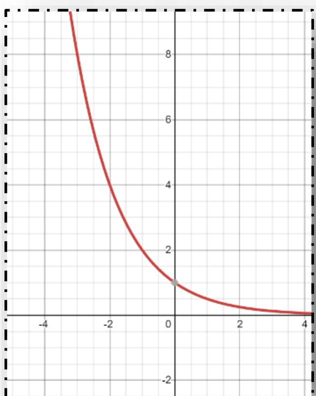
4.1-2 Exponential functions

The general form of an exponential function is $f(x) = b^x$, where $b > 0$ and $b \neq 1$.

Domain: $(-\infty, +\infty)$ **Range:** $(0, +\infty)$ **y-intercept:** $(0,1)$ **asymptote:** x-axis

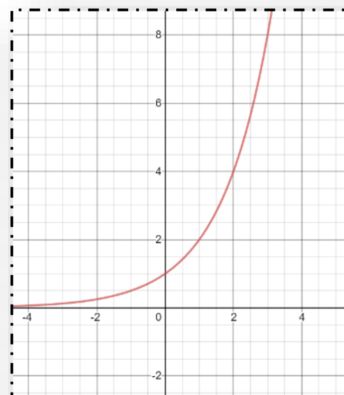
Case I $(\mathbf{0} < \boldsymbol{b} < \mathbf{1})$

Decreasing



Case II $(\boldsymbol{b} > \mathbf{1})$

Increasing



Properties:

- $b^0 = 1$
- $b^{-x} = \frac{1}{b^x}$
- $b^x \cdot b^y = b^{x+y}$
- $\frac{b^x}{b^y} = b^{x-y}$

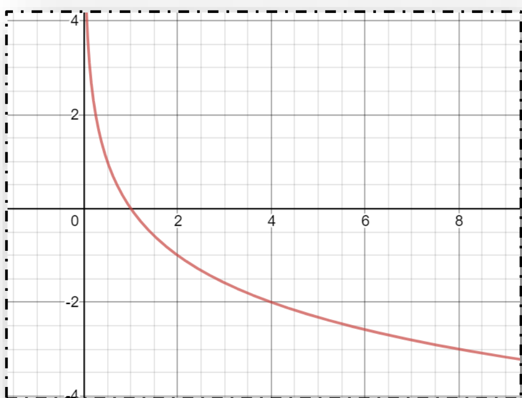
4.3-5 Logarithmic functions

The general form of a logarithmic function is $f(x) = \log_b x$, where $b > 0$ and $b \neq 1$.

Domain: $(0, +\infty)$ **Range:** $(-\infty, +\infty)$ **x-intercept:** $(1,0)$ **asymptote:** y-axis

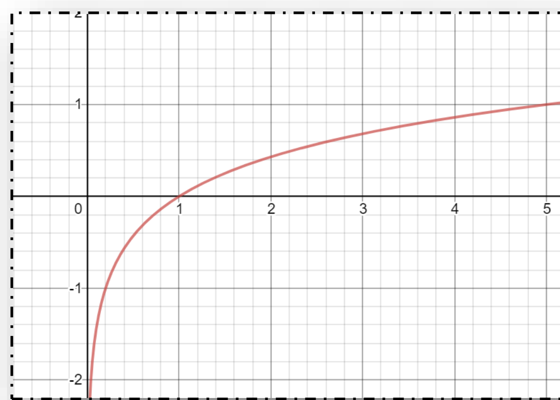
Case I $(\mathbf{0 < b < 1})$

Decreasing



Case II $(\mathbf{b > 1})$

Increasing



Relation between logarithmic and exponential functions:

$$y = \log_b x \quad \Leftrightarrow \quad b^y = x$$

Exponential form ⇔ Logarithmic form

This means that both functions are inverse of each other.

- $\log_b x = y \quad \Rightarrow \quad x = b^y$ (From logarithmic to exponential form)
- $\log_b x = y \quad \Rightarrow \quad x = b^y$ (From exponential to logarithmic form)

Remember:

- $\log_{10} x = \log x \quad (Common\ Logarithm)$
- $\log_e x = \ln x \quad (Natural\ Logarithm)$

Properties of Logarithms:

- $\log_b 0 = undefined$
- $\log_b 1 = 0$
- $\log_b b = 1$
- $\log_b b^x = x$

Laws of Logarithms:

- $\log_b(x \cdot y) = \log_b x + \log_b y$

  (Caution: $\log_b(x+y) \neq \log_b x \cdot \log_b y$)

- $\log_b(x/y) = \log_b x - \log_b y$

  (Caution: $\log_b(x-y) \neq \log_b x \,/\, \log_b y$)

- $\log_b(x^r) = r \cdot \log_b x$

  (Caution: $(\log_b x)^r \neq r \cdot \log_b x$)

Note:

$\log_b(x^2) = 2 \cdot \log_b x = \log_b x + \log_b x$ Whereas $(log_b\, x)^2 = log_b\, x \cdot log_b\, x$

Change of base formula:

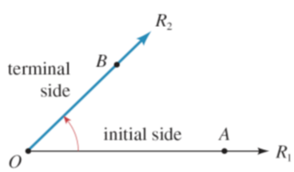
$$\log_b a = \frac{\log a}{\log b} \quad or \quad \frac{\ln a}{\ln b}$$

Some useful results:

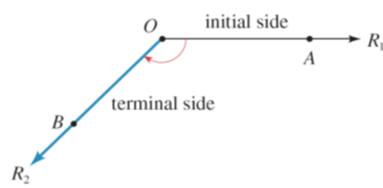
- $\log_b a = \frac{1}{\log_a b}$
- $\log_{1/b} x = -\log_b x$
- $\log_{b^n} x = \log_b x^{1/n}$

5.1 Angle Measure

- An angle is the rotation of a ray from one position (initial side) to another position (terminal side).
- **Starting side** is called the **initial side** and the **ending side** is called the **terminal side**.
- An angle is in **standard position** if its **vertex is at origin** and **initial side is along positive x-axis**.
- If the rotation is **anti-clockwise then it's a positive** angle and if the rotation is **clockwise then it is a negative angle**.



