Theoretical Framework: Optimal Low-Depth Quantum Signal Processing Estimation (QSPE)

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1 Introduction to Quantum Metrology

Quantum metrology is the field of study that leverages quantum mechanical effects—such as entanglement, superposition, and squeezed states—to perform measurements of physical parameters with a precision that exceeds classical limits.

In classical estimation (the Standard Quantum Limit), the precision of a measurement typically scales as $\Delta\theta \sim 1/\sqrt{M}$, where M is the number of resources. Quantum metrology aims to achieve **Heisenberg Scaling**, where precision scales as $1/M$. Our goal in this algorithm is to characterize two-qubit gates by converting the parameter estimation problem into a signal processing task, allowing us to reach these fundamental limits with significantly lower circuit depths.

2 Algorithm Objectives

The primary goal of this algorithm is to characterize the parameters of a two-qubit unitary gate U (such as an $fSim$ gate) within a logical subspace. Our objectives are:

1. To achieve high-precision estimation of the swap angle θ and phase ϕ at low circuit depths.
2. To saturate the **Quantum Cramer-Rao Bound (QCRB)**, which is the absolute lower bound on the variance of any unbiased estimator.
3. To experimentally verify the transition from **Pre-asymptotic scaling** ($1/d^4$ variance) to **Heisenberg scaling** ($1/d^2$ variance).

3 The General Form of the Unitary U

According to the paper (Dong et al.), the unknown gate U acting within the single-excitation subspace $\mathcal{H}_1 = \{|01\rangle, |10\rangle\}$ is parameterized as:

$$U(\theta,\phi) = e^{i\phi}|0\rangle_L \langle 0|_L + e^{-i\phi}(\cos\theta\,|1\rangle_L\langle 1|_L - i\sin\theta\,|1\rangle_L\langle 0|_L) \tag{1}$$

In the matrix representation within the logical basis, this is:

$$U = \begin{pmatrix} e^{i\phi} & 0 \\ -ie^{-i\phi}\sin\theta & e^{-i\phi}\cos\theta \end{pmatrix} \tag{2}$$

Where:

- θ **(Swap Angle):** The interaction strength between qubits; it dictates the rate of excitation exchange.
- ϕ **(Phase Drift):** The angle of the rotation axis on the logical Bloch sphere, representing unwanted detuning or phase accumulations.

4 Derivation of the Transition Probability

The core of QSPE is to convert the unitary U into a measurable probability signal $p(\omega)$. The steps are as follows:

4.1 Step 1: The Iterative Operator

We define the single-block operator $G(\omega)$ as the combination of the unknown gate and the tunable phase knob $R_z(\omega) = e^{-i\omega Z_L/2}$:

$$G(\omega) = R_z(\omega)U(\theta, \phi) = \begin{pmatrix} e^{-i\omega/2} & 0 \\ 0 & e^{i\omega/2} \end{pmatrix} \begin{pmatrix} e^{i\phi} & 0 \\ -ie^{-i\phi}\sin\theta & e^{-i\phi}\cos\theta \end{pmatrix} \quad (3)$$

4.2 Step 2: State Evolution

Applying this block d times results in the total unitary $V_d(\omega) = [G(\omega)]^d$. We prepare the initial logical state $|+\rangle_L = \frac{1}{\sqrt{2}}(|0\rangle_L + |1\rangle_L)$. The state after evolution is:

$$|\psi_d\rangle = V_d(\omega)\,|+\rangle_L \quad (4)$$

4.3 Step 3: Projection and Probability

To find $p_x(\omega)$, we measure the projection back onto the $|+\rangle_L$ basis:

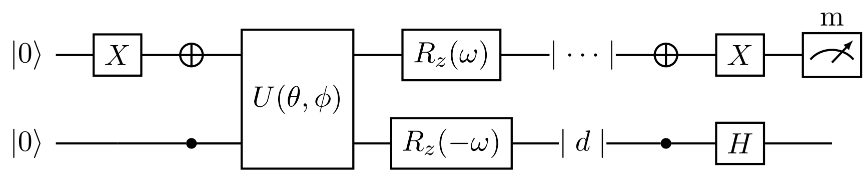
$$p_x(\omega) = |\,\langle +|_L\, V_d(\omega)\,|+\rangle_L\,|^2 = \frac{1}{2} + \frac{1}{2}\mathrm{Re}[\langle 0|_L\, V_d(\omega)\,|1\rangle_L + \langle 1|_L\, V_d(\omega)\,|0\rangle_L] \quad (5)$$

