Stochastic Model of Microcredit Interest Rate

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Outlines

- Introduction
- Yunus model
- Modelling with delays
- Default of a loan
- In-time installment and Repayment
- Random interest rate

1.1 When,Where and Who?

Microcredit program was introduced.....

- in 1974
- in Bangladesh
- by Nobel laureate Prof. Dr. Muhammad Yunus

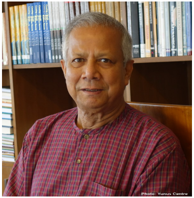


Figure: Dr. Muhammad Yunus

1.2 What is microcredit?

In general, microcredit means the loan of very small amount to people who cannot access to traditional lends.

1.3 The main characteristics of microcredit

1. The loans are very small
2. The loan is paid back with frequent(weekly) installments
3. The annual interest rate charged is usually almost 20%
4. For poor families to lift themselves out of poverty
5. The repayment rate is close to 100%
6. Micro-lending institutions do not usually take any promissory note from their clients to secure the loan

2.1 Capitalization

The main idea of capitalization is that one dollar today isn't worth one dollar tomorrow but it will be worth more.

2.1 Capitalization

Simple interest:

$$B_{n.\delta t} = B_T = B_0(1 + n\rho), \text{ after } n \text{ periods}$$

Compound interest:

$$B_{n.\delta t} = B_T = B_0(1 + \rho)^n, \text{ after } n \text{ periods}$$

If $t \in \{0, \delta t, 2\delta t, \cdots, n\delta t = T\}$ and r be the continuous interest rate, then the real number

$$e^{r\delta t} = 1 + \rho$$

Then we can rewrite the equation of compound interest as,

$$\begin{aligned} B_{k.\delta t} &= B_t = B_0 e^{r.k\delta t} \\ \implies B_t &= B_0 e^{rt} \end{aligned}$$

So if we choose, $B_0 = \$1$, the above equation says that

$$\$1 \to \$e^{rt} \implies \$e^{-rt} \to \$1$$

That is what we named *capitalization*: the fact that money gains value by accumulating interests.

2.2 Deterministic Yunus equation

We consider a loan of 1000 BDT(Bangladesh Taka) and the refund requested is 22 BDT per week during 50 weeks.

The current value of the first installment is $22e^{\frac{-r}{52}}$.

The current value of the second installment is $22e^{\frac{-2r}{52}}$.

$\vdots$

The current value of the n^{th} installment is $22e^{\frac{-nr}{52}}$.

So we obtain,

$$1000 = 22\sum_{n=1}^{50} e^{\frac{-nr}{52}} \tag{0.1}$$

Letting $y = e^{\frac{-r}{52}}$, the equation becomes

$$1000 = 22 \sum_{n=1}^{50} y^n = 22 \frac{y - y^{51}}{1 - y} \tag{0.2}$$

This reduces to,

$$f(y) := 22y^{51} - 1022y + 1000 = 0 \tag{0.3}$$

We pick a real solution $0 < q_+ = 0.9962107... < 1$ which leads to $r = 19.74\%$ (which is nearly 20%)

Bernoulli process

Let B_m be the act of repayment at time m, then

$$B_m = \begin{cases} 1, & \textit{if the borrower succeeds to payback at time m} \\ 0, & \textit{otherwise} \end{cases}$$

Also let,

$$P(B_m = 1) = p$$
$$\text{and } P(B_m = 0) = 1 - p$$

Definition 3.1 Given $(B_m)_{m\geq 1}$, the time when the k^{th} installment takes place is the sequence of random variable $(T_k)_{k\geq 0}$ defined by

$$T_0 = 0, \text{ and for } k \geq 1,\ T_k = T_{k-1} + Min\{\triangle t \geq 1 | B_{T_{k-1}+\triangle t} = 1\}$$

