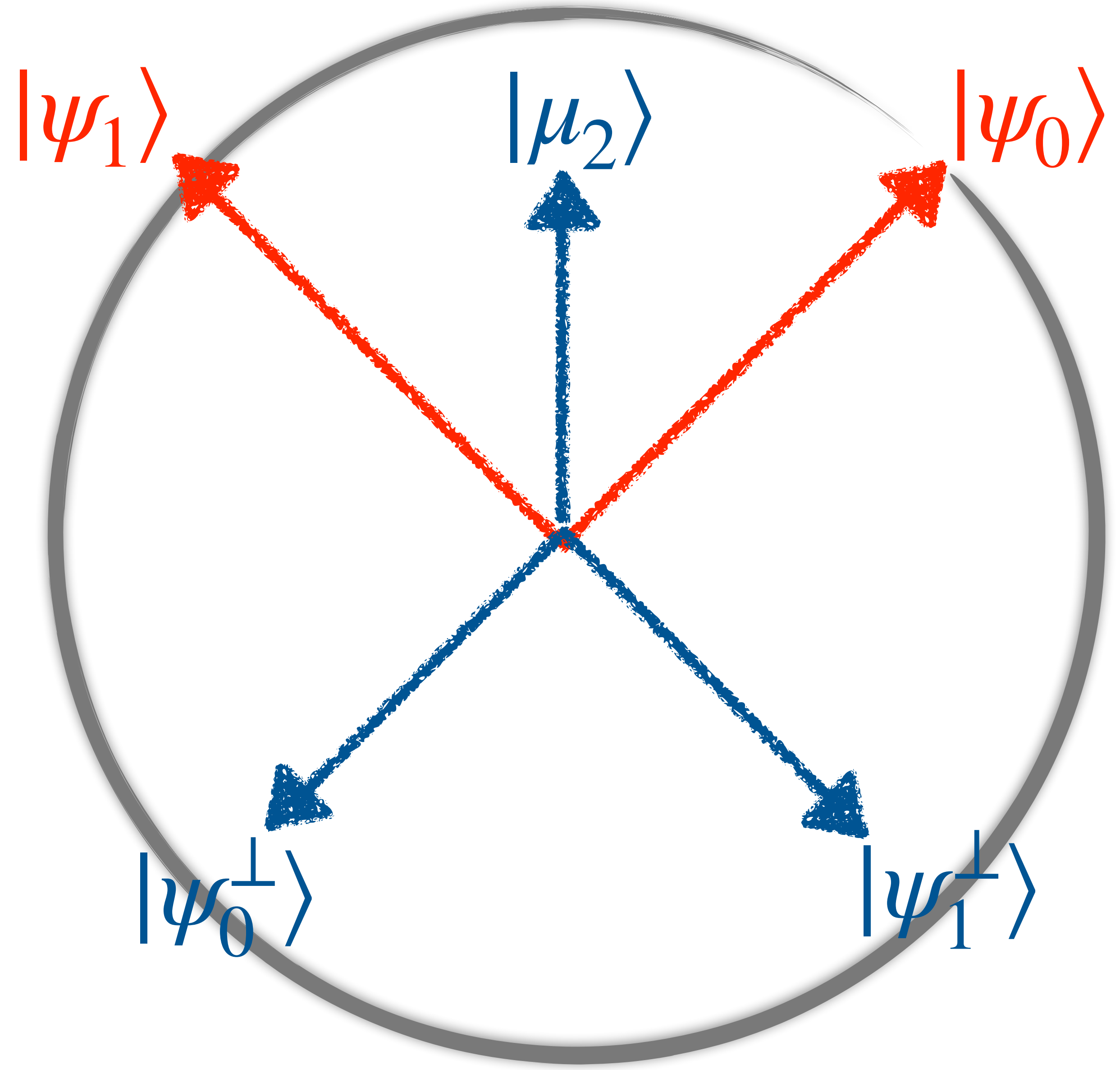
Unambiguous Discrimination

$$|\psi_0\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle \qquad \eta_0$$
$$|\psi_1\rangle = \cos\theta|0\rangle - \sin\theta|1\rangle \qquad \eta_1$$

