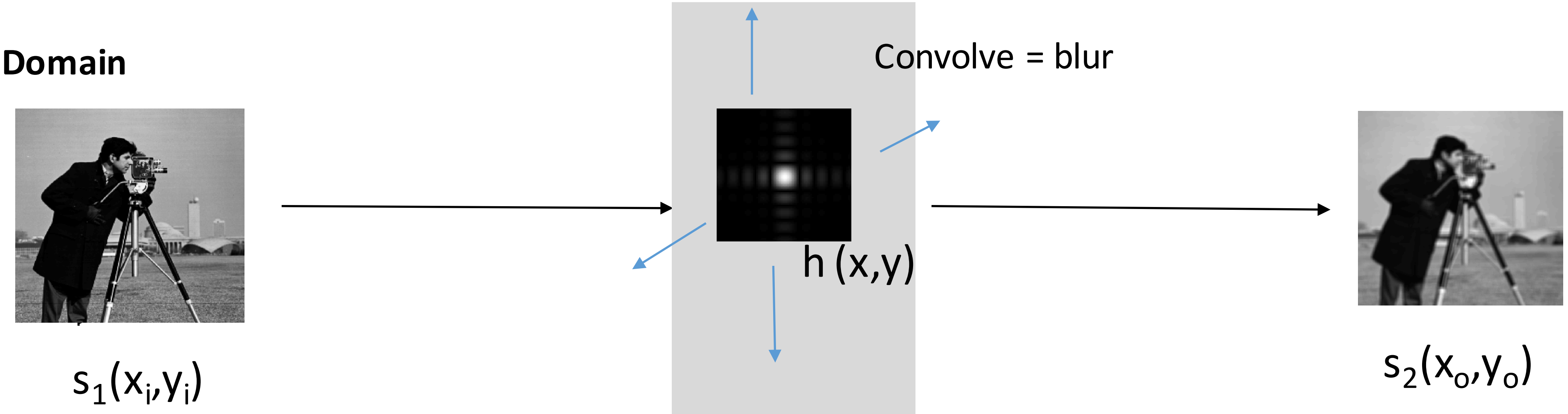
2D IFT

Two modeling choices for the camera:

Spatial Frequency Domain



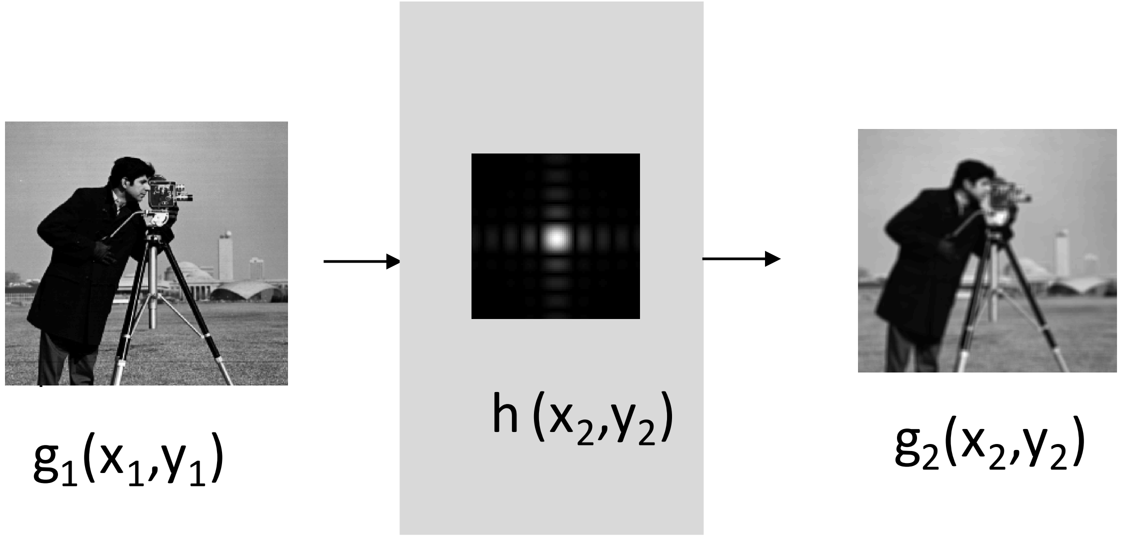
Spatial Domain



Linear systems and the black box

The optical black box system and the point-spread function:

Light $g_1(x_i,y_i)$ entering "black box" optical system modified by system point-spread function



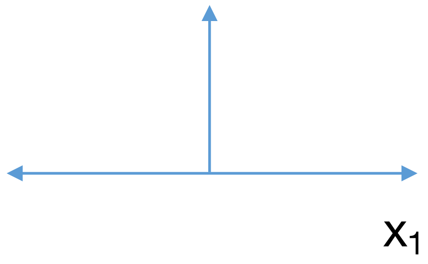
$$\boxed{g_2(x_2,y_2) = \iint_{-\infty}^{\infty} g_1(x_1,y_1)h(x_2-x_1,y_2-y_1)dx_1dy_1}$$

Assume shift invariance:
This is the system point-spread function

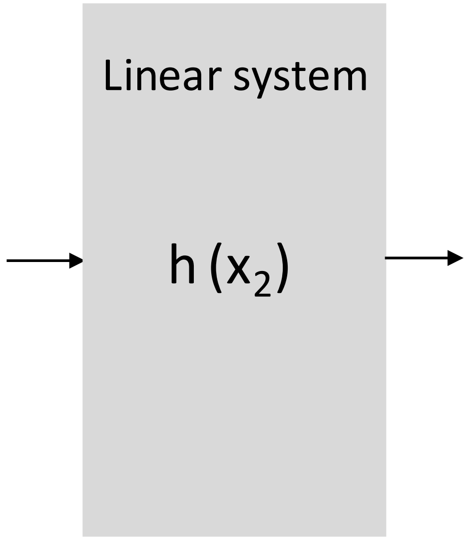
A little bit more detail about the convolution

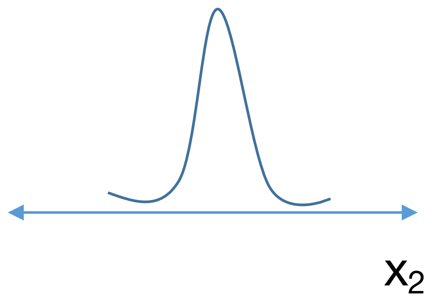
Let's send a "spike" into our linear black box:

For this black box, let's say it returns a Gaussian with a finite waist:



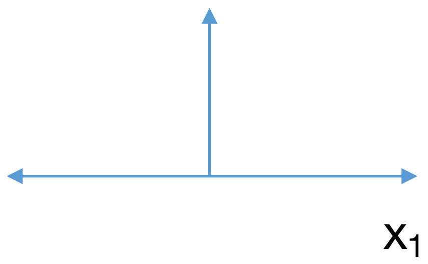
$$g_1(x_1) = \delta(x_1)$$





$$g_2(x_2) = e^{-x_2^2/\sigma^2}$$

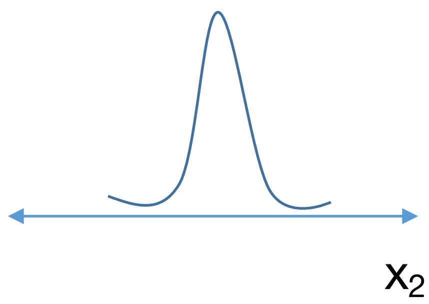
A little bit more detail about the convolution



$$g_1(x_1) = \delta(x_1)$$

Linear system

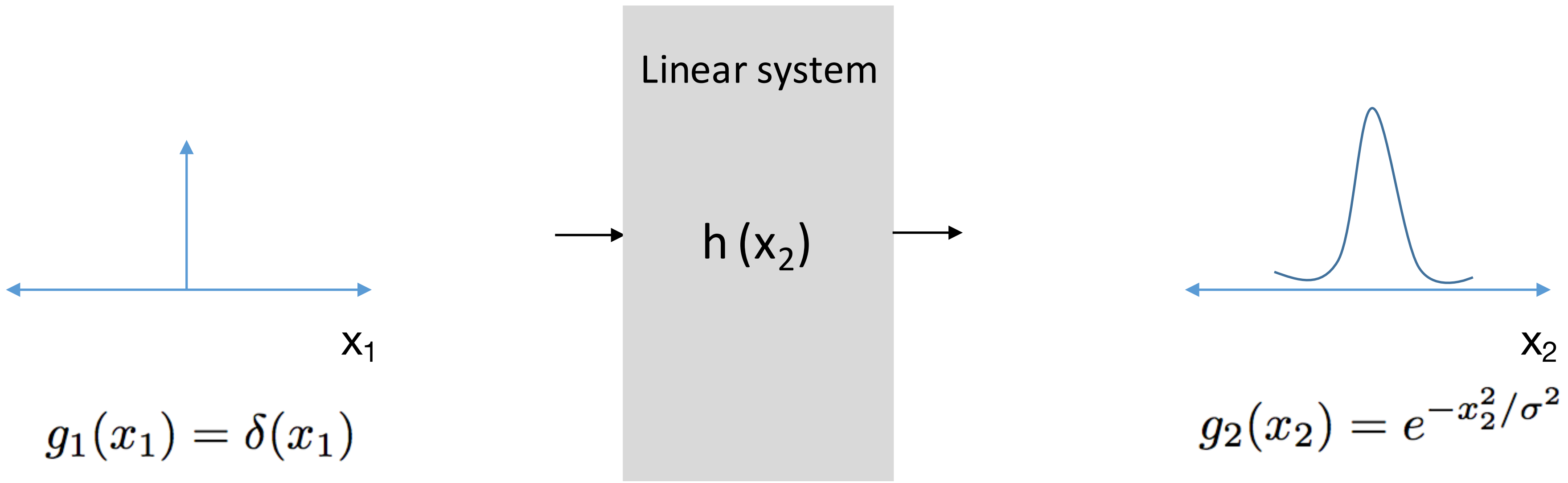
$h\,(x_2)$



$$g_2(x_2) = e^{-x_2^2/\sigma^2}$$

If we know this or measure this, then we have fully characterized the linear system and know everything we need to know about it! Why?

A little bit more detail about the convolution



$$g_1(x_1) = \delta(x_1)$$

$$g_2(x_2) = e^{-x_2^2/\sigma^2}$$

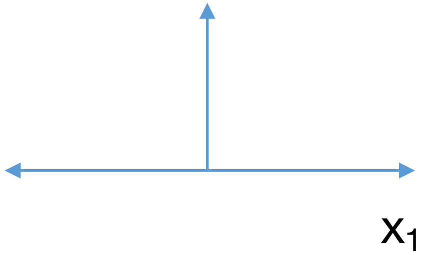
If we know this or measure this, then we have fully characterized the linear system and know everything we need to know about it! Why?

$$g_2(x_2) = \int_{-\infty}^{\infty} g_1(x_1) h(x_2 - x_1) dx_1$$

$$g_2(x_2) = \int_{-\infty}^{\infty} \delta(x_1) h(x_2 - x_1) dx_1$$

$$g_2(x_2) = h(x_2)$$ = the “impulse response”, “point-spread function” or “blur”

A little bit more detail about the convolution



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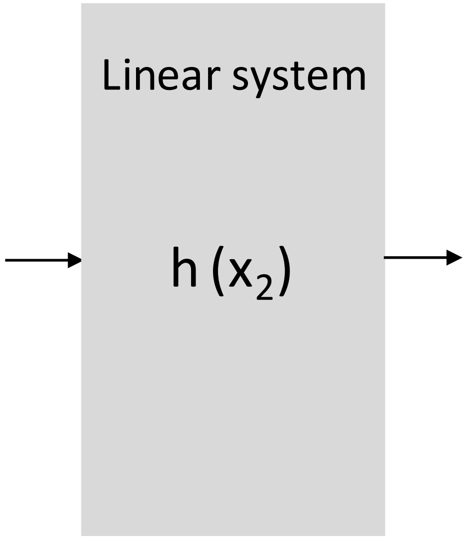
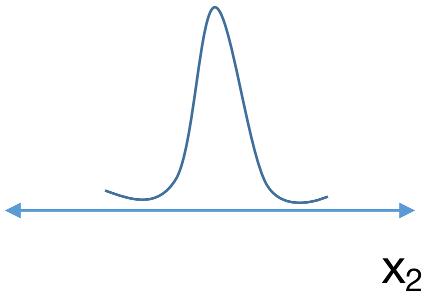
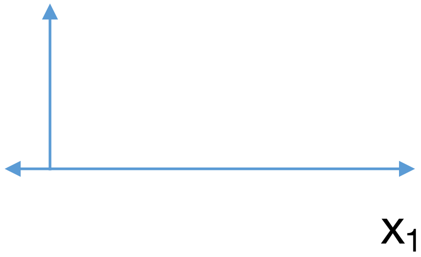


