CS 97SI: INTRODUCTION TO PROGRAMMING CONTESTS

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Last Lecture on Graph Algorithms

- Network Flow Problems
  - Maximum Flow
  - Minimum Cut
- Ford-Fulkerson Algorithm
- Application: Bipartite Matching
- Min-cost Max-flow Algorithm

Network Flow Problems

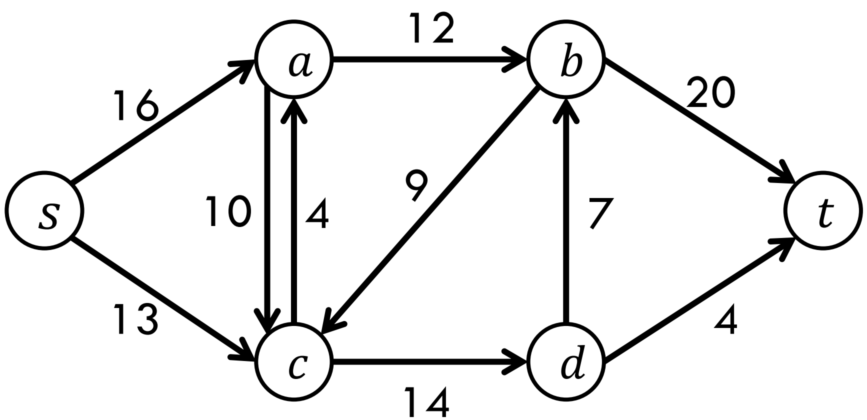
- A type of network optimization problem
- Arise in many different contexts (CS 261):
  - Networks: routing as many packets as possible on a given network
  - Transportation: sending as many trucks as possible, where roads have limits on the number of trucks per unit time
  - Bridges: destroying (?!) some bridges to disconnect s from t, while minimizing the cost of destroying the bridges

Network Flow Problems

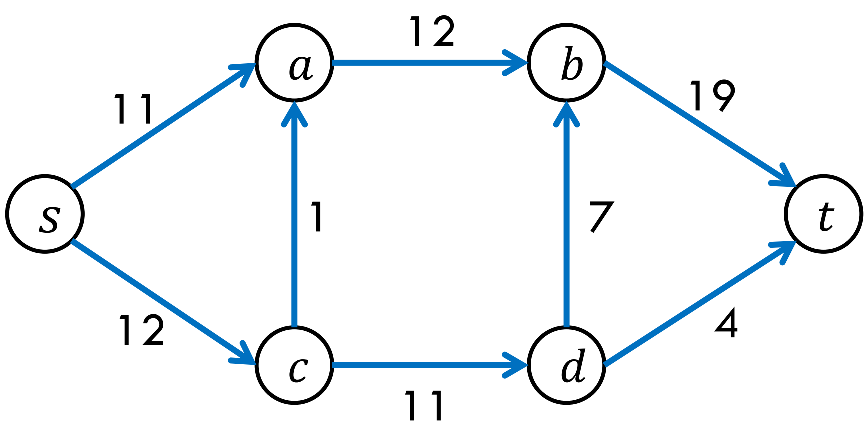
- Settings: Given a directed graph $G = (V, E)$, where each edge e is associated with its capacity $c(e) > 0$. Two special nodes *source* s and *sink* t are given ($s \neq t$)
- Problem: Maximize the total amount of *flow* from s to t subject to two constraints
  - Flow on edge e doesn't exceed $c(e)$
  - For every node $v \neq s, t$, incoming flow is equal to outgoing flow

Network Flow Example (from CLRS)

- Capacities



- Maximum Flow (of 23 units)



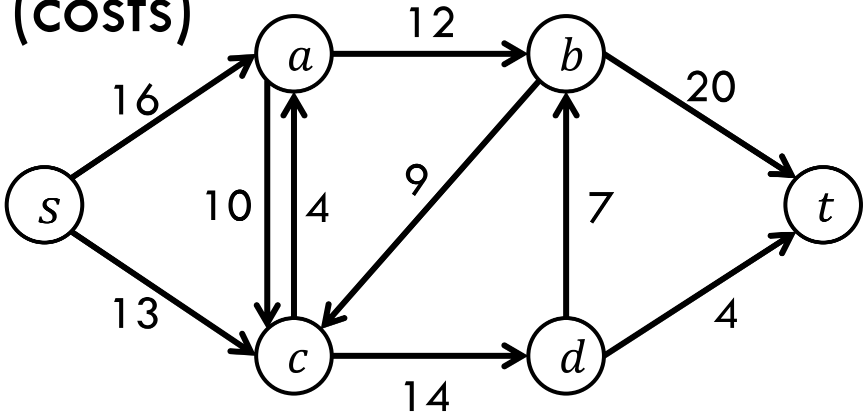
Alternate Formulation: Minimum Cut

- We want to remove some edges from the graph such that after removing the edges, there is no path from s to t
- The cost of removing e is equal to its capacity $c(e)$
- The minimum cut problem is to find a cut with minimum total cost