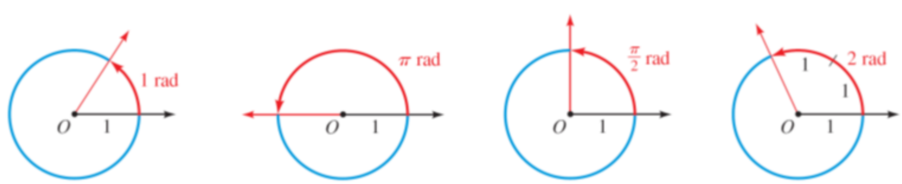
Positive angle



Negative angle

Measurement of angles:

- 1 rotation (one complete circle) = $360° = 2\pi$ radians.
- $\theta° = \theta \times \frac{\pi}{180}$ radians $\quad$ $(From\ degrees\ to\ radian\ conversion)$
- θ radians $= \left(\theta \times \frac{180}{\pi}\right)^{\circ}$ $\quad$ $(From\ radians\ to\ degree\ conversion)$



Radian measure

Types of angles:

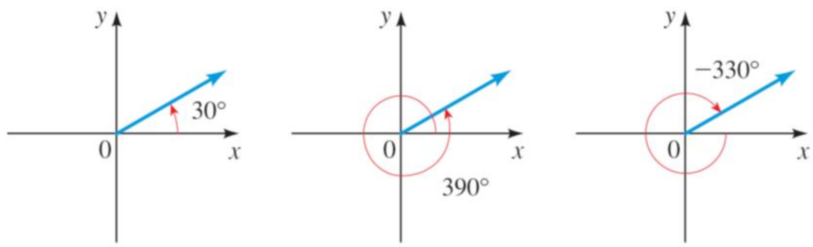
- Zero angle $\theta = 0° = 0$ radians.
- Right-angle $\theta = 90° = \frac{\pi}{2}$ radians.
- Straight-angle $\theta = 180° = \pi$ radians.
- Acute angle $0° < \theta < 90°$ or $0 < \theta < \frac{\pi}{2}$.
- Obtuse angle $90° < \theta < 180°$ or $\frac{\pi}{2} < \theta < \pi$.
- Quadrantal angles are all those angles that are along x-axis or y-axis i.e. $\theta = k \cdot 90°$ or $\theta = k \cdot \frac{\pi}{2}$ where k is any integer.
- Any two angles α and β are called complementary angles if $\alpha + \beta = 90° = \frac{\pi}{2}$ radians.
- Any two angles α and β are called supplementary angles if $\alpha + \beta = 180° = \pi$ radians.

Co-terminal angles:

Any two angles in standard position are co-terminal if their **terminal sides coincides** (matches) each other.

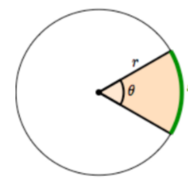
- If α is any angle measure in degrees then $\acute{\alpha}$, the co-terminal angle is given by the formula $\boldsymbol{\acute{\alpha} = \alpha + k \cdot 360°}$ where k is any non-zero integer.

- If α is any angle measure in radians then $\acute{\alpha}$, the co-terminal angle is given by the formula $\boldsymbol{\acute{\alpha} = \alpha + k \cdot 2\pi}$ where k is any non-zero integer.



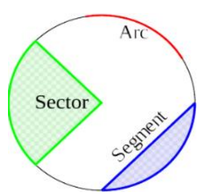
Length of a circular arc:

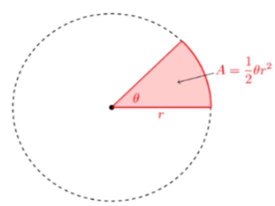
$s = r \cdot \theta$ where $s = arc\ length$, $r = radius$ and $\theta = angle$ subtended by the arc.



Area of a circular sector:

$A = \frac{1}{2} \cdot r^2 \cdot \theta$ where $A = area\ of\ the\ region$, $r = radius$ and $\theta = angle\ of\ the\ region$.





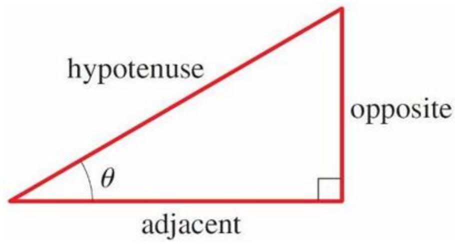
Linear and Angular Speed:

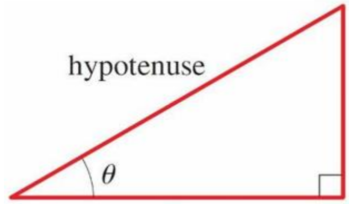
Angular and linear speeds of a point moving along a circular path are given by following formulas:

Angular speed: $\omega = \frac{\theta}{t}$ Linear speed: $v = \frac{s}{t}$

Relation between linear and angular speed: $v = r \cdot \omega$

5.2 Trigonometry of Right Triangles





Trigonometry ratios:

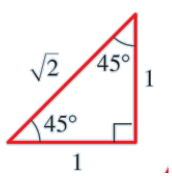
Let $p = perpendicular,\ b = base$ and $h = hypotenuse,$ then:

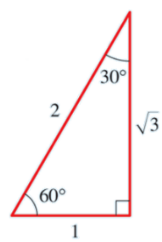
$$\sin\theta = \frac{p}{h} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{h}{p}$$

$$\cos\theta = \frac{b}{h} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{h}{b}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{p}{b} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{b}{p}$$

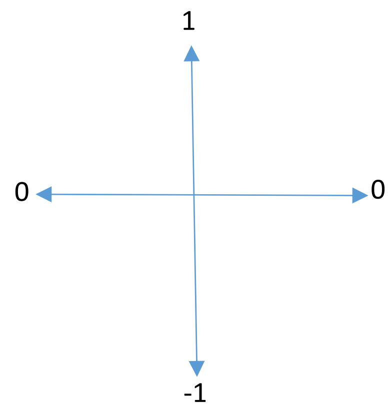
Special Triangles:



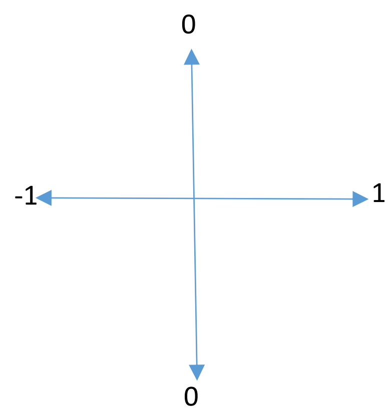


θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞

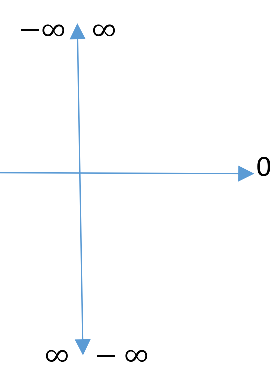
$\sin\theta$



$\cos\theta$



$\tan\theta$

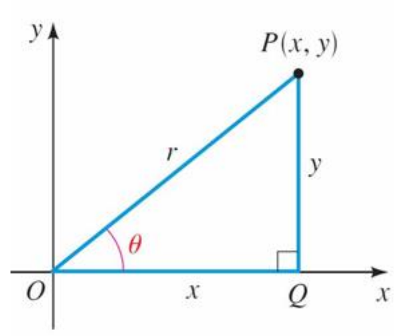


5.3 Trigonometric Functions of Angles

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{r}{y}$$

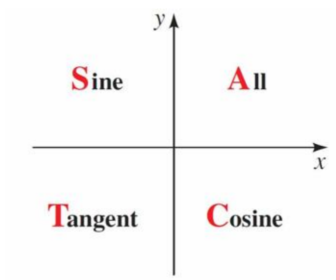
$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{r}{x}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}$$



