Hamiltonian vector fields on a space of curves on the 3-sphere

Md Ismail Hossain

Department of Mathematics
University of Manitoba

Outline

Introduction

Hamiltonian Mechanics?

- Hamiltonian mechanics is a reformulation of Classical mechanics.
- It is based on the energy concept of the system under consideration.

Classical mechanics $\to$ Hamiltonian mechanics

Example-1

$$\begin{aligned} F_i &= m_i a_i, \qquad i = 1, \ldots, n \\ \text{i.e.} \quad -\nabla V_i &= m_i \frac{d^2 q_i}{dt^2} \end{aligned} \tag{1}$$

Where,

q_i = position
m_i = mass
F_i = force
V_i = potential

Classical mechanics $\to$ Hamiltonian mechanics

Example-1 continued

Introduce, $p_i = m_i \dfrac{dq_i}{dt}$ and $H(q,p) = \displaystyle\sum_{i=1}^{n} \Big(\frac{\|p_i\|^2}{2m_i} + V_i(q_i) \Big)$.

Where,

$$q = (q_1, \ldots, q_n)$$
$$p = (p_1, \ldots, p_n)$$

Then equation (1) is equivalent to Hamilton's equations,

$$\begin{cases} \dfrac{dq_i}{dt} = \dfrac{p_i}{m_i} = \dfrac{\partial}{\partial p_i} \Big(\dfrac{\|p_i\|^2}{2m_i} + V_i(q_i) \Big) = \dfrac{\partial H}{\partial p_i} \\ \dfrac{dp_i}{dt} = m_i \dfrac{d^2 q_i}{dt^2} = -\nabla V_i(q_i) = -\dfrac{\partial H}{\partial q_i} \end{cases} \tag{2}$$

Remarks on Example-1

- $\mathbb{R}^{3n}$ is the *Configuration space*.
- $\mathbb{R}^{3n} \times \mathbb{R}^{3n}$ is the *Phase space*.
- $X_H = \left(\dfrac{\partial H}{\partial p_i}, -\dfrac{\partial H}{\partial q_i} \right)$ is the *Hamiltonian vector field*.

Generalization of the phase space

- Choose configuration space to be an m-dimensional differentiable manifold N.
- The phase space is given by $T^\star N$ is a $2m$-dimensional differentiable manifold.
- In fact, the phase space can be any symplectic manifold.

Symplectic Manifolds

Differential k-form

Let $\mathcal{M}$ be a finite dimensional smooth manifold Let $k > 0$ be an integer. A *differential k-form* ω is a smooth assignment $p \mapsto \omega_p$, where $p \in \mathcal{M}$ and ω_p is a skew-symmetric k-linear map

$$\omega_p : \underbrace{T_p\mathcal{M} \times \cdots \times T_p\mathcal{M}}_{k} \to \mathbb{R}$$

where $T_p\mathcal{M}$ denotes the tangent space to $\mathcal{M}$ at p.

Symplectic manifolds

Symplectic form

A differential 2-form ω is said to be *symplectic* if

- it is closed, i.e. $d\omega = 0$ and
- non-degenerate, i.e. $\forall p \in \mathcal{M}$, if there exist a vector $u \in T_p\mathcal{M}$ such that

$$\omega_p(u, v) = 0, \ \forall \ v \in T_p\mathcal{M}$$

then $u = 0$.

Symplectic manifolds

Symplectic manifold

A symplectic manifold is a pair $(\mathcal{M}, \omega)$,

where

- $\mathcal{M}$ is a smooth manifold and
- ω is a symplectic form

Outline

Hamiltonian vector fields

Hamiltonian vector field

A vector field X_h on $(\mathcal{M}, \omega)$ for which

$$\omega(X_h, -) = dh$$

is called a Hamiltonian vector field for the function $h : \mathcal{M} \to \mathbb{R}$.

