povm

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1 Unambiguous binary state discrimination

Given the states:

$$|\Psi_0 >= \cos(\theta)|0 > + \sin(\theta)|1 > \quad |\Psi_1 >= \cos(\theta)|0 > - \sin(\theta)|1 >$$

where: * $\theta \in [0, \pi]$ * η_0 : probability $|\Psi_0 >$ to be chosen * η_1 : probability $|\Psi_1 >$ to be chosen with probability η_0 * $|\Psi_1 >$ is chosen with probability η_1

and the POVM:

$$\pi_0 = \mu_0|\Psi_1^\perp >< \Psi_1^\perp|\pi_1 = \mu_1|\Psi_0^\perp >< \Psi_0^\perp|\pi_2 = \not{1} - \pi_0 - \pi_1$$

subject to the constraints:

$$\mu_0 \geq 0\mu_1 \geq 0\pi_2 \geq 0$$

Proof that the minimum probability of error, given by $P(\pi_2)$, is:

$$P(\pi_2)^{min} = 2\sqrt{\eta_o\eta_1} < \Psi_0|\Psi_1 >= 2\sqrt{\eta_o\eta_1}\cos(2\theta)$$

1.1 Demostration

1.1.1 Some results

Some results that will be used:

$$|\Psi_0^\perp >= \sin(\theta)|0 > - \cos(\theta)|1 > |\Psi_1^\perp >= \sin(\theta)|0 > + \cos(\theta)|1 > \ < \Psi_0^\perp|\Psi_1 >=< \Psi_1^\perp|\Psi_0 >= \sin(2\theta)$$

where: * $< \Psi_0^\perp|\Psi_0 >= (\sin(\theta) < 0| - \cos(\theta) < 1|)(\cos(\theta)|0 > + \sin(\theta)|1 >) = \sin(\theta)\cos(\theta) < 0|0 > + \sin^2(\theta) < 0|1 > - \cos^2(\theta) < 1|0 > - \cos(\theta)\sin(\theta) < 1|1 >= 0$ * $< \Psi_1^\perp|\Psi_1 >= (\sin(\theta) < 0| + \cos(\theta) < 1|)(\cos(\theta)|0 > - \sin(\theta)|1 >) = 0$ * $< \Psi_0^\perp|\Psi_1 >= (\sin(\theta) < 0| - \cos(\theta) < 1|)(\cos(\theta)|0 > - \sin(\theta)|1 >) = \sin(\theta)\cos(\theta) < 0|0 > + \cos(\theta)\sin(\theta) < 1|1 >= 2\sin(\theta)cos(\theta) = \sin(2\theta)$ * $< \Psi_1^\perp|\Psi_0 >= (\sin(\theta) < 0| + \cos(\theta) < 1|)(\cos(\theta)|0 > + \sin(\theta)|1 >) = \sin(\theta)\cos(\theta) < 0|0 > + \cos(\theta)\sin(\theta) < 1|1 >= 2\sin(\theta)cos(\theta) = \sin(2\theta)$

The probability distribution $P(\pi_i|\Psi_j)$:

	Ψ_0	Ψ_1
π_0	$\mu_0 \sin^2(2\theta)$	0
π_1	0	$\mu_1 \sin^2(2\theta)$
π_2	$1 - \mu_0 \sin^2(2\theta)$	$1 - \mu_1 \sin^2(2\theta)$

where: * $P(\pi_0|\Psi_0) = tr(\pi_0|\Psi_0 >< \Psi_0|) = tr(\mu_0|\Psi_1^\perp >< \Psi_1^\perp|\Psi_0 >< \Psi_0|) = \mu_0| < \Psi_1^\perp|\Psi_0 > |^2 = \mu_0 \sin^2(2\theta)$ * $P(\pi_1|\Psi_1) = tr(\pi_1|\Psi_1 >< \Psi_1|) = tr(\mu_1|\Psi_0^\perp >< \Psi_0^\perp|\Psi_1 >< \Psi_1|) = \mu_1| < \Psi_0^\perp|\Psi_1 > |^2 = \mu_1 \sin^2(2\theta)$

