CENG 223

Discrete Computational Structures

Fall 2023-2024

Take Home Exam 3

Due date: December 14 2023, Thursday, 23:55

Question 1

a. Use mathematical induction to show that $2^{3n} - 3^n$ is divisible by 5 for all integers $n \geq 1$.

b. Use mathematical induction to show that $4^n - 7n - 1 > 0$ for all integers $n \geq 2$.

Question 2

a. How many bit strings of length 10 have at least seven 1s in them?

b. We have 4 identical Discrete Mathematics textbooks and 5 identical Statistical Methods textbooks at hand. In how many ways can you make a collection of 4 books from these 9 textbooks with the condition that at least one Discrete Mathematics textbook and at least one Statistical Methods textbook must be in the collection?

c. How many onto functions are there from a set with 5 elements to a set with 3 elements?

Question 3

5 kids go into an open-air circus in the shape of an equilateral triangle with the side length of 500 meters. Prove that no matter how much they wander away from each other, so long as they stay in the triangle-shaped circus, there are two of them within 250 meters of each other.

Question 4

Consider the recurrence relation $a_n = 3a_{n-1} + 5^{n-1}$ with the initial condition $a_1 = 4$.

a. Determine the homogeneous solution.

b. Determine the particular solution.

c. Show by mathematical induction that the expression you found for a_n is indeed a solution to the given recurrence relation.

Practice Questions

Induction

PQ1. Use induction to prove the statement $S(m, n)$: "The number of possible (ordered) solutions to $x_1 + x_2 + \ldots + x_m = n$ is $f(m, n) = \frac{(n+m-1)!}{n!\,(m-1)!}$ where n and m are positive integers and $x_i \in \{0\} \cup \mathbb{Z}^+$".

(**Hint:** First use induction to prove $S(m, 1)$ and $S(1, n)$. Proceed, again with induction, to prove $S(m + 1, n + 1)$ using $S(m + 1, n)$ and $S(m, n + 1)$ as your basis steps.)

PQ2. Prove by induction that for $n \geq 1$

$$\sum_{k=0}^{n} k\, C(n, k) = n\, 2^{n-1}.$$

Pigeonhole principle

PQ3. Suppose that 26 positive integers are arranged in a circle. The sum of all the numbers is 75. Prove that you can always choose a consecutive sequence of numbers which sum to 50.

PQ4. A closed hemisphere of a sphere is a hemisphere including the circle on its boundary. Prove that for any configuration of 5 points on a sphere, a closed hemisphere can be found such that it contains at least 4 of the points.

Recurrence relations

PQ5. A **string** over an alphabet Σ is an ordered tuple of elements of Σ. The **length** of such a string is the number of elements it has. For example, the string 001101 is a string over $\{0, 1\}$ and its length is 6.

a. Find the recurrence relation for the number of strings over the alphabet $\Sigma = \{0, 1, 2\}$ of length n that contain two consecutive symbols that are the same.

b. State the initial conditions of the recurrence relation.

c. Solve the recurrence relation by finding both the homogeneous and particular solutions.

Regulations

1. Your submission should be a single vector-based PDF document with the name `the3.pdf`.
2. **Late Submission:** Not allowed.
3. **Cheating: We have zero tolerance policy for cheating**. People involved in cheating will be punished according to the university regulations.
4. **Updates & Announces:** You must follow the odtuclass for discussions and possible updates. You can ask your questions in the Student Forum on the course page in odtuclass.
5. **Evaluation:** Your `.pdf` file will be checked for plagiarism automatically using "black-box" technique and manually by assistants, so make sure to obey the specifications.

Submission

Submission will be done via odtuclass. You will submit a single PDF file, `the3.pdf`.