Chapter-3
Scan Conversion

Scan Conversion/rasterization: It is a process to convert each primitives from its geometric definition into a set of pixels

Graphical primitives:

- points
- lines
- circles
- filled polygons

Scan Conversion of a point:

- A mathematical point (x, y)
- x and y are real numbers
- scan converted to a pixel location (x', y')

→ $x' = \text{Floor}(x)$
→ $y' = \text{Floor}(y)$ → $x' \leq x < x'+1$, $y' \leq y < y'+1$

→ places the origin of a continuous co-ordinate system for (x, y) ~~at~~ at the lower left corner of the pixel grid.

→ $x' = \text{Floor}(x + 0.5)$
→ $y' = \text{Floor}(y + 0.5)$ → $x'-0.5 \leq x < x'+0.5$, $y'-0.5 \leq y < y'+0.5$

→ places the origin of the coordinate system for (x, y) at the center of pixel $(0, 0)$

Scan converting a line:

Line:

$P_1(x_1, y_1)$
$P_2(x_2, y_2)$

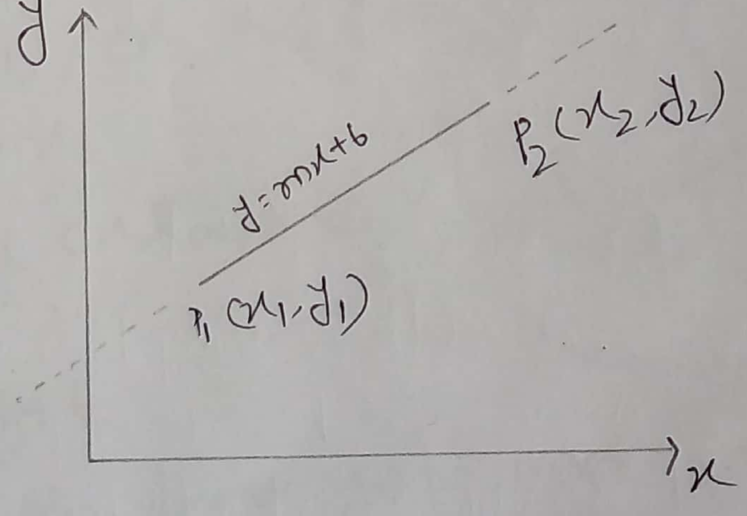
Defined by $\boxed{\text{two endpoints}}$ and the line equation $y = mx + b$,

where, m = slop

b = y intercept of the line.

~~two end point~~

→ portion of a straight line that extends indefinitely in opposite directions



Line drawing algorithm:

Arbitrary line drawing algorithm:

- direct use of the Line Equation
- DDA algorithm
- Bresenham's Line Algorithm

- Horizontal line drawing "
- vertical " "
- Diagonal " "

Direct line drawing algorithm:

- natural method of generating a straight line.
- first calculate (m) > slop
- then " y-intercept (b)

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$b = y_1 - m x_1$$

Pseudo-Code:

$x = x_1$

$y = y_1$

$m = (y_2 - y_1) / (x_2 - x_1)$

$b = y_1 - m x_1$

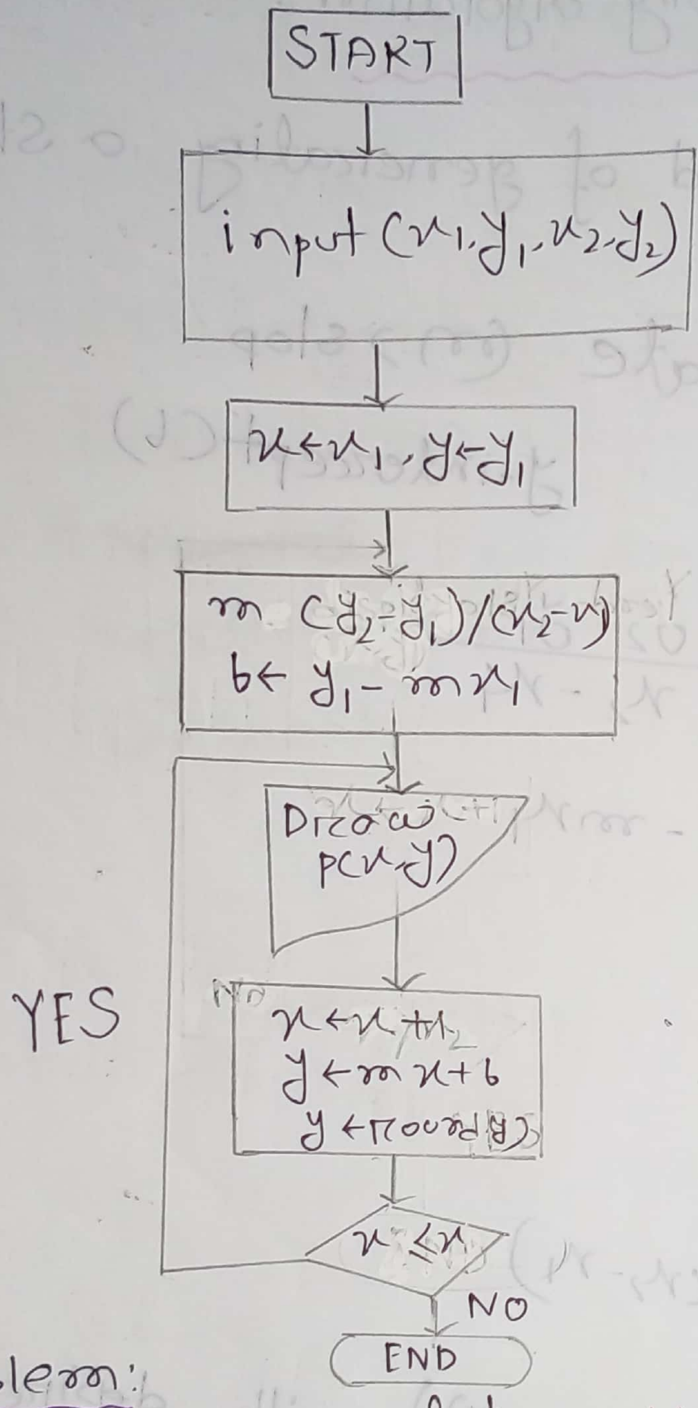
Next: Setpixel $(x, \text{Round}(y))$ with desired color

$x = x + 1$

$y = mx + b$

if $x \leq x_2$ then goto Next

End.



Problem:

2 fundamental problems.

→ required more time to draw the line.

→ lines with a slop ($m > 1$) whose absolute value is greater than 1.

$\underline{\underline{Sol^n}}$

→ if $|m|<1$. increment x value one unit & calculate corresponding value of y using the equation and scan-convert (x,y)

→ if $|m|>1$ increment y value one unit & calculate corresponding value of x using the equation.

(Sol^n of

Simple Digital Differential Analyzer (DDA)

→ incremental scan-conversion method

→ performing calculations at each step using result from the preceding step.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

Now $x_{i+1} - x_i = 1$

$$\therefore m = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} = \frac{y_{i+1} - y_i}{1}$$

$$\therefore \boxed{y_{i+1} = m + y_i}$$

Case-1 For abs(m) < 1 and $x_1 < x_2$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

if (abs(m) < 1 and $x_1 > x_2$) then

Swap endpoints $x_1 \leftrightarrow x_2$ and $y_1 \leftrightarrow y_2$

end if

Setpixel (x_1, y_1) with desired color

if abs(m) < 1 then

$y = y_1$

$x = x_1 + 1$

Next: $y = y + m$

Setpixel (x, Round(y)) with desired color

$x = x + 1$

if $x \leq x_2 - 1$ then goto Next

end if

Set pixel (x_2, y_2) with desired color

Case 2:

For abs(m) > 1 and $y_1 < y_2$

$m = (y_2 - y_1)/(x_2 - x_1)$

if (abs(m) > 1 and $y_1 > y_2$) then

swap endpoints

$x_1 \leftrightarrow x_2$ and

$y_1 \leftrightarrow y_2$

end if

setpixel (x_1, y_1) with desired color

if abs(m) > 1 then

$m = 1/m$

$y = y_1 + 1$

$x = x_1$

Next: $x = x + m$

setpixel (Round (x), y) with desired color

$y = y + 1$

if $y \leq y_2 - 1$ then goto Next

end if

setpixel (x_2, y_2) with desired color if $|m| < 1$

→ Start with $x = x_1$, $y = y_1$

→ set $\Delta x = 1$

→ $y_{i+1} = y_i + m$

→ y coordinate of each successive point

if $|m| > 1$

→ Start with $x = x_1$, $y = y_1$

→ set $\Delta y = 1$

→ $x_{i+1} = x_i + 1/m$

→ x coordinate of each successive point

Problem:

→ 1 floating-point addition.

→ cumulative error

due to → limited precision in the floating point representation

Bresenham's Line Algorithm

→ highly efficient incremental method

→ uses → integer addition

→ ,, subtraction

→ multiplication by 2

↳ simple arithmetic shift operation.

Let,

$0 < m < 1$

and two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$

Once a pixel is chosen at any ~~po~~ step.

next pixel is either the

→ one to its right (T) or the

→ ,, ,, its right & up (S)

if S is chosen $x_{i+1} = x_i + 1$

$y_{i+1} = y_i$

if T is ,, $x_{i+1} = x_i + 1$

$y_{i+1} = y_i + 1$

Actual y coordinate of the line at

$x = x_{i+1}$ is $y = m x_{i+1} + b$

$= m(x_i + 1) + b$

Distance from S to the actual line is

$s = y - y_i$

" " T " " actual line is

$t = (y_i + 1) - y$

if $s < t$, closest pixel is S

if $s > t$, " " " T

$\therefore s - t = (y - y_i) - [(y_i + 1) - y] = y - y_i - y_i - 1 + y$

$= 2y - 2y_i - 1$

$= 2m(x_i + 1) + 2b - 2y_i - 1$

Substituting m by $\Delta y / \Delta x$,

$\therefore s - t = \cancel{2m}\ 2\frac{\Delta y}{\Delta x}(x_i + 1) + 2b - 2y_i - 1$

$\cancel{\Delta x}(s - t) = 2\Delta y(x_i + 1) + (2b - 2y_i - 1)\Delta x$

$\Rightarrow d_i = 2\Delta y + 2\Delta y x_i + (2b - 2y_i - 1)\Delta x$ [decision variable

$d_i = \cancel{\Delta x}(s - t)$]

$\cancel{\Rightarrow s - t}$

$$\Rightarrow s-t = 2\Delta y + 2\Delta y x_i + 2b\Delta x - 2y_i\Delta x - \Delta x$$
$$= 2\Delta y x_i - 2\Delta x y_i + 2\Delta y + 2b\Delta x - \Delta x$$
$$= 2\Delta y x_i - 2\Delta x y_i + 2\Delta y + \Delta x(2b-1)$$
$$= 2\Delta y x_i - 2\Delta x y_i + c$$
$$[\because c = 2\Delta y + \Delta x(2b-1)]$$

$$\therefore d_i = 2\Delta y x_i - 2\Delta x y_i + c$$

$$\therefore d_{i+1} = 2\Delta y x_{i+1} - 2\Delta x y_{i+1} + c$$
$$\therefore d_{i+1} - d_i = 2\Delta y(x_{i+1} - x_i) - 2\Delta x(y_{i+1} - y_i)$$
$$\Rightarrow d_{i+1} = d_i + 2\Delta y - 2\Delta x(y_{i+1} - y_i) \quad [\because x_{i+1} = x_i + 1]$$

if T is chosen,

$$\boxed{d_{i+1} = d_i + 2\Delta y - 2\Delta x} \quad [\because y_{i+1} = y_i + 1]$$

if S is chosen

$$\boxed{d_{i+1} = d_i + 2\Delta y} \quad [\because y_{i+1} = y_i]$$

$$\therefore d_{i+1} = \begin{cases} d_i + 2(\Delta y - \Delta x), & \text{if } d_i \geq 0 \\ d_i + 2\Delta y & \text{if } d_i < 0 \end{cases}$$

$$d_1 = \Delta x\,[2m(x_1+1) + 2b - 2y_1 - 1]$$

$$= \Delta x\,[2(mx_1 + b - y_1) + 2m - 1]$$

As $[\,mx_1 + b - y_1 = 0$

$$\cancel{d_1 = 2\Delta y - \Delta x}$$

$$\Rightarrow d_1 = \Delta x\,[(2\times 0) + 2\,\Delta y/\Delta x - 1]$$

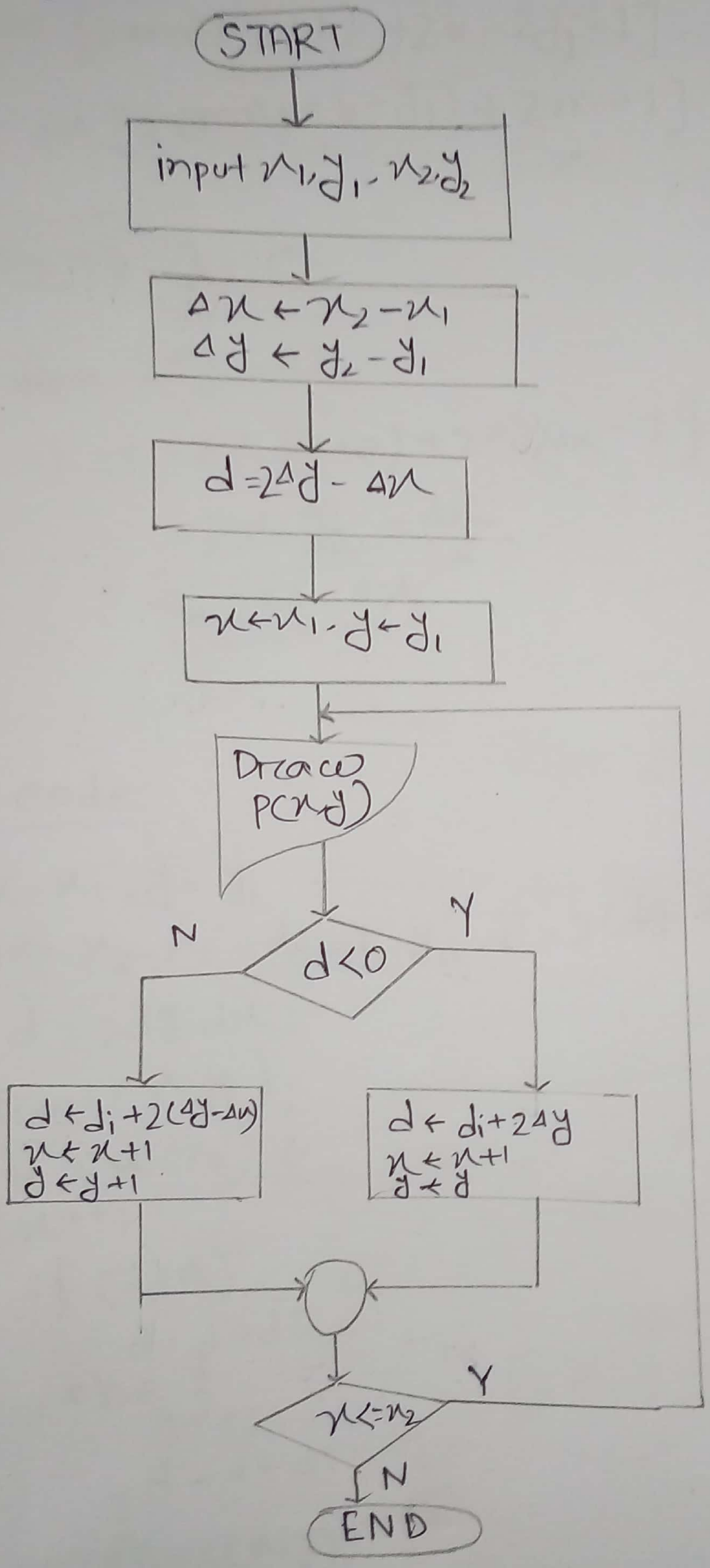
$$= 2\Delta x \frac{\Delta y}{\Delta x} - \Delta x$$

$$\boxed{= 2\Delta y - \Delta x}$$

pseudo code:

```
int x=x1, y=y1
int dx= x2-x1 , dy=y2-y1, dT=2(dy-dx), dS=2dy;
int d = 2dy-dx;
setPixel(x,y);
while (x<x2) {
    x++;
    if (d<0)
        d=d+dS;
    else {
        y++;
        d =d+dT;
    }
    setPixel(x,y);
}
```