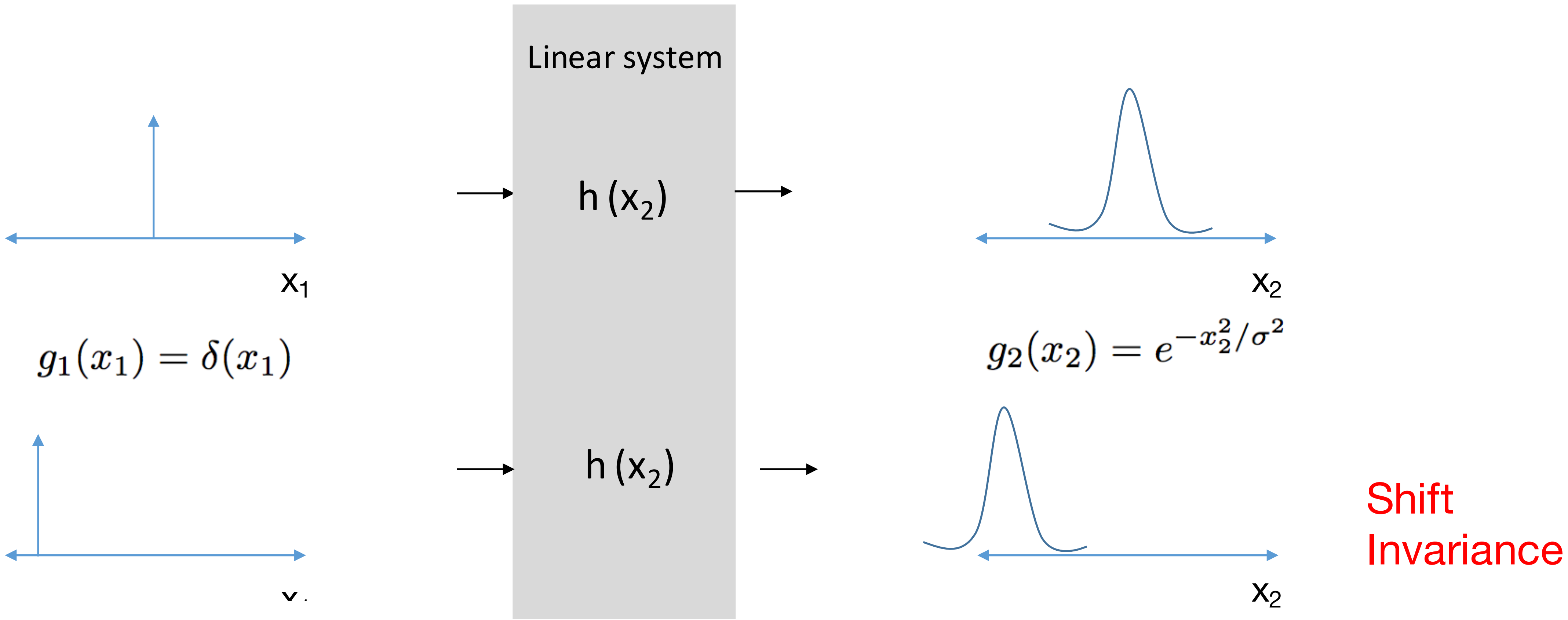


$$g_2(x_2) = e^{-x_2^2/\sigma^2}$$



$$g_1(x_1) = \delta(x_1 - a)$$

A little bit more detail about the convolution



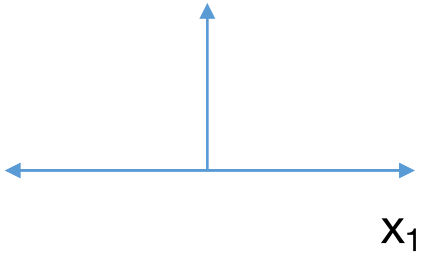
$$g_1(x_1) = \delta(x_1)$$

$$g_2(x_2) = e^{-x_2^2/\sigma^2}$$

$$g_1(x_1) = \delta(x_1 - a)$$

$$g_2(x_2) = e^{-(x-a)^2/\sigma^2}$$

A little bit more detail about the convolution



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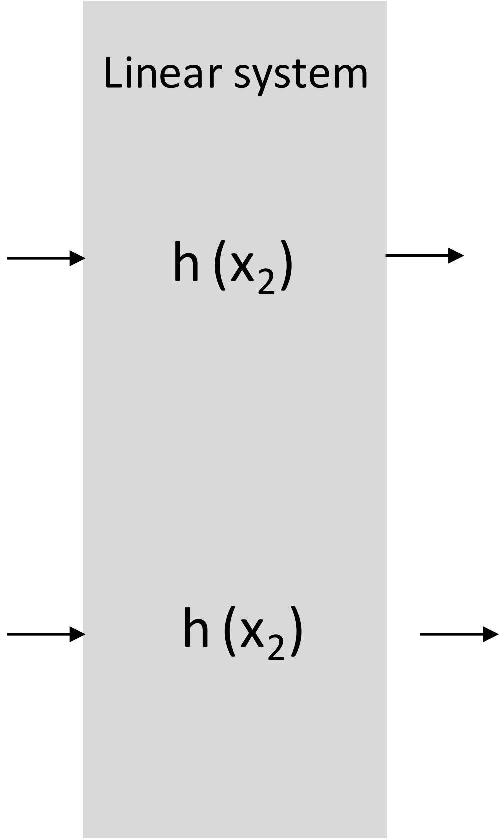

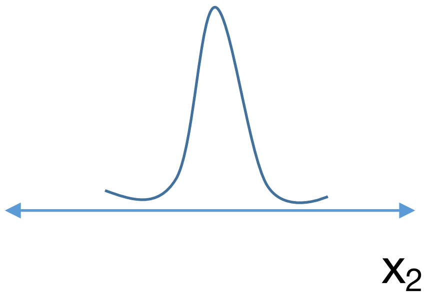

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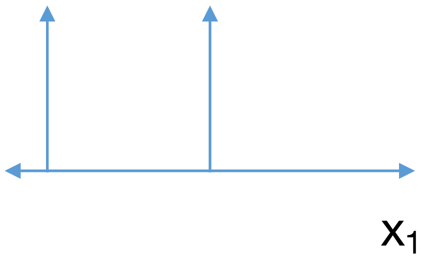
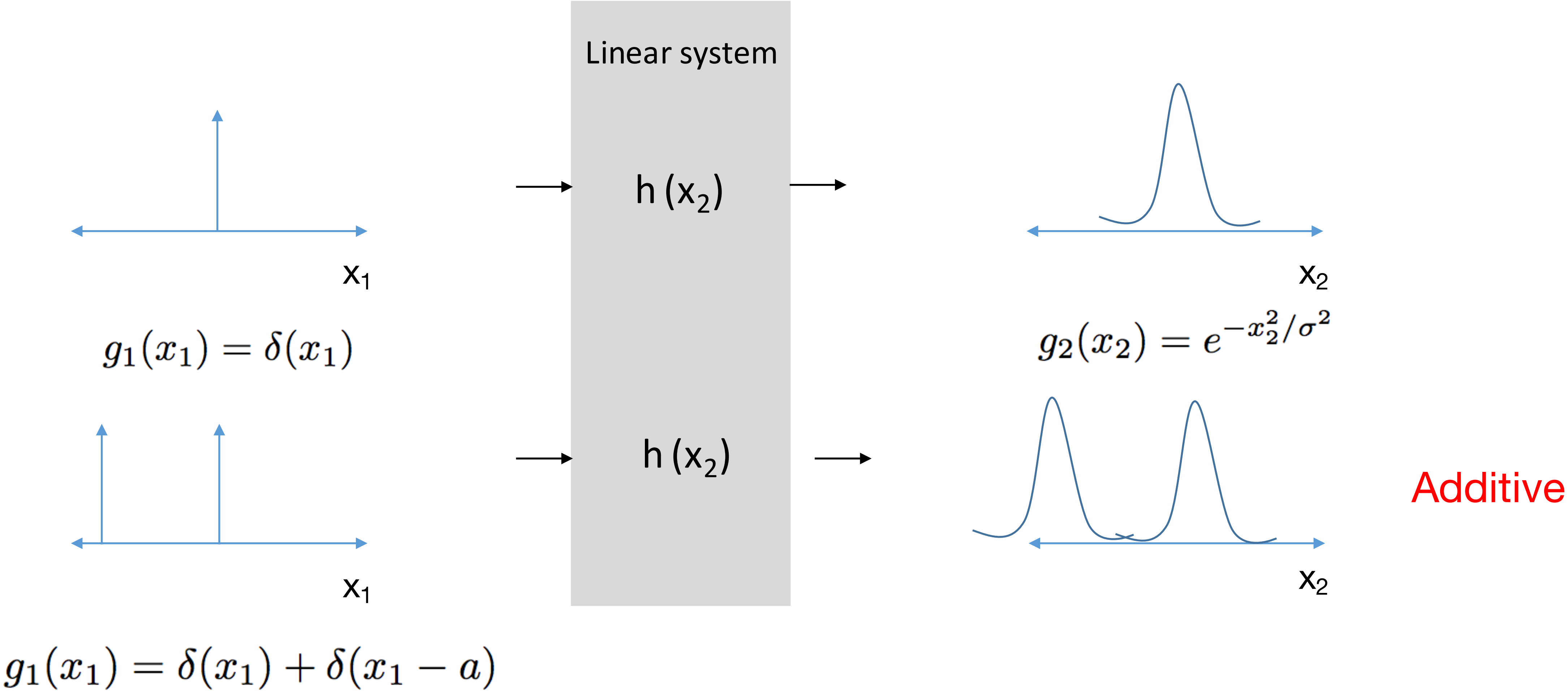
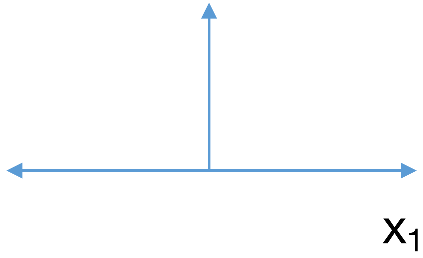


$$g_1(x_1) = \delta(x_1) + \delta(x_1 - a)$$

A little bit more detail about the convolution



A little bit more detail about the convolution



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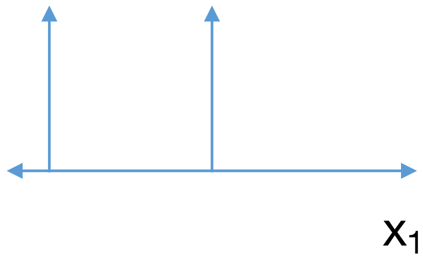
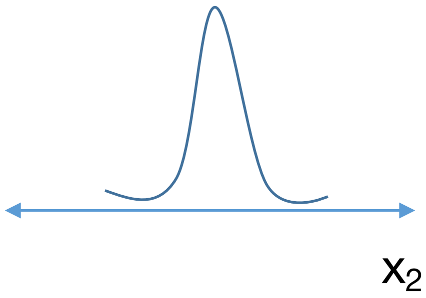


$$g_1(x_1) = \delta(x_1) + \delta(x_1 - a)$$

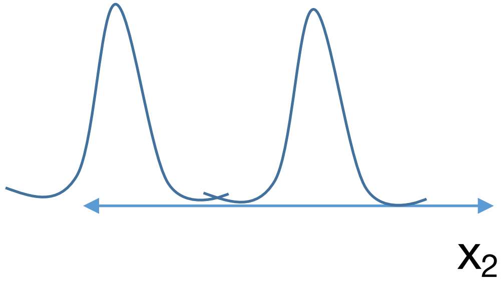
Linear system

$\rightarrow$ h (x_2) $\rightarrow$

$\rightarrow$ h (x_2) $\rightarrow$



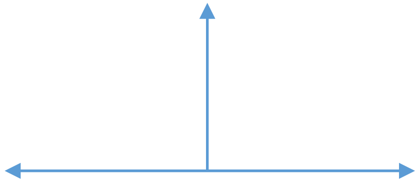
$$g_2(x_2) = e^{-x_2^2/\sigma^2}$$



Additive

$$g_2(x_2) = e^{-x_2^2/\sigma^2} + e^{-(x_2-a)^2/\sigma^2}$$

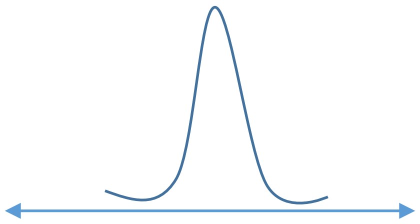
A little bit more detail about the convolution