Definition 3.2 Given $(T_k)_{k\geq 0}$. We call a sequence of inter-repayment times the sequence of random variables $(X_k)_{k\geq 1}$ defined by

$$X_k := T_k - T_{k-1}, \text{ for } k = 1, 2, \cdots$$

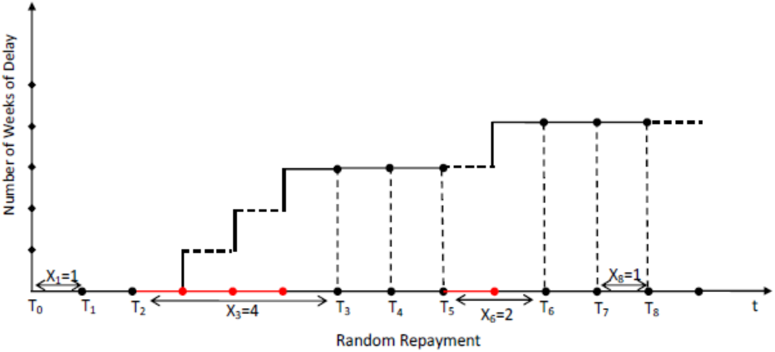


Figure: weekly repayment process with delay

Proposition 3.1 For all $k \geq 1$, the inter-repayment time, X_k follows a geometric distribution with parameter $p, G(p)$.

Proof: By definition, we have

$$X_k = Min\{\triangle t \geq 1 | B_{T_{k-1}+\triangle t} = 1\} = T_k - T_{k-1}$$

Thus,

$$\begin{aligned}
P(X_k = n) &= \sum_{m \geq k-1} P(T_{k-1} = m) P(X_k = n | T_{k-1} = m) \\
&= \sum_{m \geq k-1} P(T_{k-1} = m) \times \\
&\quad P(B_{T_{k-1}+1} = 0, \cdots, B_{T_{k-1}+n-1} = 0, B_{T_{k-1}+n} = 1 | T_{k-1} = m) \\
&= \sum_{m \geq k-1} P(T_{k-1} = m) P(B_n^{'} = 1) \prod_{i=1}^{n-1} P(B_i^{'} = 0)
\end{aligned}$$

$$
\begin{aligned}
P(X_k = n) &= \sum_{m \geq k-1} P(T_{k-1} = m) P(B_n^{'} = 1) \prod_{i=1}^{n-1} P(B_i^{'} = 0) \\
&= \sum_{m \geq k-1} P(T_{k-1} = m) p(1-p)^{n-1} \\
&= (1-p)^{n-1} p \sum_{m \geq k-1} P(T_{k-1} = m) \\
&= (1-p)^{n-1} p
\end{aligned}
$$

Therefore, for all $n \geq 1$

$$P(X_k = n) = (1-p)^{n-1} p$$

Proposition 3.2 The sequence of random variables $(X_k)_{k\geq 1}$ are independent.

Proof: For all $K \geq 1$ we have $X_k \rightsquigarrow G(p)$, that is,

$$P(X_k = n) = (1-p)^{n-1}p$$

Now, for $1 \leq n_1 \leq n_2 \cdots \leq n_k$ and B_m's are identically independent for all $m \geq 0$, we get

$$\begin{aligned} P(X_1 = n_1, X_2 = n_2, \cdots, X_k = n_k) = \\ P\Big[\{B_1 = 0, \cdots, B_{n_1-1} = 0, B_{n_1=1}\}, \\ \{B_{n_1+1} = 0, \cdots, B_{n_1+(n_2-1)} = 0, B_{n_1+n_2} = 0\}, \\ \cdots\cdots\cdots, \\ \{B_{n_{k-1}+1} = 0, \cdots, B_{n_{k-1}+(n_k-1)} = 0, B_{n_{k-1}+n_k} = 1\}\Big] \end{aligned}$$

$$
\begin{aligned}
&= (1-p)^{n_1-1}p(1-p)^{n_2-1}p\cdots(1-p)^{n_k-1}p \\
&= P(X_1=n_1)P(X_2=n_2)\cdots P(X_k=n_k)
\end{aligned}
$$

What is the default of a loan?

We say that there is a d-default if $Max\{X_1, X_2, \cdots, X_n\} \geq d$. *It means that we set a kind of tolerance limit* d *and there is a d-default when the borrower doesn't pay back one time before the* d^{th} *week.*

Relation between probability of d-fault and payback probability

Let, π_d be the probability of a d-default and p be the payback probability. Then we obtain,

$$\begin{aligned}
\pi_d(p) &= P\Big[Max\{X_1, X_2, \cdots, X_n\}\Big] \geq d \\
&= P(X_1 \geq d \cup X_2 \geq d \cdots \cup X_n \geq d) \\
&= 1 - P(X_i < d), for\ all\ i \\
&= 1 - \Big[p + p(1-p) + p(1-p)^2 + \cdots + p(1-p)^{d-1}\Big] \\
&= 1 - [1 - (1-p)^d] \\
&= (1-p)^d
\end{aligned}$$

