रुद्रदमन
घमण्ड को नाश करना or
घमण्ड का नाश करने वाला

Unit-5
(RGPV Unit-4)

PAGE NO.
DATE: / /

Discrete Numeric functions ⇒

The functions whose domain is the set of natural numbers & whose Range is the set of real numbers These functions are called discrete numeric functions.
Numeric functions are denoted by bold lower case letters a, b, ---. If a is a Numeric function then it's value at 0, 1, 2, --- r are denoted by $a_0, a_1, a_2 --- a_r ....$ written as

$$a = (a_0, a_1, a_2, ---- a_r, ----)$$

Ex $(1, 2, 9, 28, ----- r^3+1 ---)$ is the numeric function a whose r^{th} term a_r is r^3+1

or $a_r = r^3+1, \quad r \geq 0$

Ex $(0, 3, 6, 7, 15, 31, ----)$

$$a_r = \begin{cases} 3r, & 0 \leq r \leq 2 \\ 2^r-1, & r \geq 3 \end{cases}$$

Q.1) A person deposite 200/- in saving account at an interest of 8% per year compound annually. Find the numeric function a where a_r denotes the total amount in the account at the end of r^{th} year.

Soln I^{st} yr $SI = \frac{P \times R \times t}{100} = \frac{200 \times 8 \times 1}{100} = 16$

∴ Amount $= P + SI = 200 + 16 = 216/-$

DATE: / /

II^{nd} yr $\quad SI = \frac{216 \times 8 \times 1}{100} = \frac{1728}{100} = 17.28\%$

Amount $= P + SI = 216 + 17.28 = 233.28$

III^{rd} yr $\quad SI = \frac{233.28 \times 8 \times 1}{100} = \cancel{25}18.66$

Amount $= 233.28 + 18.66 = 251.94$ /

$\therefore a = (200, 216, 233.28, 251.94, \text{------})$

$a = 200\left(1 + \frac{8}{100}\right)^r$

$a = 200(1.08)^r$

$\therefore a_r = 200(1.08)^r, \quad r \geq 0$

Q.2) A ball is dropped to the floor from a height ~~of~~ of 80 m. Suppose that the ball always rebounds to each half of the height from it's falls. Find the numeric function a where a denotes the height it reaches in the r^{th} rebound.

Sol^n A ball is dropped from ~~the~~ a height ~~of~~ of 80 m above the floor.

So, the height after first rebound $= 80\left(\frac{1}{2}\right)$

$= 40$ m

DATE: / /

The height after second rebound $= 80\left(\frac{1}{2}\right)^2 = 20$

The height after third rebound $= 80\left(\frac{1}{2}\right)^3 = 10$

⋮

The height at the end of each rebound can be represented by numeric function

$$a = (80, 40, 20, 10, \ldots\ldots)$$

$$a = 80\left(\frac{1}{2}\right)^r \quad \therefore \quad a_r = 80\left(\frac{1}{2}\right)^r, \; r \geq 0$$

Manipulation of numeric functions ⇒

Sum of Numeric functions ⇒

Let a & b are two numeric functions. The sum of a & b is denoted by a+b & ~~it~~ is a numeric function whose value at r is equal to the sum of value of a & b at r.

Ex ① Let a & b are two numeric functions

$$a_r = \begin{cases} 0 & 0 \leq r \leq 2 \\ 2^{-r} + 7 & r \geq 3 \end{cases}$$

$$b_r = \begin{cases} 5 - 2^r & 0 \leq r \leq 1 \\ r + 3 & r \geq 2 \end{cases}$$

find a+b (sum)

Sol^n Let $c = a + b$

r^{th} $C_r = a_r + b_r$

$$\therefore C_r = \begin{cases} 5 - 2^r, & 0 \le r \le 1 \\ 5, & r = 2 \\ 2^{-r} + r + 10, & r \ge 3 \end{cases}$$

Multiplication of Numeric function ⇒

αa_r ← real number

$$a_r = \begin{cases} 0, & 0 \le r \le 2 \\ 2^{-r} + 7, & r \ge 3 \end{cases}$$

then $5a$

$$\therefore 5a_r = \begin{cases} 0, & 0 \le r \le 2 \\ 5(2^{-r} + 7), & r \ge 3 \end{cases}$$

