Nearest Common Ancestors

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Outline

- Distributed data structures
  - Parent labeling scheme
- Nearest common ancestor problem
- Nearest common ancestor labeling scheme
  - A first attempt
  - Heavy path decomposition
  - Alphabetic Codes

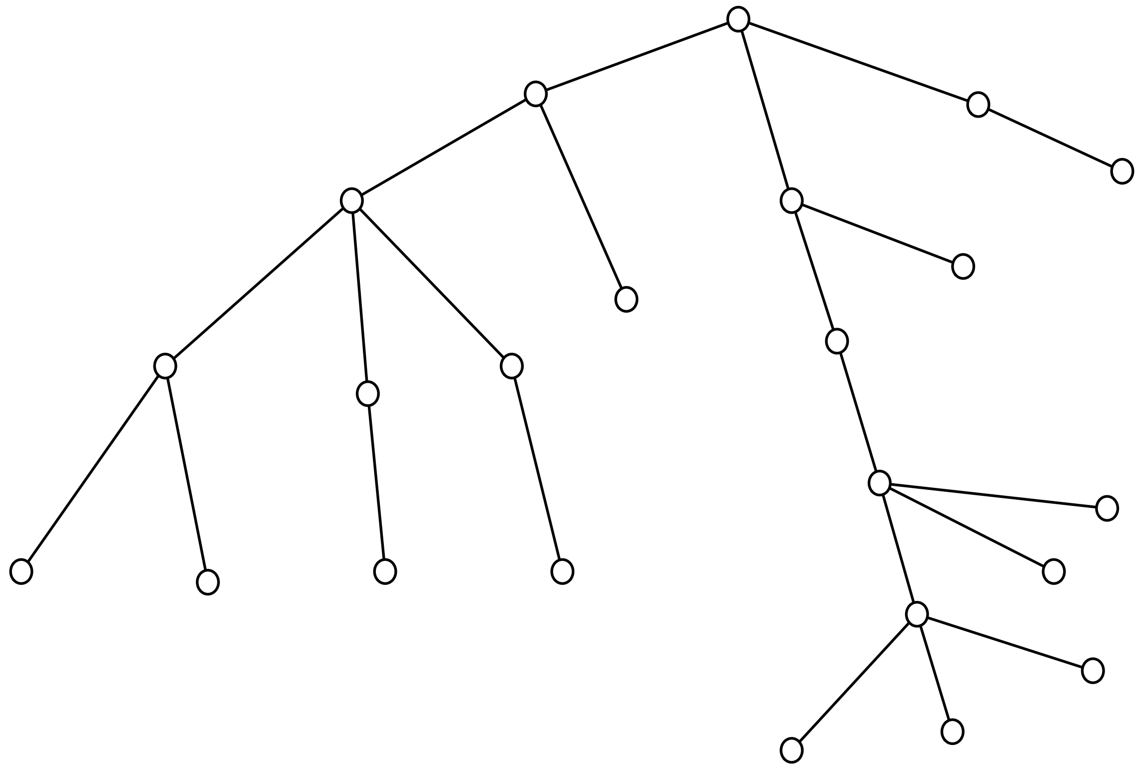
Distributed Data Structures

Distributed Data Structures

- A *labeling scheme* for a graph G supporting query q(v,w) for any pair of nodes v and w:
  - Preprocessing: To each node v, assign *label* (just a bitstring).
  - Query: Given label(v) and label(w) compute query on v and w. No other info!
- Goal:
  - Primary: Minimize the maximum size of labels.
  - Secondary: Fast queries, fast preprocessing, total space.

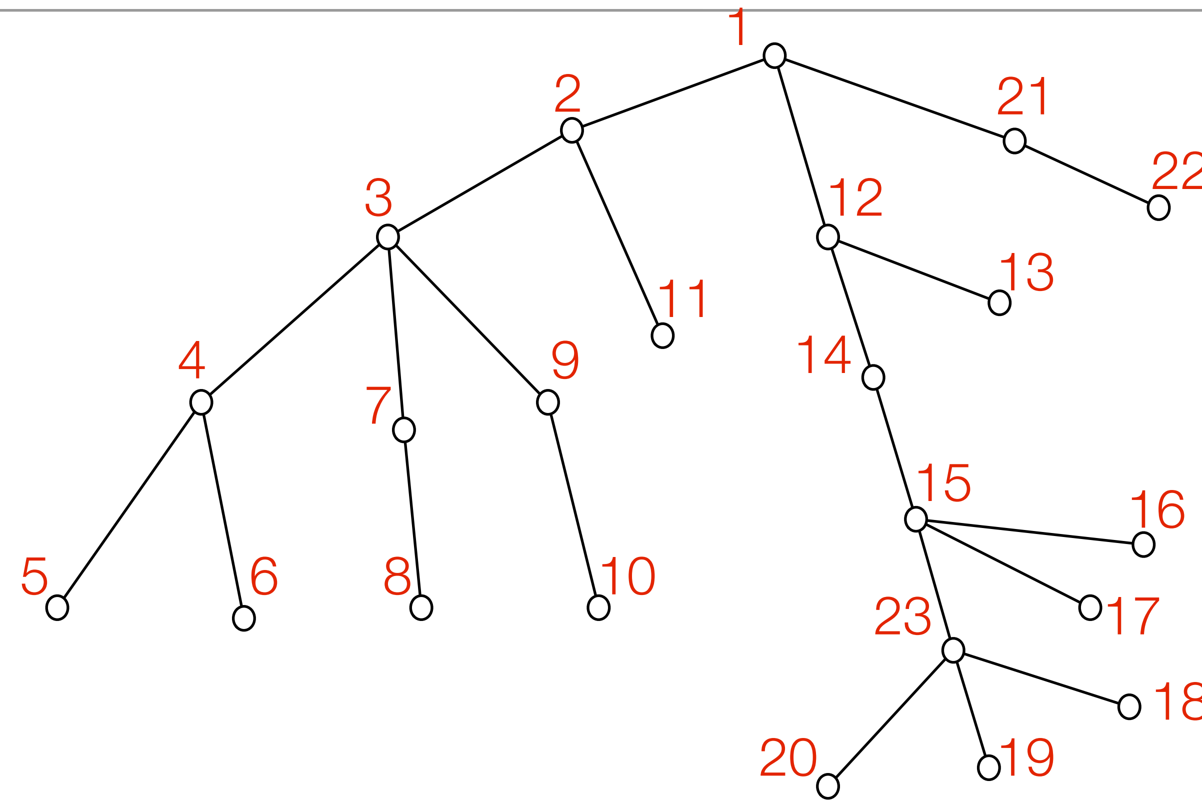
Parent Labeling Schemes for Trees

Parent Labeling Scheme for Trees



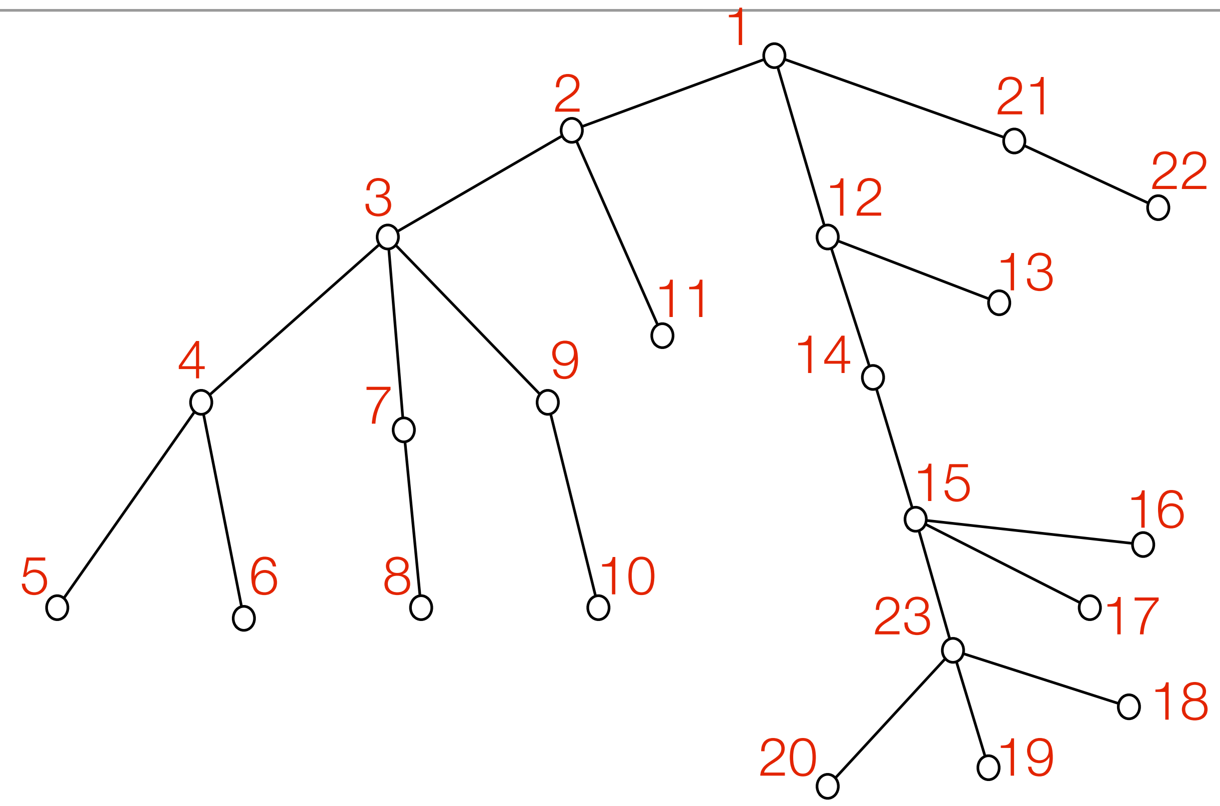
- Let T be a rooted tree with n nodes.
- Assign label to each node to support
  - lparent(label(v),label(w)): return true iff v is parent of w.
- Goal: Minimize the maximum length of a label.
- What can we store in labels to answer lparent?

Labels and Queries



- Solution: Assign unique ID to each node.
- label(v) = ID(v) · ID(parent(v))
- parent(label(v), label(w)): true iff ID(v) = ID(parent(w))
- Use $\lceil \log n \rceil$ bits to store ID => label length is $2\lceil \log n \rceil$