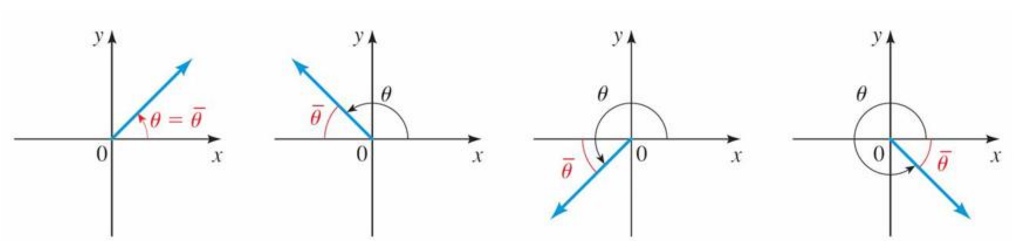
Where $\quad r = \sqrt{x^2 + y^2}$

Sign rules for trigonometric functions:



Reference Angle:

Smallest positive angle with x-axis is called a reference angle.



The reference angle $\bar{\theta}$ for an angle θ

- Quadrant I $\bar{\theta} = \theta$
- Quadrant II $\bar{\theta} = 180° - \theta = \pi - \theta$
- Quadrant III $\bar{\theta} = \theta - 180° = \theta - \pi$
- Quadrant IV $\bar{\theta} = 360° - \theta = 2\pi - \theta$

Co-function Identities:

$\sin(90^{\circ} - \theta) = \cos\theta$ $\cos(90^{\circ} - \theta) = \sin\theta$ $\tan(90^{\circ} - \theta) = \cot\theta$

$\csc(90^{\circ} - \theta) = \sec\theta$ $\sec(90^{\circ} - \theta) = \csc\theta$ $\cot(90^{\circ} - \theta) = \tan\theta$

Trigonometric Identities:

Pythagorean Identities:

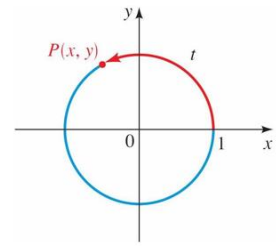
- $\sin^2\theta + \cos^2\theta = 1$
- $1 + \tan^2\theta = \sec^2\theta$
- $1 + \cot^2\theta = \csc^2\theta$

Areas of Triangles:

The area $\mathscr{A}$ of a triangle with sides of lengths a and b and with included angle θ is

$$\mathscr{A} = \tfrac{1}{2}ab\sin\theta$$

6.2 Trigonometric Functions of Real Numbers



Consider a circle of radius 1 then $s = r \cdot \theta \ \Rightarrow t = 1 \cdot \theta = \theta$, therefore:

$$\sin t = y \qquad \csc t = \frac{1}{\sin t} = \frac{1}{y}$$

$$\cos t = x \qquad \sec t = \frac{1}{\cos t} = \frac{1}{x}$$

$$\tan t = \frac{\sin t}{\cos t} = \frac{y}{x} \qquad \cot t = \frac{1}{\tan t} = \frac{x}{y}$$

Hence the coordinates of any point $P(x, y)$ on the circle are given by: $(\cos t\,, \sin t)$ where t is the angle subtended by the ray $\overrightarrow{OP}$.

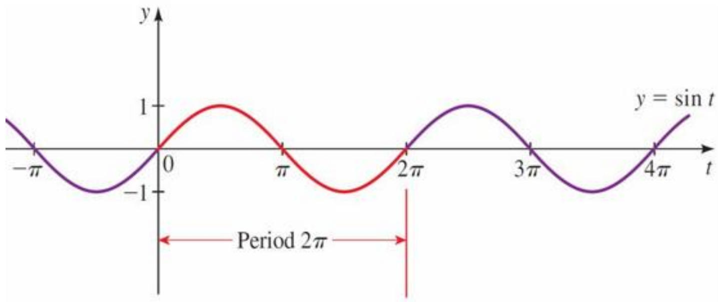
Note:

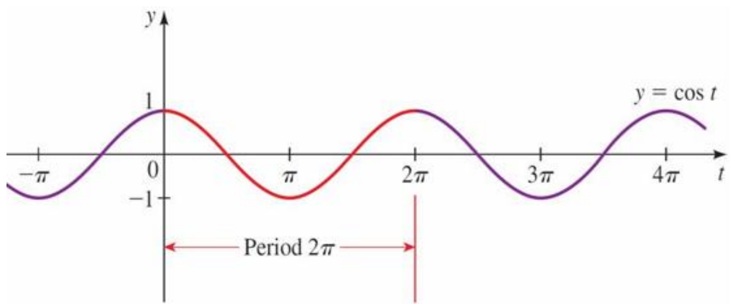
- We use reference angles to find the trigonometric functions for any real number t.
- $\sin\theta$ and $\tan\theta$ are odd functions whereas $\cos\theta$ is an even function. i.e. $\sin(-\theta) = -\sin(\theta)$, $\tan(-\theta) = -\tan(\theta)$ and $\cos(-\theta) = \cos(\theta)$.

6.3-4 Trigonometric Graphs

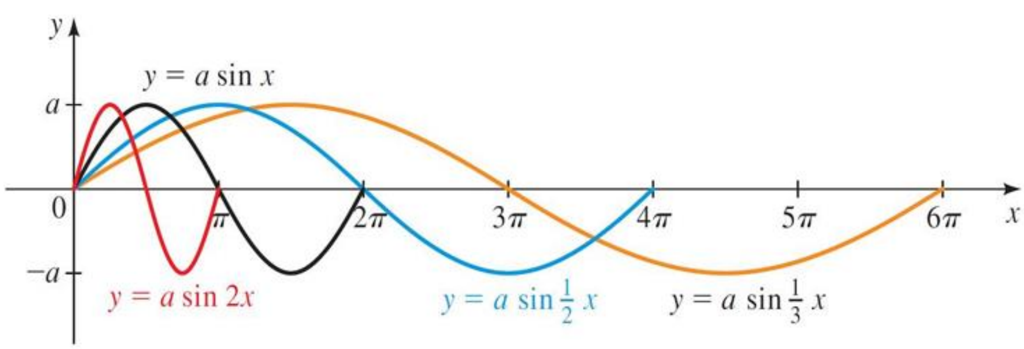
All trigonometric functions are periodic i.e. their values (and hence their graphs) repeat after certain intervals called periods. $\sin\theta$, $\cos\theta$, $\sec\theta$ and $\csc\theta$ have periods of 2π whereas $\tan\theta$ and $\cot\theta$ have period π.