Product of numeric function ⇒

$C = ab$ $\therefore C_r = a_r \cdot b_r$

$$C_r = \begin{cases} 0, & 0 \le r \le 2 \\ (2^{-r} + 7)(r+3), & r \ge 3 \end{cases}$$

a_r & b_r दिए है

Accumulated sum of numeric function ⇒

$$b_r = \sum_{i=0}^{r} a_i$$

$y(t) = \sum_{h=0} y_n t = y_0 + y_1 t + y_2 t^2 + y_3 t^3 + \cdots$
$\cdots + y_n t^n + \cdots$
↑ for initial value problems

PAGE NO.
DATE: / /

GENERATING FUNCTIONS ⇒

Generating functions provide us an alternative way to represent discrete numeric functions & solving recurrence relations. Generating functions also provide us an alternative way to solve combinatorial problems.

Let $(a_0, a_1, a_2, \cdots a_r \cdots) = a$ be a discrete numeric function then an infinite series in terms of z parameter

$$A(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_r z^r + \cdots$$

EX ① $(\overset{3^0}{1}, 3^1, 3^2, 3^3, \cdots 3^r \cdots)$ is given by

$3^0 z^0 + 3^1 z^1 + 3^2 z^2 + \quad 3^r z^r$

$$A(z) = 1 + 3z + 3^2 z^2 + 3^3 z^3 + \cdots + 3^r z^r + \cdots$$

Geometric progress — It is the G.P. (infinite G.P.) [RHS side]

$$\therefore A(z) = \frac{1}{1-3z} \quad \left(Sum_\infty = \frac{a}{1-r}\right) \quad \left(r = \frac{T_2}{T_1}\right)$$

EX ② $a_r = 1$

$$A(z) = 1 + z + z^2 + z^3 + \cdots + z^r + \cdots$$

$$\therefore A(z) = \frac{1}{1-z}$$

EX ③ $a_r = 2^r$

$$A(z) = 1 + 2z + 2^2 z^2 + \cdots + 2^r z^r + \cdots$$

$$\therefore A(z) = \frac{1}{1-2z}$$

EX ④ ~~A(z) =~~ $a_r = \alpha^r$

$$A(z) = 1 + \alpha z + \alpha^2 z^2 + \cdots + \alpha^r z^r + \cdots \qquad \therefore A(z) = \frac{1}{1-\alpha z}$$

Ex ⑤ $a_r = r$

$A(z) = z + 2z^2 + 3z^3 + \cdots + rz^r + \cdots$

$\therefore A(z) = (1 + 2z + 3z^2 + 4z^3 + \cdots) - (1 + z + z^2 + z^3 + \cdots)$

$$= \frac{1}{(1-z)^2} - \frac{1}{(1-z)} = \frac{z}{(1-z)^2}$$

Ex ⑥ $a_r = r(r+1)$

$A(z) = 1 \cdot 2z + 2 \cdot 3z^2 + 3 \cdot 4z^3 + \cdots + r(r+1)z^r + \cdots$

$$\therefore A(z) = \frac{2z}{(1-z)^3}$$

(*)

(Q.1) Let $A(z)$, $B(z)$ & $C(z)$ represent the generating functions of the numeric functions a, b, & c respectively show that

(i) If $b_r = \alpha a_r$ where α is a real constant then $B(z) = \alpha A(z)$

(ii) If $c_r = a_r + b_r$ then $C(z) = A(z) + B(z)$

(iii) If $c = a * b$ then $C(z) = A(z)\,B(z)$

Solⁿ Let $a = (a_0, a_1, a_2, a_3, \cdots a_r, \cdots)$

$b = (b_0, b_1, b_2, b_3, \cdots b_r, \cdots)$

$c = (c_0, c_1, c_2, c_3, \cdots c_r, \cdots)$

Then $A(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_r z^r + \cdots$

$B(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_r z^r + \cdots$

$C(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_r z^r + \cdots$

DATE: / /

(I) If $b_r = \alpha a_r$ then

$$B(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_r z^r + \cdots$$

$$B(z) = \alpha a_r$$

$$B(z) = \alpha a_0 + \alpha a_1 z + \alpha a_2 z^2 + \cdots + \alpha a_r z^r + \cdots$$