Example-2

Let $\mathcal{M} = \mathbb{R}^2$, $\omega = dx \wedge dy$. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be given by $h(x, y) = x^2 + y^2$. Compute Hamiltonian vector field X_h at a point $p = (a, b) \in \mathbb{R}^2$.

Hamiltonian vector fields

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Compute

$$\omega(X_h, V) = dh(V)$$

where,

- $X_h = f(x, y)\dfrac{\partial}{\partial x} + g(x, y)\dfrac{\partial}{\partial y}$
- $V = v_1(x, y)\dfrac{\partial}{\partial x} + v_2(x, y)\dfrac{\partial}{\partial y}$

Example-2 continued

Let $\gamma(t) = (a + tv_1, b + tv_2)$ be a smooth curve with $\gamma(0) = p$. Then

- $$dh_p(V) = \left.\frac{d}{dt}\right|_0 h(\gamma(t))$$
  $$= 2av_1 + 2bv_2$$

Hamiltonian vector fields

Example-2 continued

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- $$dh_p(V) = \frac{d}{dt}\bigg|_0 h(\gamma(t))$$
  $$= 2av_1 + 2bv_2$$
- $$\omega_p(X_h, V) = (dx \wedge dy)\Big(f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y},\ v_1\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y}\Big)$$
  $$= fv_2 - gv_1$$

Hamiltonian vector fields

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- Comparing, $X_h = 2b\dfrac{\partial}{\partial x} - 2a\dfrac{\partial}{\partial y}$

Outline

Hamiltonian system

A *Hamiltonian system* is a triple $(\mathcal{M}, \omega, h)$, where

- $(\mathcal{M}, \omega)$ is a symplectic manifold and

Integrability of Hamiltonian systems

Hamiltonian system

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- $(\mathcal{M}, \omega)$ is a symplectic manifold and
- $h : \mathcal{M} \to \mathbb{R}$ is a smooth function, called the *Hamiltonian*.

Integrability of Hamiltonian systems

Integrals of motion

Let $(\mathcal{M}, \omega, h)$ be a Hamiltonian system. Let $f \in C^\infty(\mathcal{M})$. Let the Poisson bracket

$$\{f, h\} = -\,\omega(X_f, X_h) = -df(X_h) = 0.$$

Integrability of Hamiltonian systems

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Which shows that,

- f is constant along the integral curves of X_h and thus said to be an integral of motion for $(\mathcal{M}, \omega, h)$.
- f Poisson commute with h.

Complete integrability

A Hamiltonian system $(\mathcal{M}, \omega, h)$ is *completely integrable* if there exists $n = \frac{1}{2}\dim(\mathcal{M})$ independent integrals of motion $f_1 = h, f_2, \cdots, f_n$ on $\mathcal{M}$, which are pairwise Poisson commuting. That is,

$$\{f_i, f_j\} = 0, \quad \forall\; i, j = 1, \ldots, n$$

Outline

A space of curves in SU_2

$S^3 = \{x \in \mathbb{R}^4 : \|x\| = 1\}$ can be viewed as $SU_2 = \{X \in \mathcal{M}_2(\mathbb{C}) : \det(X) = 1,\ XX^* = I\}$.

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Characteristics of SU_2

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Characteristics of SU_2

- SU_2 is a Lie group.
- The Lie algebra $\mathfrak{su}_2$ is isomorphic to $\mathbb{R}^3$ with the Lie bracket given by cross product.

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- $X(s)$ satisfying $X(0) = I$ will be called *anchored*.
- Tangent vectors of $X(s)$ be defined by $\dfrac{dX}{ds}(s) = X(s)\Lambda(s)$, where $\Lambda(s) \in \mathfrak{su}_2$ satisfy
  - $\langle \Lambda(s), \Lambda(s) \rangle = 1$.
  - $\Lambda(0) = \Lambda(L)$.

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The space of anchored arc length parametrized curves satisfying the above conditions will be denoted by $\mathfrak{M}$ and considered as a Fréchet manifold.