Flowchart (Bresenham's)



Horizontal line drawing algorithm:

Pseudo code: (x_1, y_1) to (x_2, y_2)
here, $y_1 = y_2$

$x = x_1$
$y = y_1$

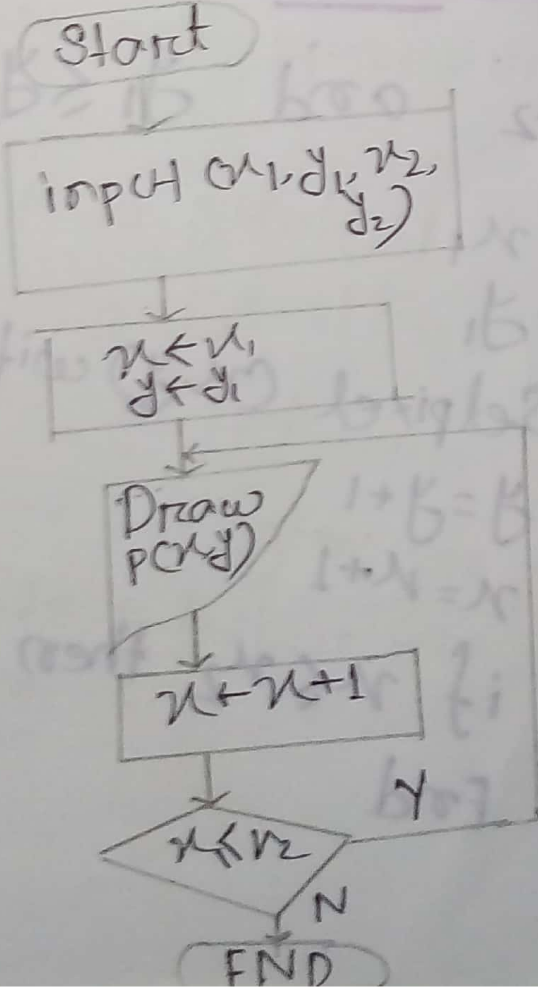
Next: set pixel (x, y) with desired color

$x = x + 1$

if $(x \leq x_2)$
then go to Next

end if

Flowchart (horizontal line)



Vertical line drawing algorithm:

Pseudocode (x_1, y_1) to (x_2, y_2)

$x_1 = x_2$

$x = x_1$

$y = y_1$

Next: Setpixel (x, y) with desired color

$y = y + 1$

if $y \leq y_2$ then go to Next

End

Diagonal line drawing algorithm

$(m = 1)$

$x_1 \leq x_2$ and $y_1 \leq y_2$

$x = x_1$

$y = y_1$

Next: Setpixel (x, y) with desired color

$y = y + 1$

$x = x + 1$

if $x \leq x_2$ then go to Next

End

Diagonal line drawing algorithm ($m=-1$)

$x_1 \leq x_2$ and $y_1 \geq y_2$

$x = x_1$

$y = y_1$

Next: setpixel (x, y) with desired color

$y = y - 1$

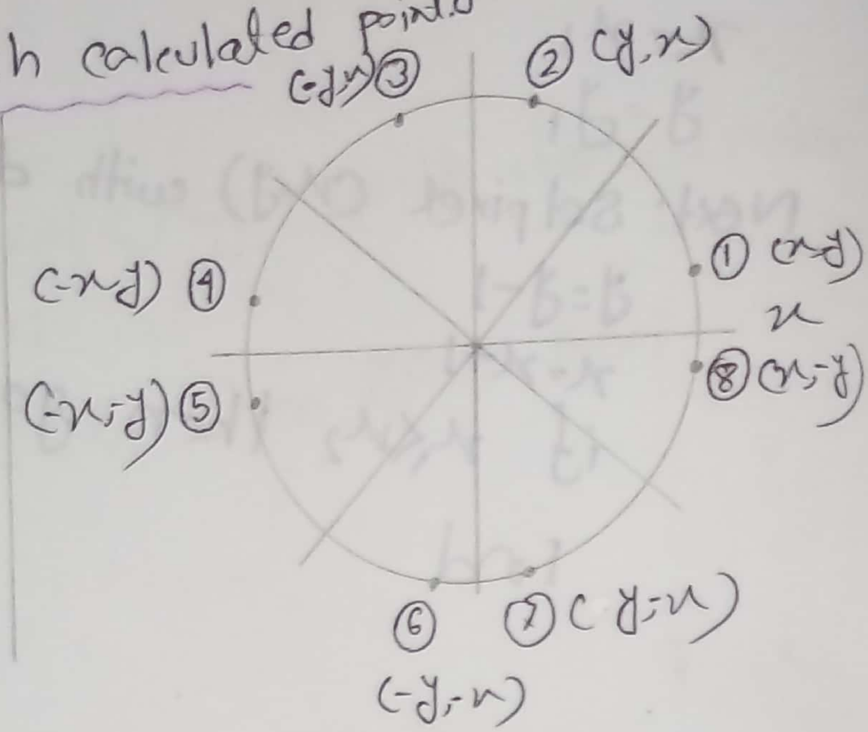
$x = x + 1$

if $x \leq x_2$ then goto Next

End

A circle is a symmetric figure

Eight way symmetry is used by reflecting each calculated point around each 45° axis.



The reflection is accomplished by

→ reversing the x,y coordinate as in point 2

→ " " " " &

reflecting about the y axis as in point 3

→ reflecting " " y " as in " 4

→ switching the signs of x and y as in point 5

→ reversing the x, y coordinates and, reflecting about y axis and x axis as in point 6

→

→ reflecting about x axis as in point 8

In summary

$P_1 = (x, y)$ $P_5 = (-x, -y)$
$P_2 = (y, x)$ $P_6 = (-y, -x)$
$P_3 = (-y, x)$ $P_7 = (y, -x)$
$P_4 = (-x, y)$ $P_8 = (x, -y)$

Circle

2 standard methods → to define a circle centered at the origin

method-1:

Circle with the 2nd order polynomial equation.

$$y^2 = r^2 - x^2$$
$$\Rightarrow x^2 + y^2 = r^2$$

where, x = the x coordinate
y = " y "
r = the circle radius

method-2

Use of trigonometric functions.

$x = r\cos\theta$

$y = r\sin\theta$

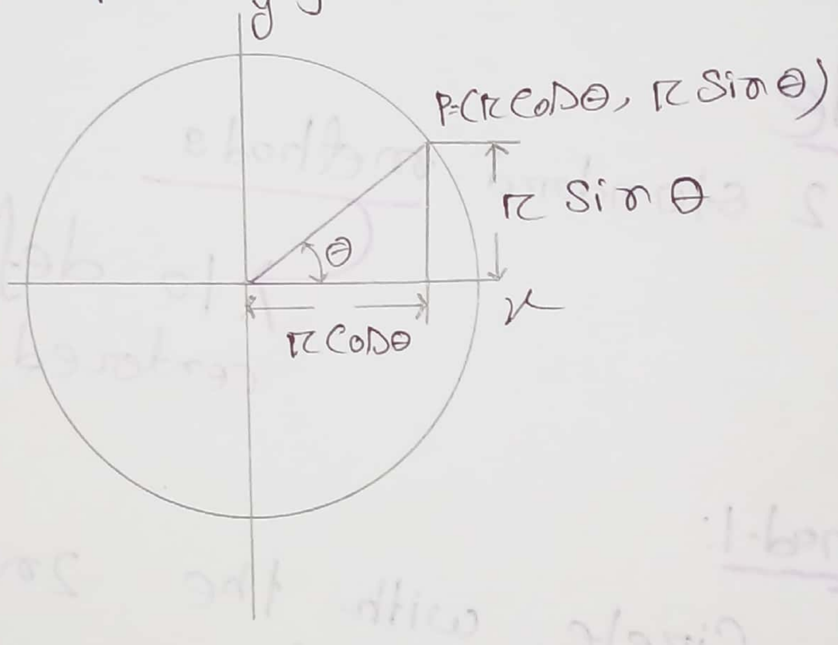
where θ = current angle

r = circle radius

x = x coordinate

y = y ,,

→ θ is stepped from 0 to $\pi/4$



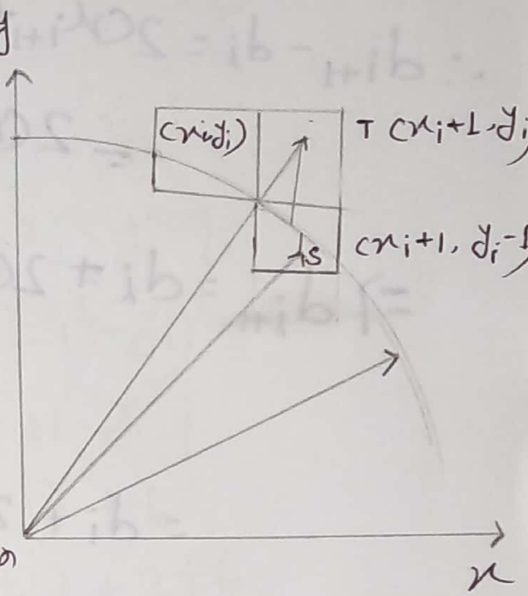
Circle drawing Algorithm:

→ Bresenham's Circle Algorithm

→ Midpoint Circle ,,

Bresenham's circle Algorithm.

Let,
(x_i, y_i) are the coordinates of the last scan converted pixel



D(T) = (distance from the origin to pixel T)^2 ~~squared~~ − (distance to the true circle)^2

D(S) = (distance from the origin to pixel S)^2 − (distance to the true circle)^2

Coordinate of T is: (x_i+1, y_i)

" of S " = (x_i+1, y_i-1)

$\therefore\ D(T) = (x_i+1)^2 + (y_i)^2 - r^2$

$D(S) = (x_i+1)^2 + (y_i-1)^2 - r^2$

D(T) is always positive,
D(S) " " negative

Decision variable $d_i = D(T) + D(S)$

if $d_i < 0$, then $|D(T)| < |D(S)|$ & pixel T is chosen
if $d_i > 0$ then $|D(T)| > |D(S)|$ & pixel S is "

$$\therefore d_{i+1} = 2(x_{i+1}+1)^2 + y_{i+1}^2 + (y_{i+1}-1)^2 - 2r^2$$

$$\therefore d_{i+1} - d_i = 2(x_{i+1}+1)^2 + y_{i+1}^2 + (y_{i+1}-1)^2 - 2r^2$$
$$- 2(x_i+1)^2 - y_i^2 - (y_i-1)^2 + 2r^2$$

$$\Rightarrow d_{i+1} = d_i + 2(x_i+1+1)^2 + y_{i+1}^2 + (y_{i+1}-1)^2$$
$$- 2(x_i+1)^2 - y_i^2 - (y_i-1)^2$$

$$= d_i + 2(x_i+2)^2 - 2(x_i+1)^2$$
$$+ y_{i+1}^2 + (y_{i+1}-1)^2$$
$$- y_i^2 - (y_i-1)^2$$

$$= d_i + 2(\cancel{x_i^2} + 4 + 4x_i - \cancel{x_i^2} - 1 - 2x_i)$$

$$= d_i + 4x_i + 6 + y_{i+1}^2 + y_{i+1}^2 \cancel{+1}$$
$$- 2y_{i+1} - y_i^2$$
$$- y_i^2 \cancel{-1} + 2y_i$$

$$= d_i + 4x_i + 2y_{i+1}^2 - 2y_i^2 + 2y_i$$
$$- 2y_{i+1} + 6$$

$$= d_i + 4x_i + 2(y_{i+1}^2 - y_i^2)$$
$$- 2(y_{i+1} - y_i) + 6$$

If T is chosen, $y_{i+1} = y_i$

$$\therefore d_{i+1} = d_i + 4x_i + 2(y_i^2 - y_i^2) - 2(y_i - y_i) + 6$$

$$= d_i + 4x_i + 6$$

If S is chosen $y_{i+1} = y_i - 1$

$$d_{i+1} = d_i + 4x_i + 2\{(y_i - 1)^2 - y_i^2\} - 2(y_i - 1 - y_i) + 6$$

$$= d_i + 4x_i + 2\{\cancel{y_i^2} - 2y_i + 1 - \cancel{y_i^2}\} - 2(-1) + 6$$

$$= d_i + 4x_i + 2(-2y_i + 1) + 2 + 6$$

$$= d_i + 4x_i + 2 - 4y_i + 8$$

$$= d_i + 4(x_i - y_i) + 10$$

$$\therefore d_{i+1} = \begin{cases} d_i + 4x_i + 6 & \text{if } d_i < 0 \\ d_i + 4(x_i - y_i) + 10 & \text{if } d_i \geq 0 \end{cases}$$

$$d_1 = 2(0+1)^2 + r^2 + (r-1)^2 - 2r^2 \quad [x_0 = 0, y_0 = r]$$

$$= 2 + r^2 + r^2 - 2r + 1 - 2r^2$$

$$\Rightarrow \boxed{d_1 = 3 - 2r}$$

Pseudocode (Bresenham's Circle)

```
int x=0, y=r, d= 3-2r
while (x<=y){
    setPixel(x,y);
    if (d<0)
        d = d+4x+6;
    else{
        d = d+4(x-y)+10;
        y--;
    }
    x++
}
```

→ incremental circle algorithm

Midpoint Circle Algorithm:

★★★

Relationship between an arbitary point (x, y) and a circle of radius r centered at origin :

$$f(x,y) = x^2 + y^2 - r^2 \begin{cases} <0 & (x,y) \text{ inside the circle} \\ =0 & (x,y) \text{ on the "} \\ >0 & (x,y) \text{ outside the circle} \end{cases}$$

Consider a point halfway between pixel T and pixel S is $\left(x_i+1,\, y_i - \frac{1}{2}\right)$

$$\therefore P_i = f\left(x_i+1, y_i - \tfrac{1}{2}\right) = (x_i+1)^2 + \left(y_i - \tfrac{1}{2}\right)^2 - r^2$$

$$\therefore P_{i+1} = (x_{i+1}+1)^2 + \left(y_{i+1} - \tfrac{1}{2}\right)^2 - r^2$$

$$x_{i+1} = x_i + 1$$

Now, $P_{i+1} - P_i = (x_i+2)^2 - (x_i+1)^2 + \left(y_{i+1} - \tfrac{1}{2}\right)^2 - \left(y_i - \tfrac{1}{2}\right)^2$

$$\Rightarrow P_{i+1} = P_i + (x_i^2 + 4 + 4x_i - x_i^2 - 1 - 2x_i) + (y_{i+1}^2 - y_i^2) - (y_{i+1} - y_i)$$

$= P_i + 2(x_i+1) + 1 + (y_{i+1}^2 - y_i^2) - (y_{i+1} - y_i)$

if T is chosen $y_{i+1} = y_i$

$P_{i+1} = P_i + 2(x_i+1) + 1$

if S is chosen $y_{i+1} = y_i - 1$

$P_{i+1} = P_i + 2(x_i+1) + 1 - (\cancel{y_i} - 1 - \cancel{y_i})$
$+ (\cancel{y_i^2} - 2y_i + 1 - \cancel{y_i^2})$

$= P_i + 2(x_i+1) + 2(-y_i+1)$
$+ 1$

$= P_i + 2(x_i+1) - 2(y_i-1) + 1$

$$\therefore P_{i+1} = \begin{cases} P_i + 2(x_i+1) + 1 & \text{if } P_i < 0 \\ P_i + 2(x_i+1) + 1 - 2(y_i-1) & \text{if } P_i \geq 0 \end{cases}$$

By simplifying we get,

$$P_{i+1} = \begin{cases} P_i + 2x_i + 3 & \text{if } P_i < 0 \\ P_i + 2(x_i - y_i) + 5 & \text{if } P_i \geq 0 \end{cases}$$

Initial value of decision

$$P_1 = (0+1)^2 + (r - \tfrac{1}{2})^2 - r^2$$

$$= 1 + \cancel{r^2} - 2\tfrac{1}{2} r + \tfrac{1}{4} - \cancel{r^2}$$

$$\therefore \boxed{P_1 = \frac{5}{4} - r}$$

Chapter-4

Two-Dimensional Transformation

Transformation:

Ability to simulate the manipulation of objects in space.