Summary

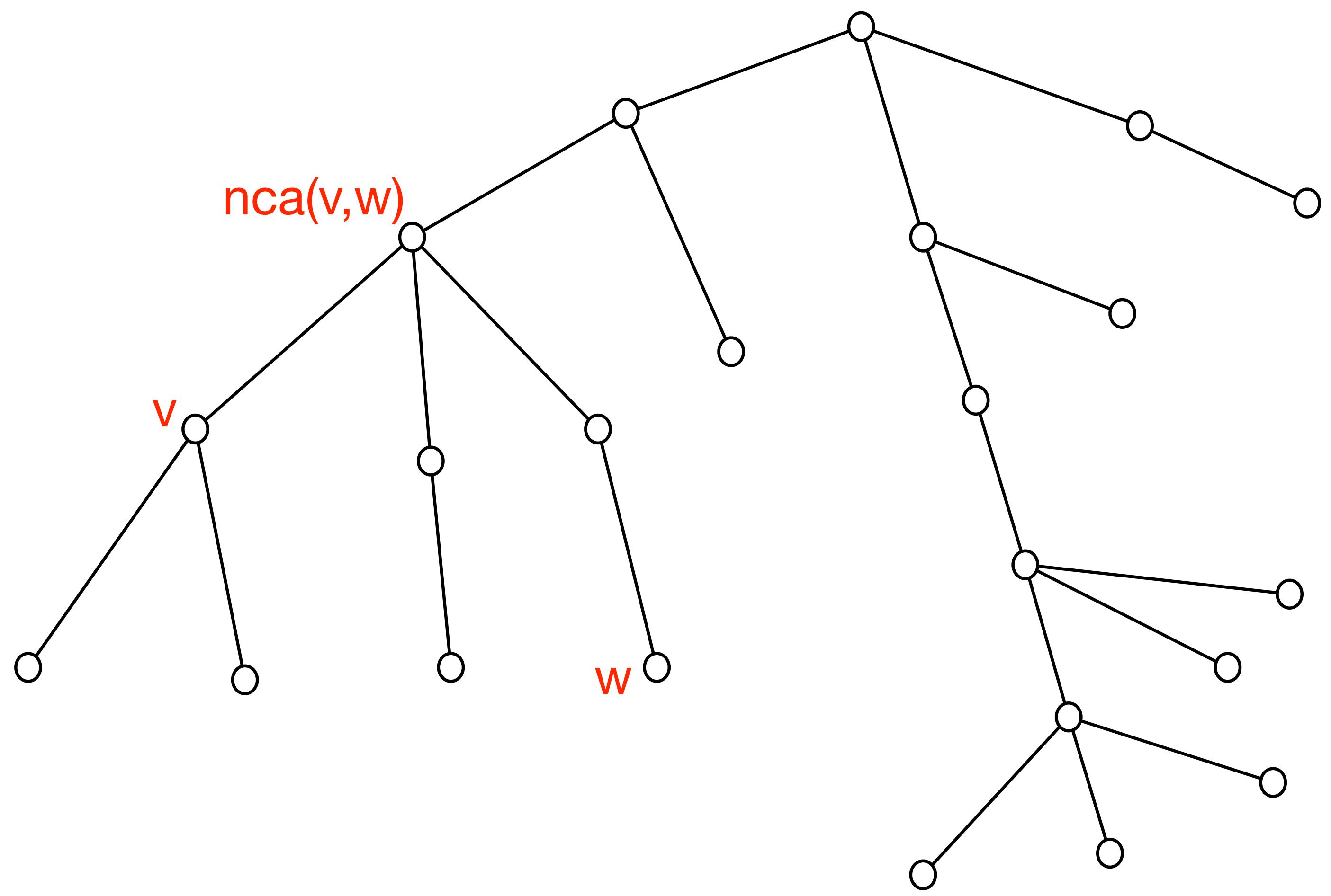


- Theorem: There is a parent labeling scheme for trees with maximum label length of $2 \lceil \log n \rceil$ bits.
- Other properties:
  - O(1) time queries
  - O(n) preprocessing and space

Applications

- Why study labeling schemes?
- Ultra compact distributed data structures:
  - Network routing, graph representation, XML search engines, ...
- Few memory accesses:
  - Limited memory access process query => Little I/O overhead.
- Graph theoretical connection:
  - Universal graphs.

Nearest Common Ancestor Problem



- T is a rooted tree with n nodes.
- The *ancestors* of node v is the set of nodes from v to the root (both inclusive)
- The *common ancestors* of nodes v and w are the ancestor of both v and w.
- The *nearest common ancestor* of nodes v and w (nca(v, w)) is the common ancestor of greatest depth.

Nearest Common Ancestor Problem

- The *nearest common ancestor problem*: Preprocess a rooted tree T with n nodes to support
- nca(v,w): return the nearest common ancestor of v and w.
- A.k.a. the *lowest common ancestor problem* or the *least common ancestor problem* (lca)
- How about *most common ancestor* (mca) as compromise between l and n?

Applications

- Key primitive in algorithms for: Weighted matching, minimum spanning trees, dominator trees, approximate string matching, dynamic planarity testing, network routing, computational geometry, computational biology, ...
    - Main point: Trees are everywhere in science and nca is a very basic primitive for trees.
- Nice illustration of data structure design techniques.

Nearest Common Ancestor Labeling Schemes

- We study algorithms for nca in labeling scheme context:
- Assign label to each node supporting *label-nca*
  - lnca(label(v),label(w)): return label(nca(v, w))
- Goal:
  - Primary: Maximum label length of O(log n) bits.
  - Secondary:
    - lnca in O(1) time.
    - O(n) preprocessing and space

Overview

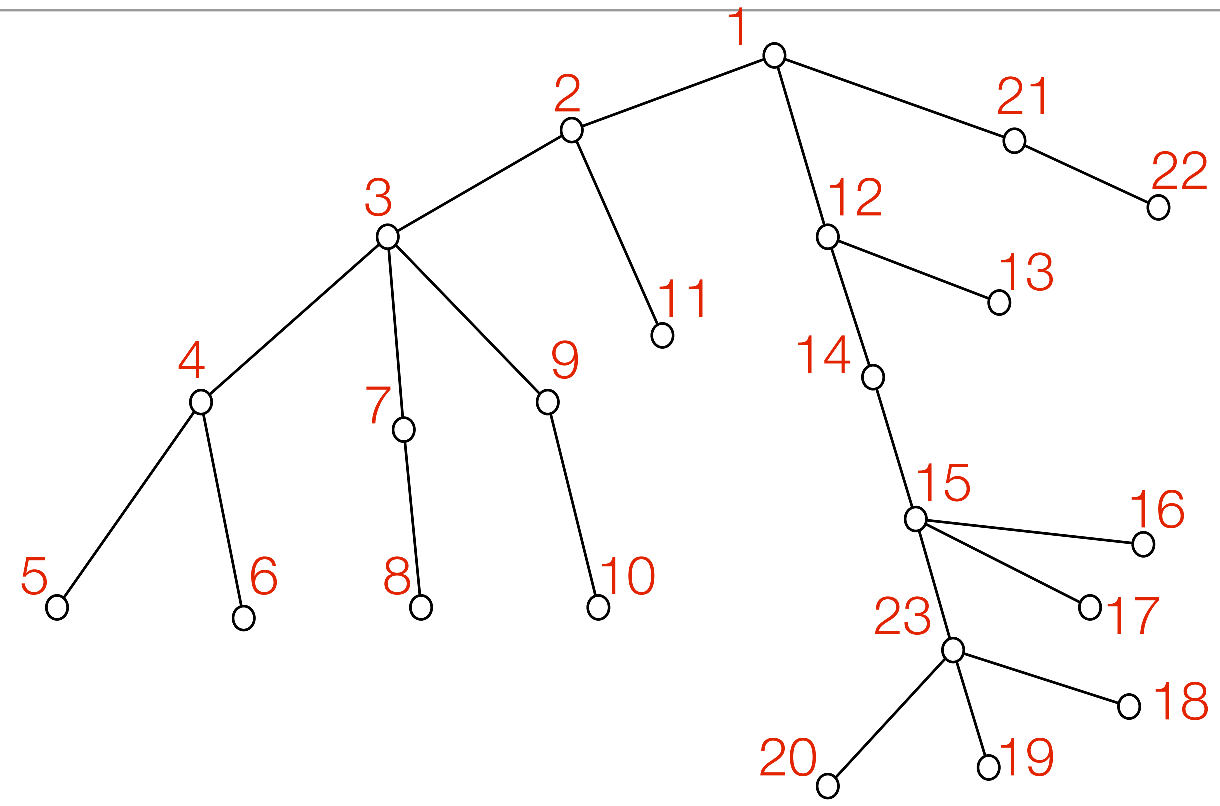
- Solution in three steps:
  - A first attempt
  - Heavy path decomposition
  - Alphabetic Codes

A First Attempt

A First Attempt

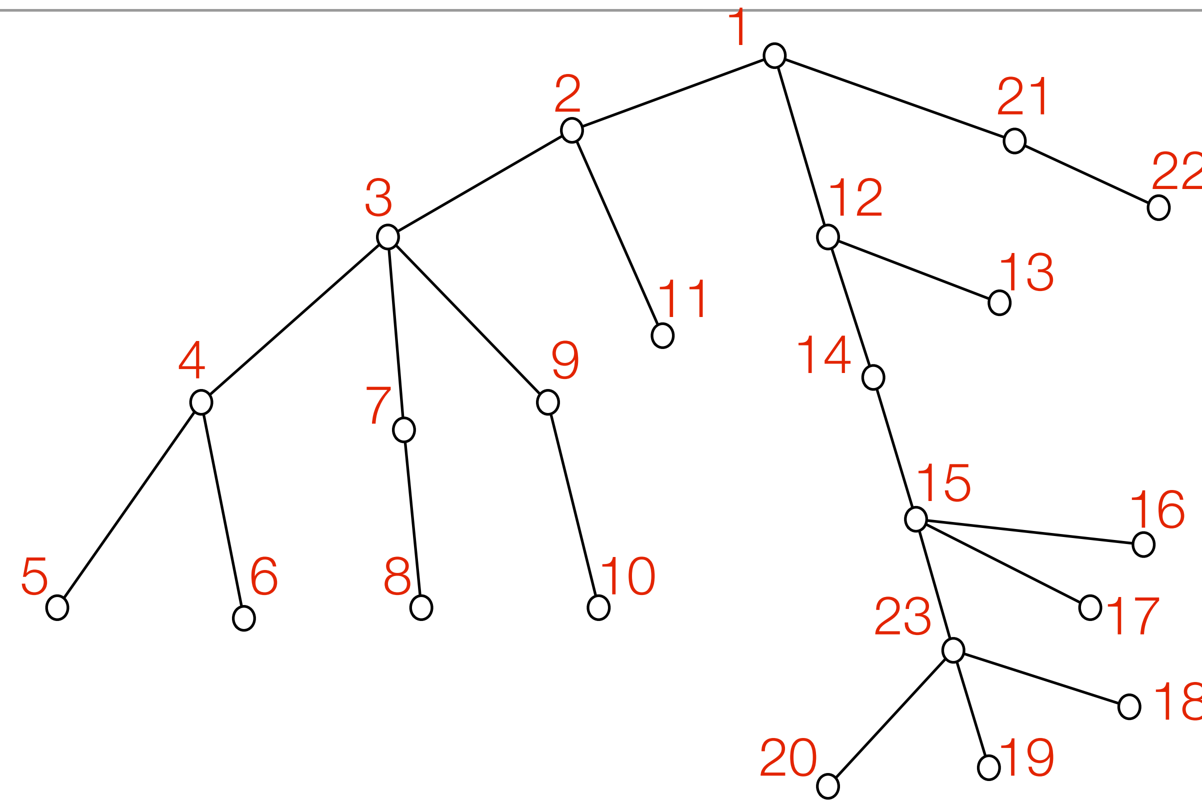
- Simplifying assumption: Ignore secondary goals (time to compute Inca, preprocessing, total space).
- Focus on maximum label length instead.
  - What information suffices to support Inca?

Labels and Queries



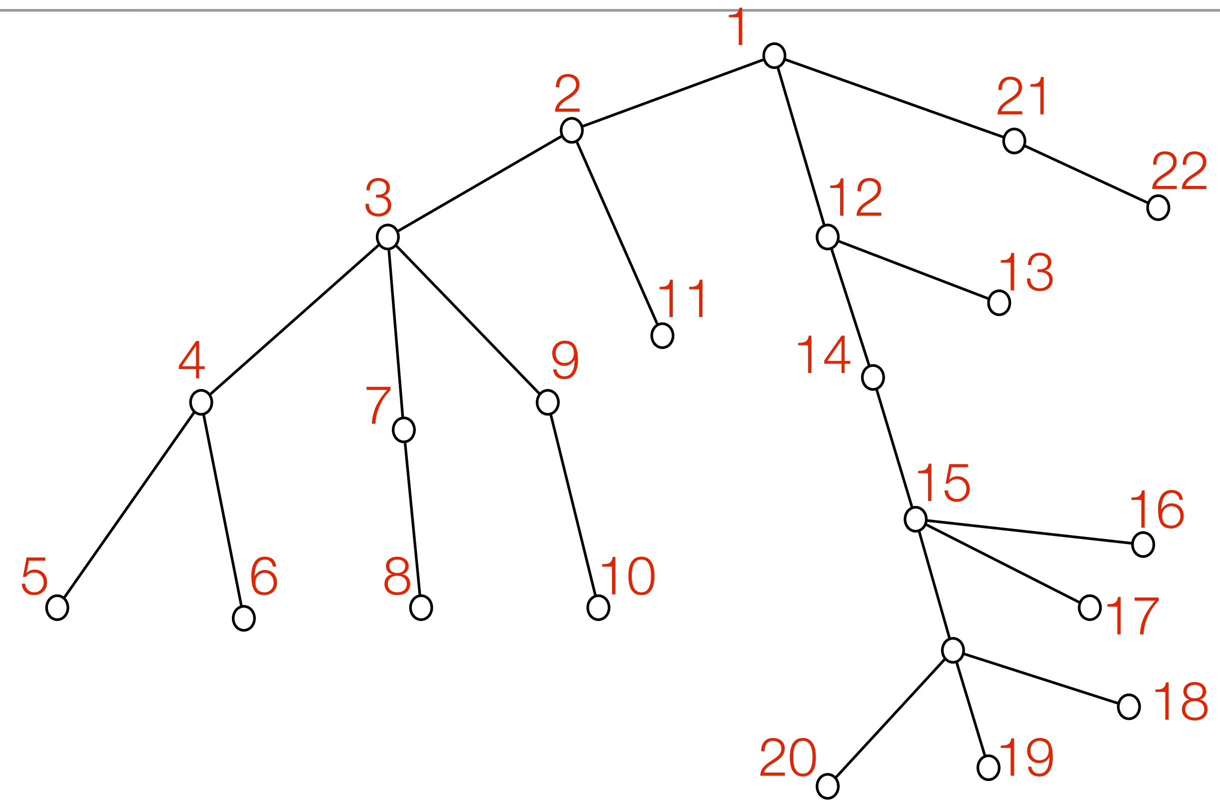
- Suppose we add unique ID to each node (as in parent labeling scheme)
- Which IDs could we store in a label to answer Inca queries?