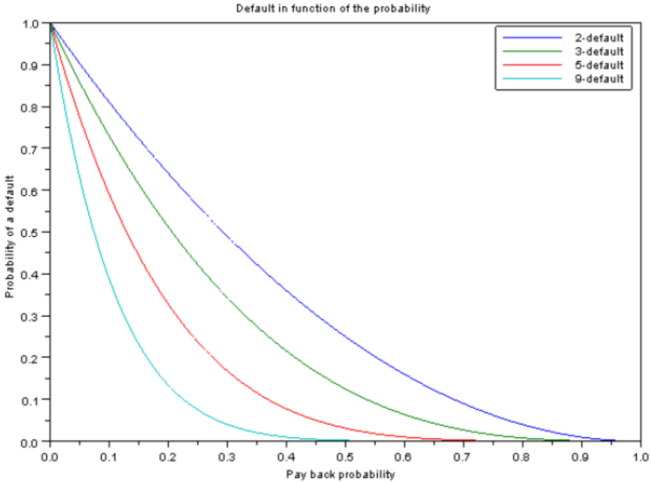


Figure: d-default probability w. r. to payback probability

Relationship between in-time installment probability and repayment rate

Here, we consider a particular case where the number of repayments takes place 50 times.
Let, γ be the repayment rate which is simply the complement of the default rate.
Then we can write,

$$\gamma = P(Max\{X_1, X_2, \cdots, X_n\} \leq d)$$

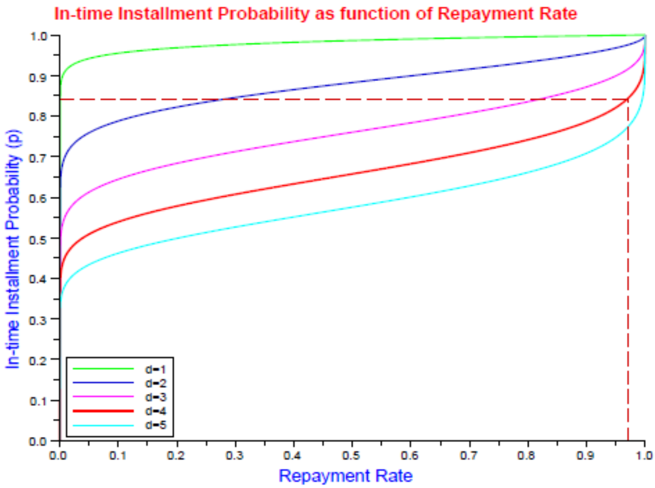
Proposition 5.1 *Given d and γ, we have the following relationship between in-time installment probability p and the repayment rate*

$$p = 1 - (1 - \gamma^{\frac{1}{50}})^{\frac{1}{d}}$$

Proof: By the definition of repayment rate and since X_i are i.i.d and $X_i \rightsquigarrow G(p)$, we obtain

$$
\begin{aligned}
\gamma &= P\left[Max\{X_1, X_2, \cdots, X_n\} \leq d\right] \\
&= P\left(\bigcap_{i=1}^{50} \{X_i \leq d\}\right) \\
&= \prod_{i=1}^{50} P(X_i \leq d) \\
&= \prod_{i=1}^{50} \left[p(1-p)^0 + p(1-p)^1 + p(1-p)^2 + \cdots + p(1-p)^{d-1}\right] \\
&= \prod_{i=1}^{50} \left[1 - (1-p)^d\right] \\
&= \left[1 - (1-p)^d\right]^{50}
\end{aligned}
$$

Therefore, we have $p = 1 - \left(1 - \gamma^{\frac{1}{50}}\right)^{\frac{1}{d}}$


In-time Installment Probability as function of Repayment Rate
In-time Installment Probability (p)
Repayment Rate
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
d=1
d=2
d=3
d=4
d=5

Proposition 6.1 *The moment generating function of any geometric random variable, $X \rightsquigarrow G(p)$, is given by*

$$M_X(t) = E[e^{tX}] = \frac{pe^t}{1-(1-p)e^t}$$

Definition 6.1 *For* $(T_k)_{k\geq 0}$, *we call actuarial interest rate* R *the random variable on probability space* (Ω, F, P), *satisfying the following equation*

$$1000 = 22\sum_{k=1}^{50} e^{\frac{-R}{52}T_k} \tag{0.4}$$

This equation is called *random Yunus equation*.