We now consider the following 3 outcome measurement

$$M_0 = \mu_0\, |\psi_1^\perp\rangle\langle\psi_1^\perp|$$
$$M_1 = \mu_1\, |\psi_0^\perp\rangle\langle\psi_0^\perp|$$
$$M_2 = 1 - M_0 - M_1$$

$$0 \leq \mu_0, \mu_1 \leq 1$$



$$p(0|0) = \mu_0|\langle\psi_1^\perp|\psi_0\rangle|^2$$
$$p(1|0) = 0$$
$$p(2|0) = 1 - p(0|0)$$

$$p(0|1) = 0$$
$$p(1|1) = \mu_1|\langle\psi_0^\perp|\psi_1\rangle|^2$$
$$p(2|1) = 1 - p(1|1)$$

Unambiguous Discrimination

$$p(0\,|\,0) = \mu_0 |\langle \psi_1^{\perp} | \psi_0 \rangle|^2 \qquad p(0\,|\,1) = 0$$
$$p(1\,|\,0) = 0 \qquad p(1\,|\,1) = \mu_1 |\langle \psi_0^{\perp} \,|\, \psi_1 \rangle|^2$$
$$p(2\,|\,0) = 1 - p(0\,|\,0) \qquad p(2\,|\,1) = 1 - p(1\,|\,1)$$

It should be clear why this measurement is called "unambiguous"; But what do we do when we get the outcome 2?

We abstain;

The goal in unambiguous discrimination is

$$\text{minimize}_{\mu_0,\mu_1}\ \eta_0\, p(2\,|\,0) + \eta_1\, p(2\,|\,1)$$

subject to
$$\mu_0 \geq 0$$
$$\mu_1 \geq 0$$
$$M_2 \geq 0$$

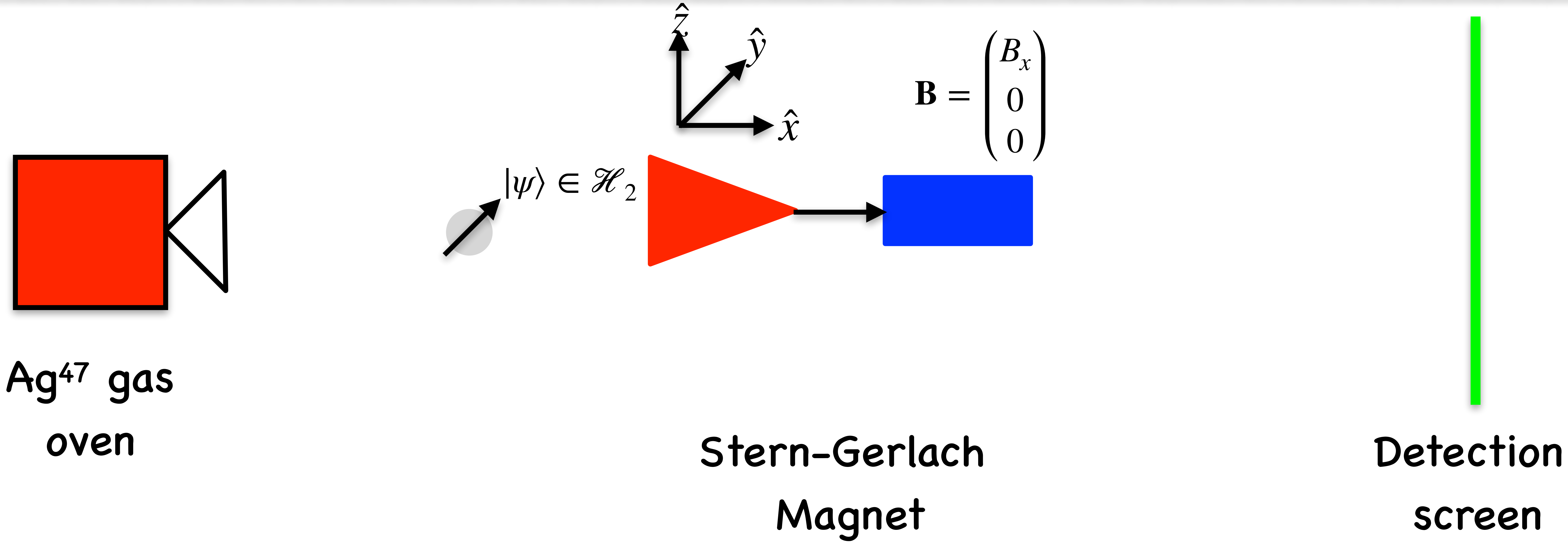
Unambiguous Discrimination

Notice that this measurement does not correspond to a projective measurement (why?)...