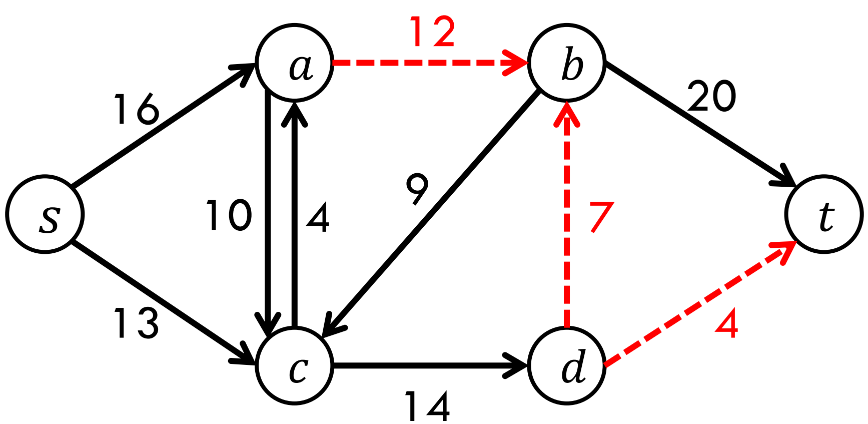
- Theorem: $(\text{maximum flow}) = (\text{minimum cut})$
  - Take CS 261 if you want to see the proof ☺

Minimum Cut Example

- Capacities (costs)

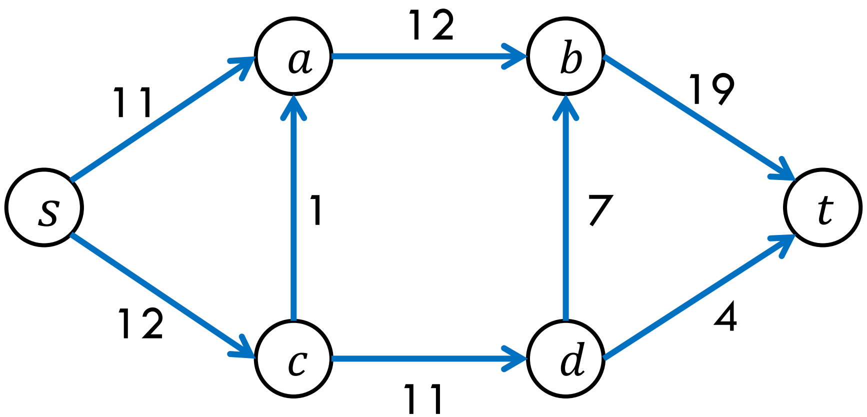


- Minimum Cut (red edges are removed)



Flow Decomposition

- Any valid flow can be decomposed into flow paths and circulations



  - $s \rightarrow a \rightarrow b \rightarrow t$: 11
  - $s \rightarrow c \rightarrow a \rightarrow b \rightarrow t$: 1
  - $s \rightarrow c \rightarrow d \rightarrow b \rightarrow t$: 7
  - $s \rightarrow c \rightarrow d \rightarrow t$: 4

Ford-Fulkerson Algorithm

- A simple and practical max-flow algorithm
- Main idea: find valid flow paths until there is none left, and add them up
- How do we know if this gives a maximum flow?
  - Proof sketch: Suppose not. Take a maximum flow $f^\star$ and subtract our flow f. It is a valid flow of positive total flow. By the flow decomposition, it can be decomposed into flow paths and circulations. These must have been found by Ford-Fulkerson. Contradiction.