$$B(z) = \alpha(a_0 + a_1 z + a_2 z^2 + \cdots + a_r z^r + \cdots)$$

$$B(z) = \alpha A(z)$$

$\because$ $b_r = \alpha a_r$ then $B(z) = \alpha A(z)$

(II) If

$C_r = a_r + b_r$ then

$$C(z) = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + \cdots + C_r z^r + \cdots$$

Similarly

$$C(z) = (a_0 + b_0) + (a_1 + b_1)z + (a_2 + b_2)z^2 + \cdots + (a_r + b_r)z^r + \cdots$$

$$C(z) = (a_0 + a_1 z + a_2 z^2 + \cdots + a_r z^r + \cdots) + (b_0 + b_1 z + b_2 z^2 + \cdots + b_r z^r + \cdots)$$

$$C(z) = A(z) + B(z)$$

$\because$ $C_r = a_r + b_r$ $\therefore$ $C(z) = A(z) + B(z)$

(III) If $c = a * b$ then

$$C(z) = A(z) * B(z)$$

Q.2) Let $A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_r z^r + \cdots$ be generating function of the numeric function $a = (a_0, a_1, a_2 \cdots a_r \cdots)$ then to prove that $\frac{1}{1-z} A(z)$ is the generating function of the numeric function b which is the accumulated sum of a

$$b_r = \sum_{k=0}^{r} a_k$$

Sol^n ⟹

$$\frac{1}{1-z} = (1-z)^{-1} = 1 + z + z^2 + \cdots + z^r + \cdots$$

there fore $\frac{1}{1-z}$ is the generating function of the numeric function $(1, 1, 1, 1, \cdots)$

Again

$$A(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_r z^r + \cdots$$

multiply by $\frac{1}{1-z}$ with $A(z)$

$$\left(\frac{1}{1-z}\right) A(z) = a_0 + (a_0 + a_1) z + (a_0 + a_1 + a_2) z^2 + \cdots + (a_0 + a_1 + a_2 + \cdots + a_r) z^r + \cdots$$

b is the accumulated sum of a

$$\therefore b_r = \sum_{k=0}^{r} a_k$$

Q.3) Find the generating function of the given numeric function.

(i) $5 \cdot 2^r \quad , \quad r \geq 0$

Sol^n ⇒ Here $a_r = 5 \cdot 2^r$

we know that generating function of 2^r

$\therefore \dfrac{1}{1-2z}$

$A(z) = \dfrac{5}{1-2z}$

(ii) $a_r = 5 \cdot 2^{r+2}$

$\therefore a_r = 5 \cdot 2^r \cdot 2^2 = 5 \cdot 2^r \cdot 4 = 20 \cdot 2^r$

$\therefore A(z) = \dfrac{20}{1-2z}$

(iii) $a_r = 2^r + 3^r$

$A(z) = \left(\dfrac{1}{1-2z}\right) + \left(\dfrac{1}{1-3z}\right) = \dfrac{1-3z+1-2z}{(1-2z)(1-3z)}$

$= \dfrac{2-5z}{(1-2z)(1-3z)}$

Q.4) Determine the generating function of the numeric function a_r

$$a_r = \begin{cases} 2^r & , \text{ if } r \text{ is even} \\ -2^r & , \text{ if } r \text{ is odd} \end{cases}$$

Sol^n I) Let generating function of a_r

$$A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$$

Where $r = 0, 1, 2, 3, 4, 5, \cdots$

We are given

$$a_r = \begin{cases} 2^r, & r \text{ is even} \\ -2^r, & r \text{ is odd} \end{cases}$$