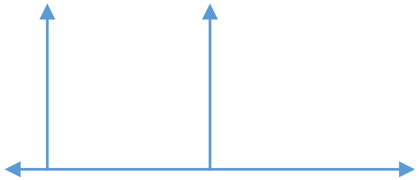
$$g_1(x_1) = \delta(x_1)$$

Linear system

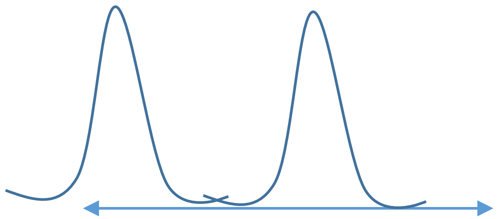
$\rightarrow$ h (x_2) $\rightarrow$



$$g_2(x_2) = e^{-x_2^2/\sigma^2}$$



$\rightarrow$ h (x_2) $\rightarrow$



Additive

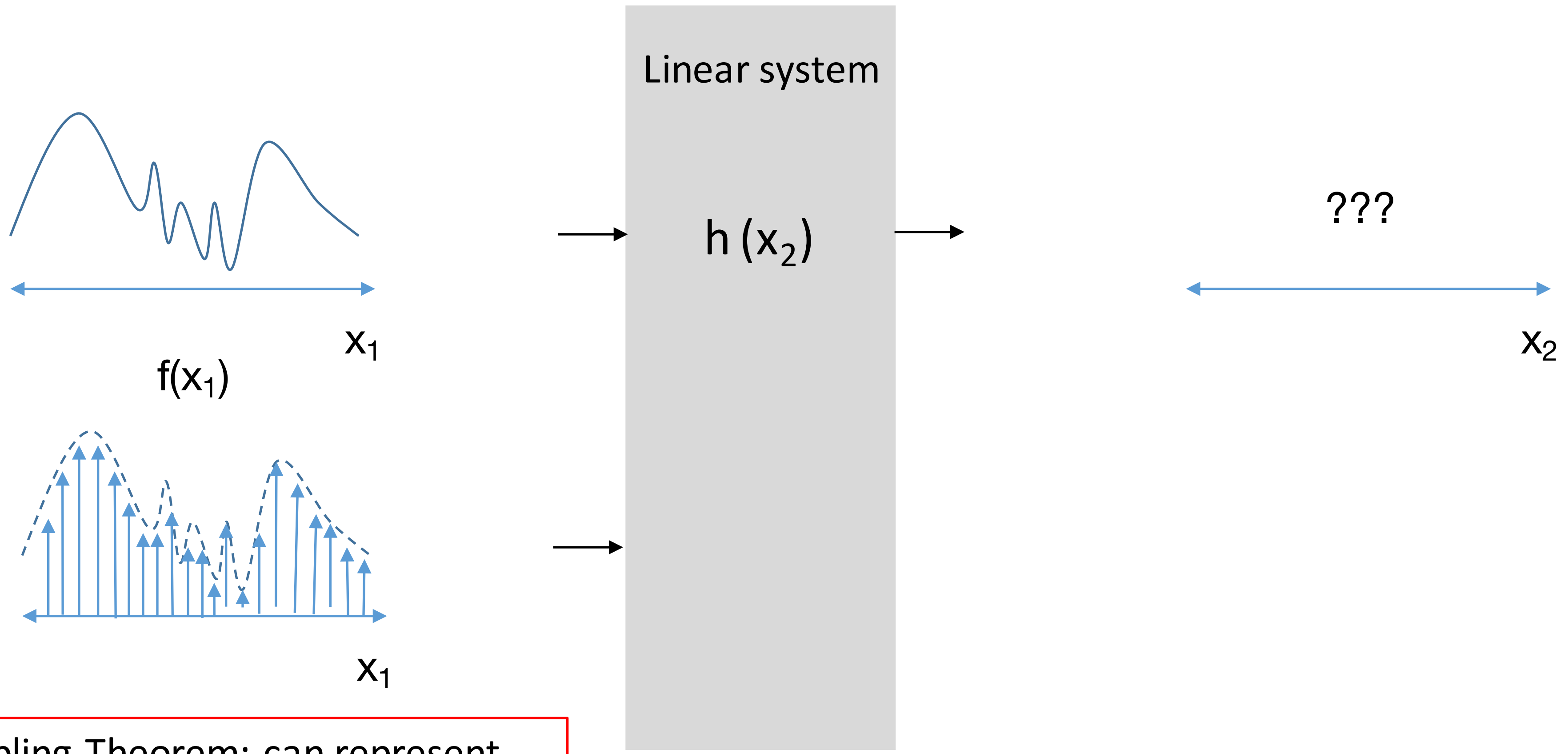
$$g_1(x_1) = \delta(x_1) + \delta(x_1 - a)$$

$$g_2(x_2) = \int^{\infty} (\delta(x_1) + \delta(x_1 - a))h(x_2 - x_1)dx_1$$

$$g_2(x_2) = \int^{\infty} \delta(x_1)h(x_2 - x_1)dx_1 + \int_{-\infty}^{\infty} \delta(x_1 - a)h(x_2 - x_1)dx_1$$