Labels and Queries



- label(v) = ID(v_1) · ID(v_2)· · · ID(v_k) (r = v_1, ..., v_k = v is path of nodes from root to v)
- lnca(label(v), label(w)): longest common prefix of IDs.
- Use ⌈log n⌉ bits to store ID
- => label length is at most h ⌈log n⌉ , where h is height of tree.

Labels and Queries



- Theorem: There is a nearest common ancestor labeling scheme for trees with height h with maximum label length of h $\lceil \log n \rceil$ bits.
- Nasty problem: h might be $\Omega(n)$ => $\Omega(n \log n)$ bit labels.
- How can we get better bounds?

Overview

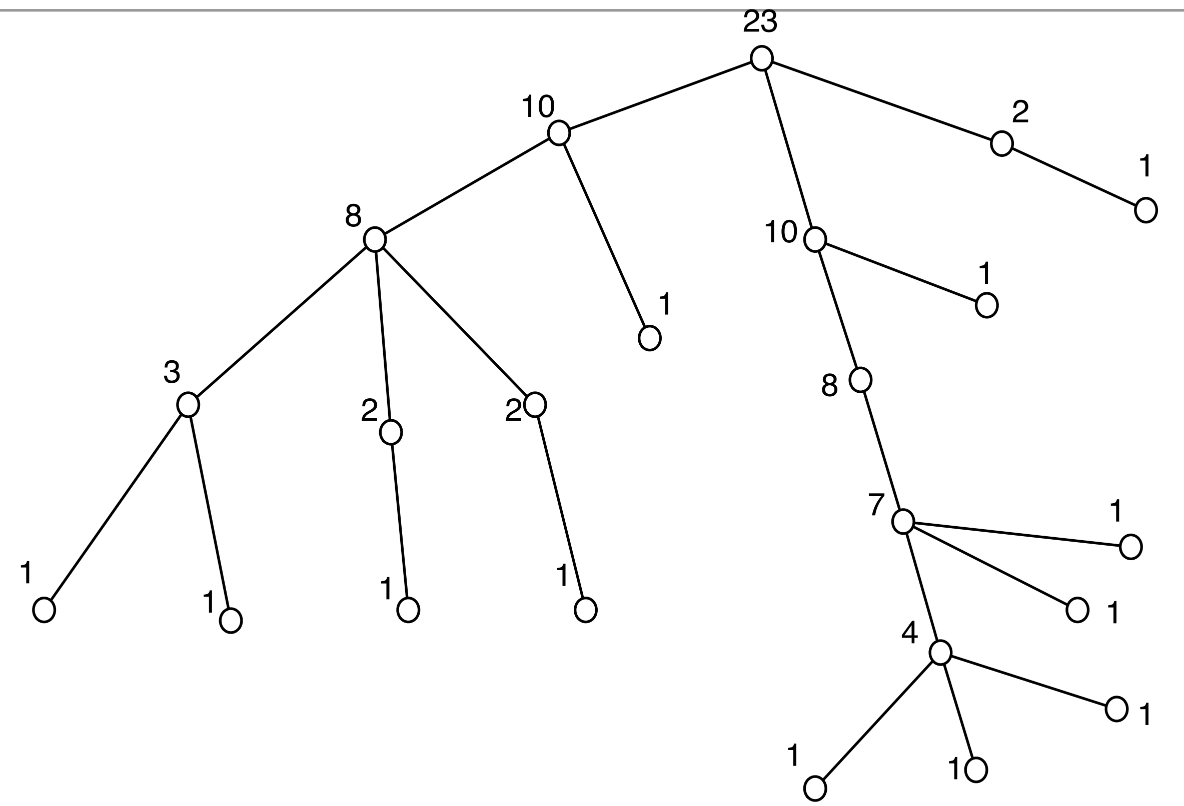
- Improve to O(log n) bit labels in two steps:
  - Heavy-path decomposition.
  - Alphabetic codes.

Heavy Path Decomposition

Heavy Path Decomposition

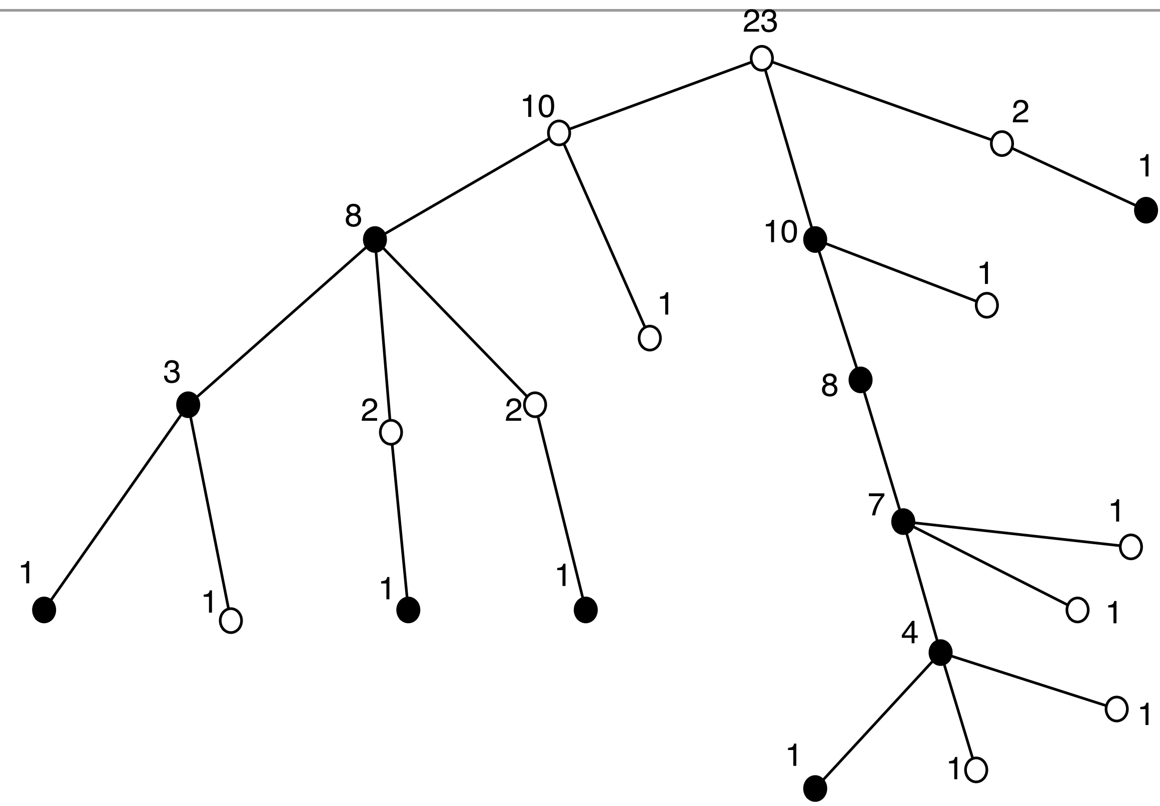
- Technique to:
  - Balance trees (sort of).
  - Reduce problems on tree to problems on *paths* with logarithmic overhead.

Subtree Size



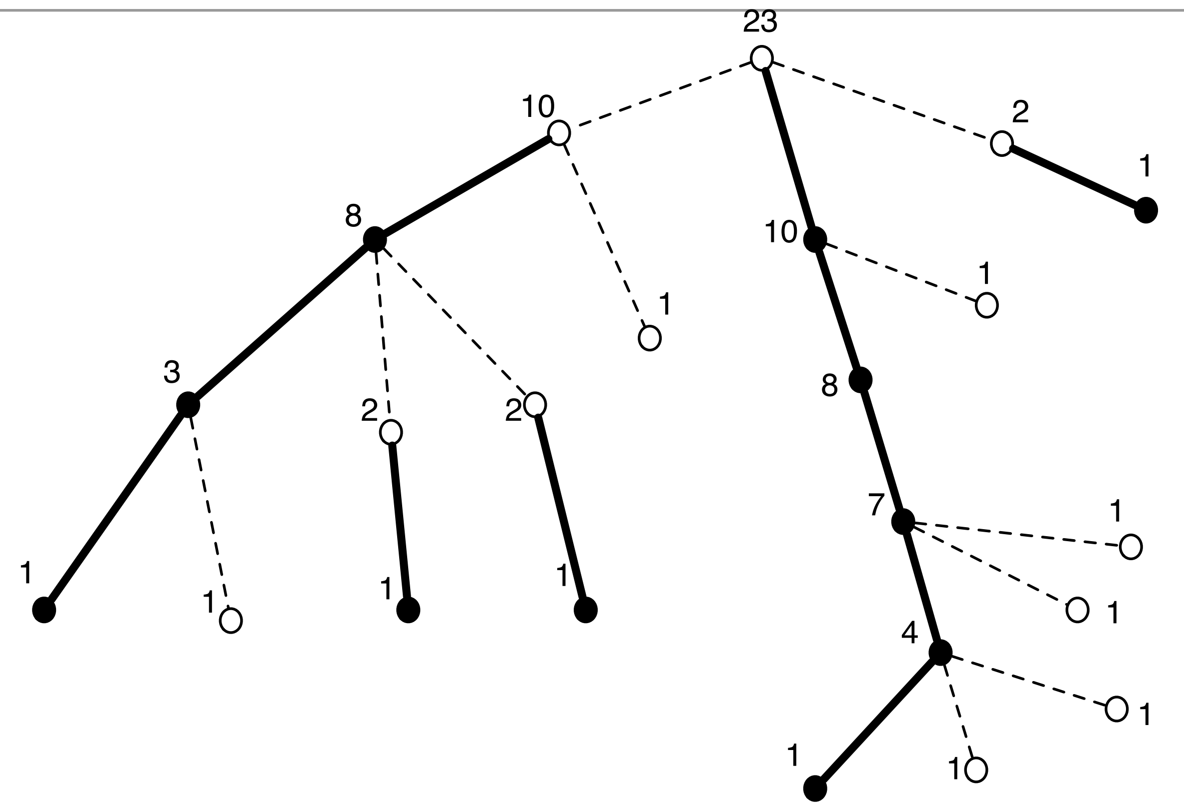
- For each node v compute:
    - size(v) = #of descendants (including v itself)