Symplectic form on $\mathfrak{M}$

The tangent vectors in $T_X\mathfrak{M}$ are of the form $v(s) = X(s)V(s)$, where $V(s)$ is the solution of

$$\frac{dV}{ds}(s) = [\Lambda(s),\ V(s)] + U(s)$$

and $U(s) \in \mathfrak{su}_2$.

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Symplectic form

Let $v_1(s) = X(s)V_1(s)$ and $v_2(s) = X(s)V_2(s)$ be two tangent vectors in $T_X\mathfrak{M}$. Then the symplectic form on $\mathfrak{M}$ is defined by

$$\omega_X(V_1, V_2) = -\int_0^L \Big\langle \Lambda(s), [U_1(s), U_2(s)] \Big\rangle ds.$$

Outline

Hamiltonian vector fields on $\mathfrak{M}$

We have four functions $f_i : \mathfrak{M} \to \mathbb{R}, \;\; i = 1,2,3,4$ given by,

- $f_1 = \displaystyle\int_0^L \tau(s) ds$
- $f_2 = \displaystyle\frac{1}{2}\int_0^L \kappa^2 \; (s) ds$
- $f_3 = \displaystyle\int_0^L (\frac{1}{2}\kappa^2(s)\tau(s) + \frac{1}{4}\kappa^2) ds$
- $f_4 = \displaystyle\int_0^L \frac{1}{2}\big((\kappa')^2(s) + \frac{1}{2}\kappa^2(s)\tau^2(s) + \frac{1}{2}\kappa^2(s)\tau(s) + \frac{1}{8}\kappa^2(s) - \frac{1}{8}\kappa^4(s)\big) ds$

Hamiltonian vector fields on $\mathfrak{M}$

Proposition

Let $\mathcal{X}_{f_1}$ be the Hamiltonian vector field corresponding to $f_1 = \int_0^L \tau(s)ds$.

- Let $g(s,t)$ denote the integral curves of $\mathcal{X}_{f_1}$ with tangent vectors

$$\frac{\partial g}{\partial s}(s,t) = g(s,t)\Lambda(s,t)$$

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Hamiltonian vector fields on $\mathfrak{M}$

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Claim: $\Lambda(s,t)$ evolves according to the *Curve Shortening equation* (CSE),

$$\frac{\partial \Lambda}{\partial t}(s,t) = \kappa(s,t)N(s,t).$$

Proof

- Rewrite f_1 as $f_1(\Lambda) = \int_0^L \langle \Lambda', \Lambda' \rangle^{-1} \langle \Lambda'', [\Lambda(s), \Lambda'] \rangle ds$.
- Let $Y(s,t)$ be a family of anchored arc length parametrized curves that satisfies $Y(s,0) = X(s)$ and $v(s) = \frac{\partial Y}{\partial t}(s,t)\Big|_{t=0} = X(s)V(s)$.
- Also let $Z(s,t)$ denote the tangent vectors defined by $\frac{\partial Y}{\partial s}(s,t) = Y(s,t)Z(s,t)$.

Hamiltonian vector fields on $\mathfrak{M}$

proof continued

Step-1:

$$df_{1X}(V) = \frac{\partial}{\partial t}\int_0^L \left\langle \frac{\partial Z}{\partial s}(s,t),\ \frac{\partial Z}{\partial s}(s,t)\right\rangle^{-1} \left\langle \frac{\partial^2 Z}{\partial s^2}(s,t),\ [Z, \frac{\partial Z}{\partial s}(s,t)]\right\rangle\, ds\bigg|_{t=0}$$

$$= I_1 + I_2 + I_3 + I_4$$

$$= -\int_0^L \left\langle G(s), U(s)\right\rangle\, ds$$

Proof continued

Step-2:

$$\omega_X(F_1, F) = -\int_0^L \left\langle \Lambda(s),\ [U_{f_1}(s),\ U(s)] \right\rangle ds$$