- $\sin(\theta + 2\pi) = \sin\theta$, $\qquad \cos(\theta + 2\pi) = \cos\theta$, $\qquad \tan(\theta + \pi) = \tan\theta$
- $\csc(\theta + 2\pi) = \csc\theta$, $\qquad \sec(\theta + 2\pi) = \sec\theta$, $\qquad \cot(\theta + \pi) = \cot\theta$





Transformation of trigonometric functions:



Consider the following two functions:

$y = a \sin k(x - b) + c$ and $y = a \cos k(x - b) + c$

These two functions have following transformations:

Amplitude $= |a|$, Period $= \frac{2\pi}{k}$, Horizontal shift $= b$, Vertical shift $= c$

- An interval on which it completes one period $= \left[b,\ b + \frac{2\pi}{k}\right]$

Graphs of tan θ and cot θ:

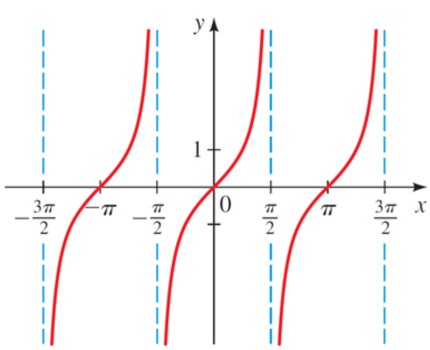
Consider the following two functions:

$y = a\tan k(x - b) + c$ and $y = a\cot k(x - b) + c$

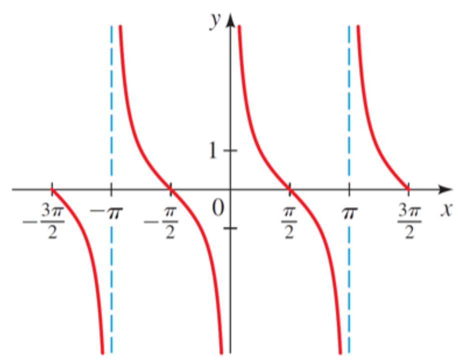
These two functions have following transformations:

Period $= \frac{\pi}{k}$, Horizontal shift $= b$, Vertical shift $= c$

- An interval on which $y = a\tan k(x - b) + c$ completes one period $= \left[b - \frac{\pi}{2k},\ b + \frac{\pi}{2k}\right]$
- An interval on which $y = a\cot k(x - b) + c$ completes one period $= \left[b,\ b + \frac{\pi}{k}\right]$



$y = \tan x$



$y = \cot x$

Graphs of $\sec\theta$ and $\csc\theta$:

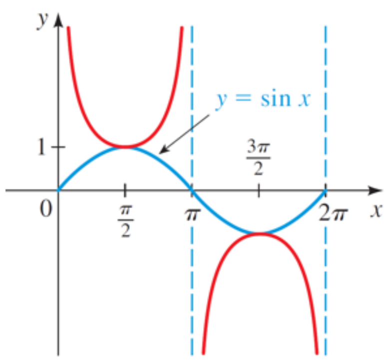
Consider the following two functions:

$y = a\sec k(x-b)+c$ and $y = a\csc k(x-b)+c$

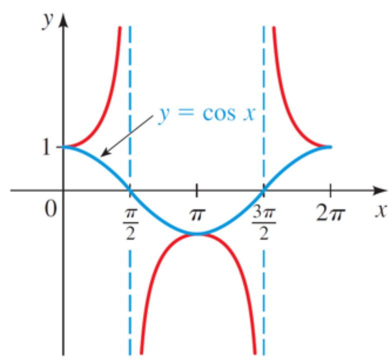
These two functions have following transformations:

Period $=\frac{2\pi}{k}$, Horizontal shift $=b$, Vertical shift $=c$

- An interval on which it completes one period $=\left[b,\ \ b+\frac{2\pi}{k}\right]$

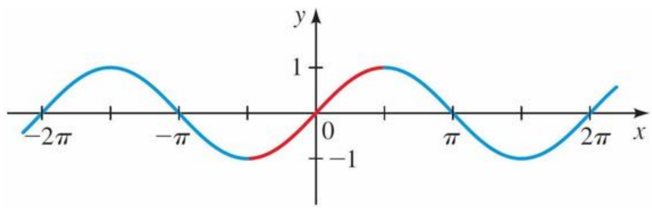


One period of $y=\csc x$

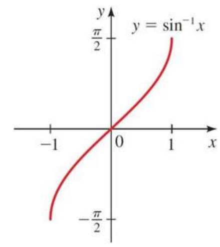


One period of $y=\sec x$

5.4, 6.5 Inverse Trigonometric Functions and Right Triangles

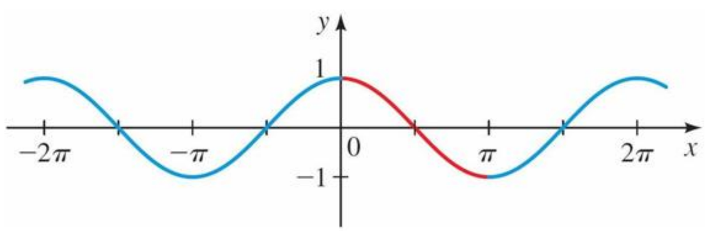


y = sin x

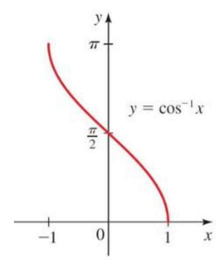


Graphs of $y = \sin^{-1}x$

$$\sin(\sin^{-1}x) = x \quad \text{for} \quad -1 \le x \le 1$$
$$\sin^{-1}(\sin x) = x \quad \text{for} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

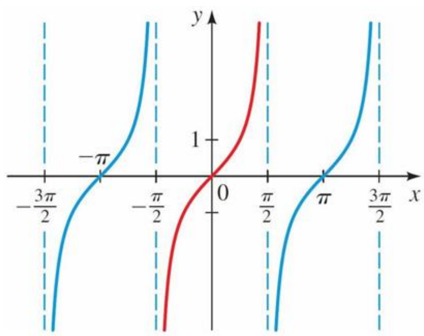


y = cos x

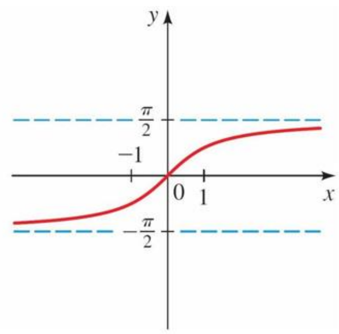


Graph of $y = \cos^{-1} x$

$$\cos(\cos^{-1}x) = x \quad \text{for} \quad -1 \le x \le 1$$
$$\cos^{-1}(\cos x) = x \quad \text{for} \quad 0 \le x \le \pi$$