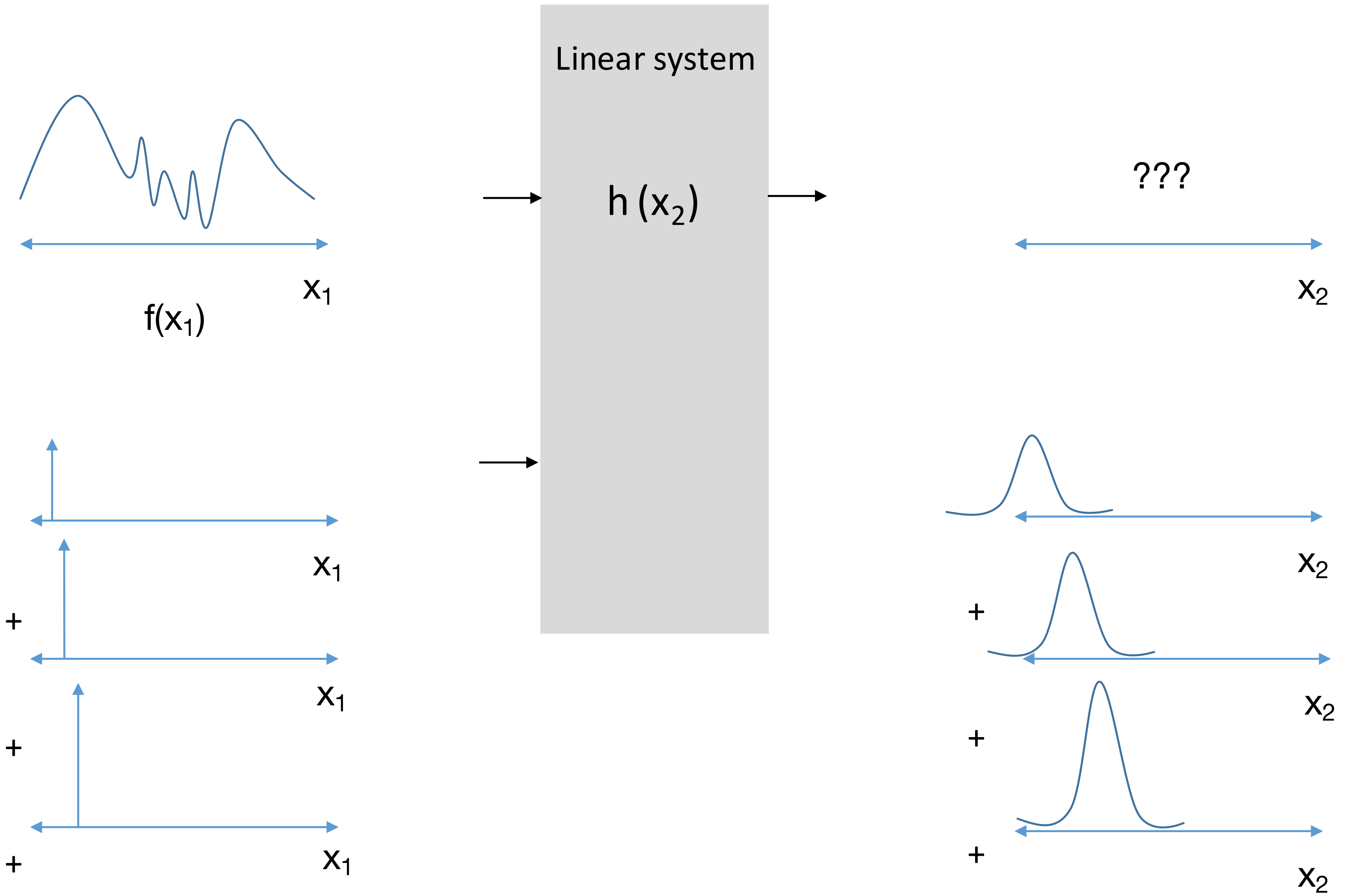
$$g_2(x_2) = h(x_2) + h(x_2 - a)$$

Let's take these ideas and apply them to an arbitrary function



Sampling Theorem: can represent any signal as a discrete set of delta functions

Let's take these ideas and apply them to an arbitrary function



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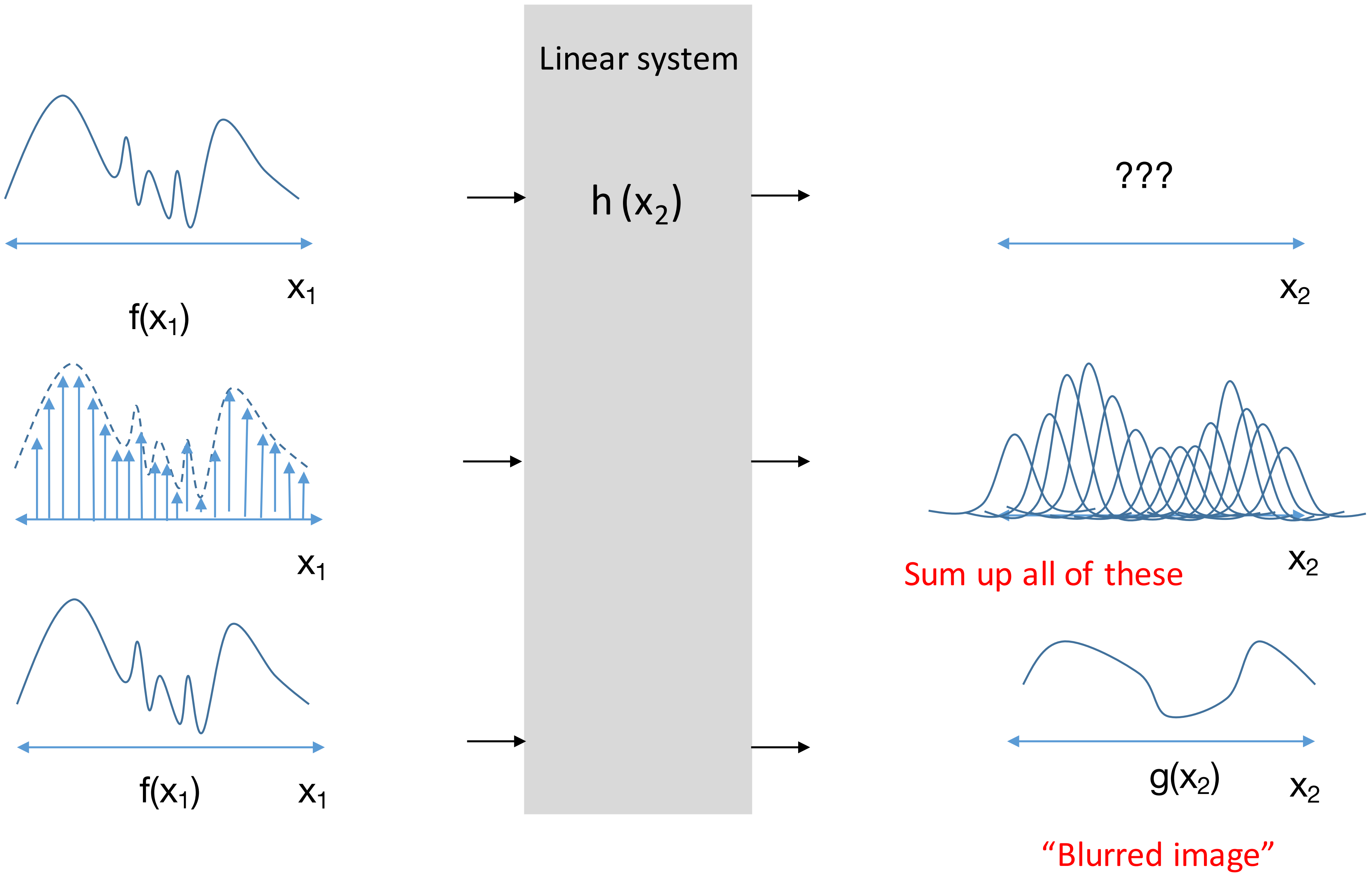

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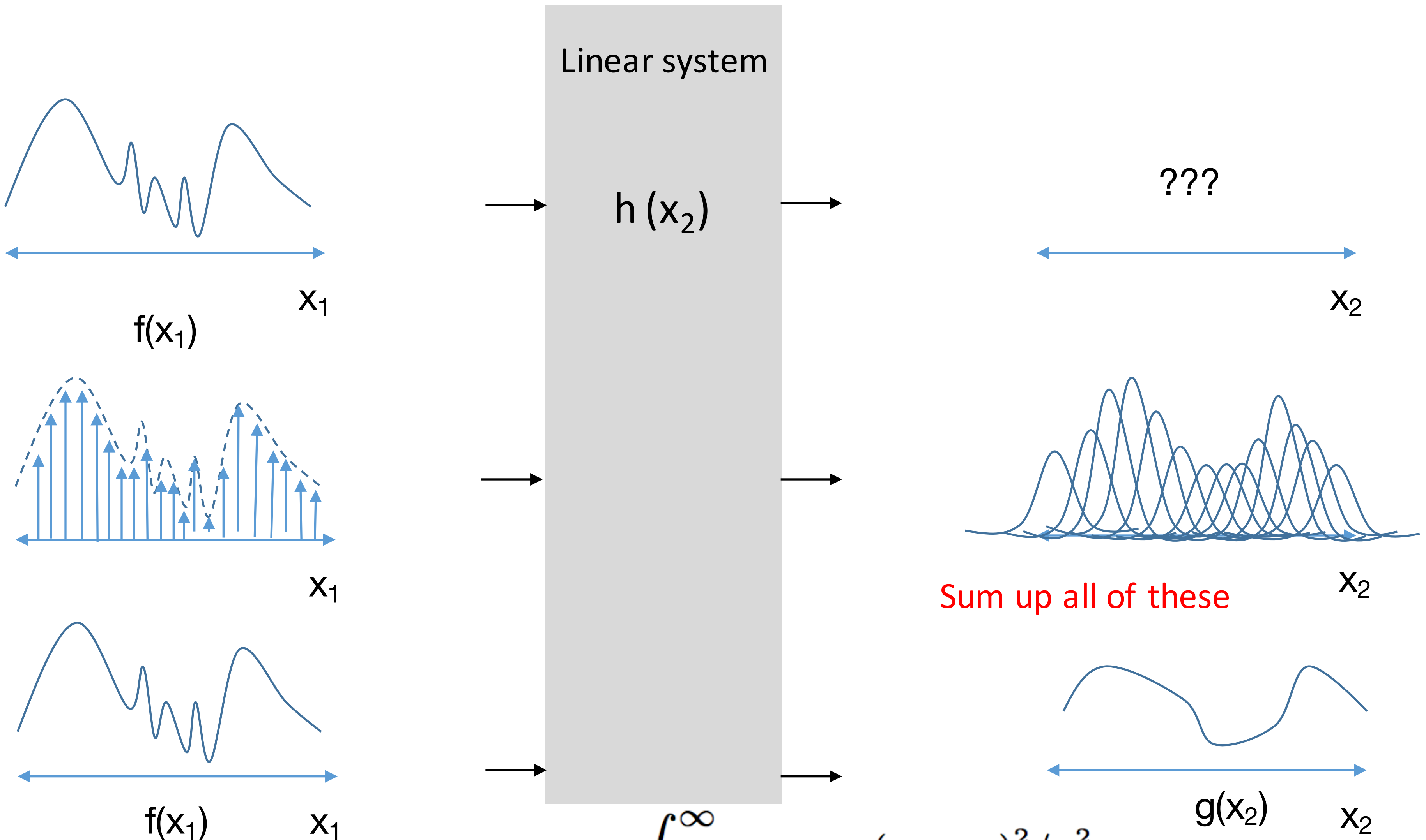
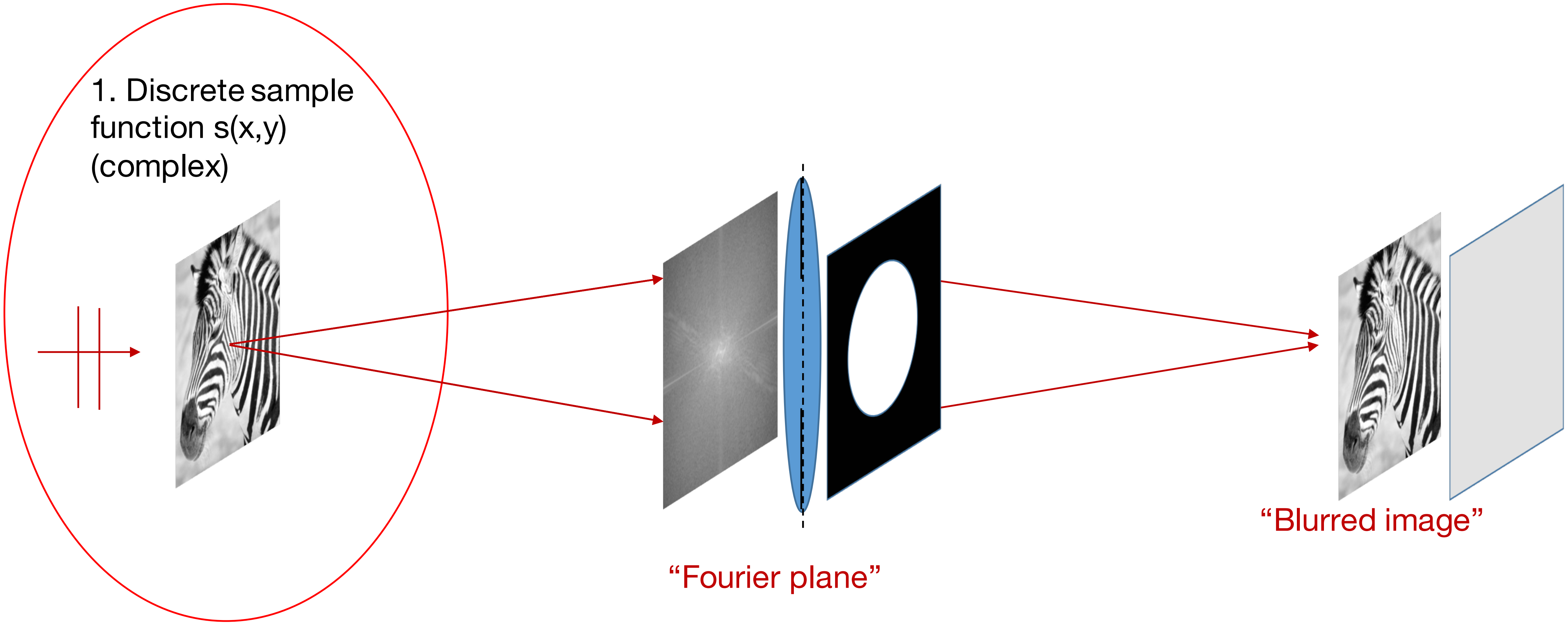


$$g_2(x_2) = \int_{-\infty}^{\infty} f(x_1)e^{(x_2-x_1)^2/\sigma^2}dx_1$$

“Blurred image”

Model of image formation for wave optics (coherent light):



What's going on between the sample and the incident light?

Microscope illumination and sample index of refraction

So far: illuminate the sample and create a field that is equivalent to the sample's absorption and phase

– why is this true???

Microscope illumination and sample index of refraction

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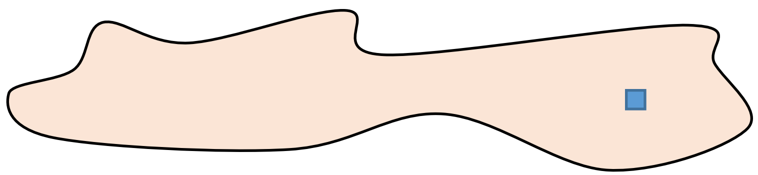
Sample index of refraction $n(x,y,z) = 1 + ia(x) + \varphi\ (x)$



$a(x=1) = .2$

$\varphi(x=1) = .1$

*For more information, see D. Paganin, Coherent X-Ray Optics, Section 2.2

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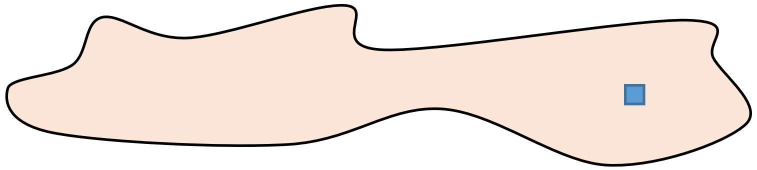

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Thin sample approximation:

Sample's effect on light is multiplication with $\exp[-ik * n(x,y)]$

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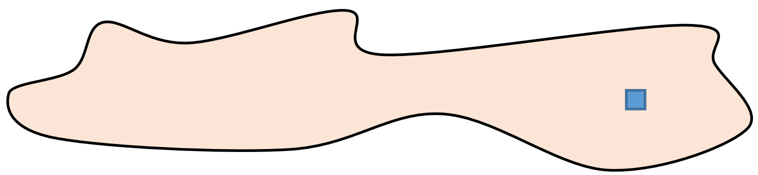

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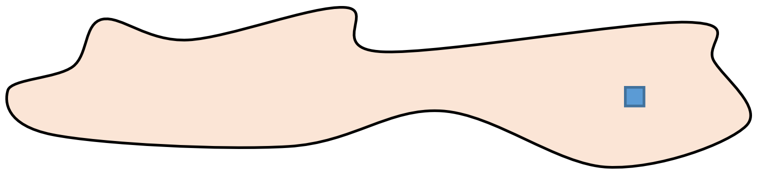

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In 1D: Emerging field $U(x)$ = incident field $U_i(x)$ * sample function $s(x)=\exp[-ik\, n(x)]$

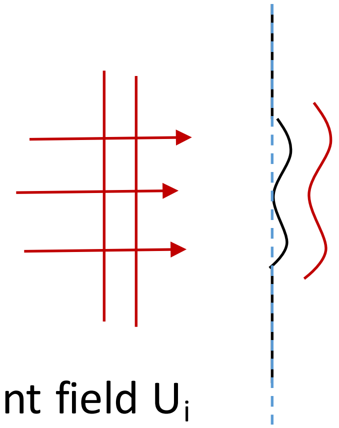
$$\boxed{U(x) = U_i(x) * \exp[-ik\, n(x)] = U_i(x)\, A(x)\, \exp[ik\varphi(x)]} \quad A(x) = \exp[k\, a(x)]$$

absorption

phase shift: new term for laser

Microscope illumination and sample index of refraction

Sample absorption = A(x)
Sample phase = exp[ikφ(x)]



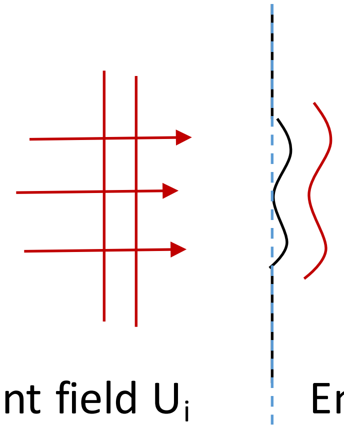
Emerging field U = incident field $U_i(x)$ * sample function s(x)

Incident field U_i

Microscope illumination and sample index of refraction

Q: When is the emerging field equal to the absorption and phase?