Back Edges

- We don't need to maintain the amount of flow on each edge but work with capacity values directly
- If f amount of flow goes through $u \to v$, then:
  - Decrease $c(u \to v)$ by f
  - Increase $c(v \to u)$ by f
- Why do we need to do this?
  - Sending flow to both directions is equivalent to canceling flow

Ford-Fulkerson Pseudocode

- Set $f_{\text{total}} = 0$
- Repeat until there is no path from s to t:
  - Run DFS from s to find a flow path to t
  - Let f be the minimum capacity value on the path
  - Add f to f_{total}
  - For each edge $u \rightarrow v$ on the path:
    - Decrease $c(u \rightarrow v)$ by f
    - Increase $c(v \rightarrow u)$ by f

Analysis

- Assumption: capacities are integer-valued
- Finding a flow path takes $\Theta(n+m)$ time
- We send at least 1 unit of flow through the path
- If the max-flow is $f^\star$, the time complexity is $O\big((n+m)f^\star\big)$
  - "Bad" in that it depends on the output of the algorithm
  - Nonetheless, easy to code and works well in practice

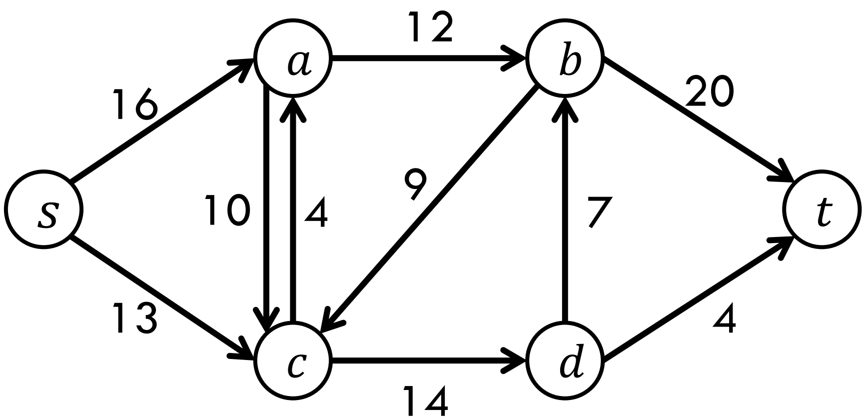
Computing the Min-Cut

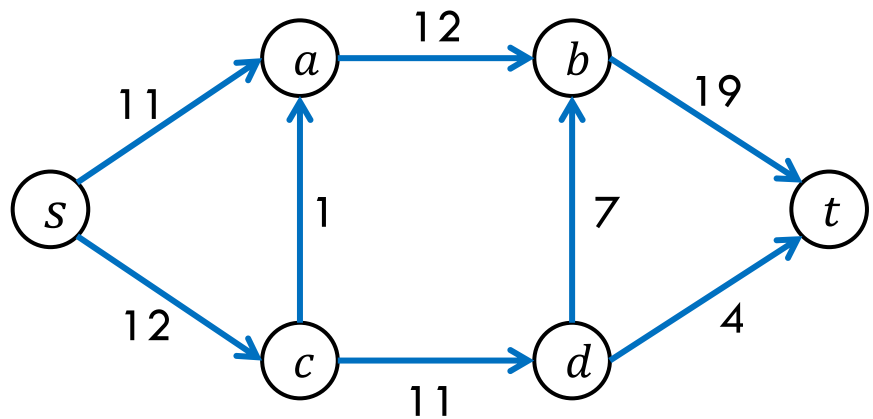
- We know that max-flow is equal to min-cut
- And we now know how to find the max-flow

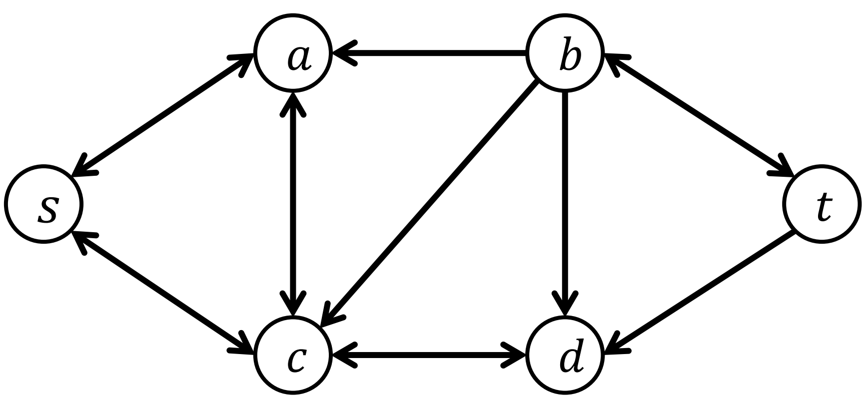
- Question: how do we find the min-cut?
- Answer: use the *residual graph*

