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Answer 1

Let's look at the G : It's connected (there is a path between every distinct pairs). The degrees are : deg(a)=2,deg(b)=2,deg(c)=4,deg(d)=4,deg(e)=4,deg(f)=2,deg(g)=4,deg(h)=4,deg(i)=4,deg(j)=4, deg(k)=2,deg(l)=4,deg(m)=2

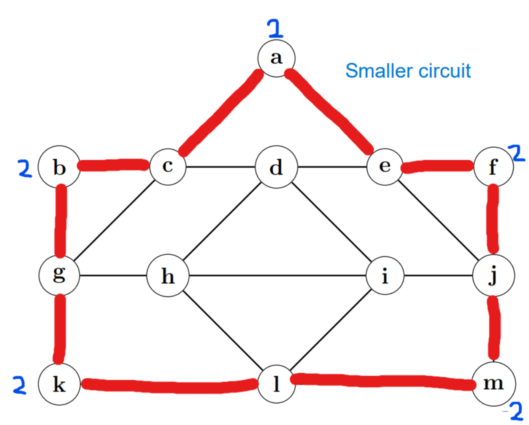
a-) According to the theorem, a simple graph has Eulerian Circuit if and only if it's connected and the number of vertices with odd degree is 0. So, G meets the requirements ($|V_o| = 0$). There is an Eulerian Circuit. Here is one example : bcaedcghdijefjmlihlkgb

b-) According to the Theorem 2 in Section 10.5, a connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree. Here, $|V_o| = 0$. As a result, G has no Eulerian Path which is not a circuit.

c-) According to the textbook (last paragraph in page 699):

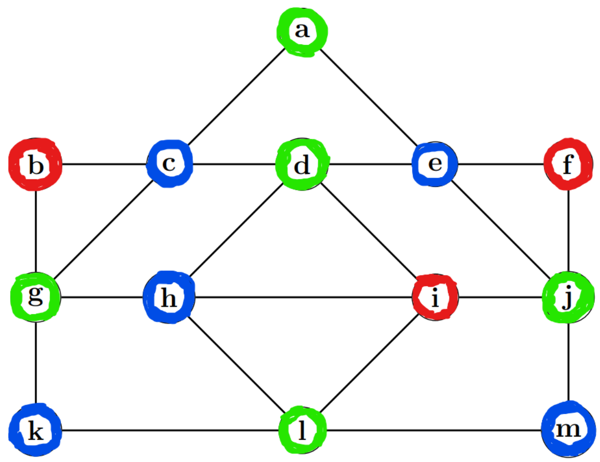
1. A graph with a vertex of degree one cannot have a Hamiltonian circuit.
2. If a vertex in the graph has degree two, then both edges that are incident with this vertex must be part of any Hamiltonian circuit.
3. A Hamiltonian circuit cannot contain a smaller circuit within it.

So, in the graph, the vertices a,b,k,m,f have degree of two. According to the second property, both edges that are incident with those vertex must be part of any Hamiltonian circuit. However, when we include these edges, we have a smaller circuit which is inside the Hamiltonian Circuit (figure is below). This contradicts third rule. A Hamiltonian circuit cannot contain a smaller circuit within it. To sum up, The graph G does not have Hamilton Circuit.

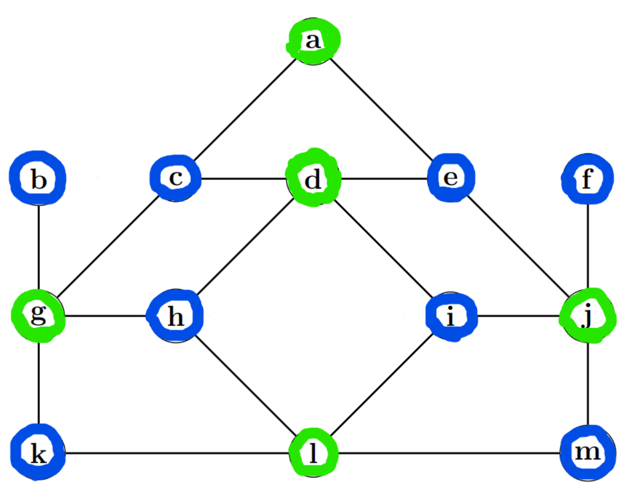


d-) According to the theorem, a simple graph has a Hamiltonian Path if for every pair of vertices in G sum of degrees of those vertices $\geq$ n-1 where n = $|V|$. Here $|V| = n = 13$. There is no pair which their sum is such that big. So, we cannot say anything from the theorem. However, we can try to find Hamiltonian Path by trying. There is Hamiltonian Path : bcdihgklmjfea