Sample absorption = $A(x)$
Sample phase = $\exp[ik\varphi(x)]$



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Incident field U_i Emerging field U

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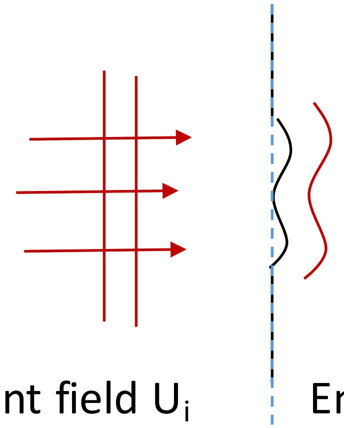
Sample phase = $\exp[ik\varphi(x)]$

A: When the incident wave = 1, means uniform in amplitude and phase:

$$U_i(x) = 1 \longrightarrow U(x)=A(x)\,\exp[ik\varphi(x)]$$



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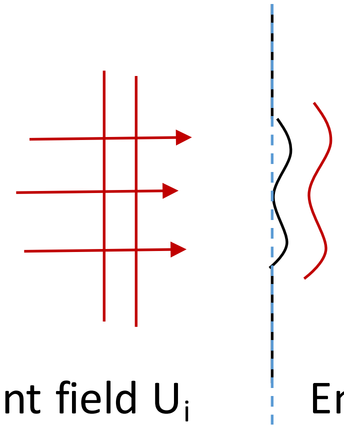


Incident field U_i

Emerging field U

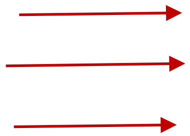
A: When the incident wave = 1, means uniform in amplitude and phase:

$$U_i(x) = 1 \longrightarrow U(x)=A(x)\ \exp[ik\varphi(x)]$$

Plane wave $U_i(x) = 1 * \exp(ik\bullet x)$

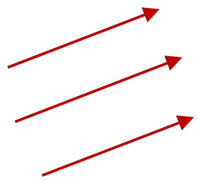
$$U_i(x) = \exp(ikx \sin(\theta))$$

$\theta=0$



1 everywhere

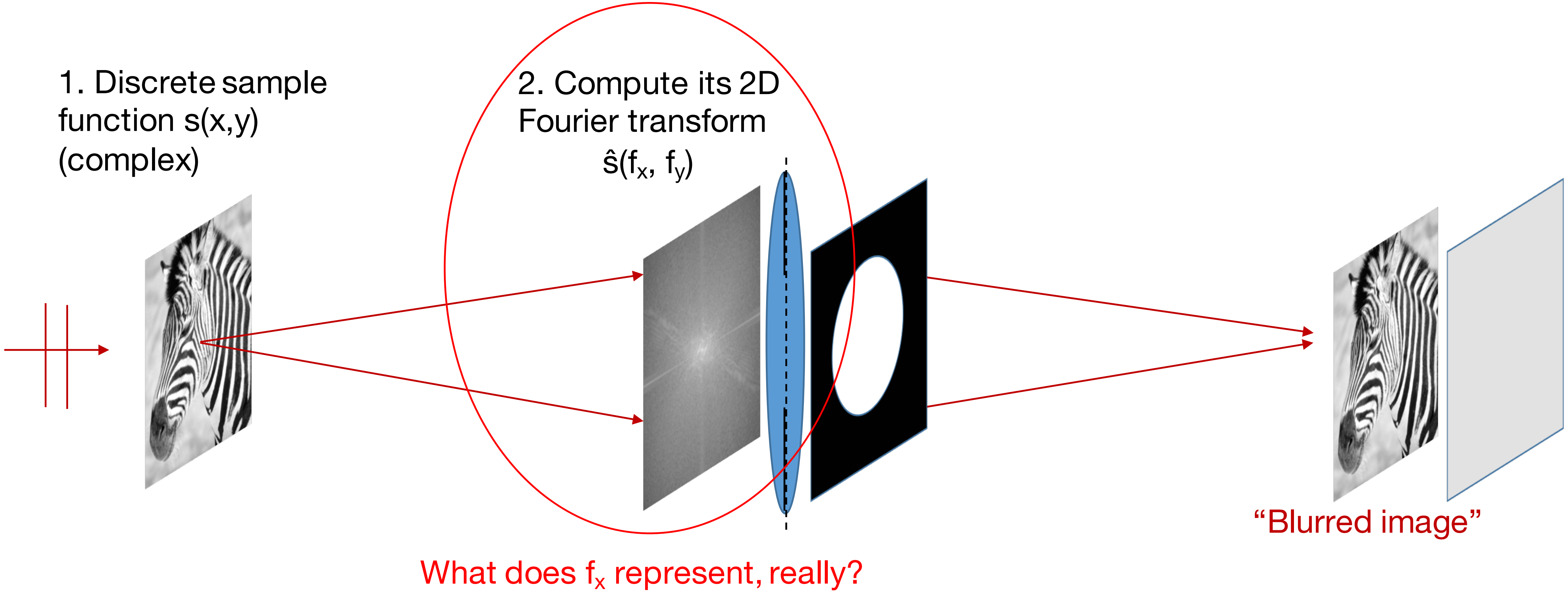
$\theta=30$



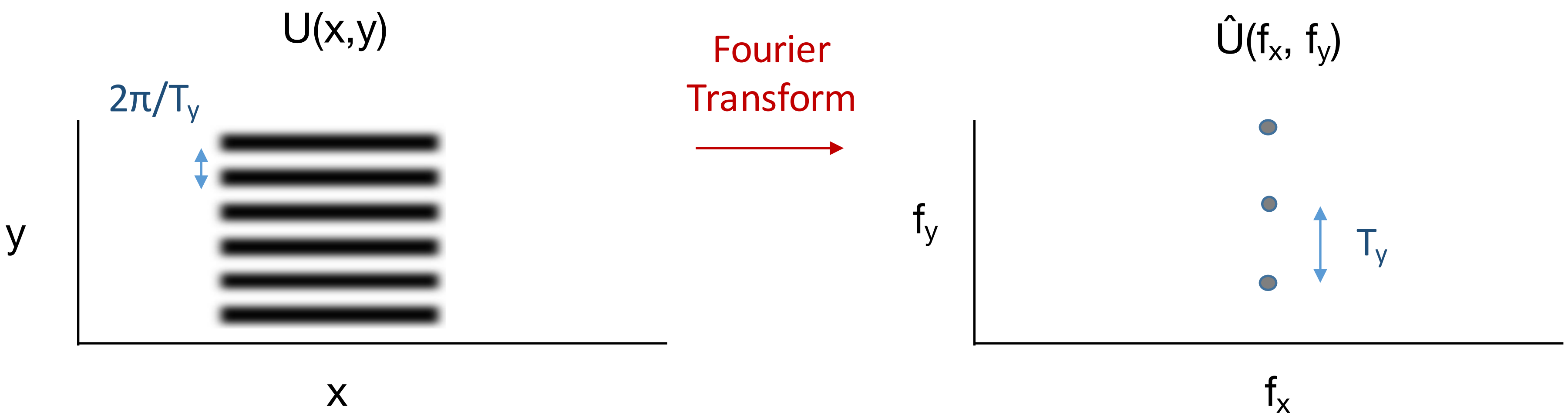
Not 1 everywhere

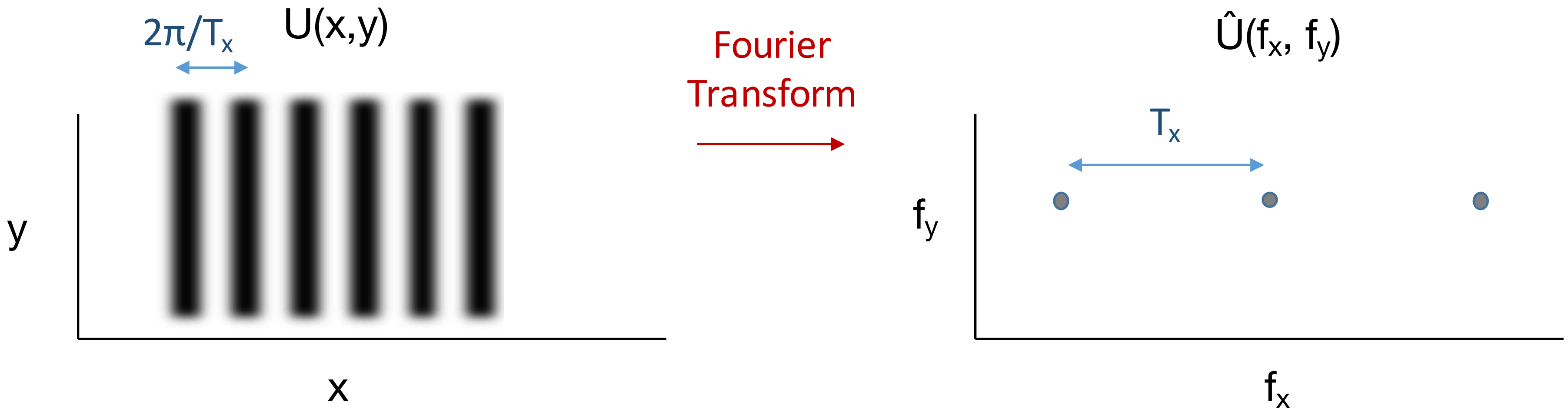
$\boxed{\text{This is when incident wave hits the sample with } \theta=0!}$

Model of image formation for wave optics (coherent light):

