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Answer 1

a-) We can prove this by induction.

- Base Case m=2 : It is already given in the question.
- For m=k≥2 : Assume that y=$\sum_{i=1}^{k} \lambda_i x_i \in C$, $\sum_{i=1}^{k} \lambda_i$=1, $\lambda_i \geq 0$,$x_i \in C$, i=1,...,k
- We will show that the statement holds for m=k+1
- Assume $x_i \in C$, i=1,...,k+1, $\sum_{i=1}^{k+1} \lambda_i$=1. We want to prove x = $\sum_{i=1}^{k+1} \in C$
- x = $\sum_{i=1}^{k+1} \lambda_i x_i$ = (1-λ_{k+1})$\sum_{i=1}^{k} \frac{\lambda_i}{1-\lambda_{k+1}} x_i + \lambda_{k+1} x_{k+1}$, where $\sum_{i=1}^{k} \frac{\lambda_i}{1-\lambda_{k+1}}$=1 and $\sum_{i=1}^{k} \frac{\lambda_i}{1-\lambda_{k+1}} x_i$=y
- If $\lambda_{k+1} = 1$, then x = $x_{k+1} \in C$. So, x is ∈ C also.
- If $\lambda_{k+1} \neq 1$, then x = (1-λ_{k+1})y + $\lambda_{k+1} x_{k+1}$ (It is explained two steps ago)
- We can use base condition m=2. y ∈ C and $x_{k+1} \in C$. The sum of the lambdas is equal to 1. It means that x ∈ C.
- Induction proves that the statement is true.

b-) Suppose g:$\mathbb{R} \to \mathbb{R}$: g(x)=x^2 and f:$\mathbb{R} \to \mathbb{R}$: f(x)=$(x-2)^2$. Lets look whether they are convex function or not.

For function g :

$\forall x_1, x_2 \in R$, t ∈ [0,1],

1. Check $g(tx_1 + (1-t)x_2) \leq tg(x_1) + (1-t)g(x_2)$
2. $(tx_1 + (1-t)x_2)^2 \leq tx_1^2 + (1-t)x_2^2$
3. $t^2x_1^2 + 2t(1-t)x_1x_2 + (1-t)^2x_2^2 \leq tx_1^2 + (1-t)x_2^2$
4. $2t(1-t)x_1x_2 \leq x_1^2t(1-t) + x_2^2t(1-t)$
5. $2x_1x_2 \leq x_1^2 + x_2^2$
6. $0 \leq x_1^2 - 2x_1x_2 + x_2^2$
7. $0 \leq (x_1 - x_2)^2$
8. It is always true. i.e, it is tautology. g is a convex function.

For function f :

1. Check $f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$
2. $t^2x_1^2 + (1-t)^2x_2^2 + 4 + 2tx_1(1-t)x_2 + 2tx_1(-2) + 2(1-t)x_2(-2) \leq t(x-2)^2 + (1-t)(x_2-2)^2$
3. $t^2x_1^2 + (1-t)^2x_2^2 + 2t(1-t)x_1x_2 - 4tx_1 - 4(1-t)x_2 + 4 \leq t(x_1^2 - 4x_1 + 4) + (1-t)(x_2^2 - 4x_2 + 4)$
4. $tx_1(tx_1 - 4) + (1-t)x_2((1-t)x_2 - 4) + 2t(1-t)x_1x_2 + 4 \leq tx_1(x_1-4) + (1-t)x_2(x_2-4) + 4$
5. $2t(1-t)x_1x_2 \leq tx_1(x_1(1-t)) + (1-t)x_2(x_2t)$
6. $2t(1-t)x_1x_2 \leq t(1-t)(x_1^2 + x_2^2)$
7. $0 \leq (x_1 - x_2)^2$
8. It is always true. i,e it is tautology. f is a convex function.

Check whether fog(x) is convex :

- $h(x) = fog(x) = x^4 - 4x^2 + 4$
- Lets check if h is convex
- $h(tx_1 + (1-t)x_2) \leq th(x_1) + (1-t)h(x_2)$,   Choose $x_1 = -1, x_2 = 1$
- $h(-t + 1 - t) \leq t + 1 - t$
- $h(1-2t) \leq 1$,   Choose t=1/2
- $h(0) \leq 1$
- $4 \leq 1$
- There is a contradiction. So, this is counter example for the statement.

c-) We need to prove both f is convex $\rightarrow$ (S is convex $\wedge$ g(t) is convex) and (S is convex $\wedge$ g(t) is convex) $\rightarrow$ f is convex.