Process of modifying and repositioning the existing graphics.

Types of transformation:

2 types

→ geometric transformation

→ coordinate "

→ applied to each point of the object

☐ geometric transformation:

Transformation in which object itself is transformed relative to

a
- → Stationary coordinate system
- → background

Ex: Moving the automobile, while keeping the background fixed.

☐ Coordinate transformation:

Object is held stationary while the coordinate system transformed → relative to the object.

Ex 2: geometric transformation
keeping the car fixed while moving the background scenery

Types of geometric transformation:

- Translation
- Rotation about the $\boxed{\text{Origin}}$
- Scaling with respect to the origin
- Mirror reflection about an Axis

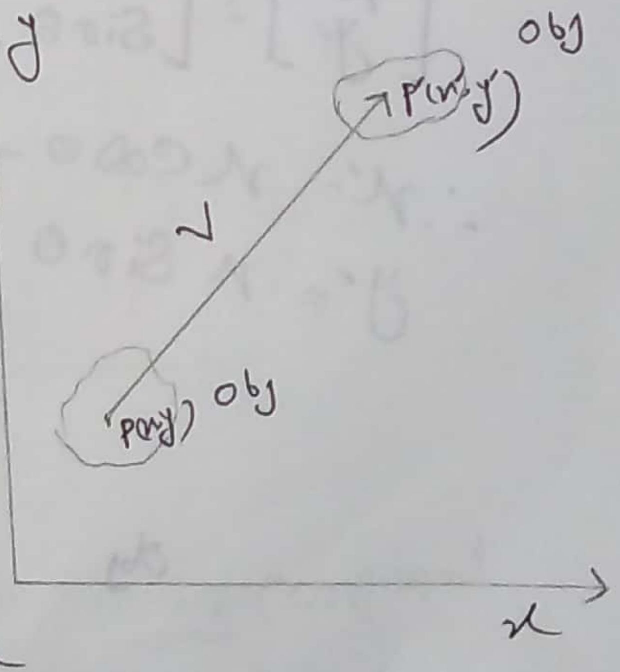
Translation

An object is displaced a given distance & direction from its original position.

displacement vector,
$v = t_x I + t_y J$

original point, $P(x, y)$
new object point, $P'(x', y')$

$\therefore \boxed{P' = T_v(P)}$



$\therefore \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$

$= \begin{bmatrix} x + t_x \\ y + t_y \end{bmatrix}$ $\therefore x' = x + t_x$
$y' = y + t_y$

Rotation about an origin:

→ Object is rotated $\theta°$ about the origin

→ ~~direction~~

→ If θ is positive,
direction of rotation is counter clockwise

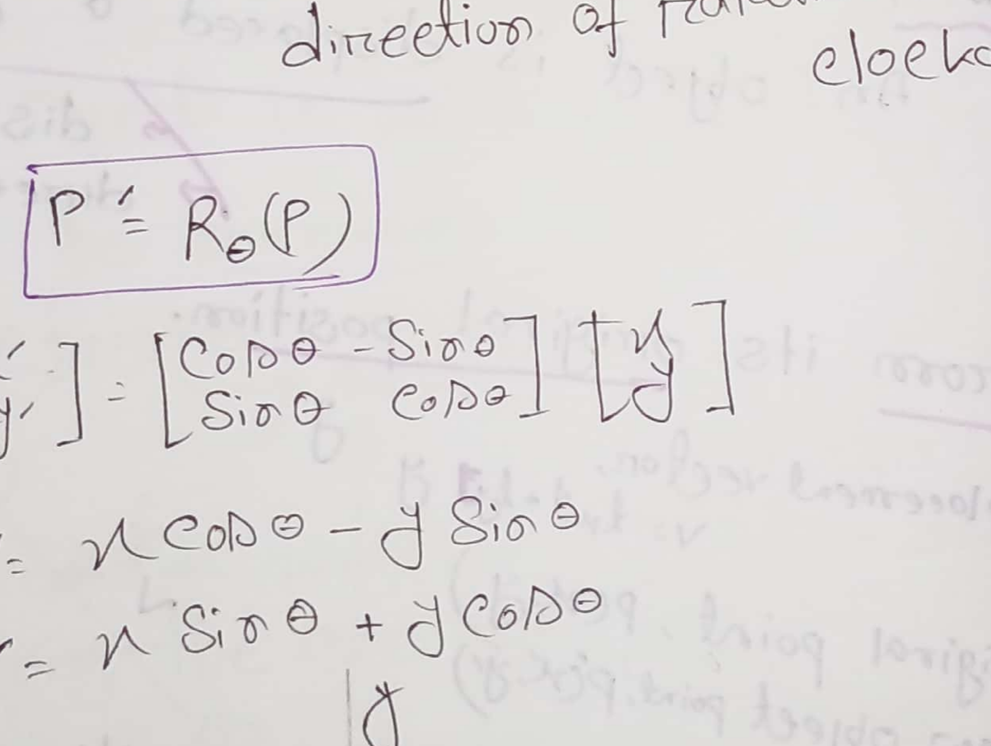
→ If θ is negative,
direction of rotation is clockwise

$$\boxed{P' = R_\theta(P)}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\therefore x' = x\cos\theta - y\sin\theta$

$y' = x\sin\theta + y\cos\theta$



Sealing with respect to the Origin:

Process of expanding or compressing the dimension of an object.

→ Scaling constant greater than one ↳ expansion of length

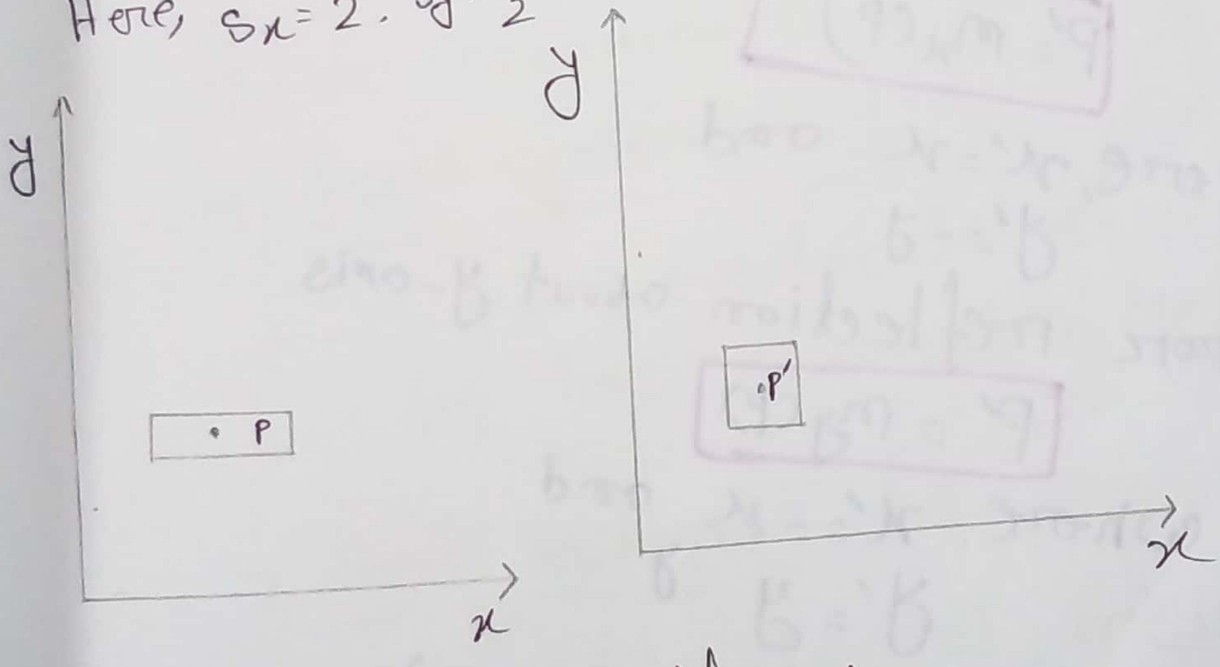
→ " " less " one ↳ compression of length

$$\boxed{P' = S_{s_x, s_y}(P)}$$

$$\boxed{x' = s_x x}$$

$$\boxed{y' = s_y y}$$

Here, $s_x = 2$, $s_y = \frac{1}{2}$



* Homogeneous/uniform:
If both sealing constant have same value

magnification:

If $s > 1$

reduction

If $s < 1$

Mirror reflection about an axis:

If either the x or y axis is treated as mirror, the object has mirror image or reflection.

Mirror reflection about x-axis.

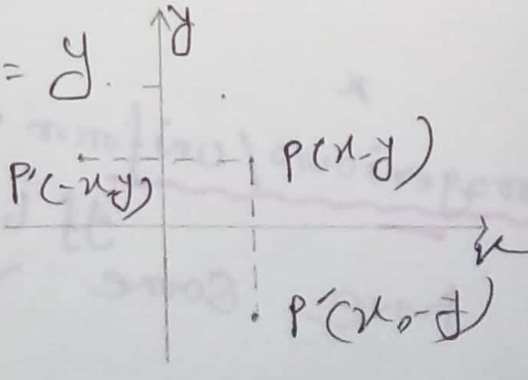
$$\boxed{P' = m_x(P)}$$

where, $x' = x$ and
$y' = -y$

Mirror reflection about y-axis

$$\boxed{P' = m_y(P)}$$

where $x' = -x$ and
$y' = y$.



Ms - $m_x = S_{1,-1}$
$m_y = S_{-1,1}$

Inverse geometric transformation:

- Translation $T_v^{-1} = T_{-v}$
- Rotation: $R_\theta^{-1} = R_{-\theta}$
- Scaling: $S_{s_x, s_y}^{-1} = S_{1/s_x, 1/s_y}$
- mirror reflection: $M_x^{-1} = M_x$ and $M_y^{-1} = M_y$

Coordinate transformation:

→ two coordinate system

→ first system:

→ origin is located at O

→ has coordinates axes xy

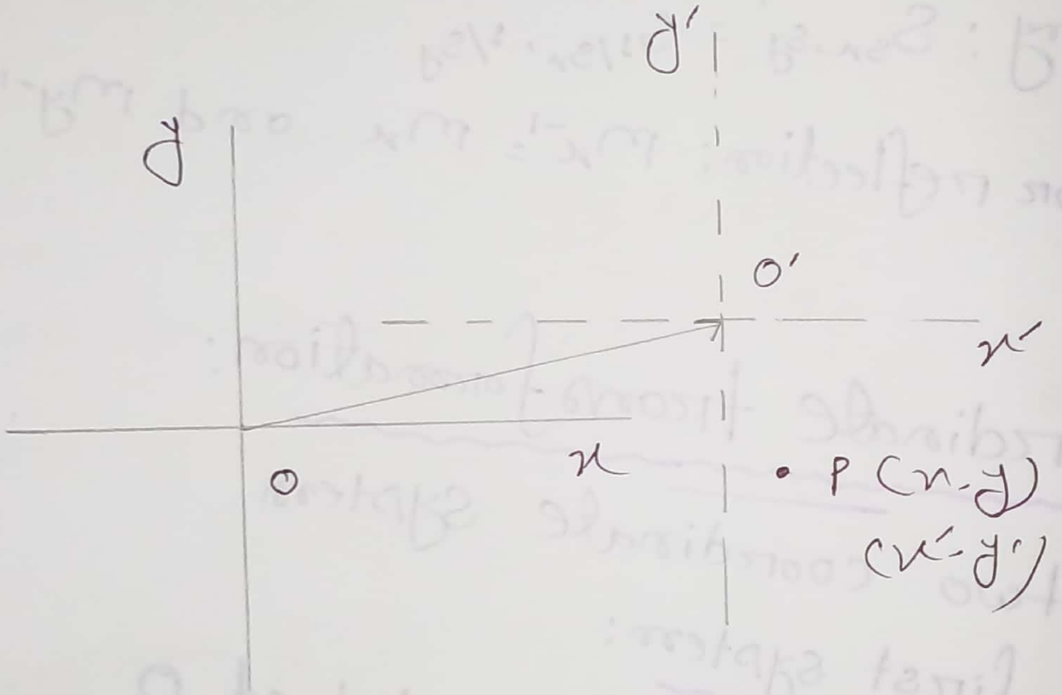
→ Second system:

→ located at origin O'

→ has coordinate axes x'y'.

2nd system $x'y'$ as arising from a transformation applied to the first system xy.

we say that a coordinate transformation has been applied.



