Exercises

Mathematical Economics Class

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Exercise 1: Value function recursion (T=2)

Context. Consider the cake-eating problem with utility $u(c) = \ln c$ and discount factor $\beta = 0.9$. With two periods remaining ($T = 2$) and current cake $W > 0$, the recursive formulation is

$$V_T(W) \;=\; \max_{0<c\leq W} \Big\{ u(c) \;+\; \beta\, V_{T-1}(W-c) \Big\}, \qquad V_1(W) = \ln W.$$

Tasks.

1. Define $V_T(W)$ as a recursive problem and the terminal condition $V_1(W)$.
2. Compute $V_2(W)$ by backward induction (set up and solve the one-step problem).
3. Derive the optimal first-period policy $c_1(W)$ for this parametrization.
4. Show you obtain the same results solving the sequence problem (i.e. using a Lagrangian).

Solution key.

1. By definition,
$$V_T(W) = \max_{0<c\leq W} \{\ln c + \beta\, V_{T-1}(W-c)\}, \qquad V_1(W) = \ln W.$$

2. For $T = 2$,
$$V_2(W) = \max_{0<c\leq W} \{\ln c + \beta \ln(W-c)\}\,.$$

   FOC (interior optimum):
$$\frac{1}{c} - \frac{\beta}{W-c} = 0 \implies W - c = \beta c \implies c_1^\star = \frac{W}{1+\beta}, \quad c_2^\star = W - c_1^\star = \frac{\beta W}{1+\beta}.$$

   The value at the optimum is
$$V_2(W) \;=\; \ln\!\Big(\tfrac{W}{1+\beta}\Big) + \beta\,\ln\!\Big(\tfrac{\beta W}{1+\beta}\Big) \;=\; (1+\beta)\ln W + \beta\ln\beta - (1+\beta)\ln(1+\beta).$$

3. The optimal first-period policy is

$$\boxed{c_1(W) = \frac{W}{1+\beta}} \quad \text{so with } \beta = 0.9: \quad \boxed{c_1(W) = \frac{W}{1.9}}.$$

4. **Sequential (Lagrangian) check.** Solve

$$\max_{c_1,c_2 \geq 0} \ \ln c_1 + \beta \ln c_2 \quad \text{s.t. } c_1 + c_2 = W.$$

Lagrangian: $\mathcal{L} = \ln c_1 + \beta \ln c_2 + \lambda(W - c_1 - c_2)$. FOCs:

$$\frac{1}{c_1} - \lambda = 0, \qquad \frac{\beta}{c_2} - \lambda = 0, \qquad c_1 + c_2 = W.$$

From the first two, $\frac{1}{c_1} = \frac{\beta}{c_2} \Rightarrow c_2 = \beta c_1$. Using the constraint:

$$c_1 + \beta c_1 = W \ \Rightarrow \ c_1^* = \frac{W}{1+\beta}, \qquad c_2^* = \frac{\beta W}{1+\beta}.$$

With $\beta = 0.9$: $c_1^* = W/1.9$, $c_2^* = 0.9\,W/1.9$. These coincide with the recursive solution above, establishing equivalence.

Exercise 2: Infinite-Horizon Dynamic Programming (stationary policy & comparative statics)

Context. Consider the infinite-horizon cake-eating problem with utility $u(c) = \ln(c)$ and discount factor $\beta = 0.95$. The law of motion is $W' = W - c$ with $0 \leq c \leq W$.
Tasks.

1. Write down the Bellman equation.

2. Derive the first-order condition (FOC) and Euler equation.

3. Guess a stationary policy of the form $c = \alpha W$. Find α.

4. (*Comparative statics*) In the problem above, how does the optimal saving rate $1 - \alpha$ depend on β? Interpret the result.

Solution key.