1.
   - We must show that if f(x) is convex $\rightarrow$ (S is convex $\wedge$ g(t) is convex)
   - Assume that f:$S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex function.
   - Thanks to the definition of the convex function, $\forall x_1, x_2 \in S, \lambda \in [0,1]$
     $f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$
   - The domain of the function f is S and this domain includes $\lambda x_1 + (1-\lambda)x_2$
   - It means that $\forall x_1, x_2 \in S$, $\lambda \in [0,1]$, $\lambda x_1 + (1-\lambda)x_2 \in S$
   - It is the definition of the being convex set for S. (Completed)
   - Now,we must show that g(t)=f(x+tv) is a convex function $\forall t \in \mathbb{R}$ such that x+tv $\in$S
   - We can write g(λt_1+(1-λ)t_2) = f($x + \lambda t_1 v + (1-\lambda)t_2 v$) by substituting into t
   - Then, we can write x = $\lambda x + (1-\lambda)x$ for common paranthes operation
   - f($x + \lambda t_1 v + (1-\lambda)t_2 v$) = f($\lambda(x + t_1 v) + (1-\lambda)(x + t_2 v)$)
   - f($\lambda(x+t_1v)+(1-\lambda)(x+t_2v)$) $\leq \lambda f(x+t_1v)+(1-\lambda)f(x+t_2v)$ (convexity of f, $x+tv \in S$)
   - $\lambda f(x+t_1v) + (1-\lambda)f(x+t_2v) = \lambda g(t_1) + (1-\lambda)g(t_2)$ (because $g(t_1) = f(x+t_1v)$ and $g(t_2) = f(x+t_2v)$)
   - In conlusion, we get g(λt_1+(1-λ)t_2) $\leq \lambda g(t_1) + (1-\lambda)g(t_2), \lambda \in [0,1]$

- And , this is the definition of being convex for function g. (Completed)
- Finally, we've reach that if is convex $\rightarrow$ (S is convex $\wedge$ g(t) is convex)

2.
- We must show that if (S is convex $\wedge$ g(t) is convex) $\rightarrow$ f(x) is convex.
- Assume that S is convex set and g(t)=f(x+tv) is a convex function $\forall t \in \mathbb{R}$ such that x+tv $\in$ S
- g$(\lambda t_1 + (1-\lambda)t_2) \leq \lambda g(t_1) + (1-\lambda)g(t_2), \forall t_1, t_2 \in dom(g)\ t \in [0,1]$(the definition of convexity)
- We will process each side of the inequality.
- First, take LHS, g$(\lambda t_1 + (1-\lambda)t_2)$ = f(x+$\lambda t_1 v + (1-\lambda)t_2 v$) (substituting t)
- Then, we can write x = $\lambda x + (1-\lambda)x$ for common paranthes operation
- f$(x + \lambda t_1 v + (1-\lambda)t_2 v)$ = f$(\lambda(x + t_1 v) + (1-\lambda)(x + t_2 v))$ (It's equal to LHS)
- Secondly, take RHS, $\lambda g(t_1) + (1-\lambda)g(t_2) = \lambda f(x + t_1 v) + (1-\lambda)f(x + t_2 v)$
- Because we know LHS $\leq$ RHS, we can write:
  f$(\lambda(x + t_1 v) + (1-\lambda)(x + t_2 v))$$\leq \lambda f(x + t_1 v) + (1-\lambda)f(x + t_2 v)$
- We already know x+tv $\in$ S. In addition to that, we can infer $\lambda(x+t_1 v)+(1-\lambda)(x+t_2 v)$ also $\in$ S because S is convex set. According to the definition of the convex set, if $x + t_1 v, x + t_2 v \in S$ then $\lambda(x + t_1 v) + (1-\lambda)(x + t_2 v)$ must be $\in$ S.
- To sum up, f:$S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ meets all the requirements to be convex a function.
- So, we can say that if (S is convex $\wedge$ g(t) is convex) $\rightarrow$ f(x) is convex.