$\therefore a_0 = 1, \quad a_1 = -2, \quad a_2 = 2^2 = 4$

$a_3 = -2^3 = -8, \quad a_4 = 2^4 = 16, \quad a_5 = -2^5 = -32$

$\cdots$

$\therefore A(z) = 1 - 2z + 4z^2 - 8z^3 + 16z^4 - 32z^5 + \cdots$

$= 1 - 2z + (2z)^2 - (2z)^3 + (2z)^4 - (2z)^5 + \cdots$

$= (1+2z)^{-1}$ by Binomial Theorem

$$\therefore A(z) = \frac{1}{1+2z}$$

II) $(1, -2, 3, -4, 5, \cdots (-1)^r (r+1), \cdots)$

Sol^n generatin function of given numeric function

$A(z) = 1 - 2z + 3z^2 - 4z^3 + 5z^4 \cdots + (-1)^r (r+1) z^r + \cdots$

$A(z) = (1+z)^{-2}$ By Binomial Theorem

$$\therefore A(z) = \frac{1}{(1+z)^2}$$

विजदम ⇒ [illegible] को नाश करने वाला
or
[illegible] को नाश करना

PAGE NO.
DATE: / /

(III) $a_r = \left(1, \frac{2}{4}, \frac{3}{16}, \ldots\right)$

$$A(z) = 1 + \frac{2}{4}z + \frac{3}{16}z^2 + \frac{4}{64}z^3 + \cdots + \frac{(r+1)}{4^r}z^r + \cdots$$

$$= 1 + 2\left(\frac{z}{4}\right) + 3\left(\frac{z}{4}\right)^2 + 4\left(\frac{z}{4}\right)^3 + \cdots + (r+1)\left(\frac{z}{4}\right)^r + \cdots$$

$$= \left(1 - \frac{z}{4}\right)^{-2}$$

$$A(z) = \frac{1}{\left(1 - \frac{z}{4}\right)^2} = \frac{16}{(4-z)^2}$$

(IV) $A(z) = \frac{2}{1-4z^2}$ Determine Discrete numeric function

Sol^n

$$A(z) = \frac{2}{1-4z^2} = \frac{1}{(1-2z)} + \frac{1}{(1+2z)}$$

$$= (1-2z)^{-1} + (1+2z)^{-1}$$

$$= \left(1 + 2z + 2^2z^2 + \cdots + 2^r z^r + \cdots\right) + \left(1 - 2z + 2^2z^2 + \cdots + (-1)^r 2^r z^r + \cdots\right)$$

$$= 2 + 2^3z^2 + 2^5z^4 + \cdots + \{2^r + (-2)^r\}z^r + \cdots$$

$$\therefore \quad a_r = 2^r + (-2)^r$$

$$a_r = \begin{cases} 0, & \text{if } r \text{ is odd} \\ 2 \cdot 2^r, & \text{if } r \text{ is even} \end{cases}$$

or

$$a_r = \begin{cases} 0, & r \text{ is odd} \\ 2^{r+1}, & r \text{ is even} \end{cases}$$

DATE : / /

Recurrence Relations ⇒

Recurrence relation in terms of difference b/w the consecutive terms of a sequence & hense recurrence relations are also called difference equation.

* A relation which involves an independent variable x, a dependent variable y & one or more than one differences $\Delta y, \Delta^2 y, \Delta^3 y \ldots$ is called a recurrence relation.

Degree of Recurrence Relation ⇒

The degree of a recurrence relation is defined to be the highest power of y_x

Q.1) Given $y_n = A \cdot 2^n + B \cdot 3^n$ find the corresponding recurrence relation.

Sol^n

$$y_n = A \cdot 2^n + B \cdot 3^n$$

$$y_{n+1} = A \cdot 2^{n+1} + B \cdot 3^{n+1}$$

$$y_{n+2} = A \cdot 2^{n+2} + B \cdot 3^{n+2}$$

Above three relation may be written as

Same

$y_{n+2} - 4 \cdot 2^n A - 9 \cdot 3^n B = 0$

$y_{n+1} - 2 \cdot 2^n A - 3 \cdot 3^n B = 0$

$y_n - 2^n A - 3^n B = 0$

Eliminating $A, 2^n, 3^n$

$$\begin{vmatrix} y_{n+2} & -4 & -9 \\ y_{n+1} & -2 & -3 \\ y_n & -1 & -1 \end{vmatrix} = 0$$

