A Brief Introduction to Optimal Control

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Outlines

- Basic Concepts

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- Controllability

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- The Pontryagin Maximum Principle

Notations

We use the following notations throughout the discussion,

$$\boldsymbol{f}(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix} \text{ and } \boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

Definition 1.1.

Dynamics: Let us consider a differential equation of the form

$$\begin{cases} \boldsymbol{x}'(t) = \boldsymbol{f}\left(\boldsymbol{x}(t)\right) \\ \boldsymbol{x}(0) = x_0 \end{cases} \tag{1}$$

Where, $t > 0$, $x_0 \in \mathbb{R}^n$ is the initial point and the function $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$. The unknown is the curve $\boldsymbol{x} : [0, \infty) \to \mathbb{R}^n$, which is regarded as the dynamical evolution of the state of a system.

Basic Definitions Continued

Definition 1.2.

Controlled Dynamics: Suppose the function $\boldsymbol{f}$ depends on some *control* parameters belonging to a set $A \subset \mathbb{R}^m$; so that $\boldsymbol{f} : \mathbb{R}^n \times A \to \mathbb{R}^n$. Then if we choose $a \in A$ and consider the corresponding dynamics,

$$\begin{cases} \boldsymbol{x}'(t) = \boldsymbol{f}\big(\boldsymbol{x}(t), a\big) \\ \boldsymbol{x}(0) = x_0 \end{cases} \tag{2}$$

we obtain the evolution of our system when the parameter value 'a' is fixed. We can change the parameter values as the system evolves.

Basic Definitions Continued

Definition 1.3.

Control: We call a function $\alpha : [0, \infty) \to A$ a control. Corresponding to each control, we consider the differential equation

$$\begin{cases} \boldsymbol{x}'(t) = \boldsymbol{f}\left(\boldsymbol{x}(t), \alpha(t)\right) \\ \boldsymbol{x}(0) = x_0 \end{cases} \tag{3}$$

and regard the trajectory $\boldsymbol{x}(\cdot)$ as the corresponding *response* of the system.

Basic Definitions Continued

Definition 1.4.

Admissible Control: We introduce

$$\mathcal{A} = \{\boldsymbol{\alpha} : [0, \infty) \to A \mid \boldsymbol{\alpha}(\cdot) \text{ measurable}\}$$

to denote the collection of all *admissible controls*, where

$$\boldsymbol{\alpha}(t) = \begin{pmatrix} \alpha_1(t) \\ \vdots \\ \alpha_m(t) \end{pmatrix}$$

Definition 1.5.

Payoffs: We always look forward to determine the best control for our system. For this we need to specify a specific payoff(or reward)criterion. Let us define the *payoff functional*

$$P[\boldsymbol{\alpha}(\cdot)] := \int_0^T r\big(\mathbf{x}(t), \boldsymbol{\alpha}(t)\big)\,dt + g\big(\mathbf{x}(T)\big) \qquad (P)$$

where $\mathbf{x}(\cdot)$ solves the system for the control $\boldsymbol{\alpha}(t)$. Here $r : \mathbb{R}^n \times A \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are given, and we call r the running payoff and g the terminal payoff. The terminal time $T > 0$ is also given.

Basic Control Problem

Basic Control Problem: Our goal is to find a control $\boldsymbol{\alpha}^*(\cdot)$, which *maximizes* the payoff. That is, we want

$$P[\boldsymbol{\alpha}^*(\cdot)] \geq P[\boldsymbol{\alpha}(\cdot)]$$

for all controls $\boldsymbol{\alpha}(\cdot) \in \mathcal{A}$. Such a control $\boldsymbol{\alpha}^*(\cdot)$ is called *optimal*.

Controllability

Controllability Problem: The controllability question is, "Given the initial point x_0 and a *target set* $S \subset \mathbb{R}^n$, does there exist a control steering the system to S in finite time?
In this section we consider the problem of driving the system to the origin $S = \{0\}$.

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Definition 2.1.