Heavy and Light Nodes



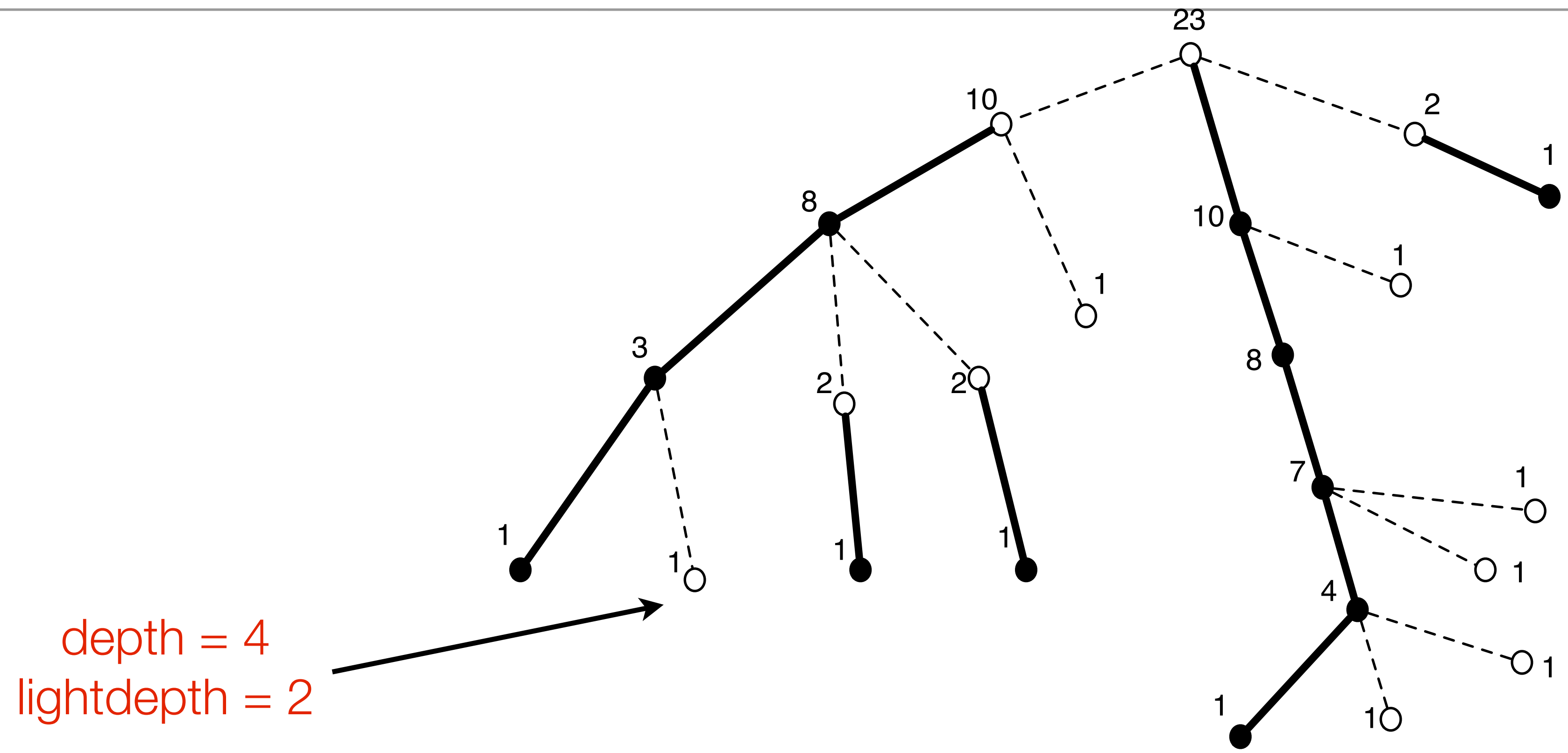
- Classify nodes as *heavy* or *light:*
  - root is light
  - For each internal node v, pick child w of maximum size and classify it as heavy. The other children are light.

Heavy and Light Edges



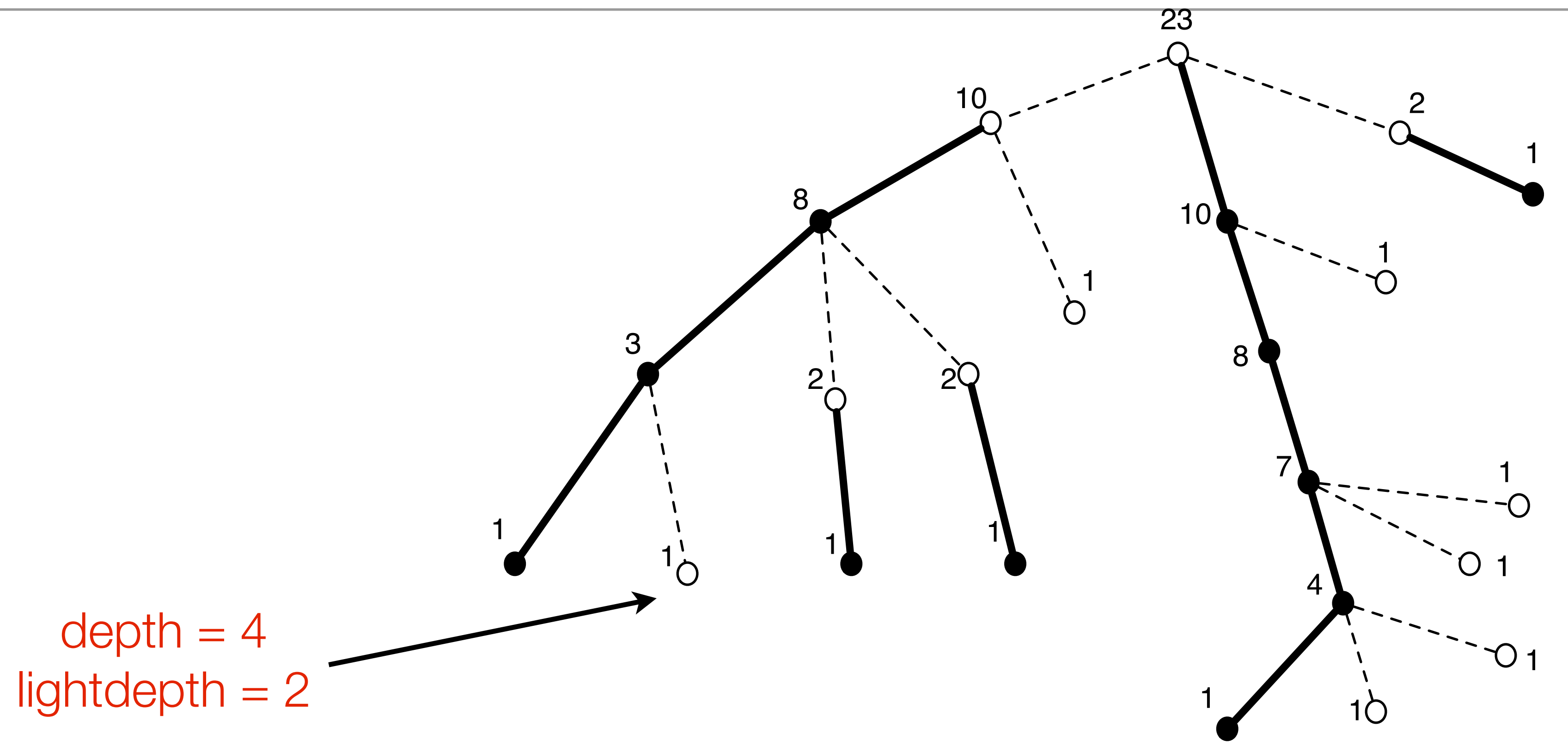
- Classify edges as *heavy* or *light:*
  - Edge to heavy child is heavy and edge to light child is light.
- If we remove light edges we partition T into *heavy paths.*

Light Depth



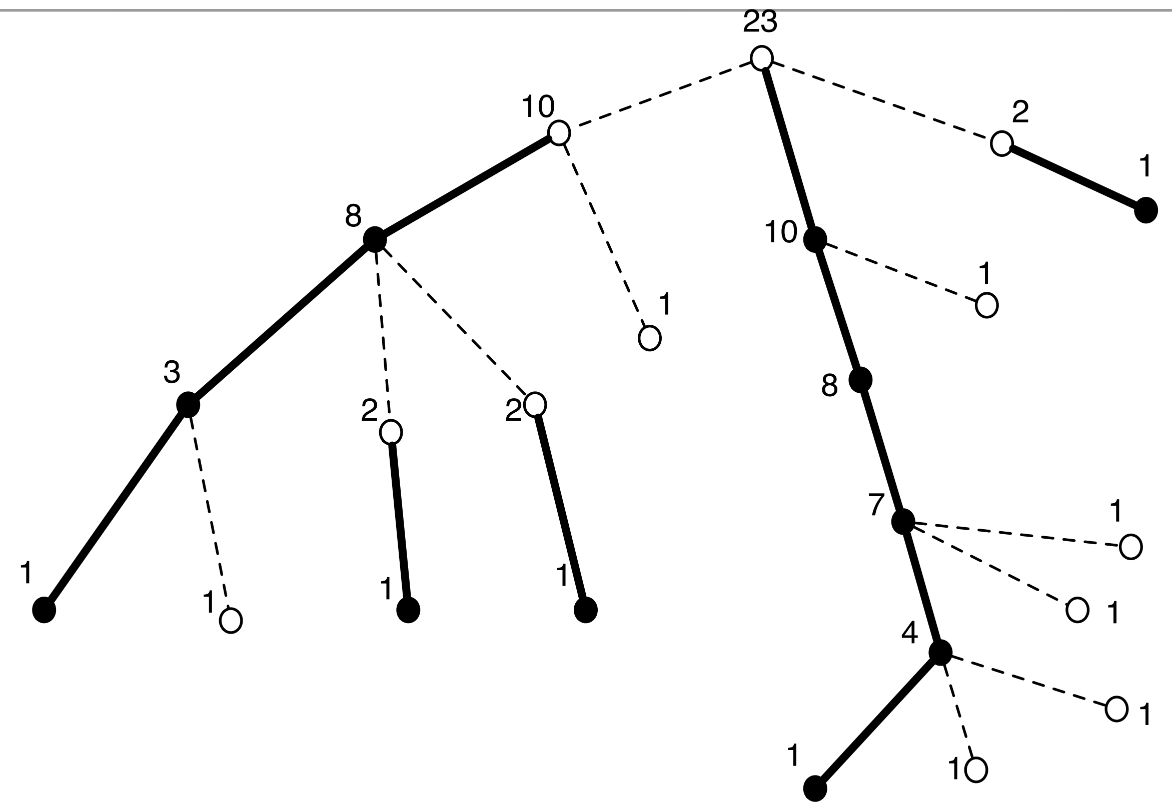
- depth(v) = #edges on path from v to root
- lightdepth(v) = #light edges on path from v to root
- depth(v) = n-1 for a worst case tree.
- What about lightdepth(v)?

Light Depth



- Each light edge on path from root decreases size by *at least half*.
- => For any node v, lightdepth(v) = O(log n).
- => Number of heavy paths on a root to leaf path is O(log n).

Heavy Path Decomposition and NCA Labeling Schemes

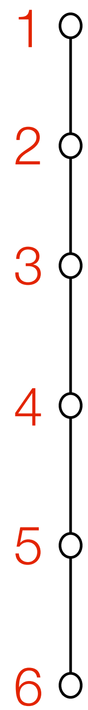


- How can we use heavy-path decomposition to improve our labeling scheme?
- Idea:
  - Find a good solution for paths.
  - Apply to the O(log n) heavy paths on root to leaf path.

NCA for Paths

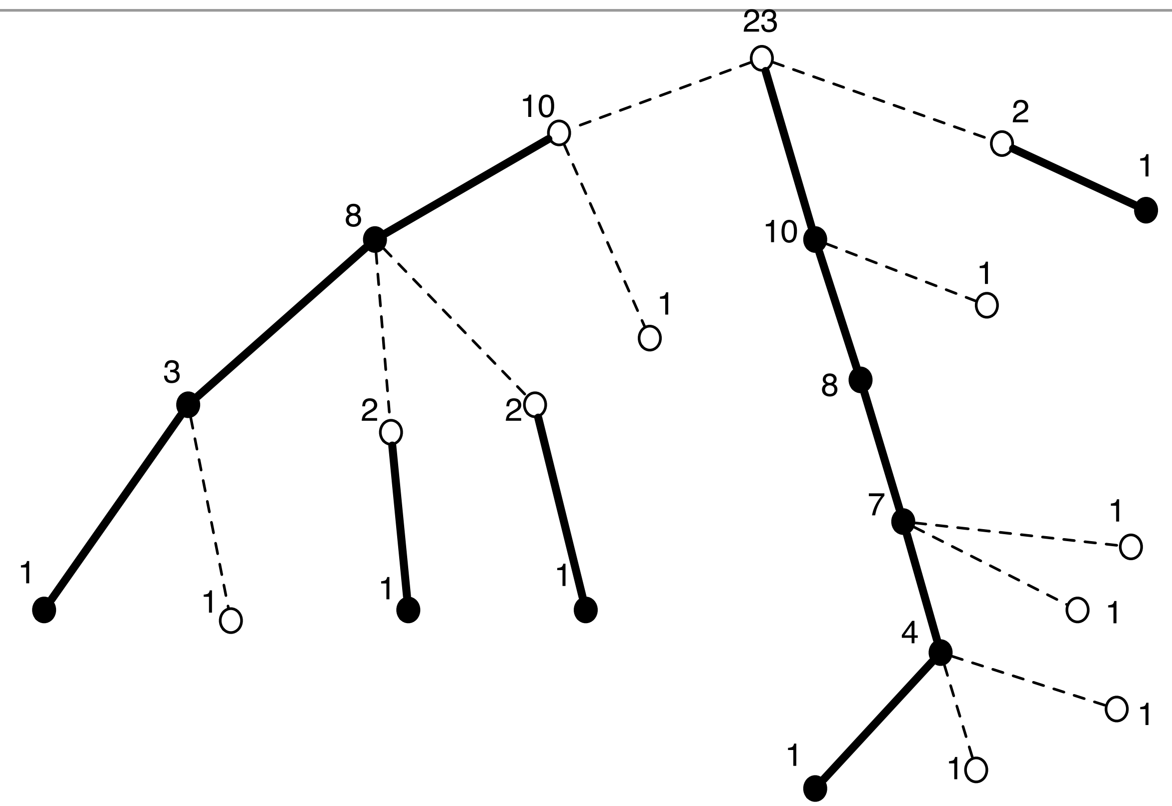
- What is nca(v,w) on a path?
- How can we label nodes for lnca queries on a path?