$y_{n+2}(2-3) - y_{n+1}(4-9) + y_n(12-18) = 0$

$y_{n+2} - 5y_{n+1} + 6y_n = 0$

Q.2) Given $y_x = ax^2 + bx$ find the corresponding recurrence relation

Sol^n)

$y_x = ax^2 + bx$ — same ^{n}

$y_{x+1} = a(x+1)^2 + b(x+1)$

$y_{x+2} = a(x+2)^2 + b(x+2)$

we may written as

$y_{x+2} - a(x+2)^2 - b(x+2) = 0$

$y_{x+1} - a(x+1)^2 - b(x+1) = 0$

$y_x - ax^2 - bx = 0$

L.R.C. (linearly recurrence relation)

$$\begin{vmatrix} y_{x+2} & (x+2)^2 & (x+2) \\ y_{x+1} & (x+1)^2 & (x+1) \\ y_x & x^2 & x \end{vmatrix} = 0$$

$$y_{x+2}\left[x(x+1)^2 - x^2(x+1)\right] - y_{x+1}\left[x(x+2)^2 - x^2(x+2)\right]$$
$$+ y_x\left[(x+2)^2(x+1) - (x+1)^2(x+2)\right] = 0$$

$$(x^2+x)\,y_{x+2} - 2x(x+2)\,y_{x+1} + (x+1)(x+2)\,y_x = 0$$

Q.) from the recurrence relation $y_n = \frac{a}{n} + b$

Sol^n

$$y_n = \frac{a}{n} + b$$

$$y_{n+1} = \frac{a}{n+1} + b$$

$$y_{n+2} = \frac{a}{n+2} + b$$

same

$$\begin{vmatrix} y_{n+2} & \frac{1}{n+2} & 1 \\ y_{n+1} & \frac{1}{n+1} & 1 \\ y_n & \frac{1}{n} & 1 \end{vmatrix} = 0$$

$$(n+2)\,y_{n+2} - 2(n+1)\,y_{n+1} + n\,y_n = 0$$

Linear Recurrence solⁿ with constant coefficients
general form of L. R. R.

$c_0 a_r + c_1 a_{r-1} + c_2 a_{r-2} + \cdots + c_k a_{r-k} = f(r)$ order of K

If $f(r)=0$ then called homogeneous linear recurrent relation with ~~constant coefficients~~

Sol^n of Recurrence Relation ⇒

A relation b/w the independent & dependent variable is said to be sol^n of recurrence relation. There are three types of sol^n

① General (Total sol^n) or sol^n ② Particular sol^n ③ Linear Recurrence sol^n.

→ $c_0 a_r + c_1 a_{r-1} + c_2 a_{r-2} + c_3 a_{r-3} + \cdots + c_k a_{r-k} = f(r)$

Homogeneous sol^n [Linear recurrence sol^n]

(I) Roots are real & distinct

$$a_r = c_1 m_1^r + c_2 m_2^r + \cdots + c_k m_k^r$$

(II) Some roots are Equal

$$(c_1 + c_2 r + c_3 r^2 + \cdots + c_p r^{p-1}) m_1^r$$

(III) Some roots are complex number

$$R^r (c_1 \cos r\theta + c_2 \sin r\theta)$$

or $c_1 R^r \cos(r\theta + c_2)$

(IV) Some roots are repeated complex roots

$$R^r [(c_1 + c_2 r) \cos r\theta + (c_3 + c_4 r) \sin r\theta]$$

DATE: / /

Q.1) Solve the following Recurrence relations

(1) $a_r - 6a_{r-1} + 8a_{r-2} = 0$ given $a_0 = 3$ & $a_1 = 2$

Sol^n Given recurrence relations

$$a_r - 6a_{r-1} + 8a_{r-2} = 0$$

Characteristic Equation is

$$m^2 - 6m + 8 = 0$$

$$(m-2)(m-4) = 0$$

$$\therefore \quad m = 2, 4$$

$$a_r = C_1 2^r + C_2 \cdot 4^r \quad \text{(A)}$$

Putting $r = 0$ we get

$$a_0 = C_1 + C_2 = 3 \quad \text{or} \quad C_1 + C_2 = 3 \quad \text{(1)}$$

Again $r = 1$

$$a_1 = 2C_1 + 4C_2 = 2 \quad \text{or} \quad 2C_1 + 4C_2 = 2$$

$$\text{or} \quad C_1 + 2C_2 = 1 \quad \text{(2)}$$

Solve Equation (1) & (2)