Translation:

distance of the displacement,

$$v = t_x I + t_y J$$

$$\boxed{(x', y') = \bar{T}_v(x, y)}$$

where, $\begin{cases} x' = x - t_x \\ y' = y - t_y \end{cases}$

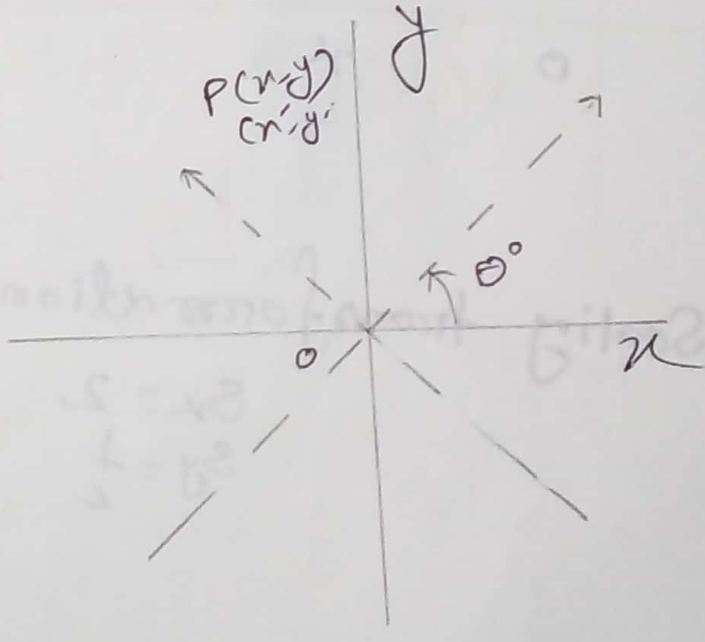
Rotation about the origin

→ xy system is rotated θ°
↳ about the origin

→ Rotation transformation: $\bar{R}_\theta$

$$\boxed{(x', y') = \bar{R}_\theta (x, y)}$$

where, $x' = x\cos\theta + y\sin\theta$
$y' = -x\sin\theta + y\cos\theta$

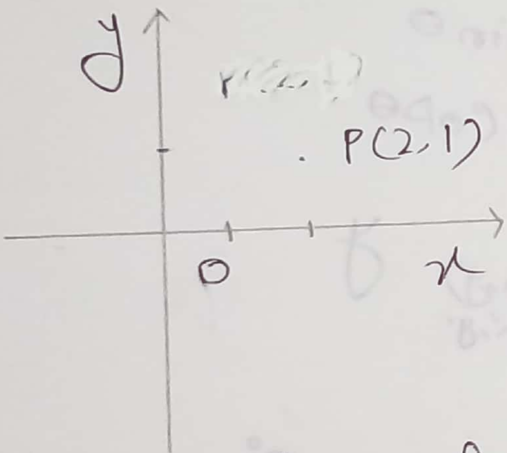


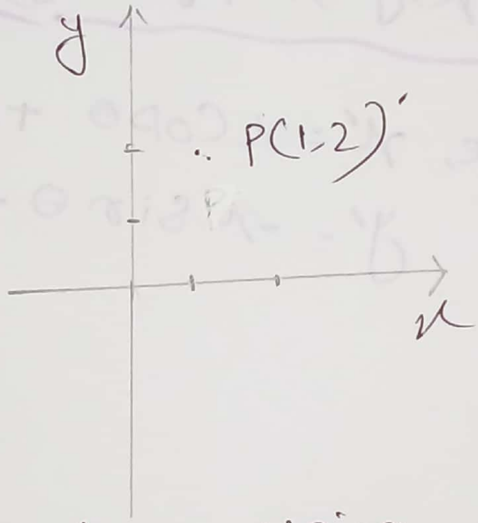
Scaling with respect to the origin:

$$(x', y') = \bar{S}_{s_x, s_y}(x, y)$$

$$x' = (1/s_x)\, x$$

$$y' = (1/s_y)\, y$$





Scaling transformation using

$S_x = 2$,

$S_y = \frac{1}{2}$

Mirror Reflection about an axis:

→ Reflecting the old system about either x or y axis

Coordinate transformation: $\overline{M}_x, \overline{M}_y$

→ For reflection about x-axis.

$$(x', y') = \overline{M}_x(x, y)$$

$$\boxed{x' = x,\ y' = -y}$$

$y' = -y$

→ " " about y-axis.

$$(x', y') = \overline{M}_y(x, y)$$

$$\boxed{x' = -x,\ y' = y}$$

Matrix description of the basic transformation.

Geometric transformation

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$S_{s_x, s_y} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

$$m_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$m_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Coordinate transformation

$$\overline{R}_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\overline{S}_{s_x, s_y} = \begin{bmatrix} \frac{1}{s_x} & 0 \\ 0 & \frac{1}{s_y} \end{bmatrix}$$

$$\overline{m}_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\overline{m}_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

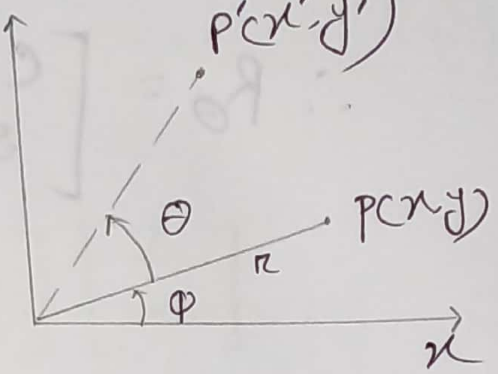
Composite transformation

A number of transformations or sewence of transformations can be combined into single one called as composition.

Problem 4.1 ☆☆

Transformation that rotates an object point θ° about the origin:

According to the definition of the trigonometric function,



$$\cos\phi = \frac{x}{r} \quad \Rightarrow x = r\cos\phi \qquad \Big| \qquad \sin\phi = \frac{y}{r} \quad \therefore y = r\sin\phi$$

$$\cos(\theta+\phi) = \frac{x'}{r} \quad \Rightarrow x' = r\cos(\theta+\phi) \qquad \Big| \qquad \sin(\theta+\phi) = \frac{y'}{r} \quad \Rightarrow y' = r\sin(\theta+\phi)$$

Now, $r\cos(\theta+\phi) = r(\cos\theta\cos\phi - \sin\theta\sin\phi)$
$= x\cos\theta - y\sin\theta$

Again $r(\sin\theta+\phi) = r(\sin\theta\cos\phi + \sin\phi\cos\theta)$
$= \cancel{y\cos\theta}$
$= x\sin\theta + y\cos\theta$

$\therefore x' = x\cos\theta - y\sin\theta$
$y' = x\sin\theta + y\cos\theta$

ϕ We can write,

$$P' = R_\theta \cdot P$$

$$\cancel{P' = \begin{pmatrix} x' \\ y' \end{pmatrix}} \quad P' = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad P = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\therefore R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Ans

~~Pro~~

Problem 8.4.2

a Given that,

$$\theta = 30^\circ$$

$$R_\theta =)$$

we know $\Rightarrow$, $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{3}/2 & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Ans

6 Given that,

$\cancel{P = (}$

$P(2, -4)$

$P' = ?$

$\therefore$

$$P' = R_\theta P$$

$$= \begin{bmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{3}/2 & -\frac{1}{2} \\ \frac{1}{2} & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

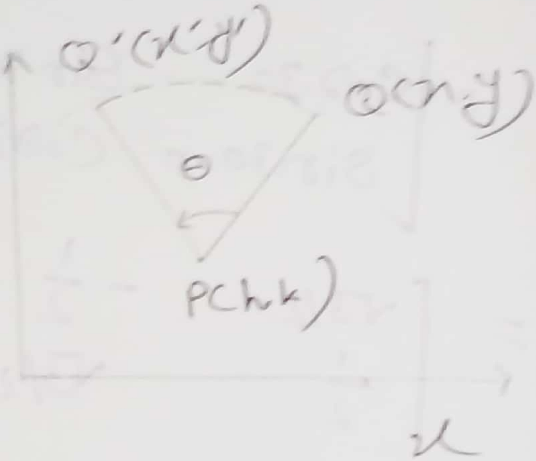
$$= \begin{bmatrix} \sqrt{3} + 2 \\ 1 - 2\sqrt{3} \end{bmatrix}$$

Ans

Problem 4.3

transformation that rotates an object point $Q(x,y)$, $\theta°$ about a fixed center of rotation $P(h,k)$

We can determine the transformation $R_{\theta,P}$ in three steps



→ $\boxed{\text{translate}}$ so that the center of rotation $\boxed{P \text{ is at the origin}}$

→ perform a $\boxed{\text{rotation}}$ of $\theta°$ about the origin

→ $\boxed{\text{translate}}$ P back to (h,k)

~~V~~ =: translation vector, $v = hI + kJ$.

$$\boxed{R_{\theta,P} = T_v \cdot R_\theta \cdot T_v^{-1}}$$

Problem 4.1

Matrix representation about a point $P(h,k)$

we know,

$$R_{\theta,P} = T_V \cdot R_\theta \cdot T_{-V}$$

Where, $V = hI + kJ$

$$\therefore R_{\theta,P} = \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -h \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix}$$

$(3\times3)\times3\times3$

$(3\times3)\times3\times3$

3

$$= \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & -h\cos\theta + k\sin\theta \\ \sin\theta & \cos\theta & -h\sin\theta - k\cos\theta \\ 0 & 0 & \cancel{-h-k}+1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & -h\cos\theta + k\sin\theta + h \\ \sin\theta & \cos\theta & -h\sin\theta - k\cos\theta + k \\ 0 & 0 & 1 \end{bmatrix}$$

Ans

Problem 4.5

a) 45° rotation of triangle $A(0,0)$, $B(1,1)$, $C(5,2)$ about the origin.

matrix representation of triangle is

$$\begin{bmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

Now

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

I

$$R_{45^\circ} = \begin{bmatrix} \cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\frac{2}{2}$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$0, 1, \frac{2}{2}, 3$

So coordinate A'B'C' of rotated triangle ABC can be found as,

$$[A' \; B' \; C'] = R_{45^\circ} \cdot [A \; B \; C]$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} + 0 & \frac{5}{\sqrt{2}} - \frac{2}{\sqrt{2}} + 0 \\ 0 & \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 0 & 5/\sqrt{2} + 2/\sqrt{2} + 0 \\ 0+0+1 & 0+0+1 & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 3/\sqrt{2} \\ 0 & \sqrt{2} & 7/\sqrt{2} \\ 1 & 1 & 1 \end{bmatrix}$$

$\therefore A' = (0, 0)$

$B' = (0, \sqrt{2})$

$C' = (3/\sqrt{2}, 7/\sqrt{2})$ Ans

6. 45° rotation of triangle A(0, 0), B(1, 1), C(5, 2) about P(-1, -1)

From (a) $[A\ B\ C] = \begin{bmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

$$R_{45^\circ} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_{\theta, P} = T_v \cdot R_\theta \cdot T_v^{-1}$ where $v = -I - J$

$$\Rightarrow R_{45^\circ, P} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \sqrt{2}-1 \\ 0 & 0 & 1 \end{bmatrix}$$

Now $[A' \; B' \; C'] = R_{45^\circ, P} \, [A \; B \; C]$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & -1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \sqrt{2}-1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{2}} & -\frac{\sqrt{2}}{\sqrt{2}} & -1 \\ \sqrt{2}/\sqrt{2} & \sqrt{2}/\sqrt{2} & \sqrt{2}-1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & \sqrt{2}/\sqrt{2} - \sqrt{2}/\sqrt{2} - 1 & 10/\sqrt{2} - 4/\sqrt{2} - 1 \\ \sqrt{2}-1 & 2/\sqrt{2} + 2/\sqrt{2} + \sqrt{2}-1 & 10/\sqrt{2} + 4/\sqrt{2} + \sqrt{2} - 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & \frac{6}{\sqrt{2}} - 1 \\ \sqrt{2}-1 & 4/\sqrt{2} + \sqrt{2} - 1 & \end{bmatrix}$$

$$= \begin{bmatrix} -1 & \sqrt{2}/2 - \sqrt{2}/2 - 1 & \frac{5\sqrt{2}}{2} - \frac{2\sqrt{2}}{2} - 1 \\ \sqrt{2}-1 & \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \sqrt{2} - 1 & \frac{5\sqrt{2}}{2} + \frac{2\sqrt{2}}{2} + \sqrt{2} - 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & \frac{3}{2}\sqrt{2} - 1 \\ \sqrt{2}-1 & 2\sqrt{2}-1 & \frac{9}{2}\sqrt{2} - 1 \\ 1 & 1 & 1 \end{bmatrix}$$

So $A' = (-1, \sqrt{2}-1)$

$B' = (-1, 2\sqrt{2}-1)$

$C' = (3/2\sqrt{2}-1, 9/2\sqrt{2}-1)$

Ans

Problem 4.7

General form of scaling matrix w.r.t. ~~to~~ a fixed point $P(h,k)$

Now transformation vector,

$V = hI + kJ$

$$\therefore S_{a,b,p} = T_v \cdot S_{a,b} \cdot T_v^{-1}$$

$$= \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -h \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & -ah \\ 0 & b & -kb \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a & 0 & -ah+h \\ 0 & b & -bk+k \\ 0 & 0 & 1 \end{bmatrix}$$

Ans

Problem 4.8

Given that

A(0, 0)

B(1, 1)

C(5, 2)

Now magnify the triangle twice while keeping C(5, 2) fixed.

S: Now transformation with,

$V = 5I + 2J$ as

$$S_{2,2,C} = T_V \cdot S_{22} \cdot T_V^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -10 \\ 0 & 2 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & -10+5 \\ 0 & 2 & -4+2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & -5 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[A' \; B' \; C'] = R\, S_{2,2,C} \, [A \; B \; C]$$

$$= \begin{bmatrix} 2 & 0 & -5 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 2-5 & 10-5 \\ -2 & 2-2 & 4-2 \\ 1 & 1 & 1 \end{bmatrix}$$

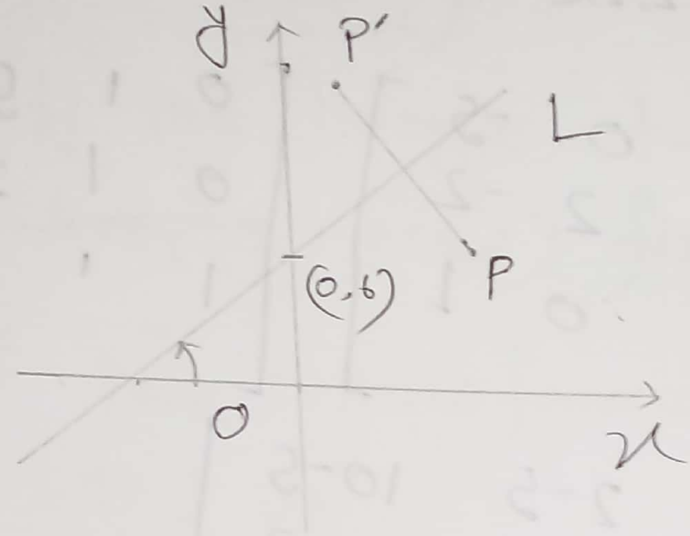
$$= \begin{bmatrix} -5 & -3 & 5 \\ -2 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$\therefore A' = (-5, -2)$
$B' = (-3, 0)$
$C' = (5, 2)$

Ans

Problem 4.9

transformation M_L - reflects an object about a line L,



Let line L have y intercept (0,b) and an angle of ~~inter~~ inclination $\theta°$ (w.r.to the x-axis)

1. Translate the intersection point B to the origin.
2. Rotate by $-\theta°$, so that line L aligns with the x-axis

3. mirror reflect about the x-axis

4. Rotate back by θ°

5. Translate B back to (0, b)

$$\therefore \boxed{M_L = T_v \cdot R_\theta \cdot M_x \cdot R_{-\theta} \cdot T_v^{-1}}$$