NCA for Paths



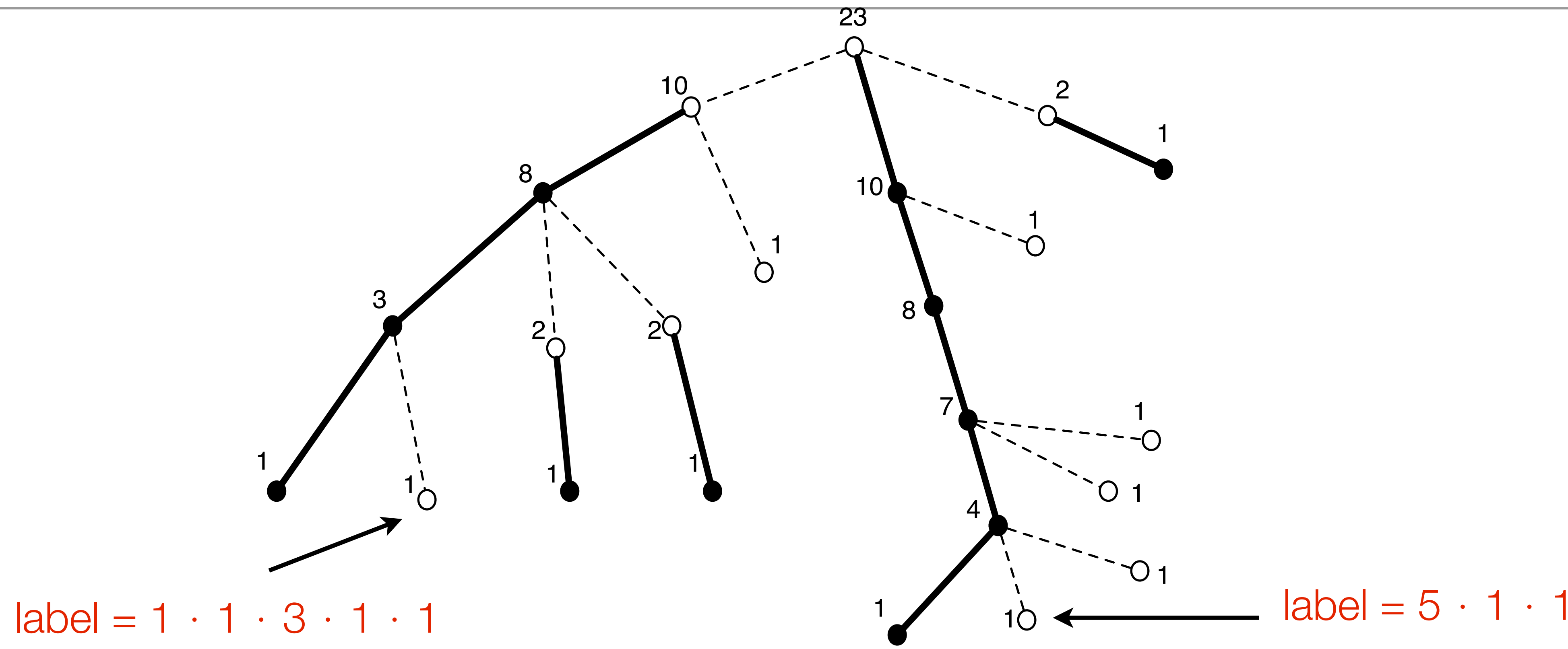
- Assign increasing numbers to nodes from root to leaf.
- lnca(label(v), label(w)) = min(label(v), label(w))

Heavy Path Decomposition and NCA Labeling Schemes



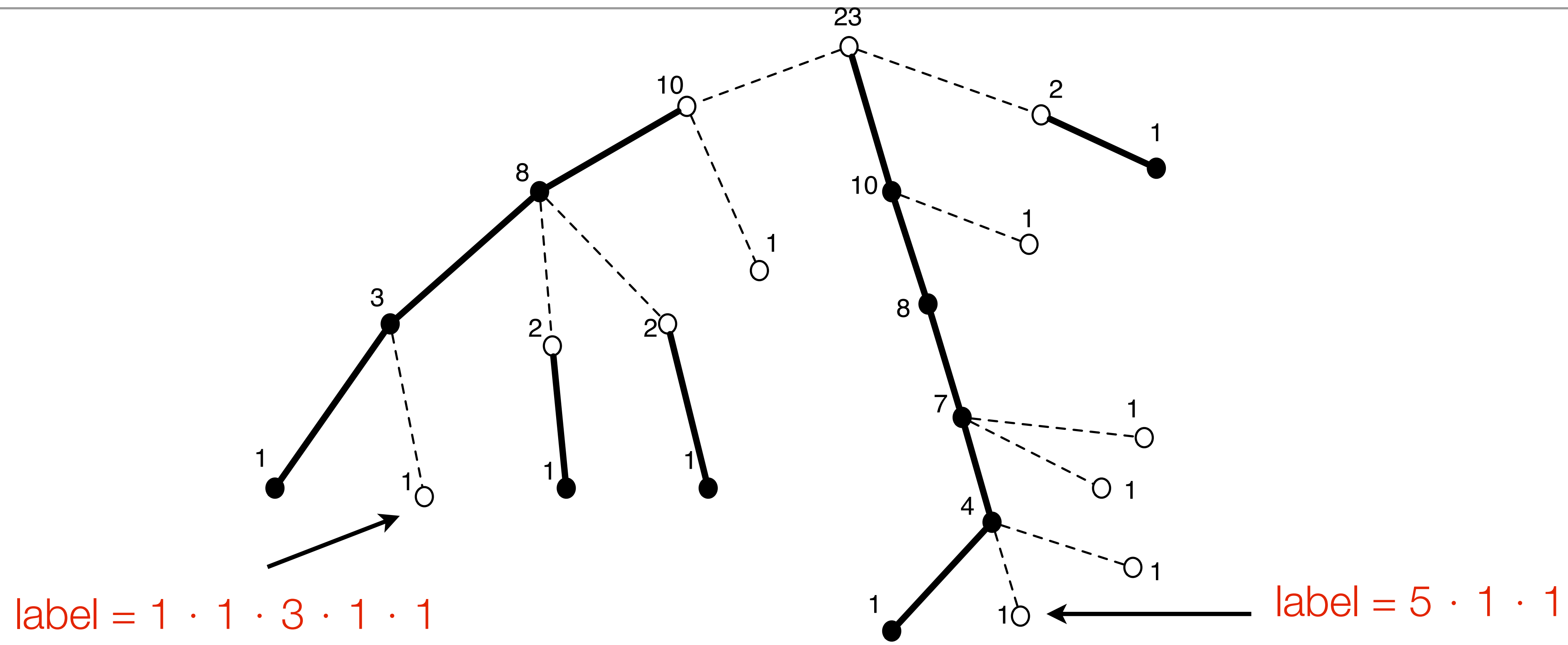
- How can we use path idea to get to $O(\log^2 n)$ bit labels?
- (E.g. $O(\log n)$ bits per heavy path on the path from root to node)

Light and Heavy IDs



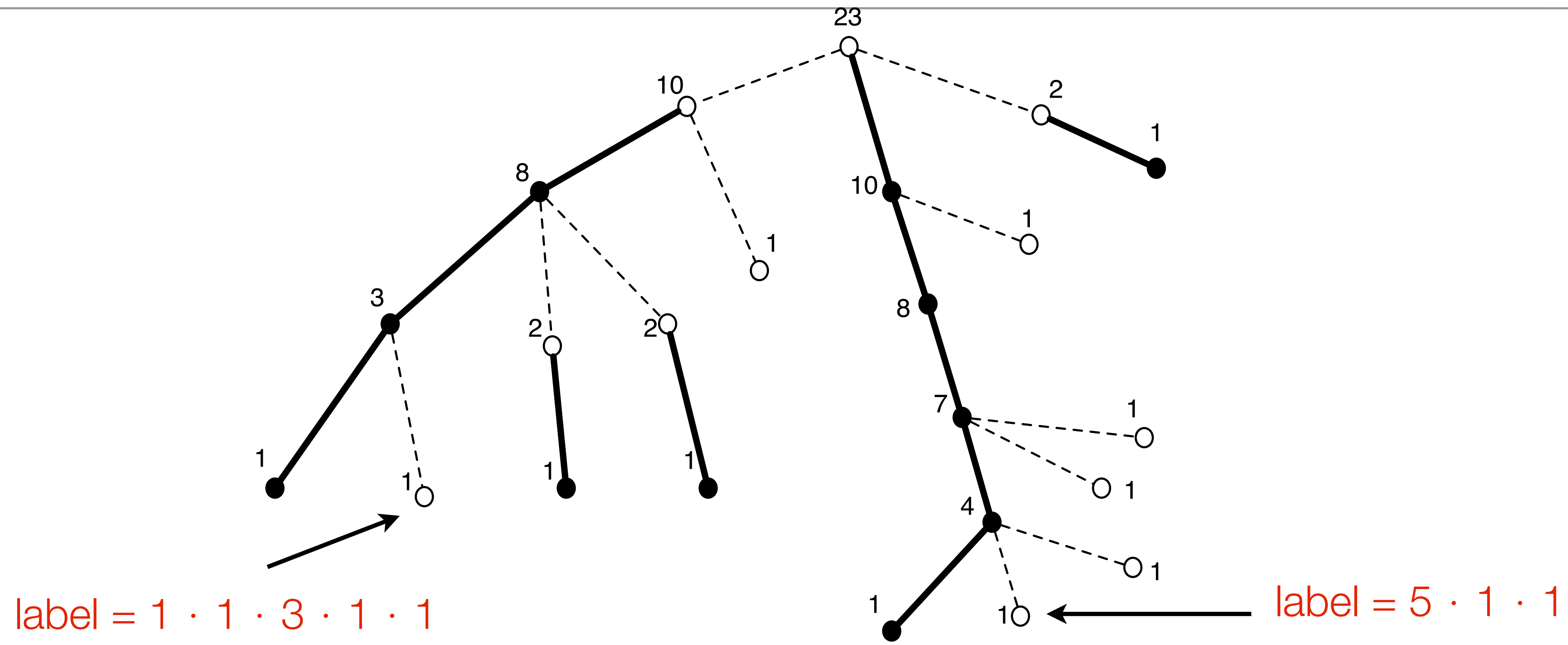
- For each heavy path $h_1 \cdots h_k$ from root to v store:
  - HeavyID: The final node on heavy path (where we exit to light descendant or stop if final heavy path)
  - LightID: The light child we exit to among the other children. E.g. number in a left-to-right ordering.
- label(v): heavyID(h_1) · lightID(h_1) · heavyID(h_2) · · · lightID(h_{k-1}) · heavyID(h_k)