Let, $\bar{r}$ be the *actuarial expected rate* corresponding to the expectation of random Yunus equation (5.1)

Then we have the following proposition:

Proposition 6.2 *Let us denote by $\bar{r}$ the positive real number which satisfies the equation*

$$1000 = E\big(22\sum_{k=1}^{50} e^{\frac{-\bar{r}}{52}T_k}\big)$$

Then we have

$$\bar{r} = 52\log\left(1 + p\Big(\frac{1}{q_+} - 1\Big)\right)$$

where q_+ is the positive non trivial solution of the deterministic Yunus equation (2.2).

Proof: For all $k \geq 1$, we have $T_k = X_1 + X_2 + \cdots + X_k$ and $X_1, \cdots, X_k$ are independent. We have

$$1000 = E\left(22 \sum_{k=1}^{50} e^{\frac{-\bar{r}}{52}(X_1+X_2+\cdots+X_k)}\right)$$

$$= 22 \sum_{k=1}^{50} E\left(e^{\frac{-\bar{r}}{52}X_1}\right) \cdots E\left(e^{\frac{-\bar{r}}{52}X_k}\right)$$

As the X_i are i.i.d and $X_i \rightsquigarrow G(p)$, then $v = E\left(e^{\frac{-\bar{r}}{52}X_1}\right) \cdots E\left(e^{\frac{-\bar{r}}{52}X_k}\right)$. Thus we get,

$$1000 = 22 \sum_{k=1}^{50} v^k$$

$$= 22 \frac{v - v^{51}}{1 - v}$$

which is the deterministic Yunus equation. Let q_+ be the positive real solution of this equation, where $0 < q_+ < 1$.

Also $v = E\left(e^{\frac{-\bar{r}}{52}X}\right) = M_X\left(\frac{-\bar{r}}{52}\right)$ is the moment generating function of $X \rightsquigarrow G(p)$,

$$v = M_X\left(\frac{-\bar{r}}{52}\right) = \frac{pe^{\frac{-\bar{r}}{52}}}{1-(1-p)e^{\frac{-\bar{r}}{52}}}$$

replacing v by q_+, we have

$$q_+ = \frac{pe^{\frac{-\bar{r}}{52}}}{1-(1-p)e^{\frac{-\bar{r}}{52}}}$$

Therefore, we deduce

$$\bar{r} = 52 \log\left(1 + p\left(\frac{1}{q_+} - 1\right)\right)$$

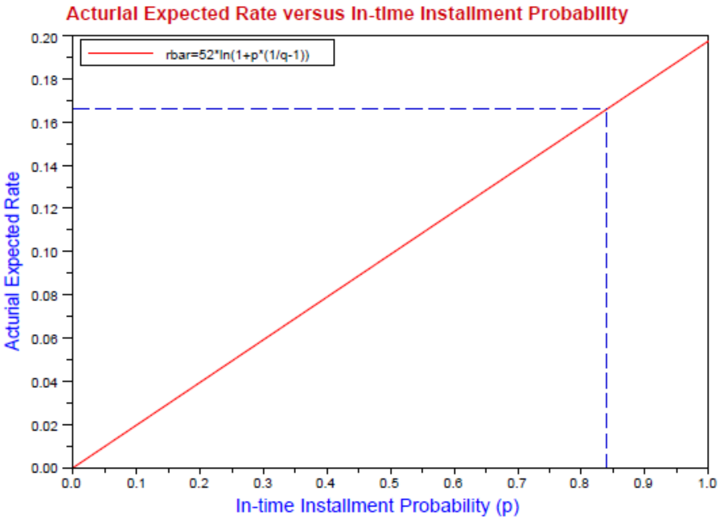


Figure: Actuarial expected rate, $\bar{r}$, as a function of in-time installment probability

Difference between deterministic and random Yunus equation

Deterministic Yunus equation:

- The k-th installment takes place at time $T_k = k$ for all k and consequently
- The interest rate is the fixed real number r

Random Yunus equation:

- The k-th installment takes place at random time T_k with possibly $T_k > k$
- Since the installments are possibly delayed, the interest rate is no longer a fixed real number but a random variable R

Thank You for Your attention