$$C_1 = 5, \quad C_2 = -2$$

Putting C_1 & C_2 in equation (A)

$$a_r = 5 \cdot 2^r - 2 \cdot 4^r$$

(II) $2a_r - 5a_{r-1} + 2a_{r-2} = 0$ given $a_0 = 0$, $a_1 = 1$

Sol^n we given recurrence relation is

$$2a_r - 5a_{r-1} + 2a_{r-2} = 0$$

characteristic Equation is

$$2m^2 - 5m + 2 = 0 \quad \text{——— (1)}$$

$$\therefore \ m = \frac{1}{2}, 2$$

general sol^n of equation (1) is given by

$$a_r = c_1 \left(\frac{1}{2}\right)^r + c_2 \cdot 2^r \quad \text{——— (2)}$$

put $r = 0$

$$a_0 = c_1 + c_2 \qquad \therefore \ c_1 + c_2 = 0 \quad \text{——— (3)}$$

put $r = 1$ $\quad a_1 = c_1 \left(\frac{1}{2}\right)^1 + c_2 (2)^1 \qquad \therefore \ \frac{c_1}{2} + 2c_2 = 1$

$$\text{or} \qquad c_1 + 4c_2 = 2 \quad \text{——— (4)}$$

solve (3) & (4) Equation

$$c_1 = \frac{-2}{3} \qquad c_2 = \frac{2}{3}$$

$$\therefore \ \boxed{a_r = -\frac{2}{3}\left(\frac{1}{2}\right)^r + \frac{2}{3}(2)^r}$$

(III) $a_r + a_{r-2} = 0$ ——— (1) given $a_0 = 0$, $a_1 = 1$

sol^n characteristic equation

$$m^2 + 1 = 0$$

$$m = \pm i \qquad \text{or} \qquad m = \left(\cos\frac{\pi}{2} \pm i \sin\frac{\pi}{2}\right)$$

$$\theta = \frac{\pi}{2} \qquad R = 1$$

$$a_r = c_1 R^r \cos(r\theta + c_2)$$

DATE : / /

$$a_r = c_1 \cos\left(r\frac{\pi}{2} + c_2\right) \quad \text{(2)}$$

put $r = 0$

$$a_0 = c_1 \cos c_2$$

$$0 = c_1 \cos c_2$$

$\therefore \cos c_2 = 0$

$\cos c_2 = \cos \pi/2$

$$\boxed{c_2 = \pi/2}$$

$r = 1$ put

$$a_1 = c_1 \cos\left(\frac{\pi}{2} + c_2\right)$$

$$c_1 \cos\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 1$$

$$c_1 \cos 180 = 1 \qquad c_1(-1) = 1 \qquad \boxed{\therefore c_1 = -1}$$

$$\therefore a_r = -\cos\left(r\frac{\pi}{2} + \frac{\pi}{2}\right)$$

or $a_r = \sin\left(r\frac{\pi}{2}\right)$

(IV) $a_r + 2a_{r-1} + 2a_{r-2} = 0$ — (1) given $a_0 = 0$, $a_1 = -1$

Solⁿ

characteristic equation

$$m^2 + 2m + 2 = 0$$

$$m = -1 \pm i$$

$$m = \sqrt{2}\left[-\frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}\right] = \sqrt{2}\left[\cos\frac{3\pi}{4} \pm i \sin\frac{3\pi}{4}\right]$$

$$\therefore R = \sqrt{2} \qquad \theta = \frac{3\pi}{4}$$

DATE: / /

$$a_r = (\sqrt{2})^r \left[C_1 \cos \frac{3\pi}{4} r \pm C_2 \sin \frac{3\pi}{4} r \right]$$