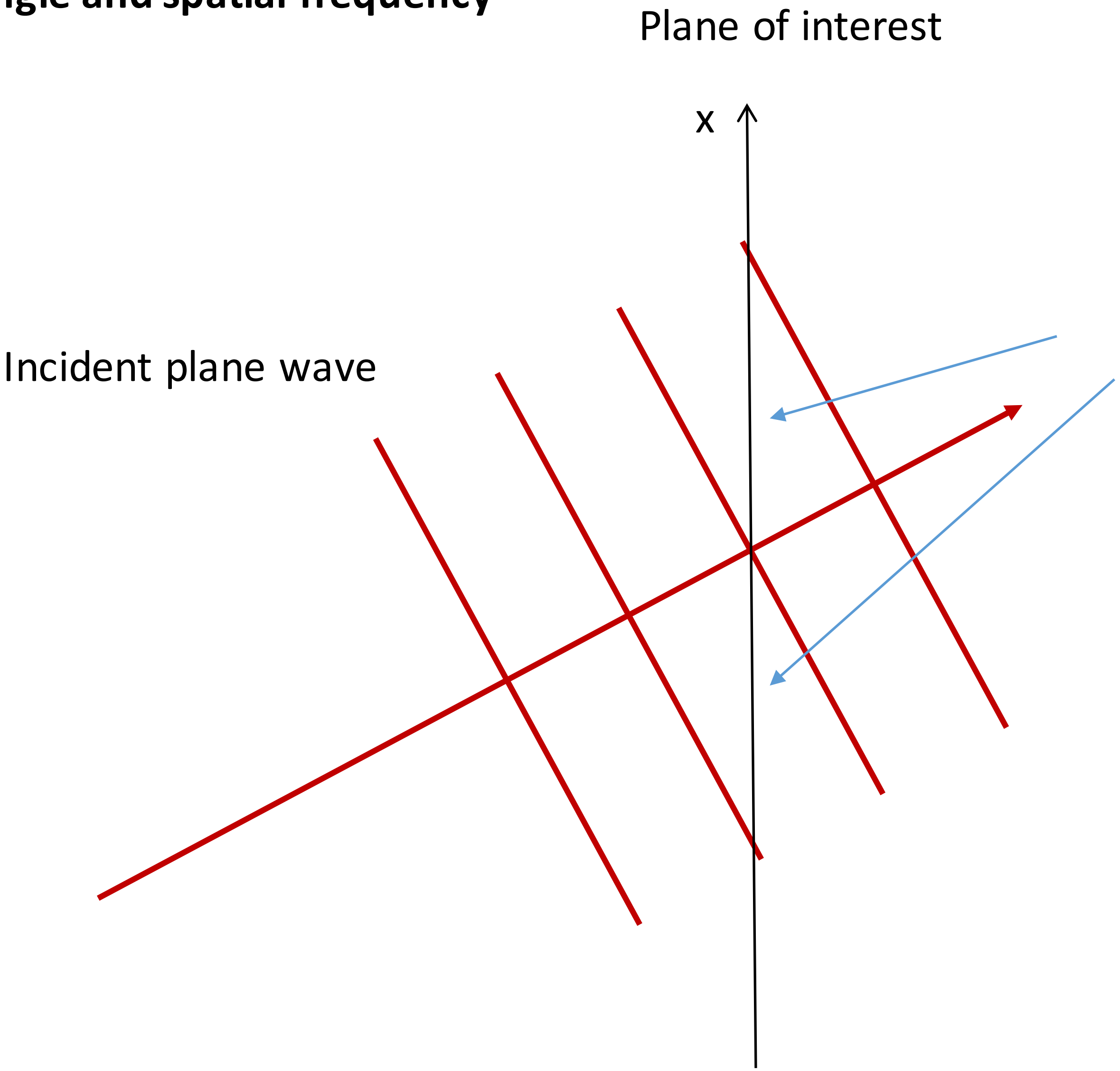


From before: Spatial frequencies = "stripes" within each image





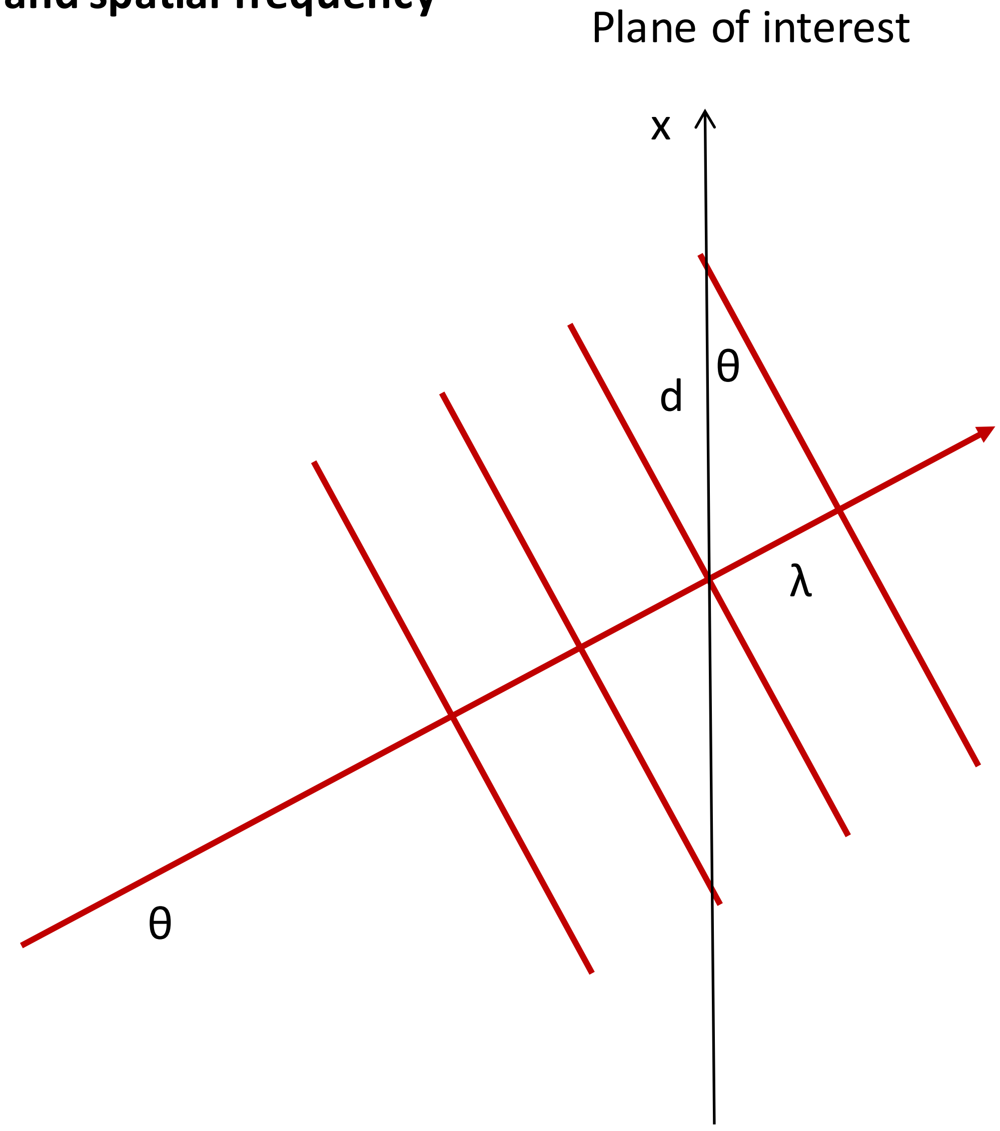
Ray angle and spatial frequency



Stripes are for complex fields!

Proportional to distance between subsequent peaks of wave along plane of interest

Ray angle and spatial frequency

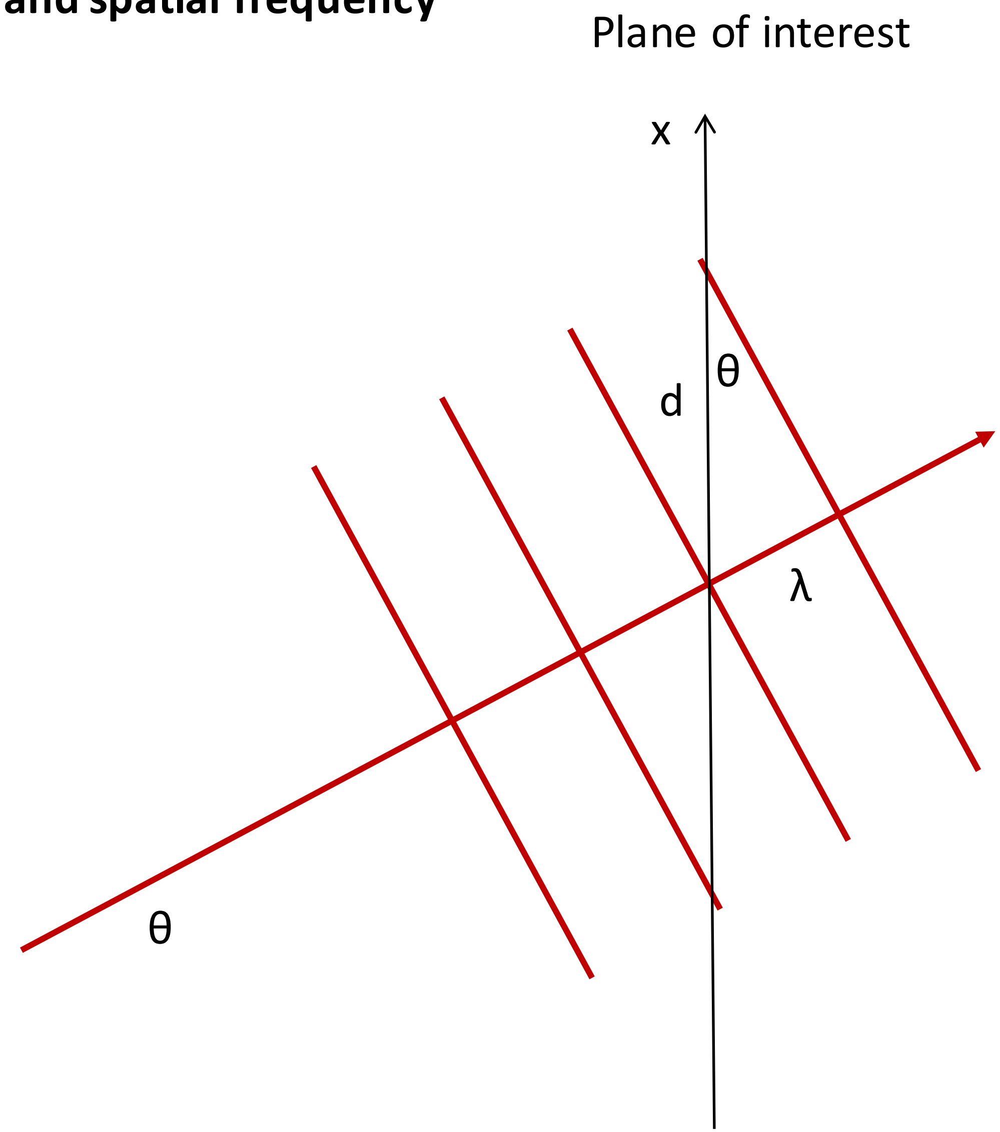


Distance to two crests = spatial period

$\sin(\theta) = \lambda/d$

$d = \lambda/\sin(\theta)$

Ray angle and spatial frequency



Distance to two crests = spatial period

$\mathrm{Sin}(\theta) = \lambda/d$

$d = \lambda/\ \mathrm{Sin}(\theta)$

Spatial frequency = 1/spatial period
(number of periods per unit length)