1.1.2 Calculate $P(\pi_2)$

$P(\pi_2) = P(\pi_2|\Psi_0)P(\Psi_0)+P(\pi_2|\Psi_1)P(\Psi_1) = \eta_0(1-\mu_0 \sin^2(2\theta))+\eta_1(1-\mu_1 \sin^2(2\theta)) = \eta_0+\eta_1-(\eta_0\mu_0+\eta_1\mu_1)\sin^2(2\theta)$

This is the quantity we want to minimize. We know the quantities η_0, η_1 and θ and the unknown are μ_0 and μ_1 and we want to find the values that minimize this function and this minimum value.

The next step is to try to find a relation between μ_0 and μ_1 using the constraints we know and specially the fact that $\pi_2 \geq 0$

1.1.3 Use the fact $\pi_2 \geq 0$

The fact that $\pi_2 \geq 0$ means its eigenvalues are ≥ 0

$$\pi_2 = \not{\mathbb{1}}-\pi_0-\pi_1 = \not{\mathbb{1}}-\mu_0|\Psi_1^\perp >< \Psi_1^\perp|-\mu_1|\Psi_0^\perp >< \Psi_0^\perp| = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}-\mu_0\begin{pmatrix}\sin(\theta)\\ \cos(\theta)\end{pmatrix}\begin{pmatrix}\sin(\theta) & \cos(\theta)\end{pmatrix}-\mu_1\begin{pmatrix}\sin(\theta)\\ -\cos(\theta)\end{pmatrix}(\mathrm{s}$$

If we compute the eigenvalues:

$$\lambda_1 = \frac{1}{2}\left(-\sqrt{\mu_0^2 + 2\mu_0\mu_1\cos(4\theta) + \mu_1^2}+2-(\mu_0+\mu_1)\right) \geq 0\lambda_2 = \frac{1}{2}\left(\sqrt{\mu_0^2 + 2\mu_0\mu_1\cos(4\theta) + \mu_1^2}+2-(\mu_0+\mu_1)\right) \geq 0$$

We can see easily the relation between μ_0 and μ_1 if we write the inequalities as

$$-A + B \geq 0A + B \geq 0$$

from what we easily deduce that the only possibility is

$$A = B$$

and that means in our case:

$$\sqrt{\mu_0^2 + 2\mu_0\mu_1\cos(4\theta) + \mu_1^2} = 2-(\mu_0+\mu_1)\mu_0^2+2\mu_0\mu_1\cos(4\theta)+\mu_1^2 = 4+\mu_0^2+\mu_1^2+2\mu_0\mu_1-4\mu_0-4\mu_1 2\mu_0\mu_1\cos(4\theta) = 4+2$$

Now let's replace this relation in our expression for $P_{\pi_2}(\mu_0, \mu_1)$ so we get an expression depending only of one variable:

$$P_{\pi_2}(\mu_0, \mu_1) = 1-(\eta_0\mu_0+\eta_1\mu_1)\sin^2(2\theta) = 1-\left(\eta_0\left(\frac{2-2\mu_1}{\mu_1\cos(4\theta)-\mu_1+2}\right)+\eta_1\mu_1\right)\sin^2(2\theta) = \left(\frac{\mu_1\cos(4\theta)-\mu_1+2+}{}\right.$$

So get get an expression that only depends of one parameter μ_1. Now the nest steps is to find the minimum of this function with

$$\frac{\partial P_{\pi_2}(\mu_1)}{\partial \mu_1} = 0$$

Once we resolve this equation, we get a value for μ_1^{min} that we can verify if it is in fact a minimum using its second derivative. Once we get it we replace in P_{π_2} to get the desired result:

$$P(\pi_2)^{min} \equiv P_{\pi_2}(\mu_1^{min}) = 2\sqrt{\eta_o\eta_1}\cos(2\theta)$$