Put $r = 0$

$$a_0 = C_1 \qquad \therefore \boxed{C_1 = 0}$$

Put $r = 1$

$$a_1 = \sqrt{2}\left[0 + C_2 \sin \frac{3\pi}{4}\right] = -1 \qquad \therefore \boxed{C_2 = -1}$$

$$\therefore a_r = -(\sqrt{2})^r \sin\left(\frac{3\pi r}{4}\right)$$

(V) $a_r = a_{r-1} + a_{r-2}$ given $a_0 = 1$, $a_1 = 1$

sol^n $a_r - a_{r-1} - a_{r-2} = 0$ —— (1)

C.E. $m^2 - m - 1 = 0$

$$m = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$\therefore$ Homogenous sol^n (general) is

$$a_r = C_1 \left[\frac{1+\sqrt{5}}{2}\right]^r + C_2 \left[\frac{1-\sqrt{5}}{2}\right]^r$$

Put $r = 0$ & 1

$$a_0 = C_1 + C_2 \qquad \therefore C_1 + C_2 = 1 \text{ —— (2)}$$

$$a_1 = C_1\left(\frac{1+\sqrt{5}}{2}\right) + C_2\left(\frac{1-\sqrt{5}}{2}\right) \qquad \therefore \left(\frac{1+\sqrt{5}}{2}\right)C_1 + \left(\frac{1-\sqrt{5}}{2}\right)C_2 \text{ —— (3)}$$

Sol^n (2) & (3)

$$C_1 = \frac{\sqrt{5}+1}{2\sqrt{5}} \qquad C_2 = \frac{\sqrt{5}-1}{2\sqrt{5}}$$

$$\therefore a_r = \left(\frac{\sqrt{5}+1}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^r + \left(\frac{\sqrt{5}-1}{2\sqrt{2}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^r$$

or

$$a_r = \frac{1}{2^{r+1} \cdot \sqrt{5}}\left[(\sqrt{5}+1)^{r+1} + (-1)^r(\sqrt{5}-1)^{r+1}\right]$$

(General solution)
or
Total sol^n ⇒

A non-homogenous linear recurrence relation of order k with constant coefficients

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \cdots + C_k a_{r-k} = f(r)$$

Where $C_0, C_1, C_2 \cdots C_k$ are constants & both C_0 & C_k are non zero. $f(r)$ is a function of r.

Total sol^n consists of two parts
1. homogenous sol^n
2. Particular sol^n

Total sol^n

$$\boxed{\begin{aligned} a_n &= \text{homogenous} + \text{Particular sol}^n. \\ a_n &= a_n^{(h)} + a_n^{(P)} \end{aligned}}$$

Particular sol^n

There is no general procedure for determining the particular sol^n of a recurrence relation. There are two general method for particular sol^n.

1. Method of inspection or Undetermined coefficients
2. operator method.

	$f(r)$ Terms	Trial sol^n a_r
1	b^r	$A \cdot b^r$
2	degree k in r	$A_0 + A_1 r + A_2 r^2 + \cdots + A_k r^k$
3	b^r & degree k in r	$b^r \cdot (A_0 + A_1 r + A_2 r^2 + \cdots + A_k r^k)$
4	$\sin br$ or $\cos br$	$A \sin br + B \cos br$
5	$a^r \sin br$	$a^r (A \sin br + B \cos br)$
6	$a^r \cos br$	"

Where $a, b, A, B, A_0, A_1, A_2 \cdots A_k$ are unknown coefficients

PAGE NO.
DATE: / /

Q.1) solve $a_r - 5a_{r-1} + 6a_{r-2} = 5^r$

Sol^n

$$a_r - 5a_{r-1} + 6a_{r-2} = 5^r \quad \text{(1)}$$

C.E. $m^2 - 5m + 6 = 0$

$(m-2)(m-3) = 0$

$m = 2, 3$

Homogenous sol^n $(a_r^{(h)})$ is given by

$$a_r^{(h)} = c_1 \cdot 2^r + c_2 \cdot 3^r \quad \text{(2)}$$

Particular sol^n corresponding to the term on RHS of (1) is $A \cdot 5^r$

$$a_r^{(p)} = A \cdot 5^r \quad \text{(3)}$$

substiting in (1) of (3)