Hamiltonian vector fields on $\mathfrak{M}$

Proof continued

Step-2:

$$\omega_X(F_1, F) = -\int_0^L \left\langle \Lambda(s),\ [U_{f_1}(s),\ U(s)] \right\rangle ds$$

Step-3: By definition,

$$df_{1X}(V) = \omega_X(F_1, F)$$

Hamiltonian vector fields on $\mathfrak{M}$

Proof continued

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Step-4: Simplifying,

$$U_{f_1} = -\left[\Lambda(s),\ G(s)\right]$$

Proof continued

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Step-5: Expressing $G(s)$ in terms of Serret-Frenet equations,

$$G(s) = \kappa B + \tau \Lambda$$

Hamiltonian vector fields on $\mathfrak{M}$

Proof continued

Step-2:

$$\omega_X(F_1, F) = -\int_0^L \left\langle \Lambda(s),\ [U_{f_1}(s),\ U(s)] \right\rangle\, ds$$

Step-3: By definition,

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Step-6:

$$U_{f_1} = -[\Lambda,\ \kappa B + \tau \Lambda] = \kappa N$$

Hamiltonian vector fields on $\mathfrak{M}$

proof continued

The integral curves $t \to X(s,t)$ of $\mathcal{X}_{f_1}$ are the solutions of

$$\frac{\partial X}{\partial t} = X(s,t)F_1(s,t) \quad \text{and} \quad \frac{\partial X}{\partial s} = X(s,t)\Lambda(s,t)$$

where $F_1(s,t)$ is the solution of

$$\frac{\partial F_1}{\partial s}(s,t) = [\Lambda(s,t),\ F_1(s,t)] + \kappa N \tag{3}$$

proof continued

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According to a Lemma we get,

$$\frac{\partial \Lambda}{\partial t}(s,t) - \frac{\partial F_1}{\partial s}(s,t) + [\Lambda(s,t), F_1(s,t)] = 0 \tag{4}$$

Proof continued

Using (3) in (4),

$$\frac{\partial \Lambda}{\partial t}(s,t) = \kappa(s,t)N(s,t)$$

Outline

Complete integrability

The Hamiltonian system $(\mathfrak{M}, \omega, h)$ is completely integrable if there exists infinitely many integrals of motion $f_1 = h, f_2, f_3, \ldots, \ldots$ on $\mathfrak{M}$, which pairwise Poisson commute. That is,

$$\{f_i, f_j\} = 0, \quad \forall\ i, j = 1, 2, 3, \ldots$$

Theorem:

The functions f_1, f_2, f_3, f_4 on $\mathfrak{M}$ pairwise Poisson commute.

Integrability of Hamiltonian systems

Theorem:

The functions f_1, f_2, f_3, f_4 on $\mathfrak{M}$ pairwise Poisson commute.

proof:

Let's see,

$$\{f_2, f_4\} = 0$$

proof continued:

Step-1:

$$\{f_2, f_4\} = \omega_X(F_2(s), F_4(s)) = -\int_0^L \langle \Lambda(s), [U_{f_2}(s), U_{f_4}(s)] \rangle ds$$

proof continued:

Step-1:

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Step-2:

- $$U_{f_2} = \kappa\tau N - \kappa' B$$
- $$U_{f_4} = (-\kappa\tau^3 + 3\kappa''\tau + 3\kappa'\tau' + \frac{3}{2}\kappa^3\tau + \kappa\tau'')N + (-\kappa''' + 3\kappa\tau\tau' + 3\kappa'\tau^2 - \frac{3}{2}\kappa^2\kappa')B$$

proof continued:

Step-3:

$$\{f_2, f_4\} = -\int_0^L \frac{d}{ds}\Big(-\kappa^2\tau^3 - \frac{3}{2}(\kappa')^2\tau - \kappa\kappa'\tau' - \frac{1}{2}(\kappa')^2\tau + \kappa\kappa''\tau\Big)\,ds$$

$$= 0$$

Thank You for Your Attention