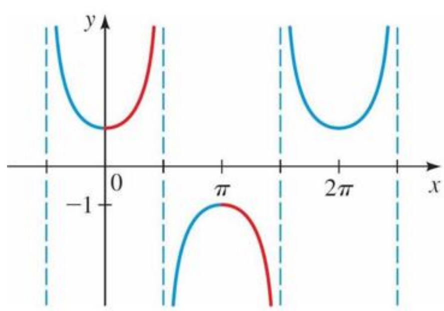


$y = \tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$

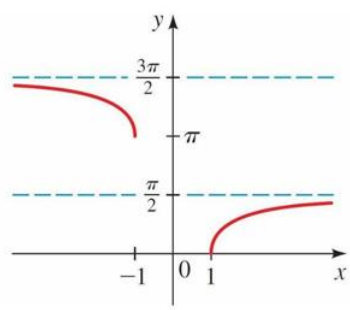


$y = \tan^{-1} x$

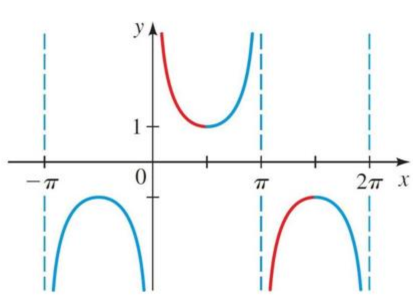
$$\tan(\tan^{-1} x) = x \quad \text{for} \quad x \in \mathbb{R}$$
$$\tan^{-1}(\tan x) = x \quad \text{for} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$



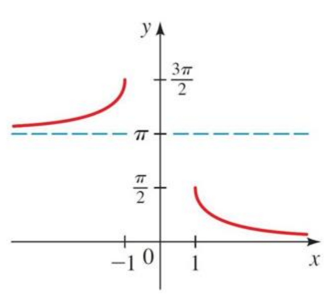
$y = \sec x,\ 0 \le x < \frac{\pi}{2},\ \pi \le x < \frac{3\pi}{2}$



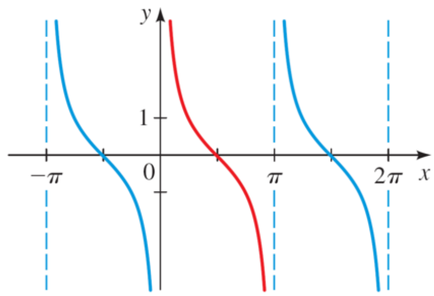
$y = \sec^{-1} x$



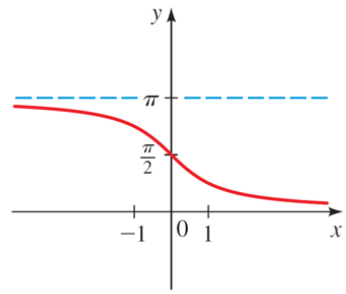
$y = \csc x,\ 0 < x \le \frac{\pi}{2},\ \pi < x \le \frac{3\pi}{2}$



$y = \csc^{-1} x$



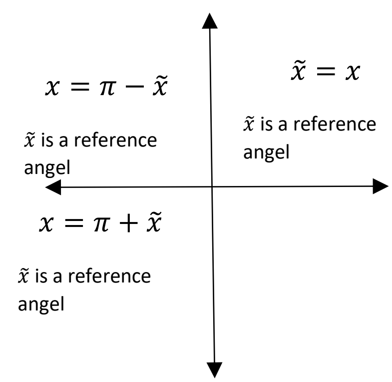
$y = \cot x,\ 0 < x < \pi$



$y = \cot^{-1} x$

Important results:

- $\sin^{-1}(-x) = -\sin^{-1}(x)$ because $\sin^{-1}(-x) \in$ Q-IV
- $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$ because $\cos^{-1}(-x) \in$ Q-II
- $\tan^{-1}(-x) = -\tan^{-1}(x)$ because $\tan^{-1}(-x) \in$ Q-IV
- $\cot^{-1}(-x) = \pi - \cot^{-1}(x)$ because $\cot^{-1}(-x) \in$ Q-II
- $\csc^{-1}(-x) = \pi + \csc^{-1}(x)$ because $ccs^{-1}(-x) \in$ Q-III
- $\sec^{-1}(-x) = \pi + \sec^{-1}(x)$ because $\sec^{-1}(-x) \in$ Q-III



Important identities:

- $\cot^{-1}(x) = \tan^{-1}\left(\frac{1}{x}\right)$ where $x > 0.$
- $\csc^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right)$ where $x > 0.$
- $\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right)$ where $x > 0.$

7.1 Trigonometry Identities

FUNDAMENTAL TRIGONOMETRIC IDENTITIES

Reciprocal Identities

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

Even-Odd Identities

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x \qquad \tan(-x) = -\tan x$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x \qquad \tan\left(\frac{\pi}{2} - x\right) = \cot x \qquad \sec\left(\frac{\pi}{2} - x\right) = \csc x$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x \qquad \cot\left(\frac{\pi}{2} - x\right) = \tan x \qquad \csc\left(\frac{\pi}{2} - x\right) = \sec x$$

7.2 Addition and Subtraction Formulas

ADDITION AND SUBTRACTION FORMULAS

Formulas for sine:

$$\sin(s + t) = \sin s \cos t + \cos s \sin t$$

$$\sin(s - t) = \sin s \cos t - \cos s \sin t$$

Formulas for cosine:

$$\cos(s + t) = \cos s \cos t - \sin s \sin t$$

$$\cos(s - t) = \cos s \cos t + \sin s \sin t$$

Formulas for tangent:

$$\tan(s + t) = \frac{\tan s + \tan t}{1 - \tan s \tan t}$$

$$\tan(s - t) = \frac{\tan s - \tan t}{1 + \tan s \tan t}$$

SUMS OF SINES AND COSINES

If A and B are real numbers, then

$$A \sin x + B \cos x = k \sin(x + \phi)$$

where $k = \sqrt{A^2 + B^2}$ and ϕ satisfies

$$\cos \phi = \frac{A}{\sqrt{A^2 + B^2}} \quad \text{and} \quad \sin \phi = \frac{B}{\sqrt{A^2 + B^2}}$$

7.3 Double Angle and Half Angle Formulas

DOUBLE-ANGLE FORMULAS

Formula for sine: $\sin 2x = 2 \sin x \cos x$

Formulas for cosine:
$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= 1 - 2\sin^2 x \\ &= 2\cos^2 x - 1 \end{aligned}$$

Formula for tangent: $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$

FORMULAS FOR LOWERING POWERS

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

HALF-ANGLE FORMULAS

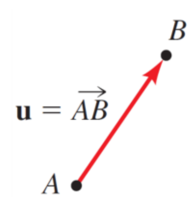
$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}} \qquad \cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$$

$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\cos u}$$

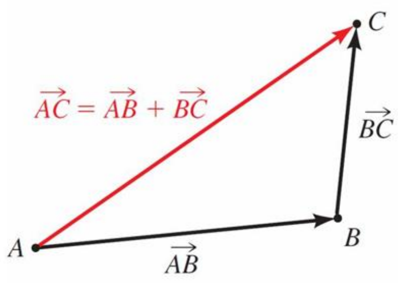
The choice of the $+$ or $-$ sign depends on the quadrant in which $u/2$ lies.