...and there does not exist an observable on the Hilbert space of the quantum system for implementing it.

So how do we implement such a measurement?

The Stern-Gerlach experiment

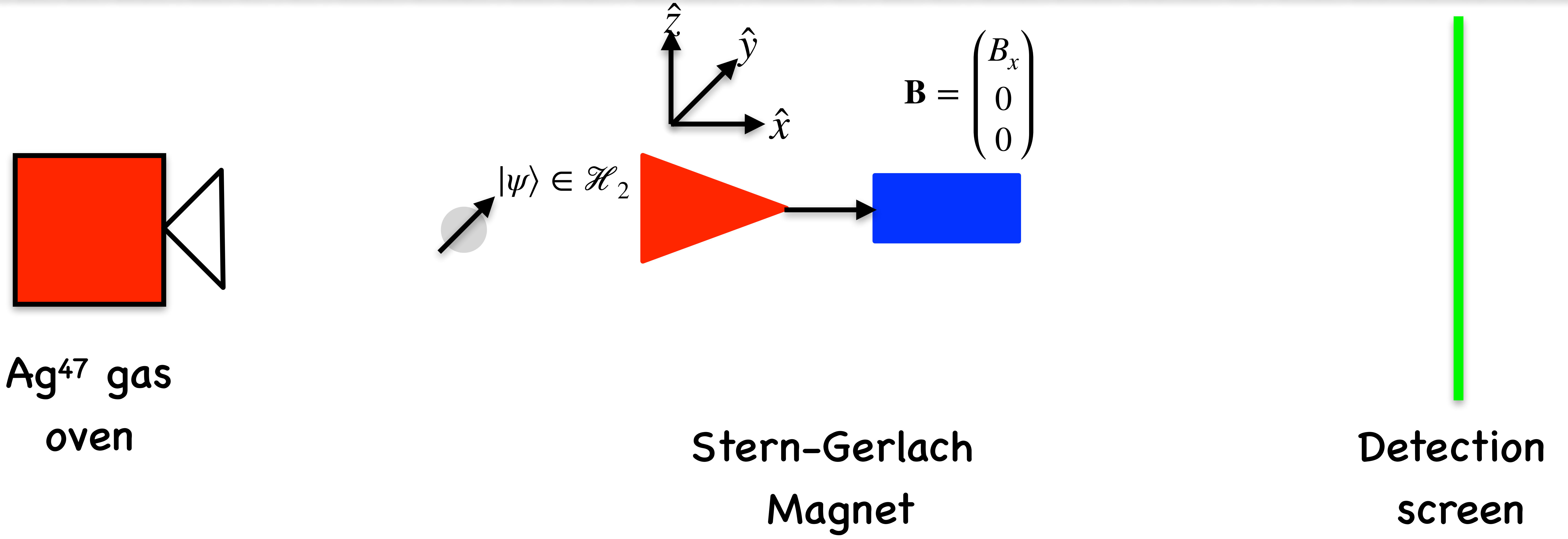


The Ag^{47} atom is a qubit in some state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

The whole Stern-Gerlach apparatus (excluding the screen) is there to measure the atom along the x-direction, i.e. to implement the projective measurement