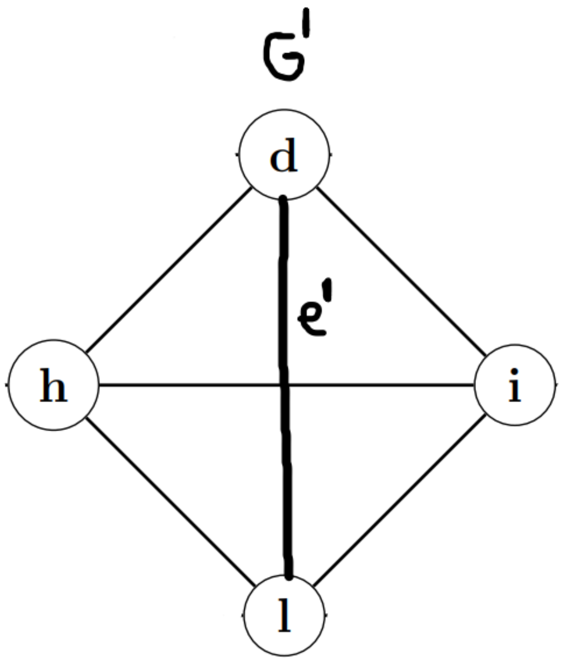
e-) In graph G, no edges cross each other. So, G is planar graph. According to the Four Color Theorem in Section 10.8, the chromatic number of a planar graph is no greater than four. 1 and 2 colors are not enough to color the graph, but 3 is enough. $\chi(G) = 3$ and here is the coloring way:



f-) According to the Theorem 4 in Section 10.2, a simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color. However, our graph is not 2-colorable so, it is not bipartite now. We must make it 2-colorable. In order to eliminate red color, we must delete 3 edges which are (b,c) , (e,f) , (h,i). Then we can color b,f,i with blue color. The graph is bipartite now. Partition is [b,c,h,k,e,f,i,m] and [a,d,g,j,l]. Here is the new graph with deleted edges:



g-) There is no complete graph with at least four nodes as a subgraph in G. If we look at h,d,i,l. h and i has three edges with others. If we add the edge $e' =$ (d,l), it will be complete graph with at least 4 nodes as a subgraph. The subgraph G' will be :



Answer 2

Let's look at the graph invariants. Both graphs have 8 vertices and 16 edges. In the both graphs, every vertex has 4 degree. Because they are same, we cannot infer anything from these information. Let's try to create a bijection between G and H. Let F be a 1-1 and onto function from G to H. $F(a) = a', F(b) = c', F(c) = e', F(d) = g', F(f) = h', F(e) = b', F(h) = f'$. Matrix representation of both must be same with bijection. Let's look at their matrix representations:

M_G:
Label Sequence : a-b-c-d-e-f-g-h

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

M_H:
Label Sequence : $a' - c' - e' - g' - b' - h' - d' - f'$

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

As a result, we found a bijection between G and H, their matrix representation is same. It means that G and H are isomorphic.

Answer 3

a-) We can color the cycle graph like this: RBRBRBRBRB (R:red, B:blue). However, this is only valid for cycle graphs with even number of vertices. In the cycle graphs with odd number of vertices, we need three colors because in the end and the beginning of the cycle two same colors come side by side with 2 colors. We can say that if our grap is 2-colorable, it is bipartite. As a result, the cycle graphs with even number of vertices is bipartite, but the cycle graphs with odd number of vertices is not partite.

b-) In the cube graphs $\chi(G) = 2$. We can prove this by induction.

- P(n) : In the cube graphs with n dimensions $\chi(G) = 2$.
- Basis Step $\rightarrow$ n=1 : P(1) is true.
- Inductive step : Inductive hypothesis : P(k) is true for k$\geq$2.
- Let's check whether $\chi(Q_{k+1}) = 2$ or not.
- We can construct Q_{k+1} with two Q_k's. While constructing the new cube graph, we put one Q_k normally, but the other Q_k must be anti-colored (for example in the one of them 00 is Red colored, but in the other 00 is Blue colored) in order to avoid color crossing.
- In this way, $\chi(Q_{k+1}) = 2$.
- By induction, $\chi(Q_n) = 2$ for n$\geq$1 .

