# HOW TO DIFFERENTIATE BETWEEN NON-ORTHOGONAL STATES

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An infinite sequence of generalized measurements, recently proposed by Ivanovic, can be performed in a single step. It is shown that the final result is the optimal one.

In a recent Letter with the same title as above, Ivanovic [1] considers the following problem: A mixture of quantum systems is prepared by choosing at random from a set in state $|P\rangle$ and a set in state $|Q\rangle$. How well can we retrodict the origin of each system? The simplest method is a direct measurement, for example one that distinguishes $|P\rangle$ from the orthogonal state $|\bar{P}\rangle$ (only the two-dimensional subspace spanned by $|P\rangle$ and $|Q\rangle$ need to be considered). If $|\bar{P}\rangle$ is found, we can be sure that the original state was not $|P\rangle$ and therefore was $|Q\rangle$. The probability for this to occur is $\frac{1}{2}|\langle\bar{P}|Q\rangle|^2$, which will be written as $\frac{1}{2}\sin^2 A$.

Ivanovic [1] shows that better results can be obtained by an indirect measurement method. First, the system under investigation is correlated to a probe by means of a unitary evolution. One then selects those probes which are in a specified state, and a direct measurement is performed only on the corresponding systems. The same process can then be repeated on the remaining (untested) systems, with new probes. As the number of iterations tends to infinity, the probability for unambiguous retrodiction converges to $1-\cos A$.

The purpose of this note is to show how a more efficient choice of the unitary evolution can lead to the same result in a single step. The initial state is, with Ivanovic's notations ($\cos A = 2a-1$)

$$\begin{pmatrix} a^{1/2} \\ \pm(1-a)^{1/2} \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$=\pm(1-a)^{1/2}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}+a^{1/2}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}.$$

The $\pm$ sign corresponds to initial states $|P\rangle$ and $|Q\rangle$ of the system, and the second spinor represents the probe. The unitary evolution simply is a rotation in the subpace spanned by $\binom{1}{0}\binom{1}{0}$ and $\binom{0}{1}\binom{0}{1}$:

$$a^{1/2}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$\to(1-a)^{1/2}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}+(2a-1)^{1/2}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}.$$

One thereby obtains a final state

$$(1-a)^{1/2}\begin{pmatrix}1\\\pm1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}+(2a-1)^{1/2}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}.$$

The probes which are found in state $\binom{1}{0}$ are now correlated to systems in *orthogonal* states, which can then be distinguished unambiguously. This occurs in a fraction $2(1-a)=1-\cos A$ of cases. The other probes are correlated to systems which always are in state $\binom{0}{1}$, irrespective of their part history. Therefore no further improvement is possible. Any claim to the contrary would violate unitarity. This last remark answers the query at the end of Ivanovic's paper.

## Reference

[1] I.D. Ivanovic, Phys. Lett. A 123 (1987) 257.