For small θ, the result approximates a Fourier series:

$$p(\omega) \approx \frac{1}{2} + \frac{1}{2}\sum_{k=1}^{d} c_k \cos(k\omega + \Phi) \quad (6)$$

5 Quantum Circuit Implementation

The implementation maps the logical operations to physical qubits:



5.1 Subspace Mapping Steps

1. **Preparation (P):** $|00\rangle \xrightarrow{X_0} |10\rangle \xrightarrow{H_1} \frac{|10\rangle+|11\rangle}{\sqrt{2}} \xrightarrow{CNOT_{1,0}} \frac{|01\rangle+|10\rangle}{\sqrt{2}}$.
2. **Inverse Prep ($P^\dagger$):** Maps the phase-accumulated state back to the computational basis for readout on a single qubit.

6 Fourier-Domain Processing and Estimation

6.1 Construction of the Complex Signal b_j

We collect data from two experiments, p_x (starting at $|+\rangle_L$) and p_y (starting at $|i\rangle_L$), to form the complex vector:

$$b_j = \left(p_x(\omega_j) - \frac{1}{2}\right) + i\left(p_y(\omega_j) - \frac{1}{2}\right) \tag{7}$$

6.2 Discrete Fourier Transform (DFT)

The Fourier coefficients c_k are obtained via:

$$c_k = \frac{1}{N} \sum_{j=0}^{N-1} b_j e^{-ik\omega_j} \tag{8}$$

As per the paper's Algorithm 1, the parameter θ is extracted by averaging the magnitudes of the relevant coefficients:

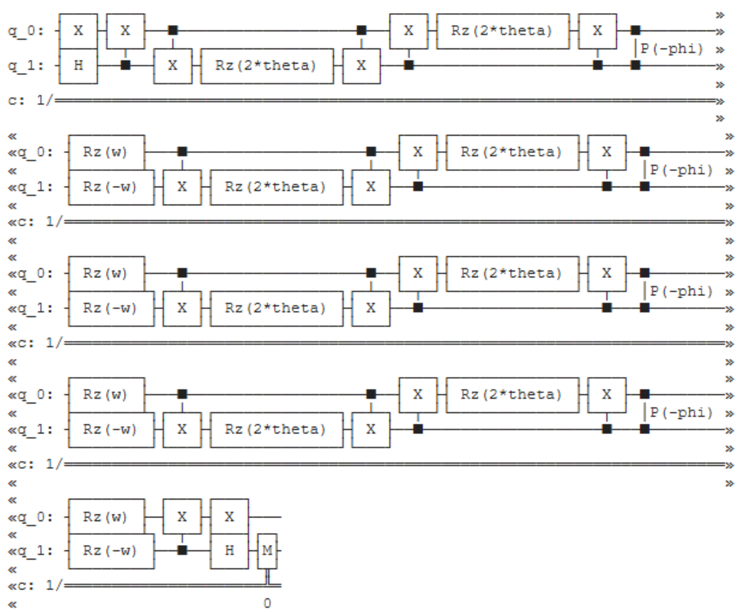
$$\hat{\theta} = \frac{1}{d} \sum_{k=0}^{d-1} |c_k| \tag{9}$$

7 Fundamental Bounds: HL vs Pre-asymptotic

- **Heisenberg Limit (HL):** Precision scales as $\Delta\theta \sim 1/d$. In variance terms, $\mathrm{Var}(\hat{\theta}) \sim 1/d^2$.
- **Pre-asymptotic Scaling:** For very low depths ($d\theta \ll 1$), the paper proves that $\mathrm{Var}(\hat{\theta}) \sim 1/d^4$, offering a massive reduction in error for short circuits.

8 Phase 1: Environment and Gate Setup

Implemented circuit



Input Parameters and Gate Definition

```
d_val = 3
num_pts = 40
q0, q1 = cirq.LineQubit.range(2)
theta_true = 0.15
u_gate = cirq.FSimGate(theta=theta_true, phi=0.1)
```

Line-by-Line Explanation:

- `d_val = 3`: Defines the circuit depth. This scales the phase accumulation linearly ($d\theta$), making the signal more sensitive to the parameter.
- `num_pts = 40`: Sets the resolution of our ω sweep. More points lead to a cleaner Fourier spectrum.
- `u_gate`: Initializes the $fSim$ gate. The goal of the code is to "rediscover" the 0.15 value through measurement.