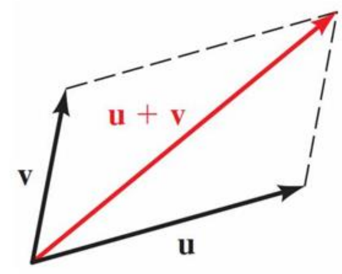
9.1 Vectors

- A vector (as shown in the adjacent figure) is a line segment with a direction.
- Vectors are usually represented by bold letters $\left(i.e.\ \boldsymbol{u} = \overrightarrow{AB}\right)$. Point A is called the initial point and point B is called the terminal point of the vector denoted by $\overrightarrow{AB}$.
- The length or magnitude of the vector is given by $\left|\overrightarrow{AB}\right|$.
- Two vector are equal if they have same magnitude as well as direction.
- Zero vector is a vector with no direction and zero magnitude.
- Unit vector is a vector whose magnitude is one.



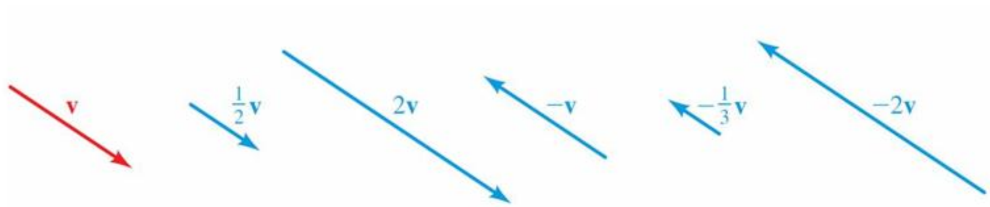
Addition of vectors:





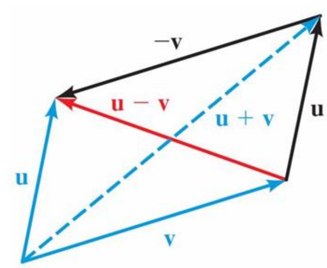
Addition of vectors

Multiplication of a vector by a scalar:



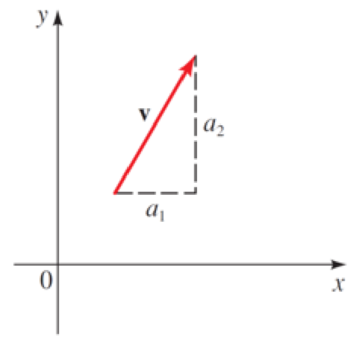
Multiplication of a vector by a scalar

Difference of two vectors:



Subtraction of vectors

Vectors in the coordinate plane:

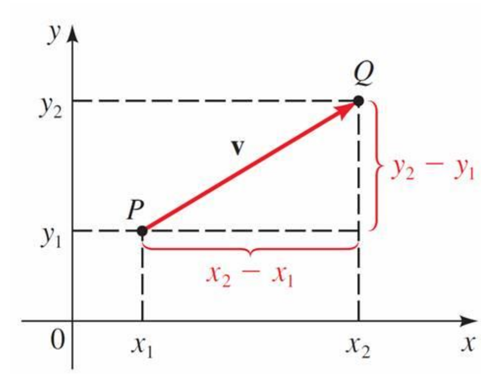


- Let $\boldsymbol{v} = \langle \boldsymbol{a_1}, \boldsymbol{a_2} \rangle$ be a vector in xy −plane.
- $\boldsymbol{a_1}$ is the horizontal component and $\boldsymbol{a_2}$ is the vertical component of the vector.

Component form of a Vector:

If $\boldsymbol{v}$ is a vector in the plane with initial point $P(x_1, \ y_1)$ and terminal point $Q(x_2, \ y_2)$ then;

$$\boldsymbol{v} = \langle x_2 - x_1, y_2 - y_1 \rangle$$



Magnitude of a Vector:

The magnitude or length of a vector $\boldsymbol{v} = \langle a, b \rangle$ is $|\boldsymbol{v}| = \sqrt{a^2 + b^2}$.

Algebraic Operations on Vectors:

Let $\boldsymbol{u} = \langle a_1, b_1 \rangle$ and $\boldsymbol{v} = \langle a_2, b_2 \rangle$ be any two vectors and c is a real number then

- $\boldsymbol{u} + \boldsymbol{v} = \langle a_1, b_1 \rangle + \langle a_2, b_2 \rangle = \langle a_1 + a_2, b_1 + b_2 \rangle$
- $\boldsymbol{u} - \boldsymbol{v} = \langle a_1, b_1 \rangle - \langle a_2, b_2 \rangle = \langle a_1 - a_2, b_1 - b_2 \rangle$
- $c\boldsymbol{u} = c\langle a_1, b_1 \rangle = \langle ca_1, cb_1 \rangle$

Properties of Vectors:

Let $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$ be any two vectors and c, d are real numbers

Vector addition

- $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$
- $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$
- $\boldsymbol{u} + \boldsymbol{0} = \boldsymbol{u}$
- $\boldsymbol{u} + (-\boldsymbol{u}) = \boldsymbol{0}$

Length of a vector

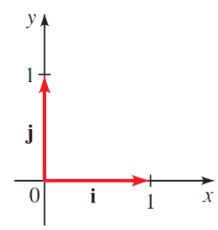
- $|c\boldsymbol{u}| = |c||\boldsymbol{u}|$

Multiplication by a scalar

- $c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$
- $(c + d)\boldsymbol{u} = c\boldsymbol{u} + d\boldsymbol{u}$
- $(cd)\boldsymbol{u} = c(d\boldsymbol{u}) = d(c\boldsymbol{u})$
- $1(\boldsymbol{u}) = \boldsymbol{u}$
- $0(\boldsymbol{u}) = \boldsymbol{0}$
- $c(\boldsymbol{0}) = \boldsymbol{0}$

Unit vector:

- A unit vector is a vector with magnitude 1. e.g. $\boldsymbol{v} = \langle \frac{\boldsymbol{3}}{\boldsymbol{5}}, \frac{\boldsymbol{4}}{\boldsymbol{5}} \rangle$
- Two useful unit vectors are $\boldsymbol{i} = \langle \boldsymbol{1}, \boldsymbol{0} \rangle$ (along x-axis) and $\boldsymbol{j} = \langle \boldsymbol{0}, \boldsymbol{1} \rangle$ (along y-axis)
- Unit vector in the direction of any vector $\boldsymbol{v}$ is given by $\widehat{\boldsymbol{v}} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|}$