$$M_\pm = |\pm\rangle\langle\pm|$$

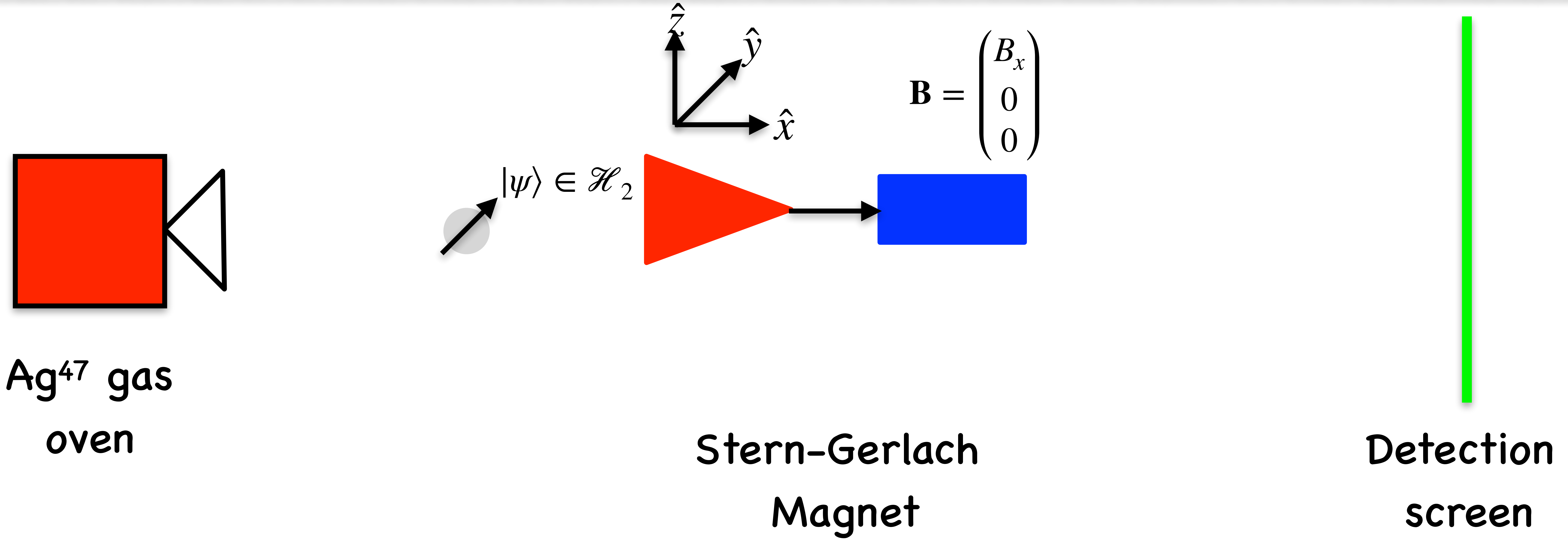
The Stern-Gerlach experiment



But the S-G magnet is a physical system too. Since it points along the x-direction lets describe it with the qubit state

$$|SG\rangle = |+\rangle$$

The Stern-Gerlach experiment



The system-plus-SG magnet interact via some unitary operator

$$U_{SA} : \mathscr{H}_S \otimes \mathscr{H}_A$$

The detection screen is us "looking" at the SG's measurement outcome.

Ancilla-Assisted Quantum Measurements

As the measurement we want to implement only has two possible outcomes, it suffices to model the measurement device as an ancilla (auxiliary) two-dimensional system—or register in some quantum state $\sigma \in \mathscr{B}(\mathscr{H}_A)$. In our case $\sigma = |+\rangle\langle+|$.

Now consider the following joint-unitary interaction on $\mathscr{H}_S \otimes \mathscr{H}_A$.

$$U_{SA} = |+\rangle_S\langle+| \otimes 1_A + |-\rangle_S\langle+| \otimes Z_A$$

Ancilla-Assisted Quantum Measurements

$$U_{SA}(|\psi\rangle_S \otimes |+\rangle_A) = U_{SA}\left(\left(\frac{\alpha+\beta}{\sqrt{2}}\right)|+\rangle_S + \left(\frac{\alpha-\beta}{\sqrt{2}}\right)|-\rangle_S\right) \otimes |+\rangle_A$$

$$= \left(\frac{\alpha+\beta}{\sqrt{2}}\right)|+\rangle_S \otimes |+\rangle_A + \left(\frac{\alpha-\beta}{\sqrt{2}}\right)|-\rangle_S \otimes |-\rangle_A$$

$$:= |\Phi\rangle_{SA}$$

Now recall that the probabilities for measuring $|\psi\rangle$ in the $|\pm\rangle$ are given by