Ans

where $V = bJ$

Problem 4.10

Matrix for reflection about a line L with slop m and y intercept (0, b)

Now $\tan\theta = m$

$V = bJ$

$\therefore M = T_v \cdot R_\theta \cdot M_x \cdot R_{-\theta} \cdot T_{-v}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & \sin\theta & -b\sin\theta \\ -\sin\theta & \cos\theta & -b\cos\theta \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & -b\sin \\ \sin\theta & -\cos\theta & b\cos \\ 0 & 0 & 1 \end{bmatrix}$$

$=$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\cos\theta\sin\theta & -2b\cos\theta\sin\theta \\ 2\sin\theta\cos\theta & \sin^2\theta - \cos^2\theta & b(\cos^2\theta - \sin^2\theta) \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\cos\theta\sin\theta & -2b\cos\theta\sin\theta \\ 2\sin\theta\cos\theta & \sin^2\theta - \cos^2\theta & b(\cos^2\theta - \sin^2\theta + 1) \\ 0 & 0 & 1 \end{bmatrix}$$

Now $\tan\theta = m$

$\Rightarrow \dfrac{\sin\theta}{\cos\theta} = m$

$\Rightarrow \sin^2\theta = m^2\cos^2\theta$

Again $\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow m^2\cos^2\theta + \cos^2\theta = 1$

$\Rightarrow (m^2+1)\cos^2\theta = 1$

$\therefore \cos^2\theta = \dfrac{1}{m^2+1}$

Now ~~$\sin^2\theta =$~~

$\therefore \sin\theta = m\cos\theta$

$= m\dfrac{1}{\sqrt{m^2+1}}$

$\sin^2\theta = \dfrac{m^2}{m^2+1}$

$2\sin\theta\cos\theta = \dfrac{2m}{m+1}$

Now, $\cos^2\theta - \sin^2\theta + 1$

$= \dfrac{1-m}{m^2+1} + 1$

$= \dfrac{1 \cancel{+m^2} + \cancel{m^2} + 1}{m^2+1}$

$= \dfrac{2}{m^2+1}$

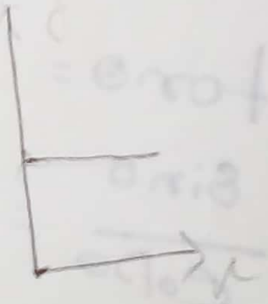
$$\therefore M_L = \begin{bmatrix} \frac{1-m^2}{m^2+1} & \frac{2m}{m^2+1} & \frac{-2bm}{m^2+1} \\ \frac{2m}{m^2+1} & \frac{m^2-1}{m^2+1} & \frac{2b}{m^2+1} \\ 0 & 0 & 1 \end{bmatrix}$$

Ans

Problem 4.11

Given that,

A(-1,0)
B(0,-2)
C(1,0)
D(0,2)



$y=2$

$$\therefore V = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

We know, $M_L = T_v R_\theta M_x R_{-\theta} I_v$

~~It makes~~

y intercept (0,2)
makes an angle 0° with
x axis. So,

$\therefore \theta = \theta^{\circ}$ and $V = 23$

Transformation of the matrix is

$M_L = \begin{bmatrix} \cancel{1/m'} \end{bmatrix}$

$$m_L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad [\because AI = I]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

To reflect the polygon we set,

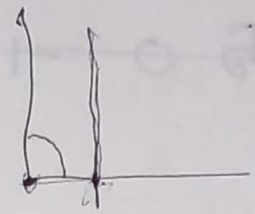
$$M_L V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 1 & 0 \\ 4 & 6 & 4 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$\therefore A'(-1, 4) \quad B'(0, 6) \quad C'(1, 4) \quad D'(0, 2)$

Ans

b $x = 2$
$v = 2y$



has no y intercept

$\therefore M_L = T_v \, M_y \, T_{-v}$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, $M_L V = \begin{bmatrix} -1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 5 & 4 & 3 & 1 \\ 0 & -2 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\therefore M_L = \begin{bmatrix} \frac{1-1}{1+1} & 2/(1+1) & -4/(1+1) \\ \frac{(2\times 1)}{1+1} & (1-1)/(1+1) & \frac{4}{(1+1)} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, $$M_L V = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -4 & -2 & 0 \\ 1 & 2 & 3 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$A'(-2,1)$
$B'(-4,2)$
$C'(-2,3)$
$D'(0,2)$ Ans

Chapter-5
2-D viewing & Clipping

Point clipping:

It is Evaluation of the following inequalities

$$x_{min} \leq x \leq x_{max}$$

$$y_{min} \leq y \leq y_{max}$$

→ x_{min}, x_{max}, y_{min} and y_{max} define the clipping window

Line clipping:

→ don't intercept the clipping window
→ completely inside the window
→ ,, outside ,,

→ intercepts the clipping window

divide by

intersection point(s) into segments

Algorithm:

→ Cohen-sutherland Algorithm
→ Liang-Barsky Algorithm

Cohen sutherland Algorithm:

3 clipping categories:

1) visible - both endpoints of line lie within the window
2) Not visible -
3) Clipping candidates -
line is in neither category 1 nor 2.

line definitely lies outside the window

$x_1, x_2 > x_{max}$ $y_1, y_2 > y_{max}$
$x_1, x_2 < x_{min}$ $y_1, y_2 < y_{min}$

line from (x_1, y_1) to (x_2, y_2)

Procedure:

→ Assign a $\boxed{\text{4 bit region code}}$ to each endpoint of line

1001	1000	1010
0001	0000	0010
0101	0100	0110

endpoint A is

$$
\begin{aligned}
\text{Bit 1} &= \text{above the window} = \text{Sign}(y - y_{max}) \\
\text{Bit 2} &= \text{below " "} = \text{Sign}(y_{min} - y) \\
\text{Bit 3} &= \text{right of " "} = \text{Sign}(x - x_{max}) \\
\text{Bit 4} &= \text{left of " "} = \text{Sign}(x_{min} - x)
\end{aligned}
$$

→ $\text{Sign}(a) = \begin{cases} 1, & a \geq 0 \\ 0, & a < 0 \end{cases}$

→ Visible: both region codes are 0000

→ not visible: bitwise logical AND of the codes is not 0000

→ Candidate for clipping: bitwise logical AND of codes is 0000

→ determine the intersection point (x_i, y_i) of ~~ω~~ line

→ bit 1 = 1 , $y_i = y_{max}$, $x_i = x_1 + \frac{y_{max} - y_1}{m}$

→ bit 2 = 1 , $y_i = y_{min}$ $x_i = x_1 + \frac{y_{min} - y_1}{m}$

→ bit 3 = 1, $x_i = x_{max}$ $y_i = y_1 + (x_{max} - x_1) m$

→ bit 4 = 1, $x_i = x_{min}$ $y_i = y_1 + (x_{min} - x_1) m$

where, $m = \frac{y_2 - y_1}{x_2 - x_1}$

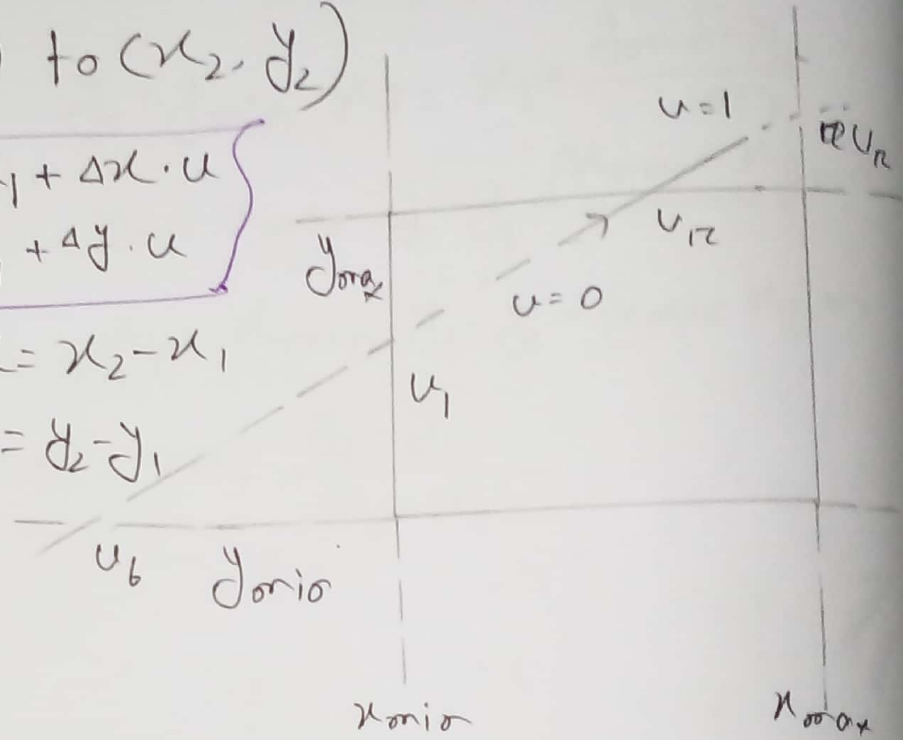
Liang-Barsky Algorithm:

→ parametric equations for a line from (x_1, y_1) to (x_2, y_2)

$$\boxed{\begin{aligned} x &= x_1 + \Delta x \cdot u \\ y &= y_1 + \Delta y \cdot u \end{aligned}}$$

where, $\Delta x = x_2 - x_1$
$\Delta y = y_2 - y_1$

$0 \leq u \leq 1$



Procedure:

→ determine P_k, q_k, $k = 1, 2, 3, 4$

$P_1 = -\Delta x$ $\quad q_1 = x_1 - x_{min}$ (left)
$P_2 = \Delta x$ $\quad q_2 = x_{max} - x_1$ (right)
$P_3 = -\Delta y$ $\quad q_3 = y_1 - y_{min}$ (bottom)
$P_4 = \Delta y$ $\quad q_4 = y_{max} - y_1$ (top)

→ if $P_k = 0$, line is parallel to the boundary

→ if $q_k < 0$, → completely outside the boundary

→ if $q_k \geq 0$ → inside the boundary

→ if $P_k < 0$, extended line proceeds from outside to the inside of the corr. boundary

→ if $P_k > 0$, line proceeds from inside to the outside of the correspo boundary

→ if $P_k \neq 0$, $\boxed{u_k = \frac{q_k}{P_k}}$

g

Process of finding the visible portion of the line:

4 steps

→ if $P_k = 0$ and $q_k < 0$, eliminate the line & stop

→ if $P_k < 0$ calculate $r_k = q_k / P_k$,

$$u_1 = \max(0, r_k)$$

→ if $P_k > 0$ calculate $r_k = q_k / P_k$,

$$u_2 = \min(1, r_k)$$

→ if $u_1 > u_2$, → completely outside the window

else, → use u_1 and u_2 too

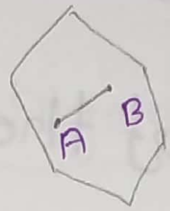
→ to calculate the end points of the clipped line.

$$x = x_1 + \Delta x\, u$$
$$y = y_1 + \Delta y\, u$$

Convex polygon:

Line joining any two interior points of the polygon

lies completely inside the window



~~Conv~~

Concave polygon:

→ A non-convex polygon.

→ Line joining any two interior points of the polygon.

touch the border

Positive oriented:

A polygon with vertices $P_1 \dots P_N$ $(P_{i-1}, P_i \dots P_N P_i)$ is said to be ~~posite~~ positively oriented:

tour of the vertices in the given order

produces a counterclockwise circuit.

☐

☐ Negative oriented

tour of the vertices in a given order

produces a clockwise circuit.

☐ left of the line segment:

→ $A(x_1, y_1)$ and $B(x_2, y_2)$ be the endpoints of a directed line segment.

if, $e = (x_2 - x_1)(y - y_1) - (y_2 - y_1)(x - x_1) \geq 0$

→ ~~point~~ $P(x, y)$ is left of the line seg

else

→ $P(x, y)$ " right of " "

Polygon clipping Algorithm:

→ Sutherland-Hodgman Algorithm

→ Weiler-Atherton "

Sutherland-Hodgman Algorithm

→ $P_1 \ldots P_N$ be the vertex list of polygon

→ edge E determined by endpoints A and B.
→ positively oriented, convex clipping polygon.

→ each edge of the polygon in turn against the edge E of the clipping polygon will be clipped

Consider the edge $\overline{P_{i-1} P_i}$

*

* → P_{i-1} and P_i are to the left of the edge

vertex P_i is placed on the vertex output list.

→ P_{i-1} and P_i are to the right of the edge

nothing is placed

→ P_{i-1} is to the left and P_i is to the right of the edge E,

intersection point I of line segment $P_{i-1} P_i$ with the extended edge E is calculated

→ P_{i-1} is to the right and P_i is to the left of the edge E.

both I and P_i are placed on the vertex output list.

NB: Special attention is necessary

→ to avoid unwanted effects

b) Weiler-Atherton Algorithm:

→ clipping window is called clip polygon

→ polygon to be clipped is called subject polygon

→ Start with an arbitrary vertex of subject polygon

→ trace around the border in the clockwise direction

Process:

⁂ → edge enter the clip polygon.

record the intersection point and continue to trace the ~~to~~ subject polygon.

→ edge leave the clip polygon.

→ record the intersection point

→ make a right turn to follow the clip polygon in the same manner

→ path of traversal forms a sub-polygon

output the sub-polygon as part of the overall result.

problem 5.6

Given that

lower-left corner L(-3,1)

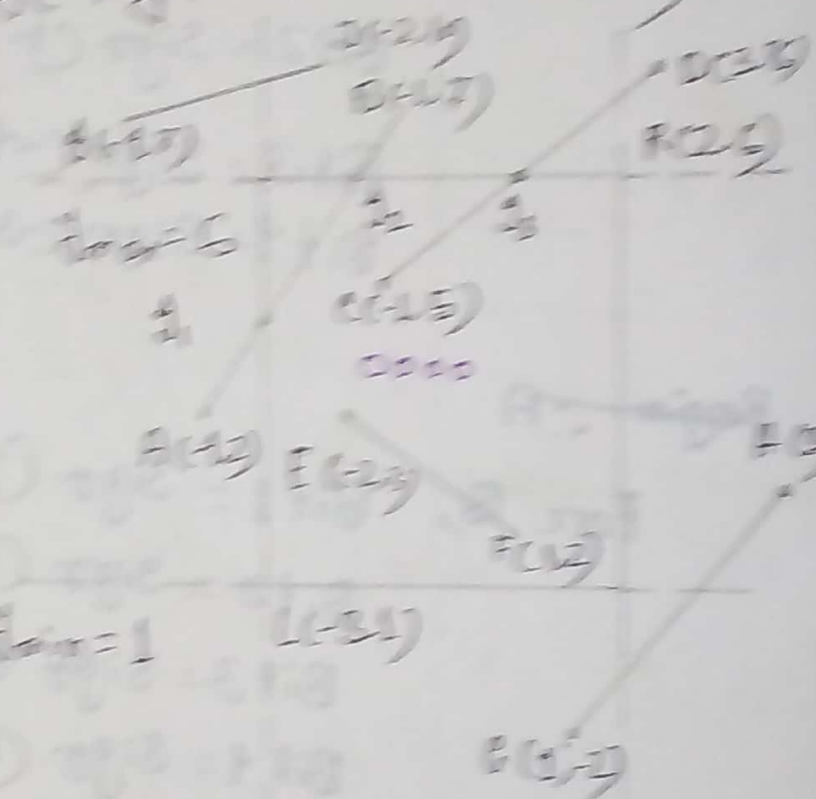
upper-right R(2,6)

$x_{min} = -3$

$x_{max} = 2$

$y_{min} = 1$

$y_{max} = 6$



Region code can be find by $x_{min} = -3$, $x_{max} = 2$

using following formula.