1. **Bellman equation.**

$$V(W) \ = \ \max_{0<c\leq W} \left\{ \ln c + \beta\, V(W-c) \right\}.$$

2. **FOC, envelope, and Euler.** Let $W' = W - c$. The FOC (interior) is

$$\frac{1}{c} - \beta\, V'(W') \ = \ 0 \quad \Longrightarrow \quad \beta\, V'(W') = \frac{1}{c}.$$

Envelope:

$$V'(W) \ = \ \beta\, V'(W').$$

Evaluate FOC at t and $t+1$, and use the envelope relation to eliminate V':

$$\frac{1}{c_t} = \beta V'(W_{t+1}), \qquad \frac{1}{c_{t+1}} = \beta V'(W_{t+2}), \qquad V'(W_{t+1}) = \beta V'(W_{t+2})$$

$$\Rightarrow \frac{1}{c_t} = \beta \cdot \beta V'(W_{t+2}) = \beta \cdot \frac{1}{c_{t+1}} \implies \boxed{c_{t+1} = \beta\, c_t}.$$

This is the Euler equation for this problem.

3. **Stationary policy guess $c = \alpha W$ and identification of α.** Under $c_t = \alpha W_t$ we have $W_{t+1} = (1-\alpha)W_t$ and hence

$$c_{t+1} = \alpha W_{t+1} = \alpha(1-\alpha)W_t.$$

Using the Euler equation $c_{t+1} = \beta c_t = \beta\alpha W_t$, we obtain

$$\alpha(1-\alpha) = \beta\alpha \implies \boxed{1-\alpha = \beta} \implies \boxed{\alpha = 1-\beta}.$$

With $\beta = 0.95$, $\boxed{\alpha = 0.05}$: consume 5% of the cake each period.

4. **Comparative statics and interpretation.** From $1-\alpha = \beta$, the optimal saving rate equals β. Thus

$$\frac{d(1-\alpha)}{d\beta} = 1 > 0,$$

so greater patience (β higher) implies a higher saving rate and a smaller contemporaneous consumption share $\alpha = 1-\beta$. Intuitively, a more patient agent is willing to defer consumption: with ln utility and no regeneration of the cake, the optimal rule saves a fraction exactly equal to β.

Exercise 3: Concavity on a p-dependent feasible set

Context. Consider the function

$$f(x,y) = -\tfrac{1}{3}\left(x^3 + y^3\right) + xy,$$

and the feasible set

$$R_p = \{(x,y) \in \mathbb{R}^2 : \ x \geq p,\ y \geq p,\ x+y \leq 10\},$$

where $p \geq 0$ is a parameter.

Tasks.

1. (*Baseline test*) Is f concave on $R_0 = \{x \geq 0,\ y \geq 0,\ x+y \leq 10\}$?
2. For which values of p is f concave on R_p?

Solution key.

1. **Concavity on R_0.** The Hessian is

$$H(x,y) = \begin{pmatrix} -2x & 1 \\ 1 & -2y \end{pmatrix}.$$

For a symmetric 2×2 matrix, $H \succeq 0$ (convex) / $H \preceq 0$ (concave) can be checked by signs of principal minors. For concavity we need

$$H_{11} = -2x \leq 0, \quad H_{22} = -2y \leq 0, \quad \det H = 4xy - 1 \geq 0.$$

On R_0 we have $x, y \geq 0$, so the first two conditions hold. However, $\det H \geq 0$ requires $xy \geq \frac{1}{4}$, which fails near the axes (e.g., at $(x, y) = (0, 1)$).

$$\Rightarrow \boxed{f \text{ is not concave on all of } R_0 \text{ (only where } xy \geq \tfrac{1}{4}\text{).}}$$

2. **Values of p ensuring concavity on R_p.** On R_p we have $x \geq p,\ y \geq p$, hence $x, y \geq 0$ when $p \geq 0$. The minimum of xy over R_p occurs at the corner (p, p), so

$$\min_{(x,y) \in R_p} xy = p^2.$$

We need $\det H = 4xy - 1 \geq 0$ for all $(x, y) \in R_p$, i.e.,

$$4p^2 - 1 \geq 0 \iff p \geq \tfrac{1}{2}.$$

Feasibility also requires $R_p \neq \varnothing$, i.e., $2p \leq 10 \Rightarrow p \leq 5$.

$$\Rightarrow \boxed{f \text{ is concave on } R_p \text{ iff } \tfrac{1}{2} \leq p \leq 5.}$$

For $p > \frac{1}{2}$, $\det H > 0$ in the interior (with $x, y > 0$), so f is strictly concave there.