Reachable Set: The reachable set for time t is defined as

$\mathcal{C}(t)$=set of initial points x_0 for which there exists a control such that $\boldsymbol{x}(t) = 0$.

and the overall reachable set is the union of all $\mathcal{C}(t)$ for all $t \geq 0$. That is,

$$\mathcal{C} = \bigcup_{t \geq 0} \mathcal{C}(t)$$

Controllability of Linear System: Let us consider that our system is linear in both the state $\boldsymbol{x}(\cdot)$ and the control $\boldsymbol{\alpha}(\cdot)$ and consequently has the form

$$\begin{cases} \boldsymbol{x}'(t) = M\boldsymbol{x}(t) + N\boldsymbol{\alpha}(t) \\ \boldsymbol{x}(0) = x_0 \end{cases} \tag{4}$$

where $t > 0$, $M \in \mathbb{M}^{n\times n}$ and $N \in \mathbb{M}^{n\times m}$. We also assume that the set of control parameters A is a cube in $\mathbb{R}^m$:

$$A = [-1, 1]^m = \{a \in \mathbb{R}^m \,|\, |a_i| \leq 1, i = 1, \cdots, m\}$$

Definition 2.2.

Controllability Matrix: The controllability matrix is defined as,

$$G = G(M, N) := [N, MN, M^2N, \cdots, M^{n-1}N]$$

and G is a $n \times (mn)$ matrix.

Theorem 2.3.

The rank of the controllability matrix G is,

$$\text{rank } G = n$$

if and only if, $0 \in \mathcal{C}^0$, where $\mathcal{C}^0$ represents the interior of the reachable set $\mathcal{C}$.

Theorem 2.4.

Controllability Criterion: *Let A be the cube $[-1,1]^n$ in $\mathbb{R}^n$. Also suppose that rank $G = n$, and Re $\lambda < 0$ for each eigenvalue λ of the matrix M. Then the system (4) is controllable.*

Controllability of Linear System: Example

Example: Let us consider the following linear system

$$\boldsymbol{x}' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \alpha$$

where,

$$M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } N = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then

$$G = [N, MN] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Therefore,

$$\text{rank } G = 2 = n.$$

Example Continued

Also, the characteristic equation of the matrix M is

$$p(\lambda) = 0 = det(\lambda I - M) = det \begin{pmatrix} \lambda & -1 \\ 0 & \lambda \end{pmatrix}$$

$$\implies \lambda^2 = 0$$

we can see that both of the eigenvalues are 0, so it fails to satisfy the hypotheses of theorem (2.4).

Theorem 2.5.

Improved Criterion for Controllability: *Assume that rank* $G = n$, *and* $Re\ \lambda \leq 0$ *for each eigenvalue* λ *of the matrix* M. *Then the system* (4) *is controllable.*

Motivation for the Pontryagin Maximum Principle

Definition 3.1.

Lagrangian: Consider a smooth function $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $L = L(x, v)$; L is called the *Lagrangian*.

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Basic Problem of Calculus of Variations: Find a curve $\boldsymbol{x}^*(\cdot) : [0, T] \to \mathbb{R}^n$ that minimizes the functional

$$I[\boldsymbol{x}(\cdot)] := \int_0^T L(\boldsymbol{x}(t), \boldsymbol{x}'(t))dt \tag{5}$$

among all functions $\boldsymbol{x}(\cdot)$ satisfying $\boldsymbol{x}(0) = x_0$ and $\boldsymbol{x}(T) = x_1$.

Motivation for the Pontryagin Maximum Principle

Theorem 3.2.

Euler-Lagrange Equation: *Let* $\mathbf{x}^*(\cdot)$ *solves the calculus of variations problem. Then* $\mathbf{x}^*(\cdot)$ *solves the Euler-Lagrange differential equations*

$$\frac{d}{dt}[\nabla_v L(\mathbf{x}^*(t), \mathbf{x}^{*\prime}(t))] = \nabla_x L(\mathbf{x}^*(t), \mathbf{x}^{*\prime}(t)) \qquad \text{(E-L)}$$

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Definition 3.3.

Generalized Momentum: For a given curve $\mathbf{x}(\cdot)$, we define the generalized momentum $\mathbf{p}(\cdot)$ as follows

$$\mathbf{p}(t) := \nabla_v L(\mathbf{x}(t), \mathbf{x}'(t)) \qquad (0 \leq t \leq T)$$

Motivation for the Pontryagin Maximum Principle

Definition 3.4.

Hamiltonian: We define the *Hamiltonian* $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ of a dynamical systems by the formula

$$H(x, p) = p \cdot \boldsymbol{v}(x, p) - L(x, \boldsymbol{v}(x, p))$$

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Theorem 3.5.