Bit 1 = $sign(y - y_{max})$

Bit 2 = $sign(y_{min} - y)$

Bit 3 = $sign(x - x_{max})$

Bit 4 = $sign(x_{min} - x)$

Here $sign(a) = \begin{cases} 1, & \text{if } a \text{ is positive} \\ 0, & \text{otherwise} \end{cases}$

~~For line~~

For line AB

$A = (-4, 2)$

$B = (-1, 7)$

$\therefore$ For A, Bit 1 = Sign $(2-6)$ = Sign (-4) = 0

Bit 2 = Sign $(1-2)$ = Sign (-1) = 0

Bit 3 = Sign $(-4-2)$ = Sign (-6) = 0

Bit 4 = Sign $(-3-(-4))$ = Sign (1) = 1

~~Region~~ $\therefore$ A

For B, Bit 1 = Sign $(7 - 6)$ = Sign(1) = 1

Bit 2 = Sign $(6-(7))$ = Sign(-1) = 0

Bit 3 = Sign $((-1)-2)$ = Sign (-3) = 0

Bit 4 = Sign $((-3)-(-1))$ = Sign (-2) = 0

For line CD

$C = (-1, 5)$

$D = (3, 8)$

For $C = (-1, 5)$

Bit 1 = Sign $(5-6)$ = 0

Bit 2 = Sign$(1-5)$ = 0

Bit 3 = Sign$((-1)-2)$ = 0

Bit 4 = Sign $((-3)-(-1))$ = 0

For D = (3, 8)

Bit 1 = Sign (8 − 6) = 1

Bit 2 = Sign (1 − 8) = 0

Bit 3 = Sign (3 − 2) = 1

Bit 4 = Sign (−3 − 3) = 0

For line EF

E = (−2, 3)

F = (1, 2)

For E,

Bit 1 = Sign (3 − 6) = 0

Bit 2 = Sign (1 − 3) = 0

Bit 3 = Sign ((−2) − 2) = 0

Bit 4 = Sign ((−3) − (−2)) = 0

~~For E~~ :

For F,

Bit 1 = Sign (2 − 6) = 0

Bit 2 = Sign (~~3~~1 − 2) = 0

Bit 3 = Sign (1 − 2) = 0

Bit 4 = Sign (−3 − 1) = 0

∴ Region code of A (−4, 2) is 0001

" " of B (−1, 7) " 1000

" " of C (−1, 5) " 0000

" " of D (3, 8) " 1010

" " of E (−2, 3) " 0000

" " of F (−1, 2) " 0000

Region code of G is 0100
" " of H " 0010
" " of g " 1001
" " of J " 1000
Ans

problem 5.8

~~For line A~~
From problem 5.G

Point	Region code
A	0001
B	1000
C	0000
D	1010
E	0000
F	0000
G	0100
H	0010
g	1001
J	1000

~~Now~~ Now for line $\overline{AB}$,

$$\begin{array}{r} 0001 \\ 1001 \\ \hline 0000 \end{array}$$

$\therefore$ Clipping candidate.

For line $\overline{CD}$

$$\begin{array}{r} 0000 \\ 0101 \\ \hline 0000 \end{array}$$

$\therefore$ Clipping candidate

For line $\overline{EF}$
visible, AS region code of both E and F is 0000

For line $\overline{GH}$,

$$\begin{array}{r} 0100 \\ 0010 \\ \hline 0000 \end{array}$$

$\therefore$ Clipping candidate.

For line $\overline{IJ}$

$$\begin{array}{r} 1001 \\ 1000 \\ \hline 1000 \end{array}$$

$\therefore$ Not visible.

$\therefore$ category 1 (visible) : $\overline{EF}$

category 2 (Not visible): $\overline{IJ}$

category 3 (clipping candidate): $\overline{AB}, \overline{CD}, \overline{GH}$

Ans

Problem 5.9

From problem 5.8 line segment. $\overline{AB}, \overline{CD}, \overline{GH}$ are clipping candidate.

For line segment $\overline{AB}$, $A(-4,2)$ $B(-1,7)$

region code of A , 0001

" " of B 1000

0000

$\therefore$

Let A intersect window at I, and B " " at B

For A Bit 4 = 1

$\therefore x_i = x_{min} = -3$

$y_i = y_1 + (x_{min} - x_1) m$

$= 2 + (-3-(-4)) \frac{5}{3}$

$= 2 + 5/3 = \frac{11}{3} = 3\frac{2}{3}$

Now m for $\overline{AB}$

$= \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{7-2}{-1-(-4)}$

$= \frac{5}{-1+4} = \frac{5}{3}$

$\therefore I_1 (-3, 3\frac{2}{3})$ Ans

For B, Bit 1 = 1

$\therefore y_i = y_{max} = 6$

$x_i = x_1 + \frac{\cancel{x_1 - x_{max}}}{m}$

$\cancel{= (-1) + \frac{-1-2}{(5/3)}}$

$\cancel{= -1 - \frac{3}{5/3}}$

$\cancel{= -1 - 3\frac{3}{5}}$

$\cancel{= -1 - \frac{9}{5}}$

$\cancel{= \frac{-5-9}{5}} = \frac{-14}{5} = -2\frac{4}{5}$

$x_i = x_1 + \frac{\cancel{y_1 - y_{max}}}{m}$

$\cancel{= (-1) + \frac{7-6}{5/3}}$

$\cancel{= -1 + \frac{1}{5/3}}$

$\cancel{= -1 + \frac{3}{5}}$

$\cancel{= \frac{-5+3}{5}}$

$= \frac{-2}{5}$

$x_i = x_1 + \frac{-y_1 + y_{max}}{m}$

$= -4 + \frac{-2+6}{5/3}$

$= -4 + \left(\frac{4}{5/3}\right)$

$= -4 + \left(4 \times \frac{3}{5}\right)$

$= -4 + \frac{12}{5}$

$= \frac{-20-12}{5} = \frac{\cancel{32}}{5} = \frac{-8}{5}$

$= -1\frac{3}{5}$

$\therefore I_2 = (-1\frac{3}{5}, 6)$

Ans

we clip $\overline{AI_1}$

~~I_1 code is~~

Now work with $\overline{I_1B}$

code(I_1) = 0000

code(B) = 1000

0000

$\therefore$ Clipping candidate

$\therefore$ we clip $\overline{BI_2B}$

$\therefore$ remaining $\overline{I_1I_2}$ is visible

$\therefore \overline{AI_1}$ and $\overline{\cancel{BI_2}B}$ is clipped

$\therefore \overline{I_1I_2}$ is visible

For line $\overline{CD}$. $C=(-1,5)$ ∴
$D=(3,8)$

region code of C, 0000

" " of D, ~~0100~~ 1010

∴ For D Bit 1 = 1

$\therefore y_i = y_{max} = 6$

$x_i = x_1 + \frac{y_{max} - y_1}{m}$

$= (-1) + \frac{6-5}{3/4}$

$= (-1) + \frac{1}{3/4}$

$= -1 + 4/3$

$= \frac{-3+4}{3} = \frac{1}{3}$

$m = \frac{8-5}{3+1}$

$= \frac{3}{4}$

For D, Bit 3 = 1

$\therefore x_i = x_{max}$

$= 2$

$y_i = y_1 + (x_{max} - x_1) m$

$= 5 + (2-(-1)) 3/4$

$= 5 + \frac{3}{4} 3$

$= 5 + \frac{9}{4}$

$= \frac{20+9}{4} = \frac{29}{4}$

$= 7\frac{1}{4}$

$(2, 7\frac{1}{4})$ is outside the window.

we clip DJ_3,

Now work with J_3C which is visible

$\therefore J_3 = (1/3, 6)$

Ans

Ans

For line $\overline{GH}$, $G(1,-2)$, $H(3,3)$

$\therefore m = \frac{3+2}{3-1} = \frac{5}{2}$

region code of G, 0 1 0 0

" " H, 0 0 1 0

For G Bit 2 = 1

$\therefore y_i = y_{min} = 1$

$x_i = x_1 + (y_{min} - y_1)/m$

$= 1 + (1+2)/(5/2)$

$= 1 + \frac{3}{5/2}$

$= 1 + \frac{3 \times 2}{5}$

$= 1 + \frac{6}{5} = \frac{11}{5} = 2\frac{1}{5}$

$\therefore J_4 = (2\frac{1}{5}, 1)$

Ans

$\therefore$ Again, for H Bit 3 = 1

$\therefore x_i = x_{max} = 2$

$y_i = y_1 + (x_{max} - x_1) m$

$= -2 + (2-1)\frac{5}{2}$

$= -2 + 5/2$

$= \frac{1}{2}$

$\therefore$ We clip $\overline{GI_4}$. Now we work with $\overline{I_4 H}$

code of I_4 is 0010

code of H is 0010

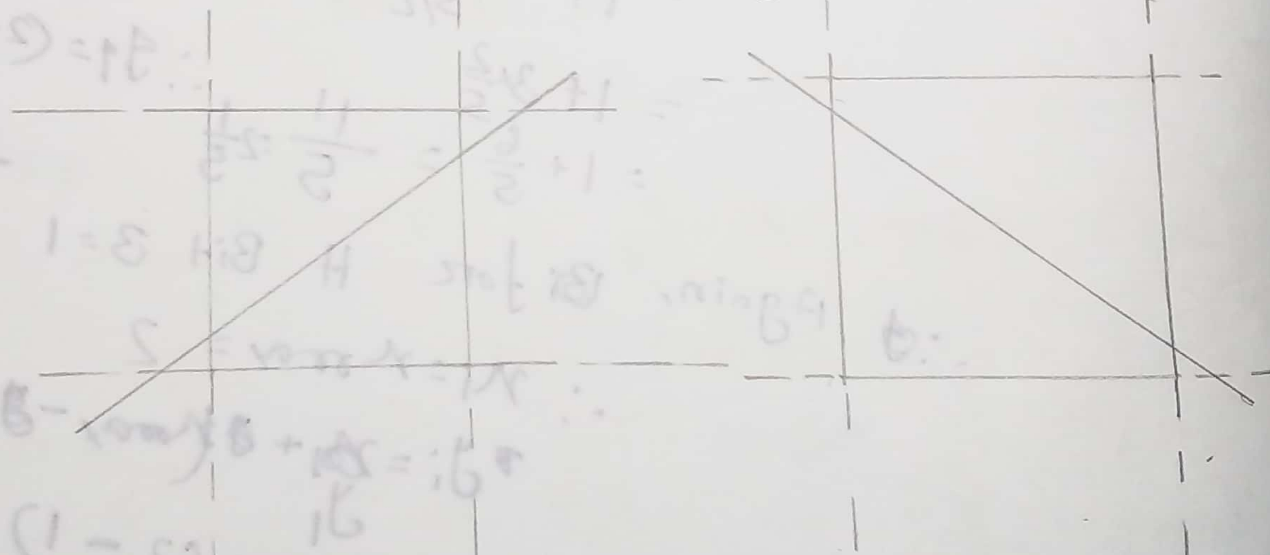
$$\frac{\begin{array}{r}0010\\0010\end{array}}{0010}$$

$\therefore$ $\overline{I_4 H}$ is not displayed.

Ans

Problem 5.11

Worst case scenario for Cohen-sutherland algorithm.



Problem 5.12

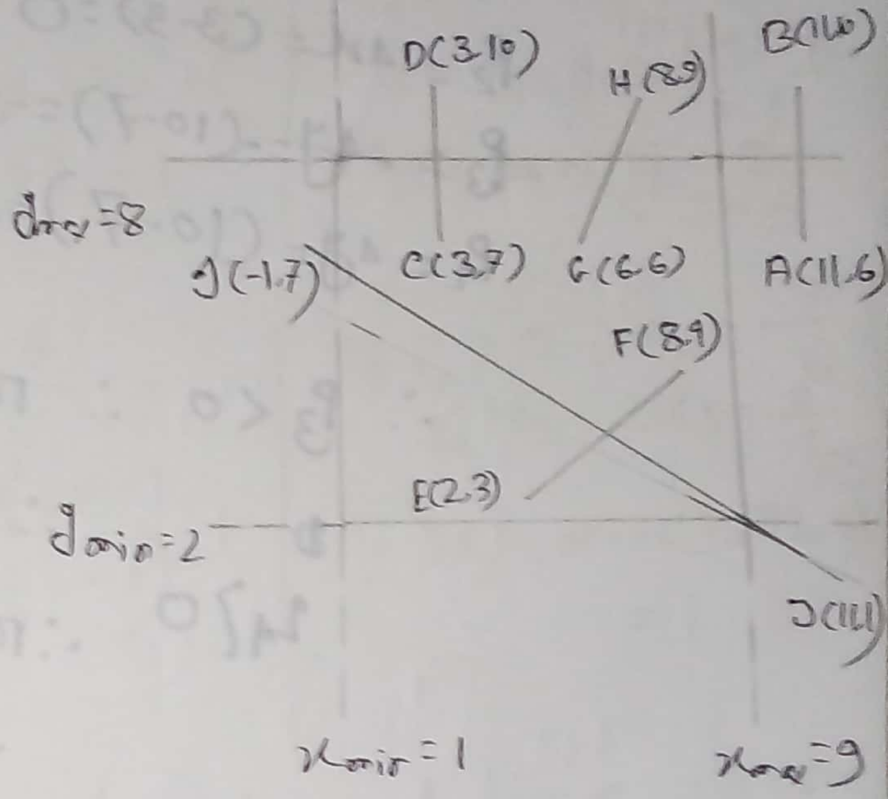
Given that

$x_{min} = 1$

$x_{max} = 9$

$y_{min} = 2$

$y_{max} = 8$



For line $\overline{AB}$

$P_1 = -\Delta x = -(11-11) = 0$ $\quad q_1 = x_1 - x_{min} = 11-1 = 10$

$P_2 = \Delta x = (11-11) = 0$ $\quad q_2 = x_{max} - x_1 = 9 - 11 = -2$

$P_3 = -\Delta y = -(10-6) = -4$ $\quad q_3 = y_1 - y_{min} = 6-2 = 4$

$P_4 = \Delta y = (10-6) = 4$ $\quad q_4 = y_{max} - y_1 = 8-6 = 2$

$q_1 = 10$ i.e $q_1 > 0$

As $P_1 = 0$

But $P_2 = 0$ and $q_2 < 0$

$\therefore$ Line is completely outside right boundary.