Label Length



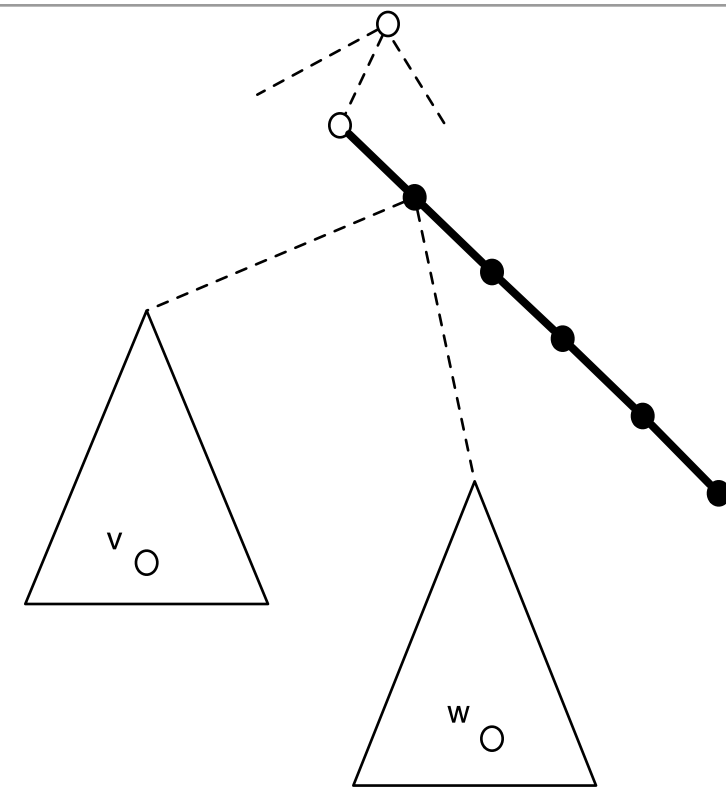
- 2 ⌈log n⌉ bits per heavy path (heavyID and lightID)
- => maximum label length is 2 ⌈log n⌉ · O(log n) = O(log^2 n)

Computing Queries



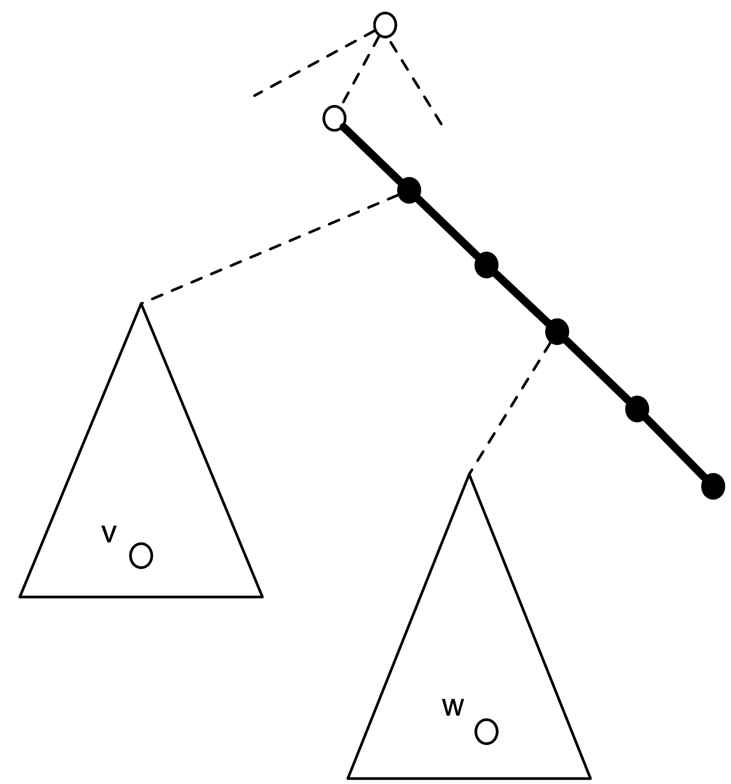
- lnca(label(v), label(w)): Almost as before
  - Compute longest common prefix L of IDs.
  - 2 cases to consider: L contains an even or odd number of IDs.

Case 1: L contains odd number of IDs



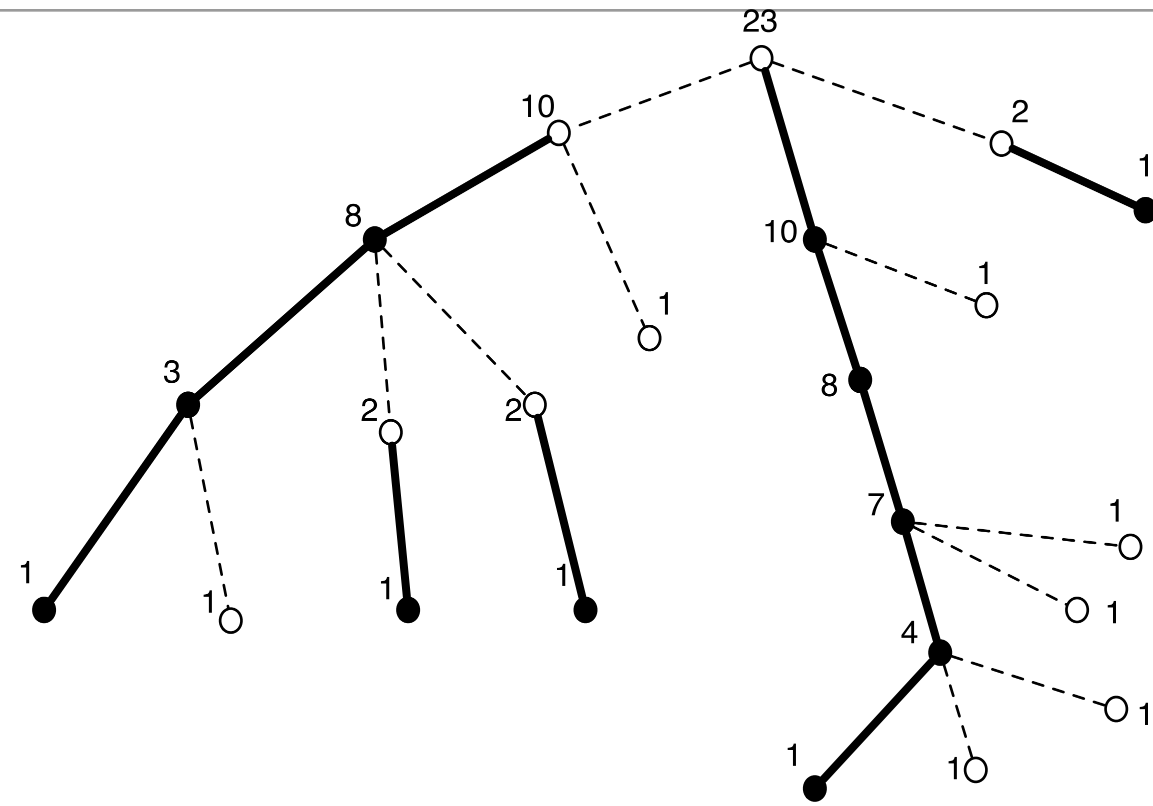
- Case 1: L contains odd number of IDs
  - => Final ID in L is a heavyID.
  - => v and w exit from same heavy path node to different light children.
  - => lnca(label(v), label(w)) = L   (the longest common prefix of IDs)

Case 2: L contains even number of IDs



- Case 2: L contains even number of IDs
    - => Final ID in L is a lightID.
    - => v and w enter same heavy path but leave at different exit points on path.
    - => lnca(label(v), label(w)) = L · min(next ID in label(v), next ID in label(w))

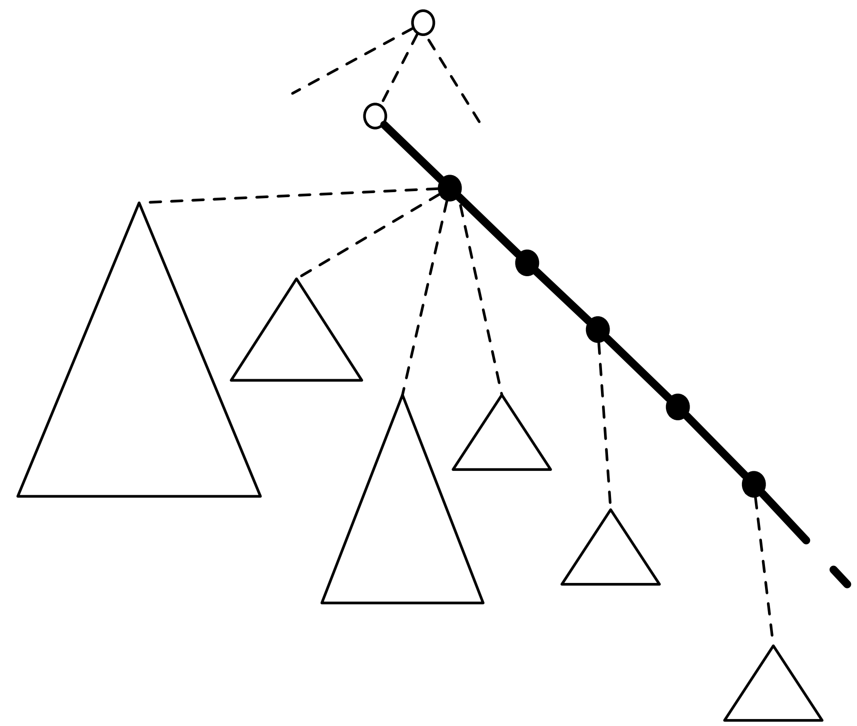
Summary



- Theorem: There is a nearest common ancestor labeling scheme for trees with maximum label length of $O(\log^2 n)$ bits.
- How do we get down to $O(\log n)$ bits?

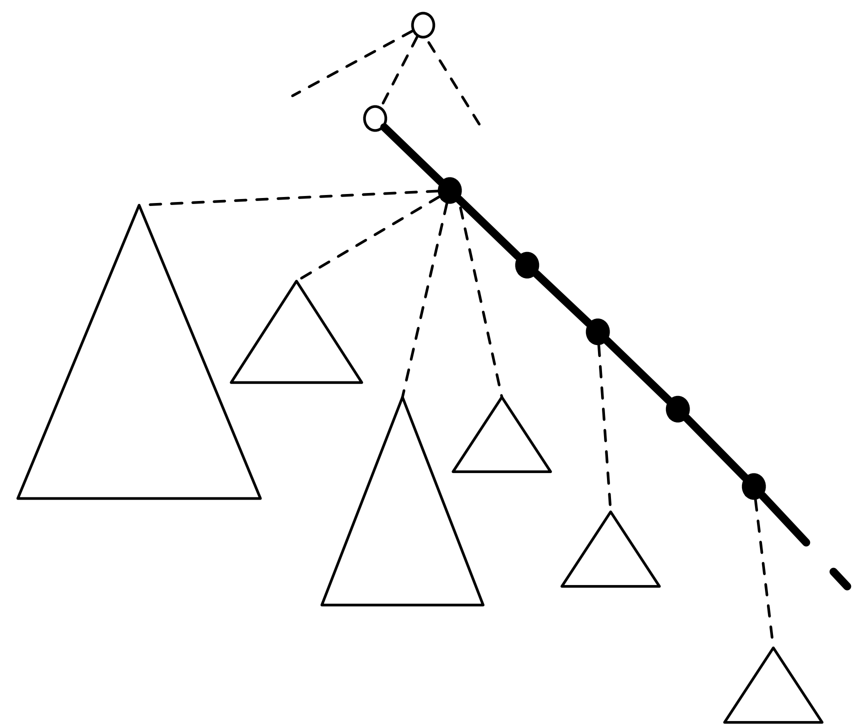
Shaving a Log

What do we need of heavyIDs and LightIDs?