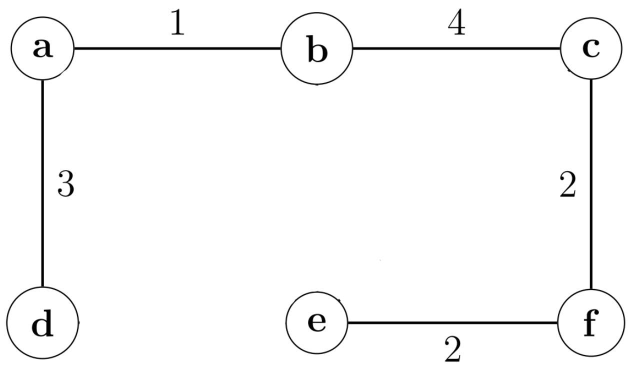
Finally, we know that $\chi(Q_n) = 2$. According to Theorem 4 in Section 10.2, we can say that Q_n is bipartite.

Answer 4

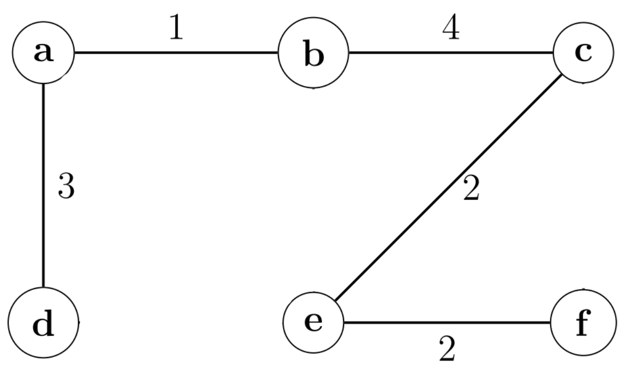
Prim's Algorithm

a-) $\{a, b\} - \{a, d\} - \{b, c\} - \{c, f\} - \{f, e\}$: Cost is 12

b-)



c-) While processing the Prim's algorithm, if there are edges with same costs, we choose one of them without any rule. This choice causes difference between minimum spanning trees. It means that although the total cost is same, the minimum spanning tree is not unique. Here the one of the other minimum spanning trees can be: $\{a, b\} - \{a, d\} - \{b, c\} - \{c, e\} - \{e, f\}$



Answer 5

a-) Full binary tree must have odd number of nodes in order to obey the definition of full binary tree. Let's define B_i is the full binary tree with 2i-1 nodes. For example B_2 is full binary tree with 3 nodes. We can prove that full binary tree with n nodes has (n+1)/2 leafs by induction. It means that, B_i has i leaf nodes. (n=2i-1)

- Basis Step : B_1 has 1 leaf : i=1.
- Inductive Step : Inductive Hypothesis : B_k has k leaf nodes.

- Let's check whether B_{k+1} has k+1 leaf nodes or not.
- When we add two leaf nodes to an full binary tree, its number of leaf nodes increases by 1. (we should both of them to same leaf node, then one leaf node goes, two new leaf nodes come). It means that $B_{k+1} = B_k + 1$.
- From the inductive hypothesis, we know that B_k is k.
- So, $B_{k+1} = k + 1$. From this, we know that B_{k+1} is also true.
- By induction, B_i has i leaf nodes.

When we transform it to the previous domain (n=2i-1 $\rightarrow$ i = (n+1)/2), full binary tree with n nodes has (n+1)/2 leaf nodes.

b-) The chromatic number of a tree is 2. We can prove this by induction.

- P(n) : Tree with n vertices have $\chi = 2$.
- Basis Step n=2 : Tree with 2 vertices can be colored with 2 colors. P(2) is true.
- Inductive Step : Inductive Hypothesis : Assume that P(k) is true : A tree with k nodes has $\chi = 2$.
- Let's check whether a P(k+1) is true or not.
- According to the theorem, a tree with 2 or more vertices has at least two leaf nodes. So, from the tree with k+1 nodes, remove one leaf and its incident edge. Let V be the neighbour of the removed leaf. and R be a removed leaf.
- The tree without R becomes a tree with k nodes, and we know that a tree with k nodes can be colored by 2 colors. Then, we add R again to our tree with opposite color to V. (if V is red R is blue, if V is blue R is red)
- Our tree is again colored by 2 colors. It means that P(k+1) is true.
- By induction P(n) is true.
- So, chromatic number of tree is 2 for n≥2 and chromatic number of a tree is 1 for n=1.

c-) In order to find upper bound for height, we must create the tree which has maximum height with minimum number of nodes. In order to achive that, In each addition of m nodes (because it must be full tree), we must increment height by adding these nodes into the leaf which has maximum depth. The figure that is explains the situation is below. Every level except root has m nodes. So, the total number of nodes is $hm + 1 = n$. This expression is the minimum number for nodes.

$$n \geq mh + 1$$

$$\frac{n-1}{m} \geq h$$

Finally, $\dfrac{n-1}{m}$ is the upper bound for height in full m-ary tree.