For $\overline{CD}$

$P_1 = -\Delta x = -(3-3) = 0$ $\quad q_1 = x_1 - x_{min} = (3-1)$

$P_2 = \Delta x = (3-3) = 0$ $\quad q_2 = x_{max} - x_1 = (9-3)$

$P_3 = -\Delta y = -(10-7) = -3$ $\quad q_3 = y_1 - y_{min} = (7-2)$

$P_4 = \Delta y = (10-7) = 3$ $\quad q_4 = y_{max} - y_1 = (8-7)$

$\therefore P_3 < 0 \quad \therefore r_3 = \frac{q_3}{P_3} = \frac{5}{-3} = -5/3$

$\therefore u_1 = \max(0, -5/3) = 0$

$q_4 > 0 \quad \therefore r_4 = q_4/P_4 = \frac{1}{3} = \frac{1}{3}$

$\therefore u_2 = \min(1, \frac{1}{3})$

$= \frac{1}{3}$

$\therefore u_1 < u_2$

$\therefore$ two end points of the clipped line

are,

$x_1 = x_1 + \Delta x\, u_1$ $\quad \Delta x = 0$

$= 3 + \Delta x\, 0$ $\quad \Delta y = 10-7 = 3$

$= 3$

$y_1 = (y_1 + \Delta y\, u_1)$ $\quad \therefore (3, 7)$

$= 7 + \Delta y\, 0$

$= 7$

and,

$(3, 7 + 3 \times \frac{1}{3})$

$= (3, 8)$

For $\overline{EF}$

$P_1 = -\Delta x = -(8-2) = -6$

$P_2 = \Delta x = (8-2) = 6$

$P_3 = -\Delta y = -(4-3) = -1$

$P_4 = \Delta y = (4-3) = 1$

$q_1 = x_1 - x_{min} = 2-1 = 1$

$q_2 = x_{max} - x_1 = 9-2 = 7$

$q_3 = y_1 - y_{min} = 3-2 = 1$

$q_4 = y_{max} - y_1 = 8-3 = 5$

$P_1 < 0 \quad \therefore r_1 = -\frac{1}{6}$

$P_3 < 0 \quad \therefore r_3 = -\frac{1}{1} = -1$

$\therefore u_1 = max(0, -\frac{1}{6}, -1) = 0$

$P_2 > 0 \quad \therefore r_2 = \frac{7}{6}$

$P_4 > 0 \quad \therefore r_4 = \frac{5}{1} = 5$

$\therefore u_2 = min(1, 7/6, 5) = 1$

$\therefore \cancel{u_1 < u_2}$

$\therefore$ two end points of ~~clipped line is~~ are

~~$(2 + 6 \times 0, 3 + 1 \times 0) = (2, 3)$~~

~~$(2 + 6 \cdot 1, 3 + 1 \cdot 1) = (8, 4)$~~ ~~Ans~~

$\therefore$ EF is completely inside the window

Ans

For $\overline{GH}$

$P_1 = -\Delta x = -(8-6) = -2$

$P_2 = \Delta x = (8-6) = 2$

$P_3 = -\Delta y = -(9-6) = -3$

$P_4 = \Delta y = (9-6) = 3$

$q_1 = x_1 - x_{min} = 6-1 = 5$

$q_2 = x_{max} - x_1 = 9-6 = 3$

$q_3 = y_1 - y_{min} = 6-2 = 4$

$q_4 = y_{max} - y_1 = 8-6 = 2$

$\therefore P_1 < 0$, $r_1 = -\frac{5}{2}$

$P_3 < 0$, $r_3 = -\frac{4}{3}$

$\therefore u_1 = max(0, -5/2, -4/3)$

$= 0$

$P_2 > 0$, $r_2 = \frac{3}{2}$

$P_4 > 0$, $r_4 = \frac{2}{3}$

$\therefore u_2 = min(1, \frac{3}{2}, \frac{2}{3})$

$= 2/3$

$\therefore u_1 < u_2$

$\therefore$ two end points of ~~the~~ clipped line is $(6+2\cdot 0, 6+3\cdot 0)$

$= (6, 3)$ and

$(6 + 2\frac{2}{3}, 6 + \cancel{3}\frac{2}{\cancel{3}})$

$= (\frac{22}{3}, 8)$

Ans

~~For $\overline{IJ}$,~~

~~$P_1 = -\Delta x = -(-1-11) = 12$~~

~~$P_2 = \Delta x = (-1-11) = -12$~~

~~$P_3 = -\Delta y = -(7-1) = -6$~~

~~$P_4 = \Delta y = (7-1) = 6$~~

~~$2/1 = r_1$~~

For $\overline{IJ}$

$P_1 = -\Delta x = -(11-(-1)) = -12$

$P_2 = \Delta x = (11-(-1)) = 12$

$P_3 = -\Delta y = -(1-7) = 6$

$P_4 = \Delta y = (1-7) = -6$

$q_1 = -1 - 1 = -2$

$q_2 = 9 + 1 = 10$

$q_3 = 7 - 2 = 5$

$q_4 = 8 - 7 = 1$

$\therefore P_1 < 0 \quad \therefore r_1 = \frac{2}{12} = \frac{1}{6}$

$P_4 < 0 \quad \therefore r_4 = \frac{-1}{6} = -\frac{1}{6}$

$\therefore u_1 = \max(0, \frac{1}{6}, -\frac{1}{6})$
$= \frac{1}{6}$

$P_2 > 0 \quad \therefore r_2 = \frac{10}{12} = \frac{5}{6}$

$q_3 > 0 \quad \therefore r_3 = \frac{5}{6}$.

$\therefore u_2 = \min(1, \frac{5}{6}, \frac{5}{6})$
$= 5/6$

$\therefore u_1 < u_2$

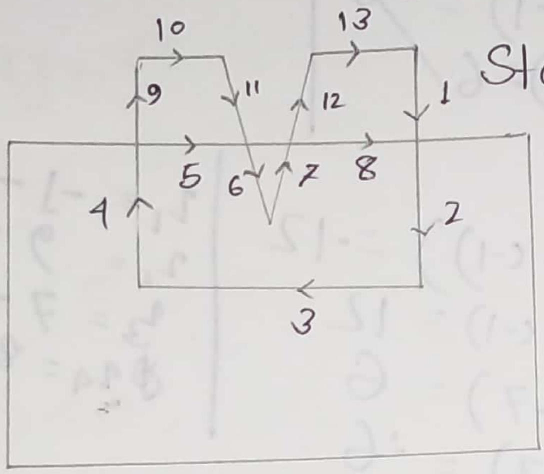
$\therefore$ two end points of clipped line

are $(-1 + 12\frac{1}{6}, 7 + (-6)\frac{1}{6}) = (1, 6)$

and $(-1 + 12\frac{5}{6}, 7 + (-6)\frac{5}{6}) = (9, 2)$ Ans

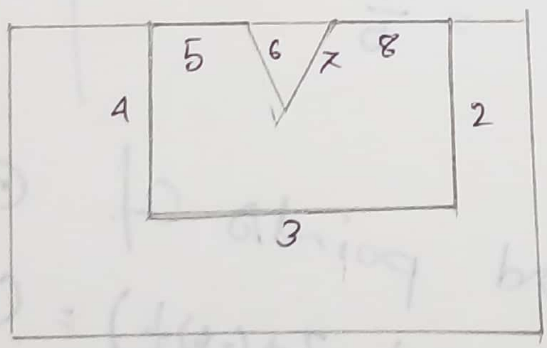
Example of weiler-Atherton Algorithm:

Exp-1

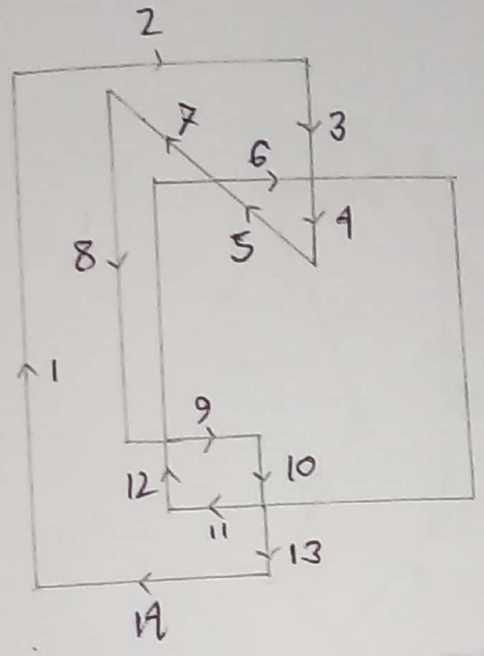


Start polygon

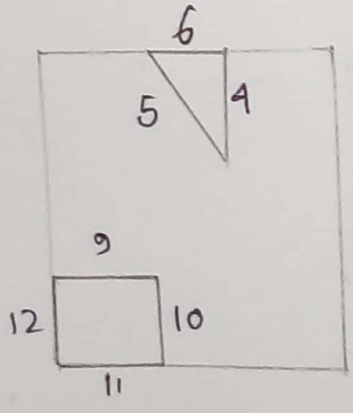
Clipped polygon



Example-2

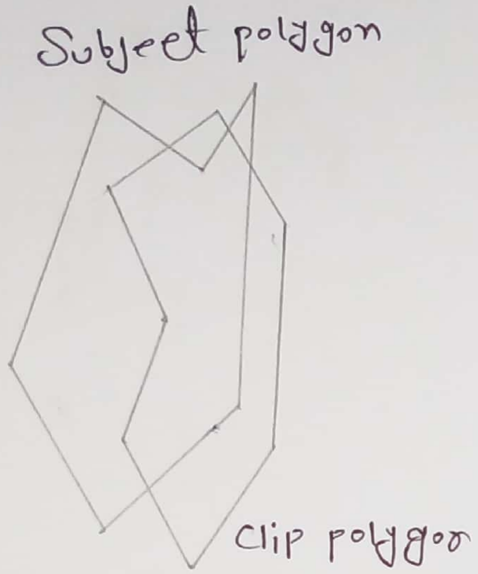


Clipped polygon.

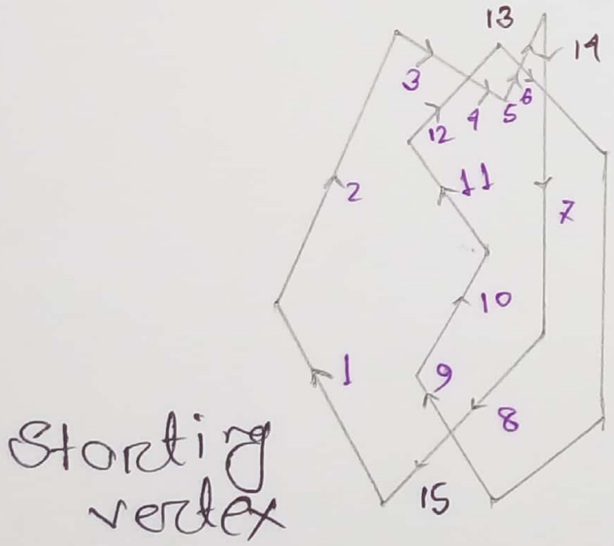


Problem 5.17

Given polygon.



polygon with traversing path



Closed polygon

Chapter-7
Mathematics of projection

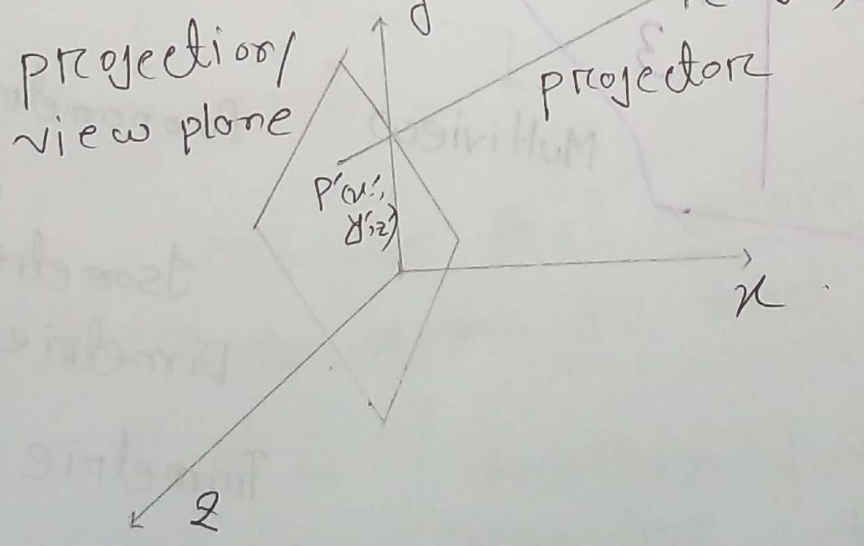
Projection:

Defined as a mapping of point P(x, y, z) onto its image P'(x', y', z') in the projection plane or view plane.

projector:

mapping is determined by a projection line.

→ passes through P and intersects the view plane.



Taxonomy of projection:

→ Two basic methods of projection
- prespective
- parallel

Projections
- Perspective
  - One point — principal vanishing point: 1
  - Two point — 2
  - Three point — 3
- Parallel
  - Orthographic
    - Multiview
    - Axonometric
      - Isometric
      - Dimetric
      - Trimetric
  - Oblique
    - General
      - Cabinet
      - Cavalier

Prespective projection:

→ eye of the artist is placed at the center of projection

→ Charaeterized by
- → prespective foreshortening
- → vanishing point

Prespective foreshortening:

** Illusion that objects and lengths appear smaller → as their distance from the center of projection increases

Vanishing point:

** Illusion that certain sets of parallel lines appear to meet at a point → another feature of prospective drawing

Principal vanishing points:

Apparent intersection of lines parallel to one of the three principal x, y or z axes.

number of principal axes intersected by the view plane.

Mathematical Description of a perspective projection:

Determined by prescribing a

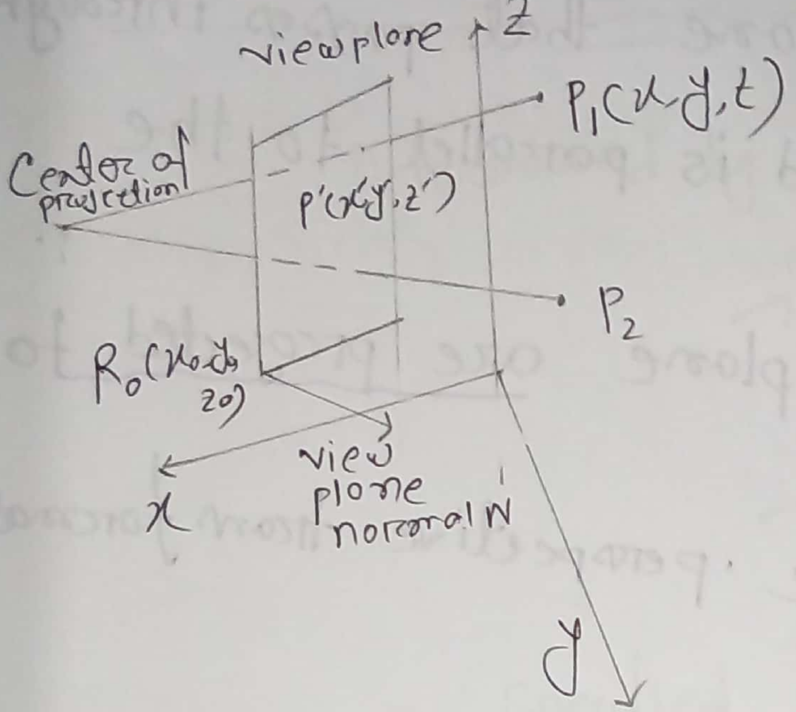
→ a center of project

→ a view plane

determined by its

→ view reference point R_0

→ view plane normal N



perspective Anomalies:

4 anomalies

→ perspective foreshortening — the farther an object is from the C.O.P. the smaller it appear

→ vanishing points — projections of lines that are not parallel to the view plane. appear to meet at some point on the view plane

→ view confusion

→ Topological distortion

Objects behind the C.O.P are projected upside down and backward onto the view plane

Topological distorction:

Consider the plane that passes through the C.O.P. and is parallel to the view plane.