$f_x = 1/d = \mathrm{Sin}(\theta)/\ \lambda$

Equivalent coordinates in the Fourier domain and at the Fourier plane

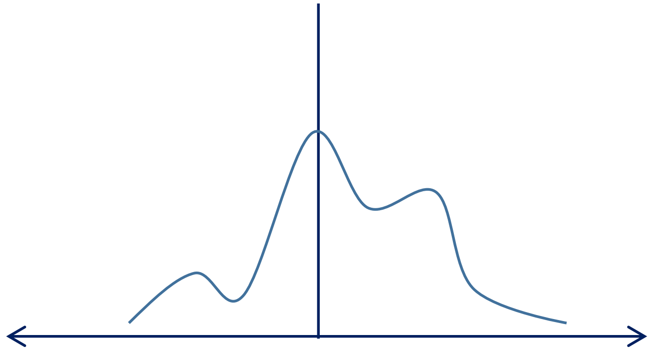


Image plane ray angle θ

$f_x = \text{Sin}(\theta)/\ \lambda$

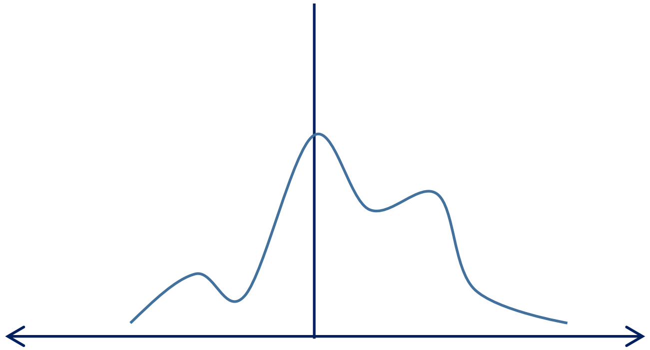


Image plane spatial frequency f_x

$k_x = 2\pi f_x = 2\pi/\ \lambda\ \text{Sin}(\theta)$

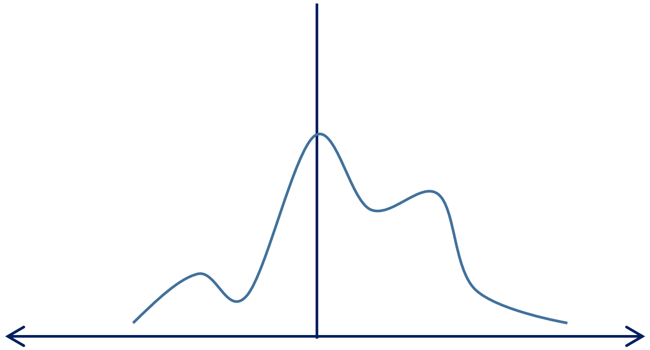
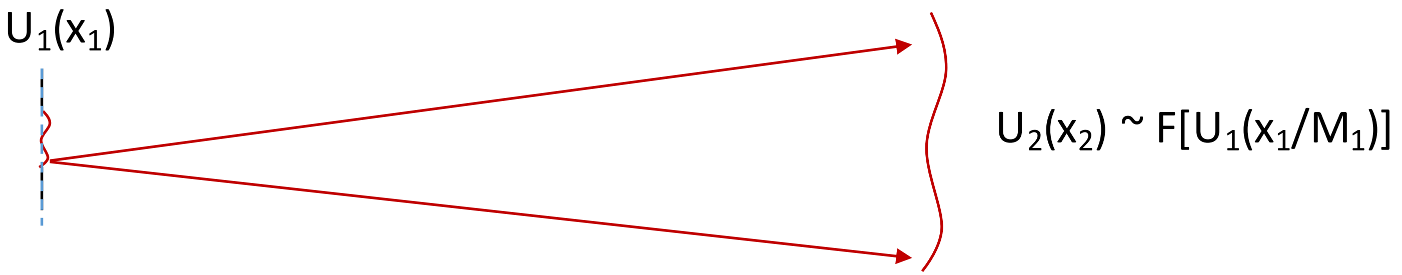


Image plane wavevectors k_x

General rules for applying the Fourier transform in optics

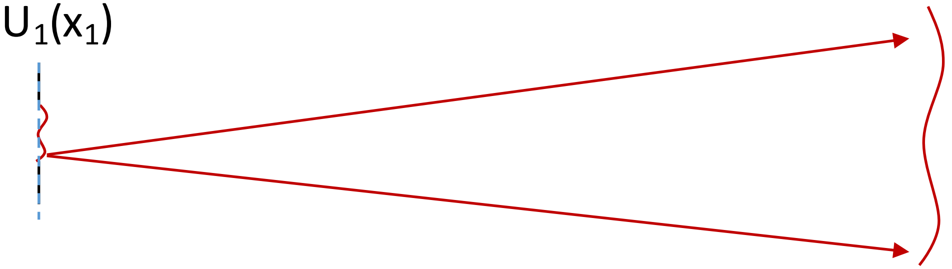
Situation 1: From an object to a plane "really far away"



General rules for applying the Fourier transform in optics

Situation 1: From an object to a plane "really far away"

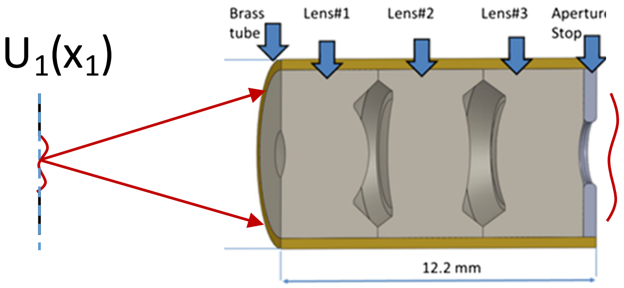
$U_1(x_1)$



$U_2(x_2) \sim F[U_1(x_1/M_1)]$

Situation 2: From an object to the back focal plane of the microscope objective lens

$U_1(x_1)$



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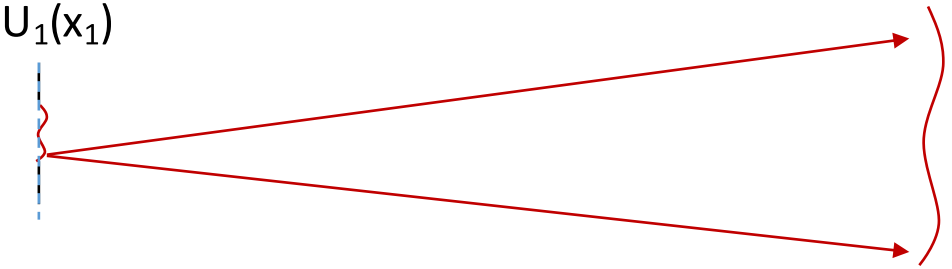

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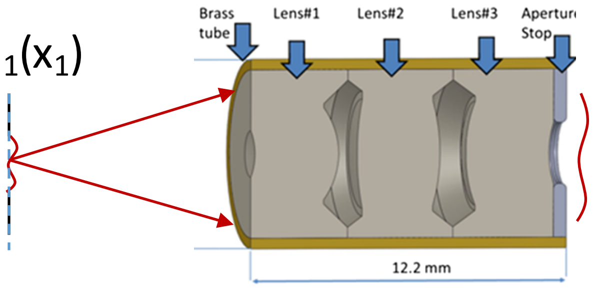
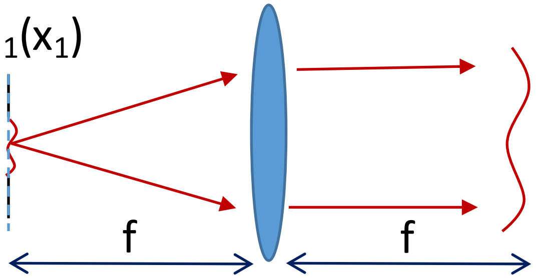


$U_2(x_2) \sim F[U_1(x_1/M_2)]$

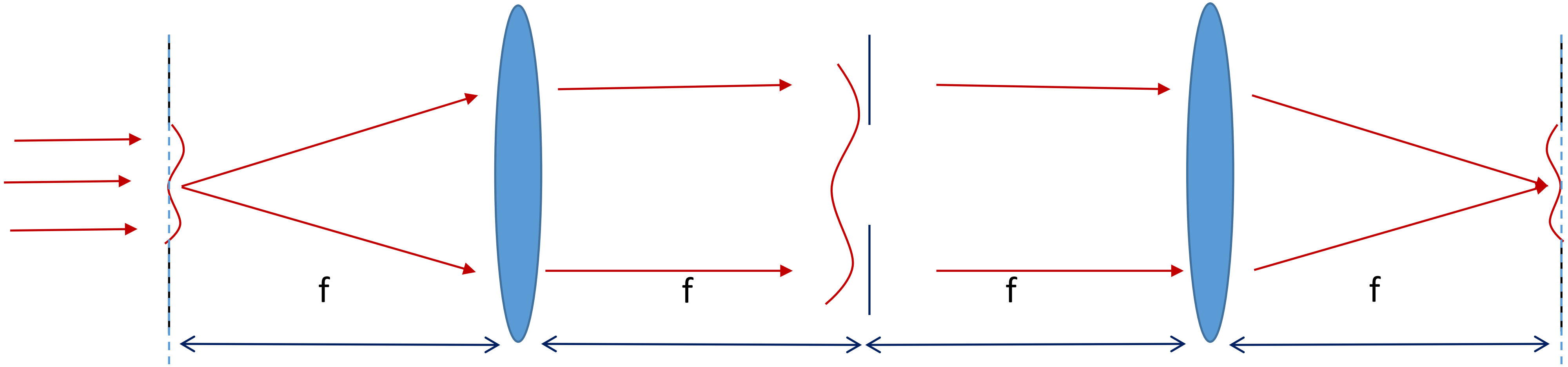
Situation 3: From an object to a plane 1 focal length away from a lens (1f-1f system)

$U_1(x_1)$



$U_2(x_2) \sim F[U_1(x_1/M_3)]$

A more exact model: the 4f optical system



$$E_s(x_s, y_s, 0)$$

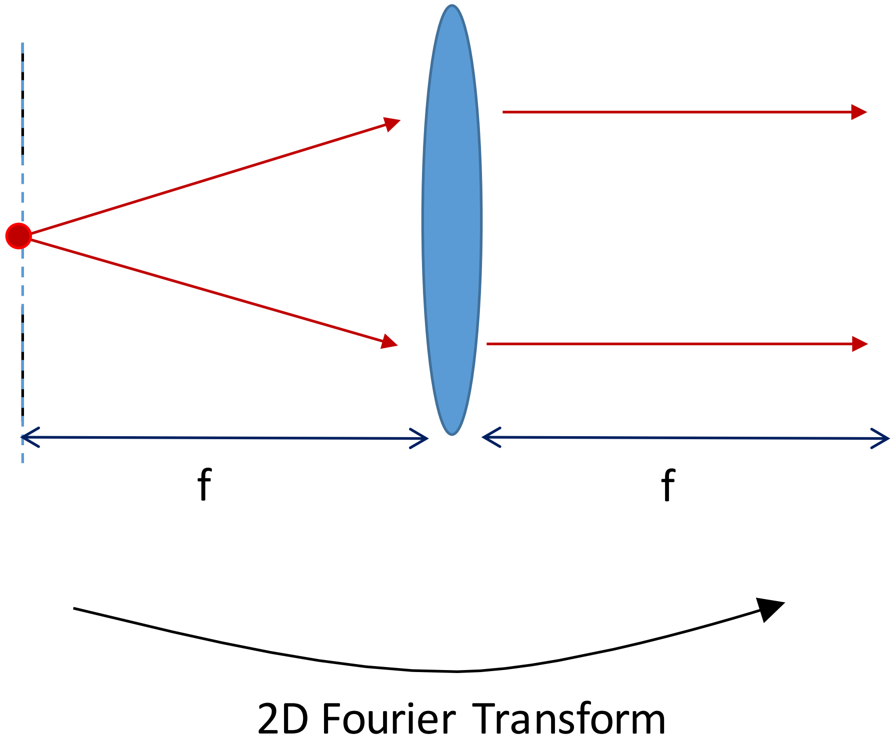
$$E'_a(x_d, y_d) = E_a(x_d, y_d)A(x_d, y_d)$$