Computing the Min-Cut

- "Subtract" the max-flow from the original graph



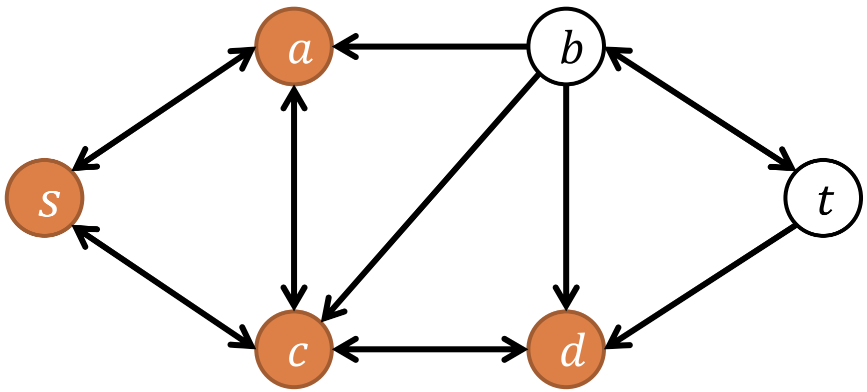




Only the topology of the residual graph is shown.
Don't forget to add the back edges!

Computing the Min-Cut

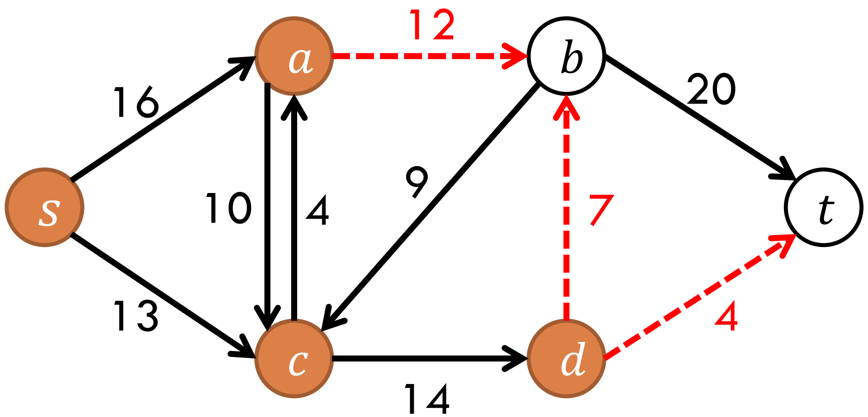
- Mark all nodes reachable from s
  - Call the set of reachable nodes A



- Now separate these nodes from the others
  - Edges go from A to $V - A$ are cut

Computing the Min-Cut

- Look at the original graph and find the cut:



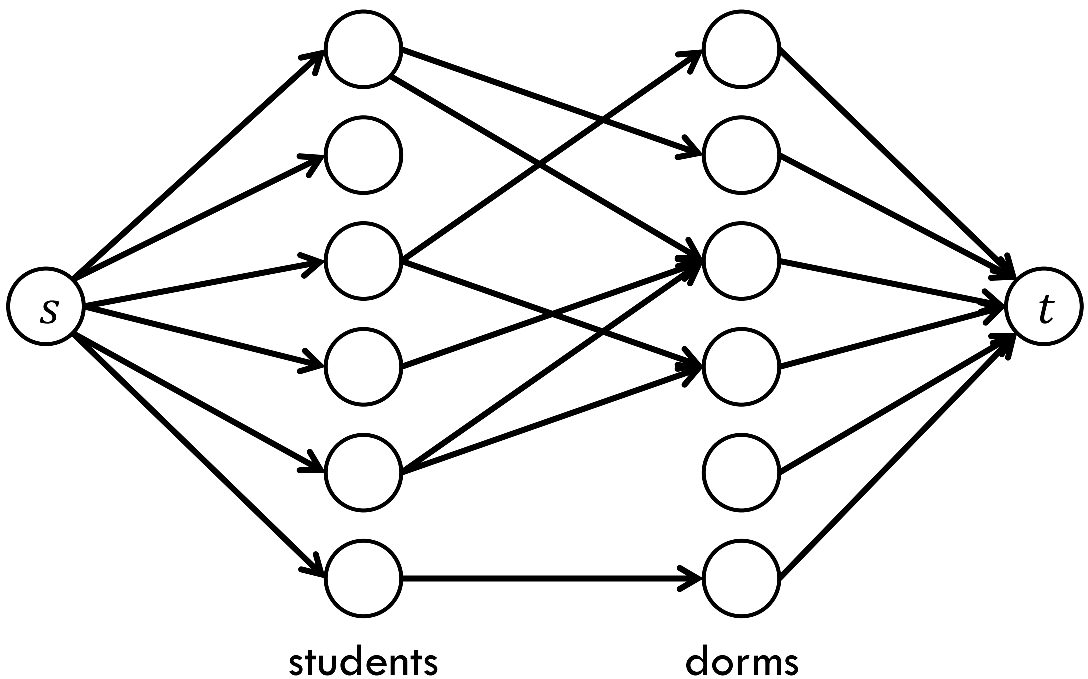
- Why isn't $b \to c$ cut?