Points of this plane are projected to infinity by the perspective transformation

Mathematical Description of a parallel projection:

Determined by prescribing

→ a direction of projection vector v

→ a view plane

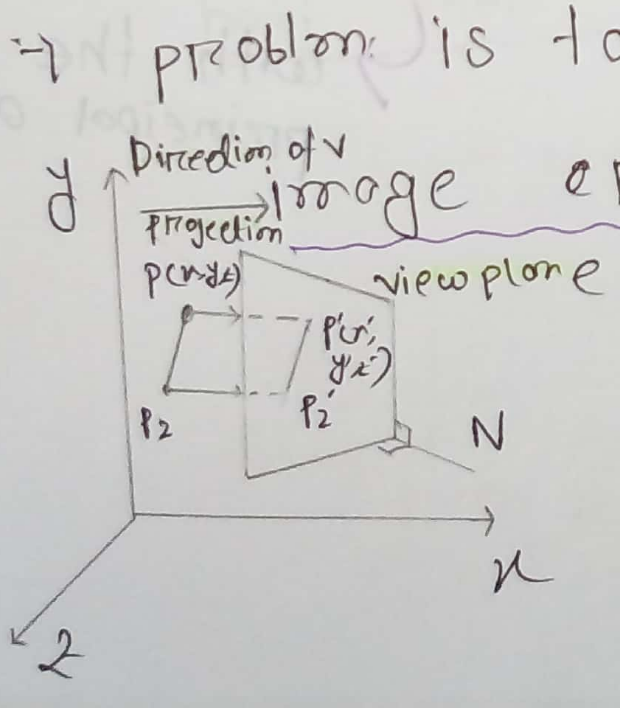
specified by it

→ view plane reference point R_0

→ view plane normal $\underline{N}$

→ object point P is located at (x, y, z) in world coordinate

→ problem is to determine the image point coordinates P′(x′, y′, z′)



Orthographic projection:

Projection vector v has the direction of the view plane normal N.

3 categories:

1) Isometric
Direction of the projection makes equal an[gle]
all three princip[al] ax[es]

2) Dimetric
makes equal angles
with exactly tw[o] of the principal axes

3) Trimetric
makes unequal angles
with the three principal axes.

Oblique projection:

If projection vector v has not the direction of the view plane normal N.

2 categories

→ **Cavalier:**
there is no foreshortening of lines perpendicular to the xy plane

→ **Cabinet**
Lines perpendicular to the xy planes are foreshorted by half their length.

Example 3

Matrix form of orthographic projection

For orthographic projection onto the xy plane.

$$P_{orth} : \begin{cases} x' = x \\ y' = y \\ z' = 0 \end{cases}$$

$\therefore$ matrix form of P_{orth} is

$$P_{orth} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Ans

NB: Standard matrix for perspective projection.

$$Per_h = \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & d \end{bmatrix}$$

Problem 7.1

Cure
Cuve matrix in terms of the homogeneous coordinates of its vertices.

$V = (ABCDEFGH)$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Standard perspective matrix is

$$Per_k = \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & d \end{bmatrix}$$

a for $d=1$

$$Per_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\therefore Per_k \cdot V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \end{bmatrix}$$

Ans

$\therefore$ ~~A'~~ ~~for~~

If these homogeneous coordinates are changed to three-dimensional coordinates, the projected image has coordinates:

$A' = (0, 0, 0)$
$B' = (1, 0, 0)$
$C' = (1, 1, 0)$
$D' = (0, 1, 0)$
$E' = (0, \frac{1}{2}, 0)$
$F' = (0, 0, 0)$
$G' = (\frac{1}{2}, 0, 0)$
$H' = (\frac{1}{2}, \frac{1}{2}, 0)$

b with $d = 10$,

$$P_{er_k} = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 10 \end{bmatrix}$$

$$\therefore P_{er_k} \cdot V = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 10 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 10 & 10 & 0 & 0 & 0 & 10 & 10 \\ 0 & 0 & 10 & 10 & 10 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 10 & 10 & 10 & 11 & 11 & 11 & 11 \end{bmatrix}$$

Am

$\therefore$ So three-dimensional coordinates are,

$A' = (0, 0, 0)$ $\quad E' = (0, 10/11, 0)$

$B' = (1, 0, 0)$ $\quad F' = (0, 0, 0)$

$C' = (1, 1, 0)$ $\quad G' = (10/11, 0, 0)$

$D' = (0, 1, 0)$ $\quad H' = (10/11, 10/11, 0)$

Problem 7.3

Let $P(x, y, z)$ be projected onto $P'(x', y', z')$

$\overline{PO}$ and $\overline{P'O}$ have the same direction.

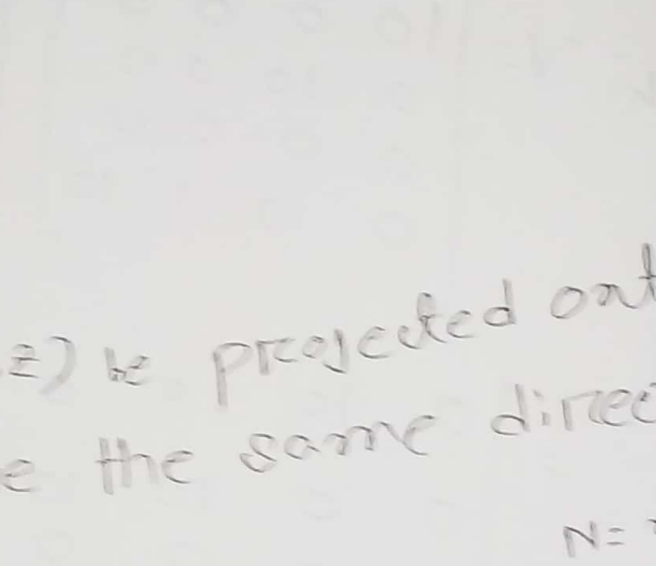
$\therefore \overline{P'O} = \alpha \overline{PO}$

$\therefore x' = \alpha x$

$y' = \alpha y$

$z' = \alpha z$

$N = n_1 i + n_2 j + n_3 k$



Since any point $P'(x', y', z')$ lying on the plane satisfies the equation

$$n_1 x' + n_2 y' + n_3 z' = d_0$$

$$\Rightarrow d_0 = n_1x' + n_2y' + n_3z'$$
$$= n_1\alpha x + n_2\alpha y + n_3\alpha z$$
$$= \alpha(n_1x + n_2y + n_3z)$$
$$\therefore \alpha = \frac{d_0}{n_1x + n_2y + n_3z}$$

$\therefore$ Projection can't be represented as a 3×3 matrix transformation.

$$\therefore Per_{N,R_0} = \begin{bmatrix} d_0 & 0 & 0 & 0 \\ 0 & d_0 & 0 & 0 \\ 0 & 0 & d_0 & 0 \\ n_1 & n_2 & n_3 & 0 \end{bmatrix}$$

$\therefore$ ~~x'~~ = $\therefore P' = (d_0x, d_0y, d_0z, n_1x + n_2y + n_3z)$

Ans

Problem 7.4:

The plane $z = d$, which is parallel to the xy plane.

$\therefore$ View plane normal vector N is the same as the normal vector K to the xy plane.

that is $N = K$.

Reference point $R_0(0,0,d)$
we identify the parameters.

$$N(n_1, n_2, n_3) = (0,0,1)$$
$$R_0(x_0, y_0, z_0) = (0,0,d)$$

$$\therefore d_0 = n_1 x_0 + n_2 y_0 + n_3 z_0$$
$$= d$$

$$\therefore Per_{K,R_0} = \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{Ans}$$

Problem 7.5

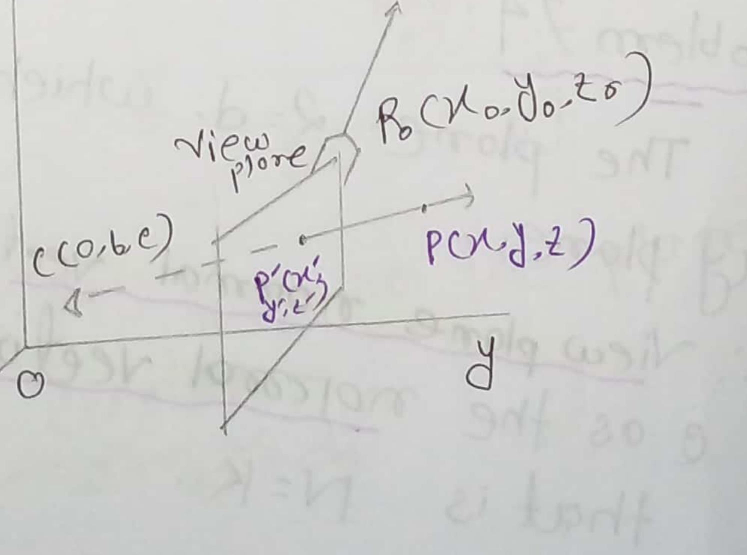
$\overline{PE}$ and $\overline{P'E}$ have same direction.

$\therefore \overline{P'E} = \alpha \overline{PE}$

$\therefore$ ~~$x' = \alpha(x-a)+a$~~

$\therefore \alpha(x-a) = x'-a$

$\Rightarrow x' = \alpha(x-a)+a$



$\alpha(y-b) = y'-b$

$\Rightarrow y' = \alpha(y-b)+b$

$\alpha(z-e) = z'-e$

$\Rightarrow z' = \alpha(z-e)+e$

$\therefore$ then, $\alpha = \dfrac{d}{n_1(x-a)+n_2(y-b)+n_3(z-e)}$

i.e. $p'(x', y', z')$ is on the view plane.

thus ~~sati~~ satisfies the view plane equation,

$$\boxed{n_1(x'-x_0)+n_2(y'-y_0)+n_3(z'-z_0) = d}$$

$\Rightarrow$ ~~$n_1 x_0$~~

Here, $d = (n_1 x_0 + n_2 y_0 + n_3 z_0) - (n_1 a + n_2 b + n_3 e)$

To find the homogeneous coordinate matrix representation, it is easiest to proceed as follows:

→ Translate so that the center of projection lies at origin

$R_0 = (x_0 - a, y_0 - b, z_0 - e)$

→ Project onto the translated plane using the origin as the center of projection.

↳ by constructing the transformation $Per_{N,R_0'}$

→ translate back.

$d_0 = n_1 x_0 + n_2 y_0 + n_3 z_0$

$d_1 = n_1 a + n_2 b + n_3 c$

$\therefore d = d_0 - d_1$

$\therefore Per_{N,R_0,C} = T_C \cdot Per_{N,R_0} \cdot T_{-C}$

$$\therefore Per_{N,R_0,C} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & 0 \\ n_1 & n_2 & n_3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -a \\ 0 & 1 & 0 & -b \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d & 0 & 0 & -ad \\ 0 & d & 0 & -bd \\ 0 & 0 & d & -cd \\ n_1 & n_2 & n_3 & -(an_1 + bn_2 + cn_3) \end{bmatrix}$$

$\uparrow$ d_1

$$= \begin{bmatrix} d & 0 & 0 & -ad \\ 0 & d & 0 & -bd \\ 0 & 0 & d & \end{bmatrix}$$

$$= \begin{bmatrix} d - an_1 \end{bmatrix} = \begin{bmatrix} d+an_1 & an_2 & an_3 & -ad-ad_1 \\ bn_1 & d+bn_2 & bn_3 & -bd-bd_1 \\ cn_1 & cn_2 & d+cn_3 & -cd-cd_1 \\ n_1 & n_2 & n_3 & -d_1 \end{bmatrix}$$

$$= \begin{bmatrix} d+an_1 & an_2 & an_3 & -ad \\ bn_1 & d+bn_2 & bn_3 & -bd \\ cn_1 & cn_2 & d+cn_3 & -cd \\ n_1 & n_2 & n_3 & -d_1 \end{bmatrix}$$

Ans

Problem 7.13

<u>a</u> A cavalier projection is an oblique projection with $f=1$.

Now for $\theta = 45^\circ$

We know,

$$P_{par,V_0} = \begin{bmatrix} 1 & 0 & f\cos\theta & 0 \\ 0 & 1 & f\sin\theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore P_{par,V_1} = \begin{bmatrix} 1 & 0 & \cos 45 & 0 \\ 0 & 1 & \sin 45 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Ans

b A cabinet projection is an oblique projection with $f = \frac{1}{2}$

we know,

$$P_{obv} = \begin{bmatrix} 1 & 0 & f\cos\theta & 0 \\ 0 & 1 & f\sin\theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

for $\theta = 30$

$$\therefore P_{obv_2} = \begin{bmatrix} 1 & 0 & \frac{1}{2} \cdot \frac{\sqrt{3}}{2} & 0 \\ 0 & 1 & \frac{1}{2} \cdot \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \frac{\sqrt{3}}{4} & 0 \\ 0 & 1 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Ans

c Matrix representation for unit cube is

$$V = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$\therefore$

$$P_{arv_2} \cdot V = \begin{bmatrix} 1 & 0 & \sqrt{3}/4 & 0 \\ 0 & 1 & 1/4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 & \sqrt{3}/4 & \sqrt{3}/4 & 1+\sqrt{3}/4 & 1+\sqrt{3}/4 \\ 0 & 0 & 1 & 1 & 1+1/4 & 1/4 & 1/4 & 1+1/4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Ans

$\therefore$ image coordinates are.

$A' = (0, 0, 0)$

$B' = (1, 0, 0)$

$C' = (1, 1, 0)$

$D' = (0, 1, 0)$

$E' = (\sqrt{3}/4, 1\frac{1}{4}, 0)$

$F' = (\sqrt{3}/4, 1/4, 0)$

$G' = (1+\sqrt{3}/4, 1/4, 0)$

$H' = (1+\sqrt{3}/4, 1\frac{1}{4}, 0)$

Ans

$$\text{Parr.}; V = \begin{bmatrix} 1 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1+\frac{1}{\sqrt{2}} & 1+\frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 1 & 1+\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1+\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

∴ coordinate of image are

$A' = (0, 0, 0)$

$B' = (1, 0, 0)$

$C' = (1, 1, 0)$

$D' = (0, 1, 0)$

$E' = (\frac{1}{\sqrt{2}}, 1 + \frac{1}{\sqrt{2}}, 0)$

$F' = (1/\sqrt{2}, \frac{1}{\sqrt{2}}, 0)$

$G' = (1 + \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$

$H' = (1 + \frac{1}{\sqrt{2}}, 1 + \frac{1}{\sqrt{2}}, 0)$

Ans