3. This is proven in both directions, which means this is true.

Answer 2

a-)

- If X is finite, then its all subsets are finite. Thus, X-U is also finite or $\emptyset$. It can be concluded that $\sum$ = P(X). $\sum$ includes all subsets of X. So, we can infer that, X is in $\sum$, $\forall A \in \sum, X - A$ is in $\sum$, union of all sets in $\sum$ is X and it is in $\sum$ already. As seen, all conditions are satisfied. This set is σ-algebra on X.
- If X is countably infinite, lets consider Z^+ as X. X-U is finite when U={2,3,4,5,6,...}, X-U ={1} and {2,3,4,5,6,...} must be in $\sum$. However, if U={1}, X-U={2,3,4,5,6,...} that is not finite or $\emptyset$, {1} is not in $\sum$. To sum up, $\sum$ includes {2,3,4,5,6,...} but not its complement ({1}), this violates second condition. We can infer that this set is NOT σ-algebra on X.
- If X is infinite, we can consider $\mathbb{R}$ as X. U=$\mathbb{R}$-{99}, then X-U is equal to {99} which is finite, so $\mathbb{R}$-{99} will be in $\sum$. When U={99}, X-U= $\mathbb{R}$-{99} which is not finite or $\emptyset$, so {99} is not in $\sum$. As seen, this pair violates the second condition $\mathbb{R}$-{99} in $\sum$ but its complement {99} is not in $\sum$. We can infer that, this set is NOT σ-algebra on X.

b-)

- If X is finite, then its all subsets are finite, and countable. Thus, X-U is also finite $\forall U \subseteq X$. It can be concluded that $\sum$ = P(X). $\sum$ includes all subsets of X, so we can infer that, X is in $\sum$, $\forall A \in \sum$ X-A is in $\sum$, and union of all sets in $\sum$ is X and it is in $\sum$ already. As seen, all conditions are satisfied. This set is σ-algebra on X.
- If X is countably infinite, we can consider $\mathbb{Z}^+$. Actually, a subset of a set cannot be strictly larger than the original set since the original set can be enumerated (it is countably infinite), and we can simply choose the elements of the subset based on their position in the enumeration. In this way, one-to-one correspondence is established between the subset and natural numbers, which makes the subset countable also. Thus, we can apply this approach into this question. We have countably infinite set X. It means that its all subsets are countable (including $\emptyset$,or all of X). We can infer $\sum$ = P(X). Lets consider the conditions: X is in $\sum$, and so its complement $\emptyset$, the complements of all subsets in $\sum$ because they're all countable, the union of all subsets is X and X is in $\sum$ already. All conditions are satisfied. This implies that this set is σ-algebra on X.
- If X is infinite, we can consider the real numbers between 0 and 1. If U = (0,1)$\cap\overline{\mathbb{Q}}$, then X-U = (0,1)$\cap\mathbb{Q}$ which is countable (can be proven by zigzag listing approach). So, (0,1)$\cap\overline{\mathbb{Q}}$ will be in $\sum$. However, if U = (0,1)$\cap\mathbb{Q}$, X-U = (0,1)$\cap\overline{\mathbb{Q}}$ which is uncountable (can be proven by Cantor's diagonal argument). So, (0,1)$\cap\mathbb{Q}$ is NOT in $\sum$. It means that, (0,1)$\cap\overline{\mathbb{Q}}$ is in $\sum$, but its complement is NOT. This violates second condition. We can infer that this set is NOT σ-algebra on X.

c-)

- If X is finite, then its all subsets are finite, or X, or $\emptyset$. Then, two possibility remains. U = X, X-U= $\emptyset$ and U = $\emptyset$, X-U=X. It means that $\sum$ = {X,$\emptyset$}. This sigma satisfies all conditions. X is in $\sum$, complements of all sets in $\sum$ are in $\sum$ already, and union of sets in $\sum$ is X and its in $\sum$. To sum up, this set is σ-algebra on X.
- If X is countably infinite, its all subsets is countable (I've already explained this in part b.2). So, there are two possibility again (the subsets cannot be infinite). U = X, X-U= $\emptyset$ and U = $\emptyset$, X-U=X.It means that $\sum$ = {X,$\emptyset$}. This sigma satisfies all conditions. X is in $\sum$, complements of all sets in $\sum$ are in $\sum$ already, and union of sets in $\sum$ is X and its in $\sum$. To sum up, this set is σ-algebra on X.
- If X is infinite, we can consider $\mathbb{R}$ as X. If U = {1,2}, X-U = $\mathbb{R}$-{1,2} which is infinite, then {1,2} is in $\sum$. However, if U = $\mathbb{R}$-{1,2}, X-U = {1,2} which is NOT finite,or NOT X, or NOT $\emptyset$, then $\mathbb{R}$-{1,2} is NOT in $\sum$. This pair violates the second condition since {1,2} is in $\sum$, but its complement is not. To sum up, this set is NOT σ-algebra on X.

Answer 3

a-) We must prove if part and vice versa.