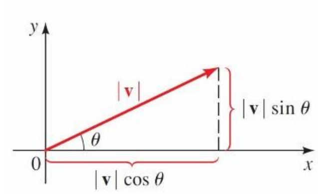


Vectors in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$:

The vector $\boldsymbol{u} = \langle a, b \rangle$ can be expressed in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$ as:

$$\boldsymbol{u} = \langle a, b \rangle = a\,\boldsymbol{i} + b\,\boldsymbol{j} = |\boldsymbol{u}| \cos\theta\ \ \boldsymbol{i} + |\boldsymbol{u}| \sin\theta\ \ \boldsymbol{j}$$

Where the reference angle $\tilde{\theta}$ of the anlge θ is given by $\tilde{\theta} = \tan^{-1}\left|\frac{b}{a}\right|$



9.2 The Dot Product

Dot Product:

Let $\boldsymbol{u} = \langle a_1, b_1 \rangle$ and $\boldsymbol{v} = \langle a_2, b_2 \rangle$ be any two vectors then their dot product is defined by

$$\boldsymbol{u} \cdot \boldsymbol{v} = a_1 a_2 + b_1 b_2$$

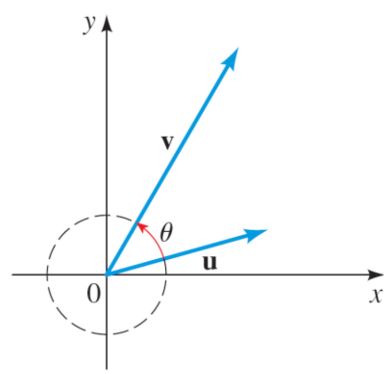
Properties of the Dot Product:

- $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$
- $(c\boldsymbol{u}) \cdot \boldsymbol{v} = c(\boldsymbol{v} \cdot \boldsymbol{u}) = \boldsymbol{u} \cdot (c\boldsymbol{v})$
- $(\boldsymbol{u} + \boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \boldsymbol{w}$
- $|\boldsymbol{u}|^2 = \boldsymbol{u} \cdot \boldsymbol{u}$

The Dot Product Theorem:

Let θ be the angle between two nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$**,** then

$$\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}| \cos \boldsymbol{\theta}$$



Angle between two vectors:

Let θ be the angle between two nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$**,** then

$$\cos \boldsymbol{\theta} = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|} = \frac{a_1 a_2 + b_1 b_2}{\sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2}}$$

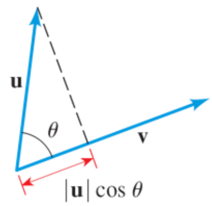
Orthogonal vectors:

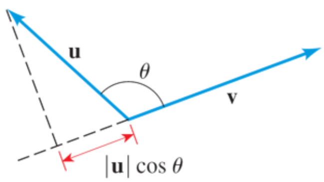
Two nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal (perpendicular) if and only if $\boldsymbol{u} \cdot \boldsymbol{v} = 0$

The component of $\boldsymbol{u}$ along $\boldsymbol{v}$:

The component of $\boldsymbol{u}$ along $\boldsymbol{v}$ (also called the component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$ or the scalar projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$) is defined to be $|\boldsymbol{u}|\cos\boldsymbol{\theta}$ where $\boldsymbol{\theta}$ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

$$\text{comp}_{\boldsymbol{v}}\boldsymbol{u} = |\boldsymbol{u}|\cos\boldsymbol{\theta} = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{|\boldsymbol{v}|}$$





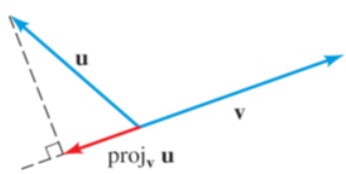
The vector projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$:

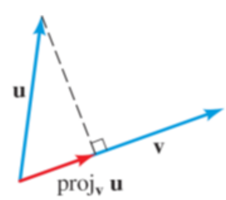
The projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$ is the vector $\text{proj}_{\boldsymbol{v}}\boldsymbol{u}$ given by

$$\text{proj}_{\boldsymbol{v}}\boldsymbol{u} = \left(\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{|\boldsymbol{v}|^2}\right)\boldsymbol{v}$$

If the vector $\boldsymbol{u}$ is resolved into $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$, where $\boldsymbol{u_1}$ is parallel to $\boldsymbol{v}$ and $\boldsymbol{u_2}$ is orthogonal to $\boldsymbol{v}$, then

$$\boldsymbol{u_1} = \text{proj}_{\boldsymbol{v}}\boldsymbol{u} \quad \text{and} \quad \boldsymbol{u_2} = \boldsymbol{u} - \text{proj}_{\boldsymbol{v}}\boldsymbol{u}$$





Work:

The work $\boldsymbol{W}$ done by a force $\boldsymbol{F}$ in moving along a vector $\boldsymbol{D}$ is $\boldsymbol{W} = \boldsymbol{F}\cdot\boldsymbol{D}$

11.4 Determinants

- Determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$
- The minor M_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting ith row and jth column of A.
- The cofactor A_{ij} of the element a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$.
- **THE DETERMINANT OF A SQUARE MATRIX**

  If A is an $n \times n$ matrix, then the **determinant** of A is obtained by multiplying each element of the first row by its cofactor and then adding the results. In symbols,

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$$

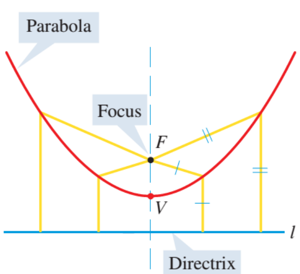
- The inverse of a square matrix A exist if and only if $\det(A) \neq 0$.
- If matrix B is obtained from matrix A by adding a multiple of one row/column another row/column then $\det(A) = \det(B)$.

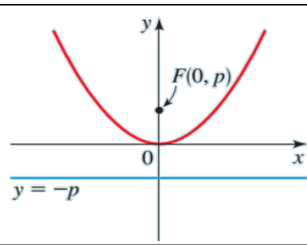
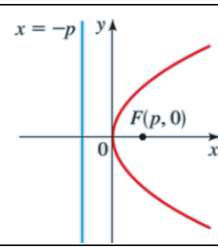
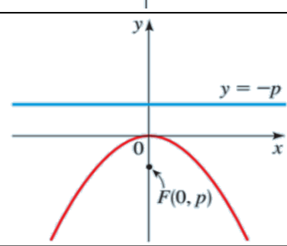
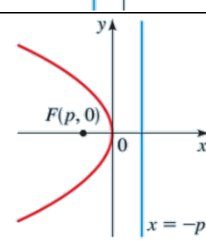
Properties of Determinants:

- If all the rows of a matrix are interchanged with its columns then the resulting determinant will remain same.
- If any two rows or columns of a matrix are interchanged then the sing of its determinant will be changed.
- If any two rows/columns of a matrix are equal then its determinant will be zero.
- If all elements in any row/column of a matrix are zero then its determinant will be zero.