Hamiltonian Dynamics: *Suppose $\boldsymbol{x}(\cdot)$ solve the Euler-Lagrange equations (E-L) and define $\boldsymbol{p}(\cdot)$ as above. Then the pair $\big(\boldsymbol{x}(\cdot), \boldsymbol{p}(\cdot)\big)$ solves the Hamilton's equations*

$$\begin{cases} \boldsymbol{x}'(t) = \nabla_p H(\boldsymbol{x}(t), \boldsymbol{p}(t)) \\ \boldsymbol{p}'(t) = -\nabla_x H(\boldsymbol{x}(t), \boldsymbol{p}(t)) \end{cases}$$

Furthermore the mapping $t \to H(\boldsymbol{x}(t), \boldsymbol{p}(t))$ is constant.

Fixed Time, Free Endpoint Problem

Fixed Time, Free Endpoint Problem: Consider the system (3), (P), admissible controls defined earlier. The basic problem is to find a control $\boldsymbol{\alpha}^*(\cdot)$ such that

$$P[\boldsymbol{\alpha}^*(\cdot)] = \max_{\boldsymbol{\alpha}(\cdot)\in\mathcal{A}} P[\boldsymbol{\alpha}(\cdot)]$$

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Definition 3.6.

The control theory Hamiltonian is the function

$$H(x,p,a) := \boldsymbol{f}(x,a)\cdot p + r(x,a) \qquad (x,p\in\mathbb{R}^n, a\in A).$$

Statement of Pontryagin Maximum Principle

Theorem 3.7.

Pontryagin Maximum Principle: *Let $\alpha^*(\cdot)$ be optimal for system* (3), (P) *and $\mathbf{x}^*(\cdot)$ be the corresponding trajectory. Then there exists a function $\mathbf{p}^* : [0, T] \to \mathbb{R}^n$ such that*

$$\mathbf{x}^{*\prime}(t) = \nabla_p H(\mathbf{x}^*(t), \mathbf{p}^*(t), \alpha^*(t)) \qquad (ODE)$$

$$\mathbf{p}^{*\prime}(t) = -\nabla_x H(\mathbf{x}^*(t), \mathbf{p}^*(t), \alpha^*(t)) \qquad (ADJ)$$

and

$$H(\mathbf{x}^*(t), \mathbf{p}^*(t), \alpha^*(t)) = \max_{a \in A} H(\mathbf{x}^*(t), \mathbf{p}^*(t), a) \quad (0 \le t \le T) \qquad (M)$$

Additionally, $\qquad H(\mathbf{x}^*(t), \mathbf{p}^*(t), \alpha^*(t)) \equiv constant$
Finally, we have the terminal condition

$$\mathbf{p}^*(T) = \nabla g(\mathbf{x}^*(T)) \qquad (T)$$

Free Time, Fixed Endpoint Problem: Consider the system (3) defined earlier. Assuming that a target point $x_1 \in \mathbb{R}^n$ is given, we introduce the following payoff functional

$$P[\boldsymbol{\alpha}(\cdot)] = \int_0^\tau r(\boldsymbol{x}(t), \boldsymbol{\alpha}(t))dt$$

Here, $\tau = \tau[\alpha(\cdot)] \leq \infty$ denotes the first time the solution of the system hits the target point x_1.

Statement of Pontryagin Maximum Principle

Theorem 3.8.

Pontryagin Maximum Principle: *Let $\alpha^*(\cdot)$ be optimal for system* (3), (*P*) *and $\mathbf{x}^*(\cdot)$ be the corresponding trajectory. Then there exists a function $\mathbf{p}^* : [0, \tau^*] \to \mathbb{R}^n$ such that*

$$\mathbf{x}^{*\prime}(t) = \nabla_p H(\mathbf{x}^*(t), \mathbf{p}^*(t), \alpha^*(t))$$

$$\mathbf{p}^{*\prime}(t) = -\nabla_x H(\mathbf{x}^*(t), \mathbf{p}^*(t), \alpha^*(t))$$

and

$$H(\mathbf{x}^*(t), \mathbf{p}^*(t), \alpha^*(t)) = \max_{a \in A} H(\mathbf{x}^*(t), \mathbf{p}^*(t), a) \quad (0 \le t \le \tau^*) \quad (M)$$

Also,

$$H(\mathbf{x}^*(t), \mathbf{p}^*(t), \alpha^*(t)) \equiv 0 \qquad (0 \le t \le \tau^*)$$

References

- Lawrence C. Evans, *An introduction to mathematical optimal control theory, Version* 0.2. Lectures at the University of Maryland, 1983. https://math.berkeley.edu/ evans/control.course.pdf

Thank You for Your Attention