Prove for $ax \equiv$b(mod p) $\rightarrow$ gcd(a,p)|b :

1. gcd(a,p)=d
2. a=dt_1, p=dt_2
3. ax=b+kp from ax $\equiv$ b(mod p)
4. Substituting a and p into the third equation gives dt_1x = b + kt_2
5. b = dt_1x - kt_2 = d(t_1x-t_2k)
6. Fifth row means that d|b
7. Thus, $ax \equiv$b(mod p) $\rightarrow$ gcd(a,p)|b

Prove for gcd(a,p)|b $\rightarrow$ $ax \equiv$b(mod p) :

1. gcd(a,p) = sa + tb , s,t are integer (from the Bezout's Identity)
2. (sa+tp) |b
3. (sa + tp)*m = sam + tpm = b
4. sam - b = tpm
5. p | (sam -b)
6. x = sm
7. p | ax - b
8. It means that , ax $\equiv$ b(mod p)
9. gcd(a,p)|b $\rightarrow$ $ax \equiv$b(mod p)

b-) gcd(a_1, p_1)| b_1, gcd(a_2, p_2)| b_2. So, we can use what we have proved in part a. We can write $x_1 \equiv b_1'$ mod(p_1) and $x_2 \equiv b_2'$ mod(p_2)

1. gcd(p_1, p_2) = sp_1 + tp_2 = 1, s and t are integers (from the Bezout's Identity)
2. x' = msp_1 + ntp_2
3. x' = msp_1 (mod p_2)
4. x' = m(1-sp_1) (mod p_2)
5. x' = m (mod p_2)
6. Similarly, x' = n (mod p_1)
7. Then, b_2' = m, b_1'= n ,and x' is our solution.
8. $\exists c$ such that x $\equiv$ c (mod p_1p_2)

c-) We can prove this by induction:

1. Base case : P(2) is proved in part b.
2. Inductive step: Assume P(k-1) is true.
3. We have to prove P(k)
4. From the assumption, we know that the system has a solution for x of the form x $\equiv$ c mod($p_1p_2....p_{k-1}$), where gcd($p_1....p_{k-1}$)=1 and gcd$(a_i, p_i)|$ b_i for some c$\in \mathbb{Z}$.
5. We know that gcd$(p_1, ..., p_k)$=1, then gcd$(p_1...p_{k-1}, p_k)$=1 (p_k is coprime with the product of previous numbers)
6. x$\equiv$ x_1 mod($p_1p_2....p_{k-1}$), x$\equiv$ b_k mod(p_k) (from the Inductive Hypothesis)
7. Then, we can use part b here. $p_1p_2....p_{k-1}$ and p_k are relatively prime.
8. So, $\exists c$ such that x$\equiv$c (mod $p_1p_2...p_k = \Pi$). There is a solution.

Answer 4

a-) In order to show that final set is uncountable, we can use Cantor's diagonal argument. We suppose it is countable and arrive a contradiction.

$\cdot w_1 = c_{11}\ c_{12}\ c_{13}\ c_{14}$ ...
$\cdot w_2 = c_{21}\ c_{22}\ c_{23}\ c_{24}$ ...
$\cdot w_3 = c_{31}\ c_{32}\ c_{33}\ c_{34}$ ...
$\cdot w_4 = c_{41}\ c_{42}\ c_{43}\ c_{44}$ ... where $c_{ij} \in X$. Then, we can form a new word with the following rule:

$\cdot w = c_1\ c_2\ c_3\ c_4$ ... where c_i = a if $c_{ii} \neq a$, else z. Thus, the new word w is not equal to any word w_1,w_2,... To sum up, a new word (cartesian product of characters) that is not in the list can be created. There is a contradiction, it is uncountable.

b-) The union of countably many list of countable many items is countable. We can prove this by zigzag listing approach.

	1	2	3	4	...
Y_1	Y_{11}	Y_{12}	Y_{13}	Y_{14}	...
Y_2	Y_{21}	Y_{22}	Y_{23}	Y_{24}	...
Y_3	Y_{31}	Y_{32}	Y_{33}	Y_{34}	...
Y_4	Y_{41}	Y_{42}	Y_{43}	Y_{44}	...
...	...	...	...	...	

As seen, we can list the items in the table in a way that first, the items whose i+j=2, then the items whose i+j=3, so on. The list order is Y_{11} - Y_{12} - Y_{21} - Y_{31} - Y_{22} - Y_{13} - .... In this way, there is a one-to-one correspondence with Z^+. It means that the cardinality of the desired set is equal to cardinality of the Z^+. So, it is countable.