9 Phase 2: Subspace State Preparation

Preparation of the Logical $|+\rangle_L$ State

```
circuit.append([cirq.X(q0), cirq.H(q1), cirq.CNOT(q1
    , q0)])
if basis == 'Y':
    circuit.append(cirq.S(q1))
```

Line-by-Line Explanation:

- `X(q0)`: Excites the first qubit to the $|1\rangle$ state.
- `H(q1) + CNOT`: Creates an entangled Bell state $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. In QSP, this is our logical zero/one superposition.
- `S(q1)`: For the Y-basis experiment, this adds a complex i phase, allowing us to measure the imaginary component of the phase rotation.

10 Phase 3: The Amplification Loop

Coherent Phase Accumulation

```
for _ in range(d):
    circuit.append(u_gate(q0, q1))
    circuit.append([cirq.rz(w)(q0), cirq.rz(-w)(q1)
        ])
```

Line-by-Line Explanation:

- `for _ in range(d)`: This loop repeats the interaction. Higher d narrows the "peaks" in our estimation, increasing precision.

- `u_gate(q0, q1)`: Applies the unknown swap rotation.
- `rz(w)(q0), rz(-w)(q1)`: This is the "Phase Knob." By rotating qubits in opposite directions, we apply a purely logical Z-rotation that interferes with the gate's swap angle.

11 Phase 4: Numerical Extraction (FFT)

Fourier Analysis of the Probabilities

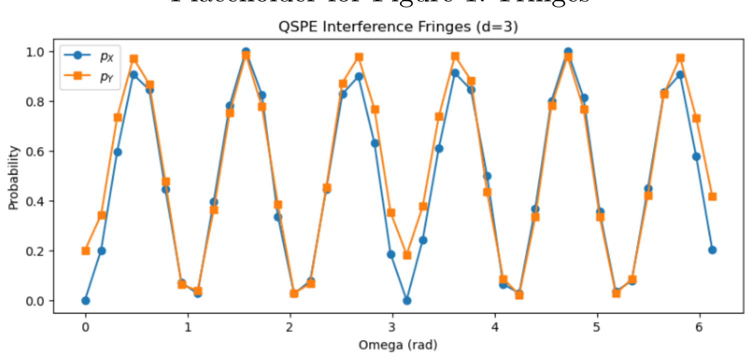
```
b = (px - 0.5) + 1j * (py - 0.5)
ck = np.fft.fft(b) / num_pts
theta_hat = np.abs(ck[d]) * 2
```

Line-by-Line Explanation:

- `b = (px - 0.5) + ...`: Combines two real experiments into one complex signal. Subtracting 0.5 centers the sine wave, ensuring the FFT doesn't have a massive peak at frequency zero.
- `np.fft.fft(b)`: Converts the time-domain (omega-domain) signal into the frequency domain.
- `theta_hat = np.abs(ck[d]) * 2`: We select the coefficient at index `d`. Because we amplified the signal d times, the information lives exactly at the d-th harmonic of our sweep.

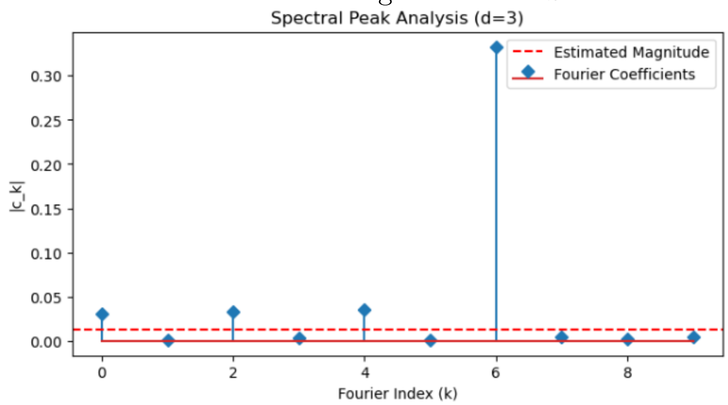
12 Visualizing the Results

Placeholder for Figure 1: Fringes



The graph above shows the sinusoidal fringes. P_x and P_y are offset, representing the real and imaginary parts of the state rotation.

Placeholder for Figure 2: FFT Stem



The stem plot shows a clear spike at index $k = 6 = 2 * 3$. This confirms the signal frequency matches our depth $d = 3$.

13 References

- Dong, Y., et al. (2024). *Optimal low-depth quantum signal-processing phase estimation.* Nature Communications.