Exercice 16 : $A = \frac{1}{4}\begin{pmatrix} -3 & 4 & 3 \\ 1 & 0 & 3 \\ -1 & 4 & 1 \end{pmatrix}$

$P_A(x) = \text{Det}(A - xI_3) = \text{Det}\left(\frac{1}{4}(4A - 4xI_3)\right) = \frac{1}{4^3}\text{Det}(4A - 4xI_3)$

$$P_A(x) = \frac{1}{4}\begin{vmatrix} -3-4x & 4 & 3 \\ 1 & -4x & 3 \\ -1 & 4 & 1-4x \end{vmatrix} \quad L_3 \leftarrow L_3 + L_2$$

$$= \frac{1}{4^3}\begin{vmatrix} -3-4x & 4 & 3 \\ 1 & -4x & 3 \\ 0 & 4-4x & 4-4x \end{vmatrix} \quad C_2 \leftarrow C_2 - C_3$$

$$P_A(x) = \frac{1}{4^3}\begin{vmatrix} -3-4x & 1 & 3 \\ 1 & -3-4x & 3 \\ 0 & 0 & 4-4x \end{vmatrix}$$

$$= \frac{1}{4^3}(4-4x)(2+4x)(4+4x)$$

$$= (1-x)(1+x)\left(\frac{1}{2}+x\right).$$

Donc A possède 3 v.p simples $\lambda_1 = -1$, $\lambda_2 = -\frac{1}{2}$, $\lambda_3 = 1$

$E_{\lambda_1} = \text{Ker}(A + I_3) = \text{Ker}\left(\frac{1}{4}(4A + 4I_3)\right)$

$E_{\lambda_1} = \{v \in \mathbb{R}^3,\ \frac{1}{4}(4A + 4I_3)v = 0\} = \{v \in \mathbb{R}^3,\ (4A + 4I_3)v = 0\}$

$$(4A + 4I_3)v = 0 \iff \begin{pmatrix} 1 & 4 & 3 \\ 1 & 4 & 3 \\ -1 & 4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{cases} x + 4y + 3z = 0 \\ x + 4y + 3z = 0 \\ -x + 4y + 5z = 0 \end{cases}$$

$$\iff \begin{cases} x + 4y + 3z = 0 \\ -x + 4y + 5z = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x = z \\ y = -z \end{cases} \quad E_{\lambda_1} = \text{Vect}\{(1, -1, 1)\}$$

Soit $v_1 = (1, -1, 1)$

$E_{\lambda_2} = \ker\left(A + \frac{1}{2} I_3\right) = \ker\left(\frac{1}{4}\left(4A + 2I_3\right)\right)$

$(4A + 2I_3)\,V = 0 \iff \begin{pmatrix} -1 & 4 & 3 \\ 1 & 2 & 3 \\ -1 & 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\iff \begin{cases} -x + 4y + 3z = 0 \\ x + 2y + 3z = 0 \\ -x + 4y + 3z = 0 \end{cases}$

$\Rightarrow \begin{cases} x = -z \\ y = -z \end{cases}$ $\quad E_{\lambda_2} = \text{Vect}\{(-1,-1,1)\}$

Soit $v_2 = (-1,-1,1)$.

$E_{\lambda_3} = \ker\left(A - \frac{I_3}{3}\right) = \ker\left(\frac{1}{4}\left(4A - 4I_3\right)\right)$

$(4A - 4I_3)\,V = 0 \iff \begin{pmatrix} -7 & 4 & 3 \\ 1 & -4 & 3 \\ -1 & 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\iff \begin{cases} -7x + 4y + 3z = 0 \\ x - 4y + 3z = 0 \end{cases}$

$\Rightarrow x = y = z \quad E_{\lambda_3} = \text{Vect}\{(1,1,1)\}$

Soit $v_3 = (1,1,1)$.

Donc $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $P = \begin{pmatrix} 1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$D = P^{-1} \cdot A \cdot P \Rightarrow A = P \cdot D \cdot P^{-1}$

$\Rightarrow A^n = P \cdot D^n \cdot P^{-1}$

$P^{-1} = ?$

$$\left(\begin{array}{ccc|ccc} \boxed{1} & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ L_2 \leftarrow L_2 + L_1 \\ L_3 \leftarrow L_3 - L_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & \boxed{-2} & 2 & 1 & 1 & 0 \\ 0 & 2 & 0 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} L_1 \leftarrow L_1 - \frac{1}{2} L_2 \\ L_2 \leftarrow -\frac{1}{2} L_2 \\ L_3 \leftarrow L_3 + L_2 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & 1 & -1 & -1/2 & -1/2 & 0 \\ 0 & 0 & \boxed{2} & 0 & 1 & 1 \end{array}\right) \begin{array}{l} \\ \\ L_3 \leftarrow \frac{1}{2} L_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & 1 & 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 1/2 & 1/2 \end{array}\right)$$

D'où alors $P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

$$A^n = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & (-1/2)^n & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} (-1)^n & -(-1/2)^n & 1 \\ -(-1)^n & -(-1/2)^n & 1 \\ (-1)^n & (-1/2)^n & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} (-1)^n + (-\frac{1}{2})^n & -(-1)^n + 1 & -(-1/2)^n + 1 \\ -(-1)^n + (-1/2)^n & (-1)^n + 1 & -(-1/2)^n + 1 \\ (-1)^n - (-1/2)^n & -(-1)^n + 1 & (-1/2)^n + 1 \end{pmatrix}$$

prenons $X = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \in \mathbb{R}^3$. on a alors

$$A^n X = \frac{1}{2}\begin{pmatrix} (-1)^n(\alpha-\beta) + (\frac{1}{2})^n(\alpha-\gamma) + (\beta+\gamma) \\ (-1)^n(-\alpha+\beta) + (-1/2)^n(\alpha-\gamma) + (\beta+\gamma) \\ (-1)^n(\alpha-\beta) + (-\frac{1}{2})^n(-\alpha+\gamma) + (\beta+\gamma) \end{pmatrix}$$

on a : $\lim\limits_{n\to+\infty} (-\frac{1}{2})^n = 0$. Donc la suite $(A^n X)_n$ cvg

si $\begin{cases} (\alpha-\beta) = 0 \\ -\alpha+\beta = 0 \\ \alpha-\beta = 0 \end{cases} \Rightarrow \alpha = \beta$ et dans ce cas

$$\lim_{n\to+\infty} (A^n X)_n = \frac{(\beta+\gamma)}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$