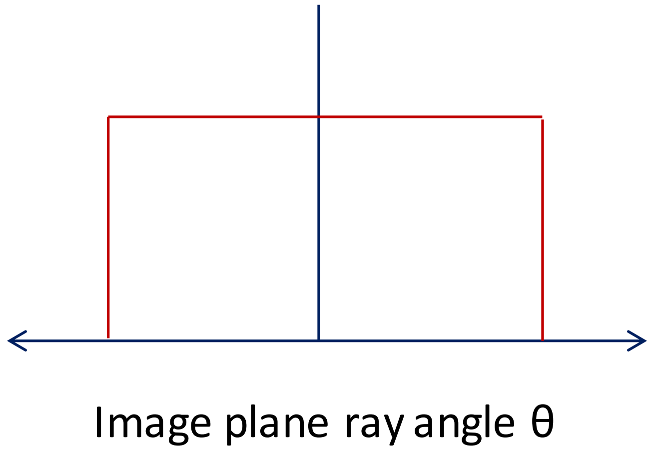
2D Fourier Transform

2D inverse Fourier Transform

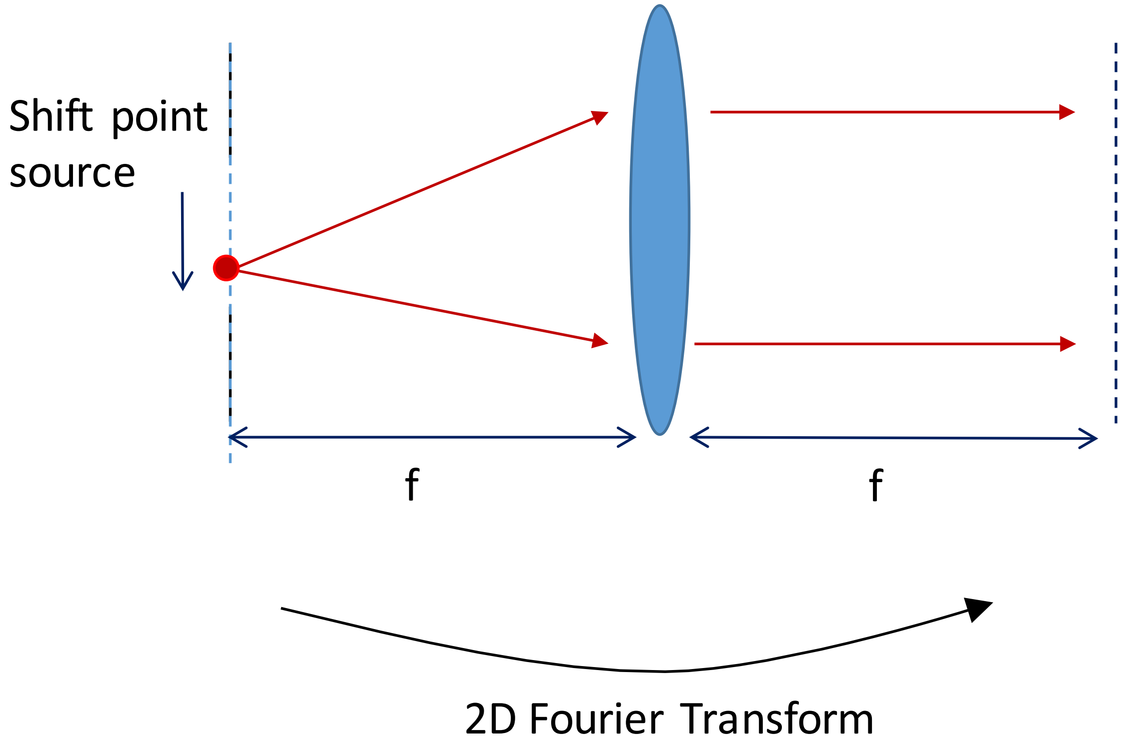
A more exact model: the 4f optical system

The Fourier plane provides a measure of the **ray angles at the image plane**



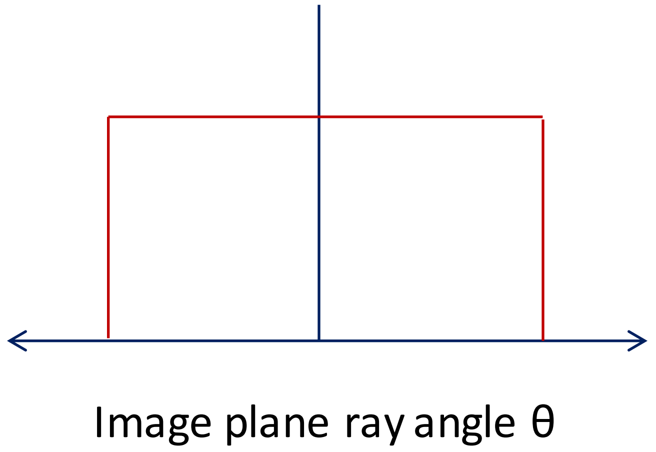


A more exact model: the 4f optical system



The Fourier plane provides a measure of the **ray angles at the image plane**

Doesn't contain info about spatial distribution light



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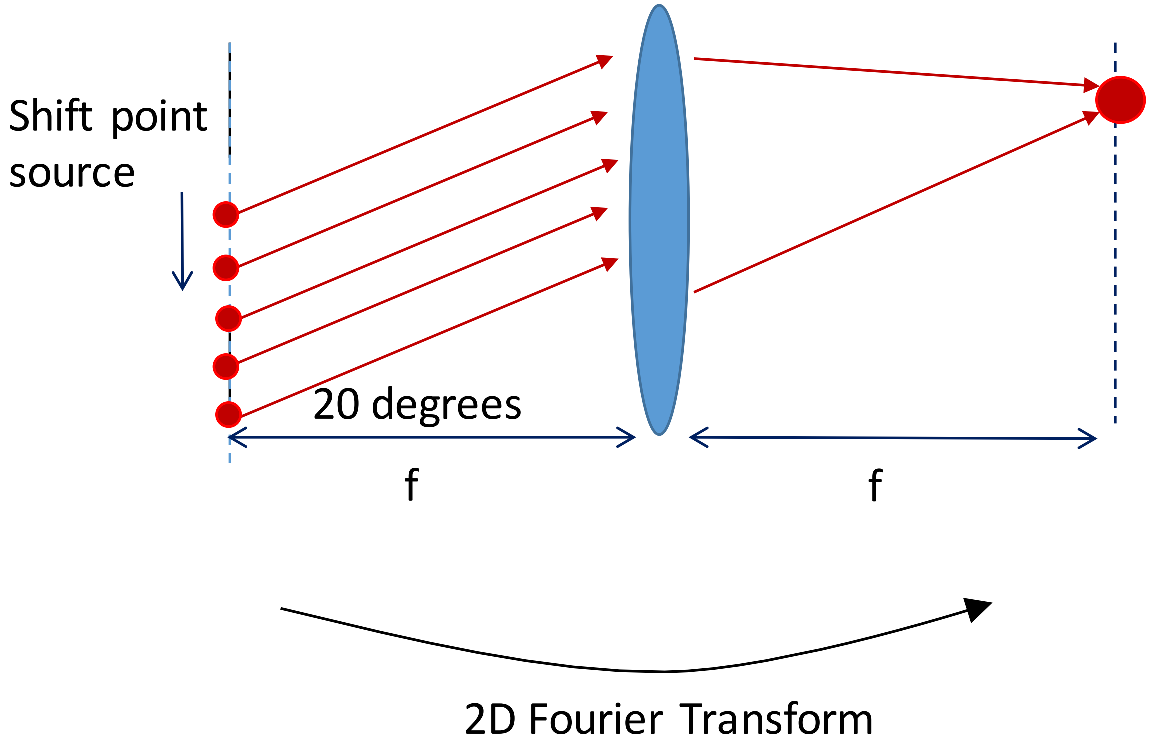
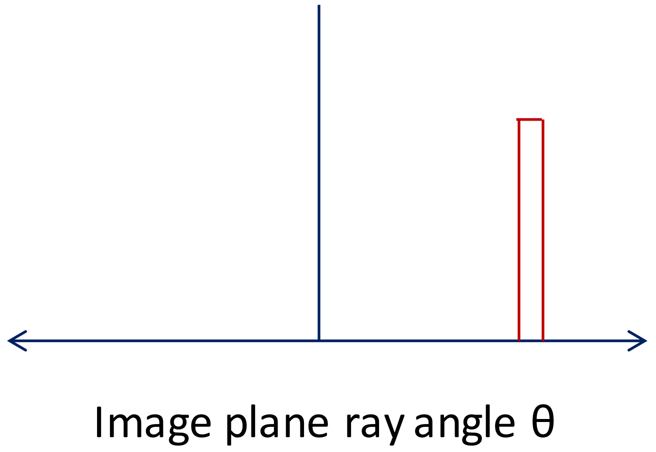
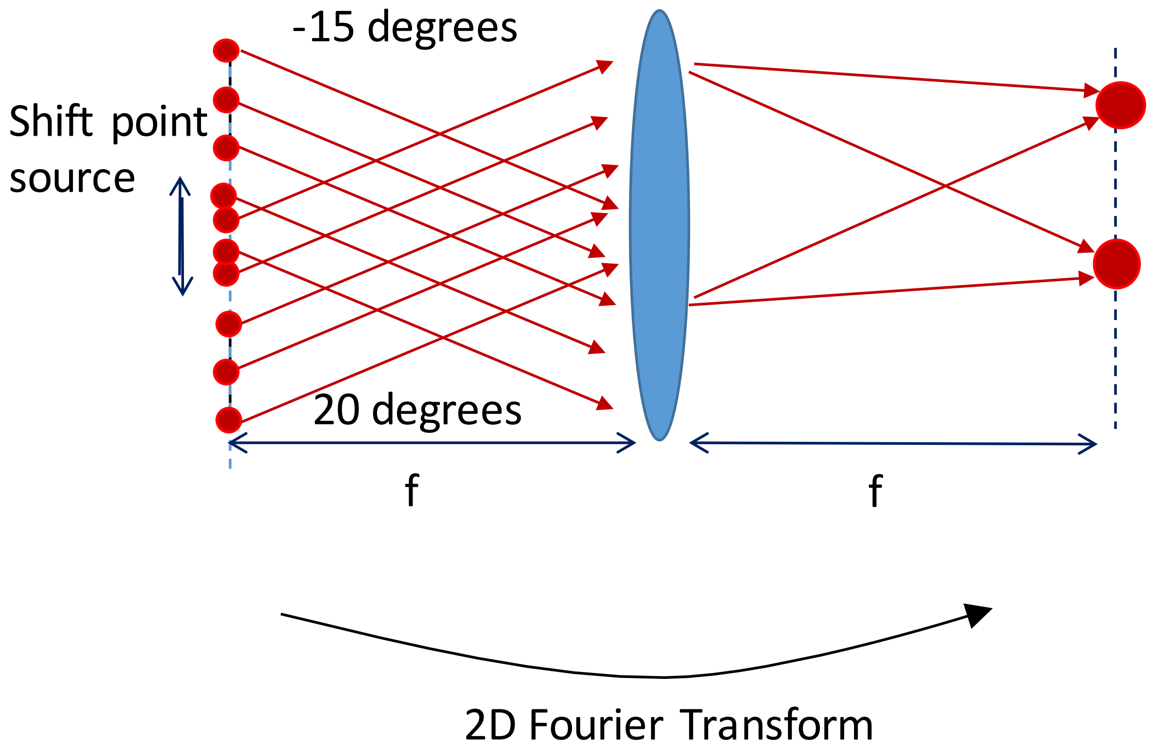


The Fourier plane provides a measure of the **ray angles at the image plane**

Rays are leaving image plane at +20 degrees

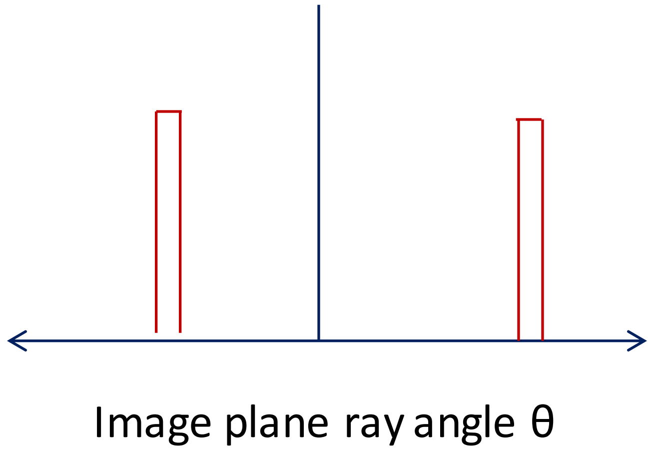


A more exact model: the 4f optical system



The Fourier plane provides a measure of the **ray angles at the image plane**

Rays are coming in at +20 degrees and -15 degrees



You typically go between 4 functions to describe one imaging system:

$$
\begin{tikzcd}
\boxed{\begin{array}{c}\text{Coherent point-}\\ \text{spread function}\\ \\ h(x)\end{array}} \arrow[r, "|.|^2"'] \arrow[d, "F[.]"] & \boxed{\begin{array}{c}\text{Incoherent point-}\\ \text{spread function}\\ \\ h_i(x)\end{array}} \arrow[d, "F[.]"] \\
\boxed{\begin{array}{c}\text{Coherent}\\ \text{transfer function}\\ \\ H(f_x)\end{array}} \arrow[r, "H*H"] & \boxed{\begin{array}{c}\text{Incoherent}\\ \text{transfer function}\\ \\ H_i(f_x)\end{array}}
\end{tikzcd}
$$