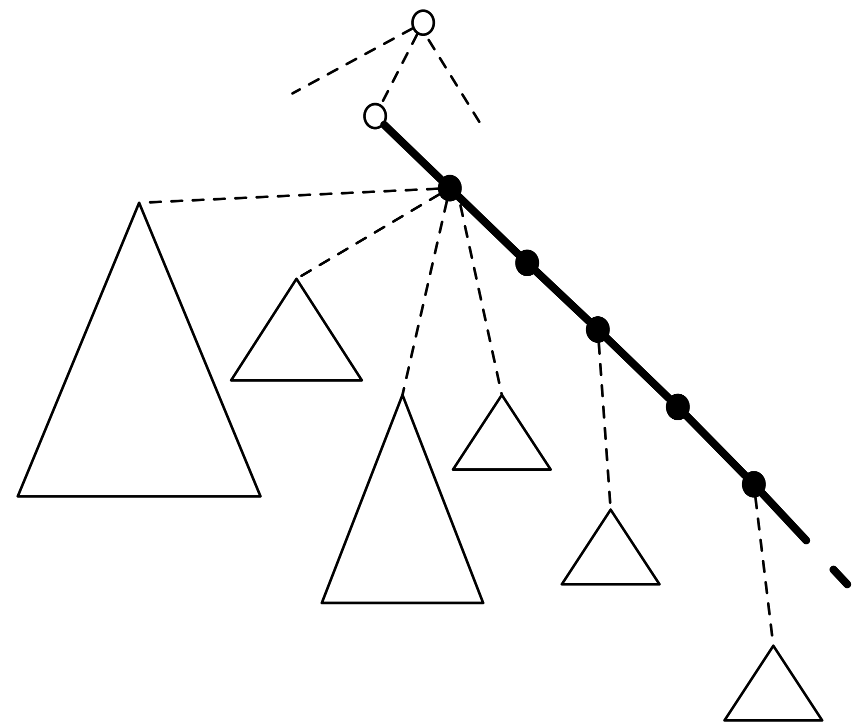
- Do we need binary $\lceil \log n \rceil$ bit numbers for heavyIDs and lightIDs?
  - LightID: Any method that assigns unique codes to distinct light siblings.
  - HeavyID: Any method that assigns unique codes to distinct nodes on heavy path and allows us to determine ordering from top to bottom.

What do we need of heavyIDs and LightIDs?



- How do we minimize the length of labels?
- Intuition:
  - Use variable length codes for heavyIDs and lightIDs to average out lengths.
  - Small subtree => long IDs and large subtree => short IDs.

What do we need of heavyIDs and LightIDs?



- Solution: Alphabetic codes
    - Variable length codes (bitstrings) preserving order (lexicographic order).

Alphabetic Codes

Lexicographic Ordering

- Let a and b be bitstrings.
- a $<_{lex}$ b if a is *lexicographically* smaller than b:
  - a is prefix of b or
  - the first bit where a and b differ is 0 in a and 1 in b.
- Example: 000 $<_{lex}$ 01 $<_{lex}$ 100 $<_{lex}$ 11 $<_{lex}$ 1111

Alphabetic Sequences

- Let $Y = y_1, y_2, \ldots, y_k$ be a sequence of integers
- An *alphabetic sequence* for Y is a sequence $B = b_1, b_2, \ldots, b_k$ such that
  - $b_1 <_{lex} b_2 <_{lex} \cdots <_{lex} b_k$
- Hence, B is an order-preserving coding (using $<_{lex}$) of Y.

- Lemma:
  - Let $Y = y_1, y_2, ..., y_k$ be a sequence of integers with $y_1 + y_2 + \cdots + y_k = s$
  - There exists an alphabetic sequence $B = b_1, b_2, ..., b_k$ for Y such that
    - For all i, $|b_i| \leq \lceil \log s \rceil - \lfloor \log y_i \rfloor = \log s - \log y_i + O(1)$
- Example: Y = 3,5,3,4,1.
- s = 3+5+3+4+1 = 16, and log s = 4
- $B = b_1, b_2, b_3, b_4$ = 000, 01, 100, 11, 1111 (000 $<_{lex}$ 01 $<_{lex}$ 100 $<_{lex}$ 11 $<_{lex}$ 1111)
- $|b_1| = 4 - \lfloor \log 3 \rfloor = 4 - 1 = 3$
- $|b_2| = 4 - \lfloor \log 5 \rfloor = 4 - 2 = 2$
- $|b_3| = 4 - \lfloor \log 3 \rfloor = 4 - 1 = 3$
- $|b_4| = 4 - \lfloor \log 1 \rfloor = 4 - 0 = 4$

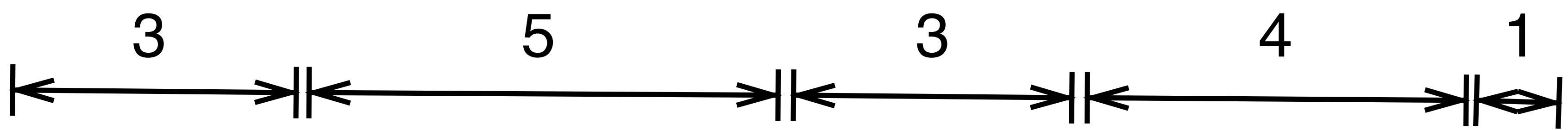
Construction

0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Y = 3,5,3,4,1 s = 16

- Consider binary representation of integers {0,..., s-1}.

Construction

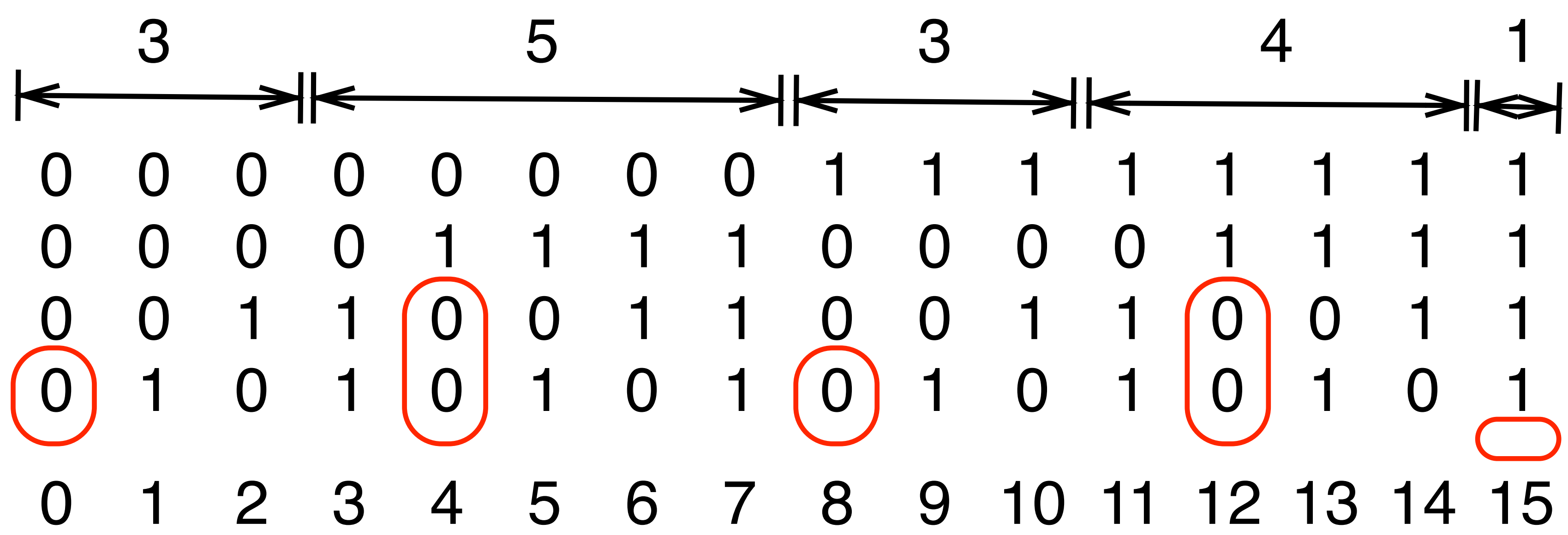


0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

$Y = 3,5,3,4,1$ $\qquad s = 16$

- Consider binary representation of integers $\{0,\ldots, s-1\}$.
- Partition into consecutive intervals of sizes $y_1, y_2, \ldots, y_k$

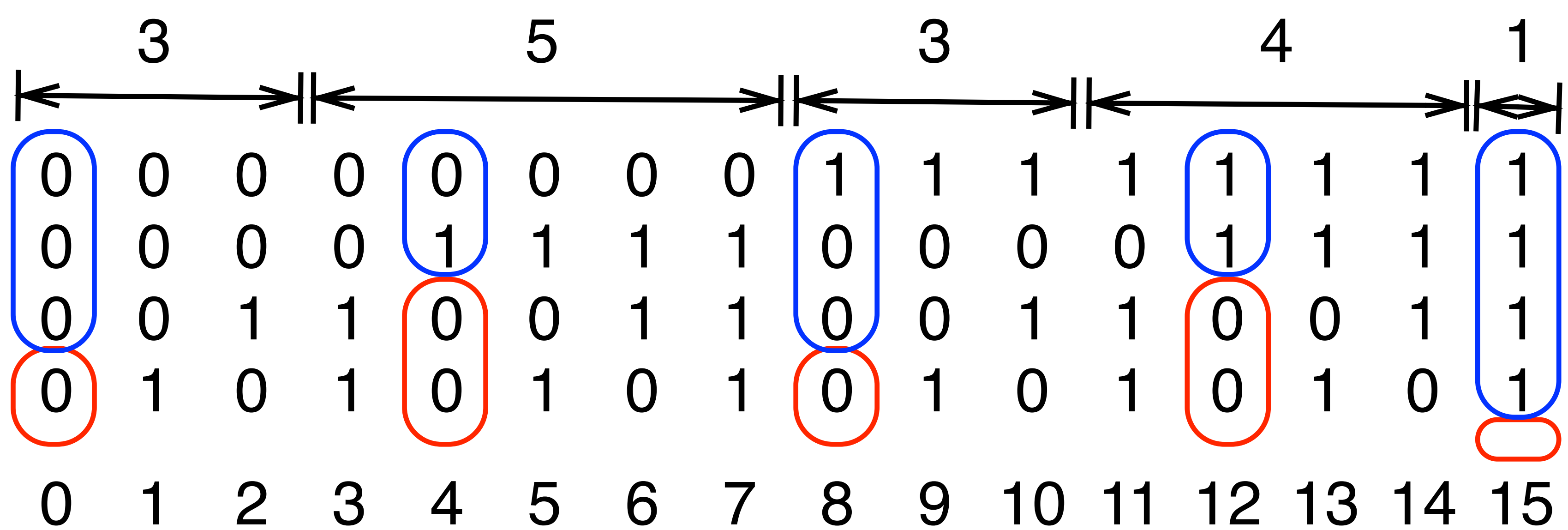
Construction



Y = 3,5,3,4,1 s = 16

- Consider binary representation of integers {0,..., s-1}.
- Partition into consecutive intervals of sizes y_1, y_2, ..., y_k
- In interval i pick number z_i with $\lfloor \log y_i \rfloor$ least significant bits all 0 (why does z_i exist?).

Construction



$Y = 3,5,3,4,1$ $s = 16$

- Consider binary representation of integers $\{0, \ldots, s-1\}$.
- Partition into consecutive intervals of sizes $y_1, y_2, \ldots, y_k$
- In interval i pick number z_i with $\lfloor \log y_i \rfloor$ least significant bits all 0 (why does z_i exist?).
- b_i is z_i with $\lfloor \log y_i \rfloor$ least significant bits removed.
- $B = b_1, b_2, b_3, b_4 = 000, 01, 100, 11, 1111$

Summary

- Lemma:
  - Let $Y = y_1, y_2, ..., y_k$ be a sequence of integers with $y_1 + y_2 + \cdots + y_k = s$
  - There exists an alphabetic sequence $B = b_1, b_2, ..., b_k$ for Y such that
    - For all i, $|b_i| \leq \lceil \log s \rceil - \lfloor \log y_i \rfloor = \log s - \log y_i + O(1)$
- B is an *alphabetic code* for Y.