- If matrix B is obtained from matrix A by multiplying one row/column by a real number k then, $\det(B) = k \cdot \det(A)$.
- If a row/column of a square matrix A is a multiple of another row/column then $\det(A) = 0$.
- If $A = [a_{ij}]$ is an $n \times n$ triangular matrix, then $|A| = a_{11}\, a_{22}\, a_{33} \cdots a_{nn}$
- Let A and B be two square matrix then $\det(A \cdot B) = \det(A) \cdot \det(B)$
- $\det(A^{-1}) = \frac{1}{\det(A)}$

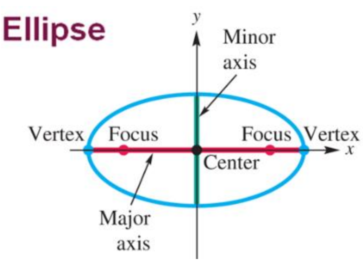
12.1 Parabola

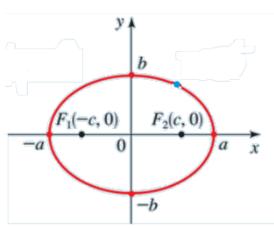
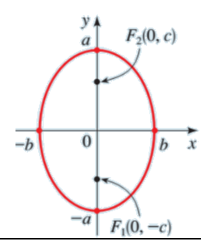


	$x^2 = 4py$ Vertex $V = (0,0)$ Focus $F = (0,p)$ Directrix $y = -p$	$y^2 = 4px$ Vertex $V = (0,0)$ Focus $F = (p,0)$ Directrix $x = -p$
$p > 0$	 	
$p < 0$	 	

	Vertical Parabola (Axis of sym is parallel to y-axis)	Horizontal Parabola (Axis of sym is parallel to x-axis)
Standard equation:	$(x-h)^2=4p(y-k)$	$(y-k)^2=4p(x-h)$
Axis of symmetry:	$x=h$	$y=k$
Vertex:	$(h,\ k)$	$(h,\ k)$
Focus:	$(h,\ k+p)$	$(h+p,\ k)$
Directrix:	$y=k-p$	$x=h-p$
If $p>0$:	parabola opens up	parabola opens right
If $p<0$:	parabola opens down	parabola opens left
Distance:	$d(P,F) = d(P,D)$	P : a point on the parabola F : the focus D : the directrix

12.2 Ellipse



	$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ 	$\dfrac{y^2}{a^2}+\dfrac{x^2}{b^2}=1$ 
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Foci	$(\pm c, 0)$	$(0, \pm c)$
Major Axis $(2a)$	Horizontal	Vertical
Minor Axis $(2b)$	Vertical	Horizontal

- $a > b$
- $c^2 = a^2 - b^2$
- Major Axis length $= 2a$
- Minor Axis length $= 2b$

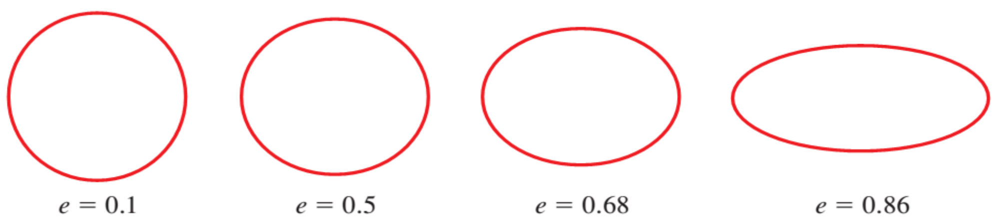
DEFINITION OF ECCENTRICITY

For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ (with $a > b > 0$), the **eccentricity e** is the number

$$e = \frac{c}{a}$$

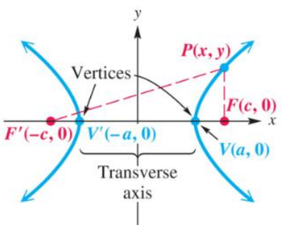
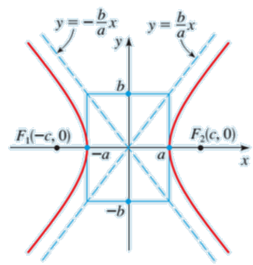
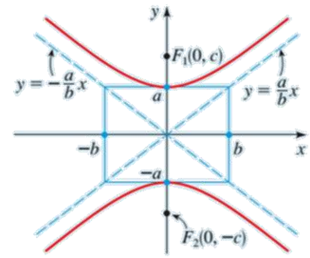
where $c = \sqrt{a^2 - b^2}$. The eccentricity of every ellipse satisfies $0 < e < 1$.

The eccentricity is a measure of how “stretched” the ellipse is.



	Horizontal Ellipse (Major axis parallel to x-axis)	Vertical Ellipse (Major axis parallel to y-axis)
Standard equation:	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, \quad a > b$	$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1, \quad a > b$
Center:	(h, k)	(h, k)
Vertices:	$(\pm a + h,\ k)$	$(h,\ \pm a + k)$
Foci:	$(\pm c + h,\ k)$ where $c^2 = a^2 - b^2$	$(h,\ \pm c + k)$ where $c^2 = a^2 - b^2$
Equation of major axis:	$y = k$	$x = h$
End points of minor axis:	$(h,\ \pm b + k)$	$(\pm b + h,\ k)$
Eccentricity:	$e = \frac{c}{a} < 1; \qquad 0 < e < 1$	

12.3 Hyperbola

	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ 	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ 
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Foci	$(\pm c, 0)$	$(0, \pm c)$
Transverse Axis $(2a)$	Horizontal	Vertical
Asymptotes	$y = \pm \frac{b}{a} x$	$y = \pm \frac{a}{b} x$

- $c^2 = a^2 + b^2$
- Transverse Axis length $= 2a$
- $e = \frac{c}{a} > 1$

	Horizontal Hyperbola (Transverse axis parallel to x-axis)	**Vertical Hyperbola** (Transverse axis parallel to y-axis)
Standard equation:	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Center:	(h, k)	(h, k)
Vertices:	$(\pm a + h,\ k)$	$(h,\ \pm a + k)$
Foci:	$(\pm c + h,\ k)$	$(h,\ \pm c + k)$
End points of conjugate axis:	$(h,\ \pm b + k)$	$(\pm b + h,\ k)$
Asymptote:	$y - k = \pm \frac{b}{a}(x - h)$	$y - k = \pm \frac{a}{b}(x - h)$
Eccentricity:	$e = \frac{c}{a} > 1$	

GENERAL EQUATION OF A SHIFTED CONIC

The graph of the equation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

where A and C are not both 0, is a conic or a degenerate conic. In the nondegenerate cases the graph is

1. a parabola if A or C is 0,
2. an ellipse if A and C have the same sign (or a circle if $A = C$),
3. a hyperbola if A and C have opposite signs.