$$p_{\pm} = |\frac{\alpha \pm b}{\sqrt{2}}|^2$$

Ancilla-Assisted Quantum Measurements

$$
\begin{aligned}
U_{SA}(|\psi\rangle_S \otimes |+\rangle_A) &= U_{SA}\left(\left(\left(\frac{\alpha+\beta}{\sqrt{2}}\right)|+\rangle_S + \left(\frac{\alpha-\beta}{\sqrt{2}}\right)|-\rangle_S\right) \otimes |+\rangle_A\right) \\
&= \left(\frac{\alpha+\beta}{\sqrt{2}}\right)|+\rangle_S \otimes |+\rangle_A + \left(\frac{\alpha-\beta}{\sqrt{2}}\right)|-\rangle_S \otimes |-\rangle_A \\
&:= |\Phi\rangle_{SA} \\
&= \sqrt{p_+}|++\rangle_{SA} + \sqrt{p_-}|--\rangle_{SA}
\end{aligned}
$$

At this point the system-plus-device are in an entangled state. More than that, they are in Schmidt form, with the Schmidt coefficients related to the probabilities of the measurement outcomes.

Also note that at this point the measurement is done. The system and apparatus are separated and no longer interact with each other.

Ancilla-Assisted Quantum Measurements

$$|\Phi\rangle_{SA} = \sqrt{p_+}\,|++\rangle_{SA} + \sqrt{p_-}\,|--\rangle_{SA}$$

$$\mathrm{tr}_A\,|\Phi\rangle_{SA}\langle\Phi| = \mathrm{tr}_S\,|\Phi\rangle_{SA}\langle\Phi| = p_+|+\rangle\langle+| + p_-|-\rangle\langle-|$$

After this interaction but before the atoms reach the screen our knowledge about either the state of the atom or the magnet is described by tracing out the relevant system.

When the atom fluoresces on the screen, we see the photons emitted and by doing so we learn what the outcome of the measurement is (also the post measurement states).

The ancillary system is called a register because it's state corresponds to the register reading on the device (red/green light).

POVMs and Naimark's Theorem

Theorem: Let $\mathscr{H}_S$ be of dimension d and let Ω be an alphabet with

$$M : \Omega \to \mathrm{Pos}(\mathscr{H}_S)$$

its corresponding POVM (i.e., $\sum_{k\in\Omega} M_k = 1_S$). Then there exists an auxiliary system, $\mathscr{H}_A = \mathbb{C}^\Omega$, and an isometry

$$V : \mathscr{H}_S \to \mathscr{H}_S \otimes \mathscr{H}_A$$

such that

$$M_k = V^\dagger \left(1_S \otimes |k\rangle\langle k|\right) V \quad \forall\, k \in \Omega$$

POVMs and Naimark's Theorem

Let V_1, V_2 be two **metric spaces** of dimension d_1, d_2, respectively. The mapping

$$f : V_1 \to V_2$$

is **distance preserving** (ισωμετρία, ίσω=equal, μέτρο=measure), if $\forall\, \mathbf{x}, \mathbf{y} \in V_1$

$$\langle f(\mathbf{x}), f(\mathbf{y}) \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$$

POVMs and Naimark's Theorem

Theorem: Let $\mathscr{H}_S$ be of dimension d and let Ω be an alphabet with

$$M : \Omega \to \mathrm{Pos}(\mathscr{H}_S)$$

its corresponding POVM (i.e., $\sum_{k \in \Omega} M_k = 1_S$). Then there exists an auxiliary system, $\mathscr{H}_A = \mathbb{C}^\Omega$, and an isometry

$$V : \mathscr{H}_S \to \mathscr{H}_S \otimes \mathscr{H}_A$$

such that

$$M_k = V^\dagger \left(1_S \otimes |k\rangle\langle k| \right) V \quad \forall\, k \in \Omega$$