Bipartite Matching

- Settings:
  - n students and d dorms
  - Each student wants to live in one of the dorms of his choice
  - Each dorm can accommodate at most one student (?!)
    - Fine, we will fix this later…
- Problem: find an assignment that maximizes the number of students who get a housing

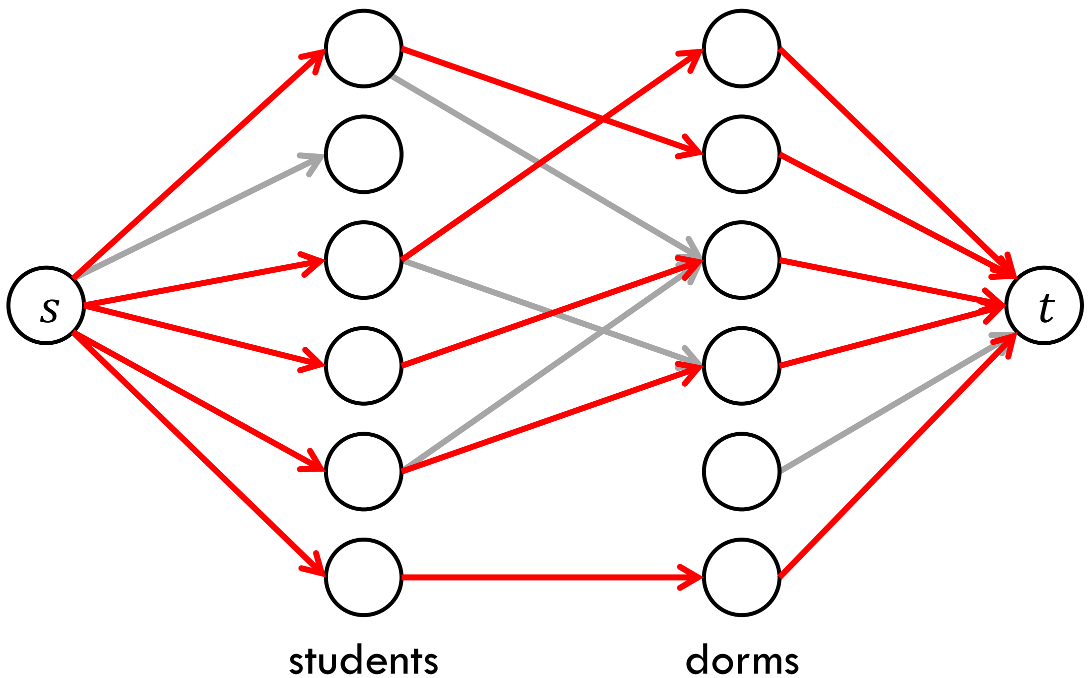
Flow Network Construction

- Add source and sink
- Make edges between students and dorms
  - All the edge weights are 1



Flow Network Construction

- Find the max-flow
- Find the optimal assignment from the chosen edges



Related Problems

- A more reasonable variant of the previous problem: dorm j can accommodate c_j students
  - Make an edge with capacity c_j from dorm j to the sink
- Decomposing a DAG into nonintersecting paths
  - Split each vertex v into v_{left} and v_{right}
  - For each edge $u \to v$ in the DAG, make an edge from u_{left} to v_{right}
- And many others…

Min-Cost Max-Flow

- A variant of the max-flow problem
- Each edge e has capacity $c(e)$ and cost $\mathrm{cost}(e)$
- You have to pay $\mathrm{cost}(e)$ amount of money per unit flow flowing through e
- Problem: find the maximum flow that has the minimum total cost
- A lot harder than the regular max-flow
  - But there is an easy algorithm that works for small graphs

Simple (?) Min-Cost Max-Flow

- Forget about the costs and just find *a* max-flow
- Repeat:
  - Take the residual graph
  - Find a negative-cost cycle using Bellman-Ford
    - If there is none, finish
  - Circulate flow through the cycle to decrease the total cost, until one of the edges is saturated
    - The total amount of flow doesn't change!
- Time complexity: very slow

Notes on Max-Flow Problems

- Remember different formulations of the max-flow problem
  - Again, $(\text{maximum flow}) = (\text{minimum cut})$!
- Often the crucial part is to construct the flow network
- We didn't cover fast max-flow algorithms
  - Refer to the Stanford Team notebook for efficient flow algorithms