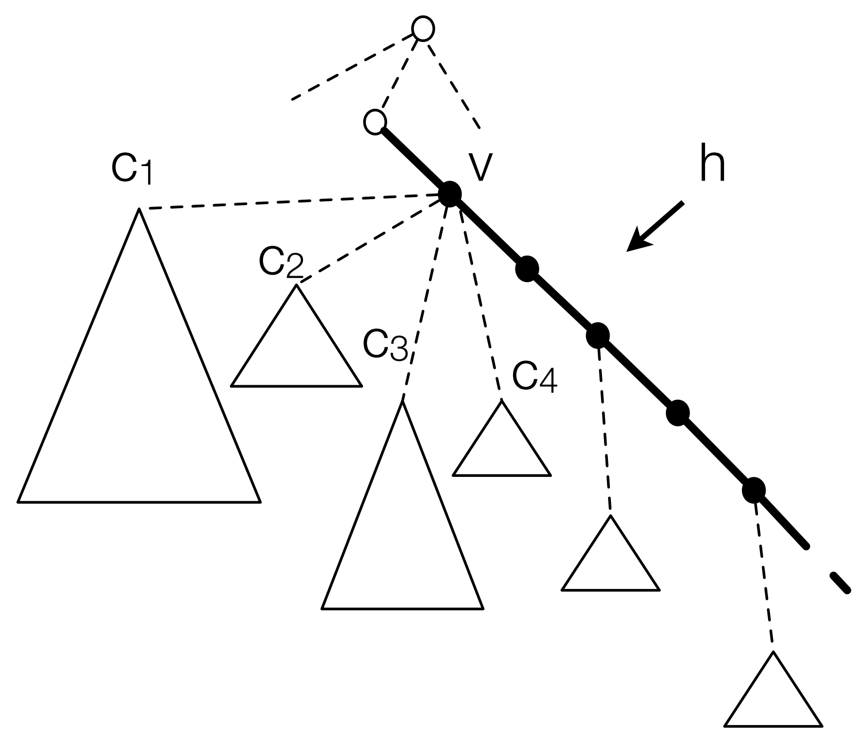
An $O(\log n)$ Labeling Scheme

Overview

- Alphabetic encoding of lightIDs and heavyIDs.
- Handling variable length encoded lightIDS and heavyIDs in labels.

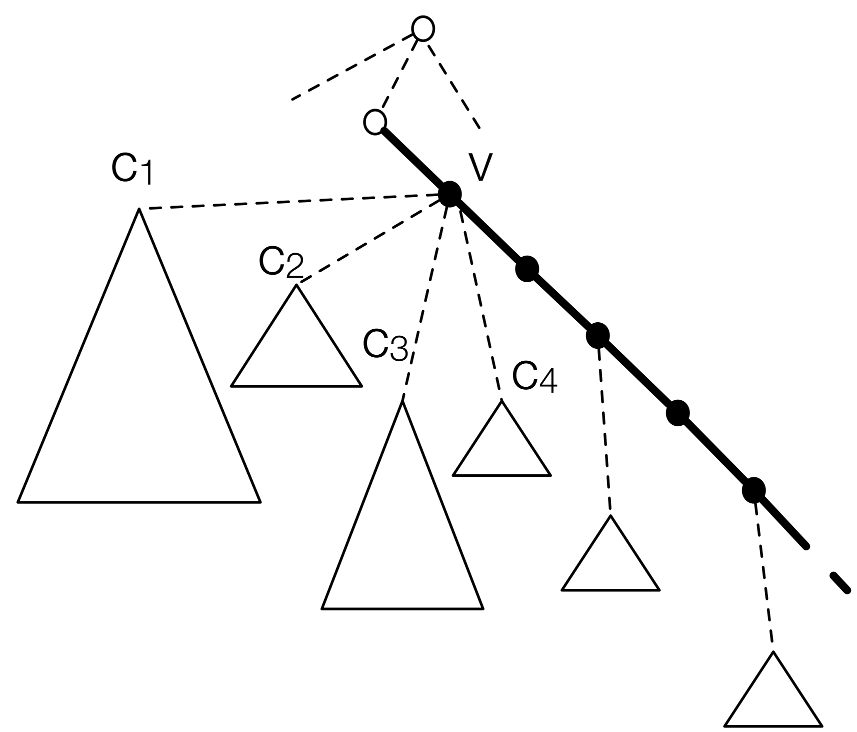
Alphabetic Encoding of LightIDs and HeavyIDs

Light Sizes



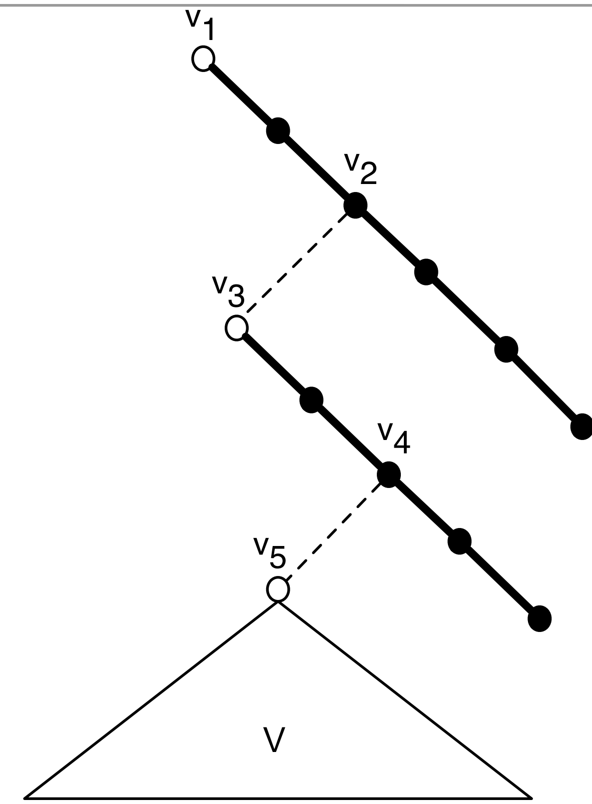
- Define lsize(v) = $\Sigma_{c \text{ is light child}}$ size(c)
- For heavy path h, define lsize(h) = $\Sigma_{v \text{ is on } h}$ lsize(v)

LightID Encoding



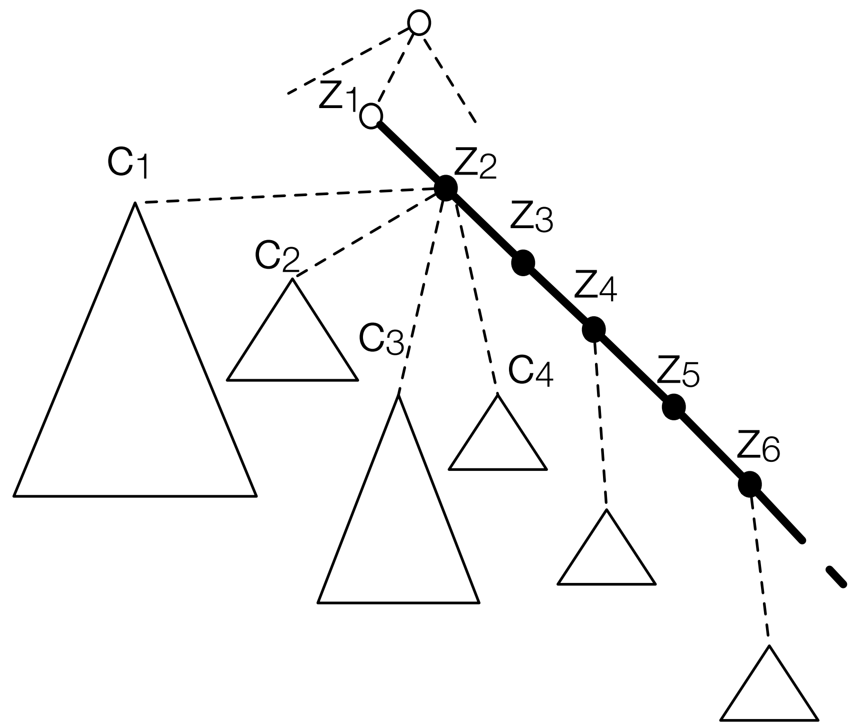
- LightIDs for children c_1, c_2, ..., c_k encoding:
  - Alphabetic code B = b_1, ..., b_k for size(c_1), ..., size(c_k).
  - We have size(c_1) + $\cdots$ + size(c_k) = lsize(v)
  - => $|b_i| \leq \log(\text{lsize}(v)) - \log(\text{size}(c_i)) + O(1)$

LightID Encoding



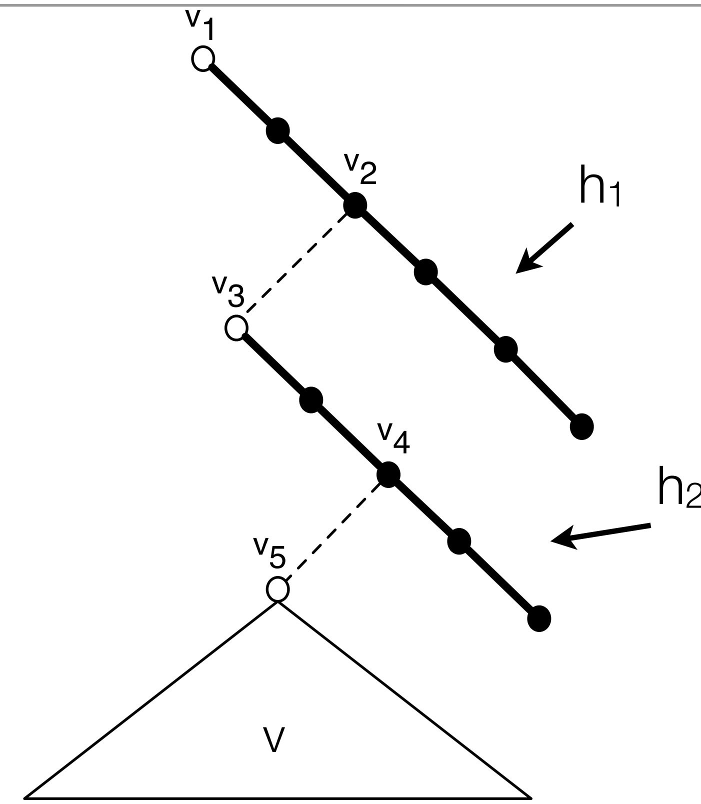
- What is the total length l of lightIDs in label(v)?
  - $l \leq \log(\mathrm{lsize}(v_2)) - \log(\mathrm{size}(v_3)) + O(1) + \log(\mathrm{lsize}(v_4)) - \log(\mathrm{size}(v_5)) + O(1) + \cdots$
  - We have $\mathrm{size}(v_3) > \mathrm{lsize}(v_4)$
  - $\Rightarrow l \leq \log(\mathrm{lsize}(v_2)) + O(1) - \log(\mathrm{size}(v_5)) + O(1) + \cdots$
  - Telescoping sum of O(log n) terms => $l = O(\log n)$

HeavyID Encoding



- HeavyIDs for nodes $z_1, z_2, \ldots, z_k$ on heavy path h encoding:
  - Alphabetic code $B = b_1, \ldots, b_k$ for $\mathrm{lsize}(z_1), \ldots, \mathrm{lsize}(z_k)$.
  - We have $\mathrm{lsize}(z_1) + \cdots + \mathrm{lsize}(z_k) = \mathrm{lsize}(h)$
  - => $|b_i| \leq \log \mathrm{lsize}(h) - \log \mathrm{lsize}(z_i) + O(1)$

HeavyID Encoding



- What is the total length l of heavyIDs in label(v)?
  - $l \leq \log(\text{lsize}(h_1)) - \log(\text{lsize}(v_2)) + O(1) + \log(\text{lsize}(h_2)) - \log(\text{lsize}(v_4)) + O(1) + \cdots$
  - We have $\text{lsize}(v_2) > \text{lsize}(h_2)$
  - $\Rightarrow l \leq \log(\text{lsize}(h_1)) + O(1) - \log(\text{lsize}(v_4)) + O(1) + \cdots$
  - Telescoping sum of $O(\log n)$ terms => $l = O(\log n)$

Summary

- The total length of lightIDs in label(v) is O(log n)
- The total length of heavyIDs in label(v) is O(log n)
- The total length of IDs in labels is O(log n)
- How do we distinguish between start and end of IDs in label(v)?

Handling Variable Length LightIDs and HeavyIDs

Variable Length Encodings

000000000001	000000000001	0000001	001	0000000000000001	00001

- Add additional *indicator label* containing 1 at the end (or start) of each ID in label.
- => Unique decoding of IDs in label.
- Doubles length of label.
- => maximum length label remains O(log n).

The Labeling Scheme

- Theorem: There is a nearest common ancestor labeling scheme for trees with maximum label length of $O(\log n)$ bits.
- Also:
  - We can compute nca in $O(1)$ time.
  - We can compute all labels in $O(n)$ time.
  - Total space is $O(n)$ ($n \cdot \log n$ bits).

Summary

- Distributed data structures
  - Parent labeling scheme
- Nearest common ancestor problem
- Nearest common ancestor labeling scheme
  - A first attempt
  - Heavy path decomposition
  - Alphabetic Codes

References

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