POVMs and Naimark's Theorem

Proof: Choose

$$V = \sum_{k\in\Omega} M_k^{\frac{1}{2}} \otimes |k\rangle$$

and note that

$$V : \mathscr{H}_S \to \mathscr{H}_S \otimes \mathscr{H}_A$$

$$V^\dagger V = \sum_{k,l} M_k^{\frac{1}{2}} M_l^{\frac{1}{2}} \otimes \langle k|l\rangle$$

$$= \sum_{k\in\Omega} M_k = 1_S$$

so V is the isometry we are looking for

POVMs and Naimark's Theorem

Corollary: Let $\mathscr{H}_S$ be of dimension d and let Ω be an alphabet with

$$M : \Omega \to \mathrm{Pos}(\mathscr{H}_S)$$

its corresponding POVM (i.e., $\sum_{k\in\Omega} M_k = 1_S$). Introduce an auxiliary system, $\mathscr{H}_A = \mathbb{C}^\Omega$, prepared in state $|u\rangle$. Then there exists a **projective measurement** $\Pi : \Omega \to \mathrm{Pos}(\mathscr{H}_S \otimes \mathscr{H}_A)$, such that

$$\mathrm{tr}\left(\Pi_k\, \rho \otimes |k\rangle\langle k|\right) = \mathrm{tr}\left(M_k\, \rho\right)$$

$\forall \rho \in \mathscr{B}(\mathscr{H}_S)$.

POVMs and Naimark's Theorem

Proof: By Naimark's theorem we know there exists $V : \mathscr{H}_S \to \mathscr{H}_S \otimes \mathscr{H}_A$. We can complete this isometry into a full unitary operation $U \in \mathbb{U}(\mathscr{H}_S \otimes \mathscr{H}_A)$ such that

$$U\left(1_S \otimes |u\rangle\right) = V$$

holds.

Remark: Note that the "completion to a unitary" is not unique. There exists many $U \in \mathbb{U}(\mathscr{H}_S \otimes \mathscr{H}_A)$ whose action can lead to V. Indeed notice the role of the arbitrary state $|u\rangle \in \mathscr{H}_A$.

POVMs and Naimark's Theorem

Now define

$$\Pi_k = U^\dagger \left(1_S \otimes |k\rangle_A\langle k|\right) U$$

Observe that

$$\begin{aligned}\Pi_k\Pi_l &= U^\dagger \left(1_S \otimes |k\rangle_A\langle k|\right) U\, U^\dagger \left(1_S \otimes |l\rangle_A\langle l|\right) U \\ &= U^\dagger \left(1_S \otimes |k\rangle_A\langle k|\right) \left(1_S \otimes |l\rangle_A\langle l|\right) U \\ &= U^\dagger \left(1_S \otimes |k\rangle_A\langle k|l\rangle_A\langle l|\right) U \\ &= U^\dagger \left(1_S \otimes |k\rangle_A\langle k|\right) U\, \delta_{kl} \\ &= \Pi_k\, \delta_{kl}\end{aligned}$$

i.e., $\{\Pi_k\}_{k=1}^{\Omega}$ are a set of projectors ($\Rightarrow\ \exists$ an observable on $\mathscr{H}_S \otimes \mathscr{H}_A$)

POVMs and Naimark's Theorem

The probability for obtaining outcome k using this construction is

$$
\begin{aligned}
\operatorname{tr}\left(\Pi_k(\rho_S \otimes |u\rangle_A\langle u|)\right) &= \operatorname{tr}\left(U^\dagger(1_S \otimes |k\rangle_A\langle k|)U(\rho_S \otimes |u\rangle_A\langle u|)\right) \\
&= \operatorname{tr}\left(U^\dagger(1_S \otimes |k\rangle_A\langle k|)U(1_S \otimes |u\rangle_A)\rho_S(1_S \otimes_A \langle u|)\right) \\
&= \operatorname{tr}\left((1_S \otimes_A \langle u|)U^\dagger(1_S \otimes |k\rangle_A\langle k|)U(1_S \otimes |u\rangle_A)\rho_S\right) \\
&= \operatorname{tr}\left(V^\dagger(1_S \otimes |k\rangle_A\langle k|)V\rho_S\right) \\
&= \operatorname{tr}\left(M_k\rho_S\right)
\end{aligned}
$$

Hence $\Pi : \Omega \to \mathrm{Pos}(\mathscr{H}_S \otimes \mathscr{H}_A)$ faithfully implements $M : \Omega \to \mathrm{Pos}(\mathscr{H}_S)$.