$A \cdot 5^r - 5 \cdot A \cdot 5^{r-1} + 6 \cdot A 5^{r-2} = 5^r$

$A[6 \cdot 5^{r-2}] = 5^r \qquad A = \frac{25}{6}$

$\frac{\cancel{5} \cdot A 5^r}{\cancel{5}}$

$A 5^r$

$\frac{5^r}{5^2} = \frac{5^r}{25}$

Putting A in (3)

$$a_r^{(p)} = \frac{25}{6} \cdot 5^r = \frac{1}{6} \cdot 5^{r+2}$$

Total sol^n

$$\boxed{a_r^{(T)} = a_r^{(h)} + a_r^{(p)}}$$

$$\boxed{a_r = c_1 \cdot 2^r + c_2 \cdot 3^r + \frac{1}{6} \cdot 5^{r+2}}$$

Ans

Q.2) Determine the Particular solⁿ for the difference equation

$$a_r - 2a_{r-1} = f(r) \quad \text{where } f(r) = 7r$$

solⁿ

$$a_r - 2a_{r-1} = 7r \quad \text{--- (1)}$$

Particular solⁿ

$$a_r^{(P)} = A_0 + A_1 r \quad \text{--- (2)}$$

Substituting (2) in (1) we get

$$(A_0 + A_1 r) - 2[A_0 + A_1(r-1)] = 7r$$

$$(-A_0 + 2A_1) + (-A_1 r) = 7r \quad \text{--- (3)}$$

comparing two sides of (3)

$$-A_1 = 7, \quad -A_0 + 2A_1 = 0 \quad \therefore$$

$$\therefore A_1 = -7 \qquad A_0 = -14$$

Putting for A_0 & A_1 in (2)

$$a_r^{(P)} = -14 - 7r$$

Q.3) Solve the recurrence relation

$$a_r - 5a_{r-1} + 6a_{r-2} = 2 + r, \quad r \geq 2$$

with boundary conditions $a_0 = 1$, $a_1 = 1$

solⁿ The given equation is

$$a_r - 5a_{r-1} + 6a_{r-2} = 2 + r \quad \text{--- (1)}$$

Characteristic equation is

$$m^2 - 5m + 6 = 0$$

$$\therefore m = 2, 3$$

DATE : / /

$$a_r^{(h)} = c_1 \cdot 2^r + c_2 \cdot 3^r \quad \text{——— (2)}$$

The particular solⁿ corresponding to the term $2+r$ on RHS of (1) is $A_0 + A_1 r$

$$a_r^{(p)} = A_0 + A_1 r \quad \text{——— (3)}$$

substituting (3) in (1) we get

$$(A_0 + A_1 r) - 5\left[A_0 + A_1(r-1)\right] + 6\left[A_0 + A_1(r-2)\right] = 2+r$$

$$(2A_0 - 7A_1) + 2A_1 r = 2 + r \quad \text{——— (4)}$$

comparing two sides

$$2A_0 - 7A_1 = 2 \qquad \therefore \; A_0 = \frac{11}{4}, \; A_1 = \frac{1}{2}$$

$$2A_1 = 1$$

Putting for A_0 & A_1 in (3) we get

$$a_r^{(p)} = \frac{11}{4} + \frac{r}{2}$$

$\therefore$ Total solⁿ $\quad a_r = a_r^{(h)} + a_r^{(p)}$

$$a_r = c_1 \cdot 2^r + c_2 \cdot 3^r + \frac{11}{4} + \frac{r}{2} \quad \text{——— (5)}$$

Now putting $r=0$ & 1 using boundary conditions in (5) we get

$$r=0 \Rightarrow 1 = c_1 + c_2 + \frac{11}{4} \qquad \therefore \; c_1 + c_2 = -\frac{7}{4} \quad \text{——— (6)}$$

$$r=1 \Rightarrow 1 = 2c_1 + 3c_2 + \frac{11}{4} + \frac{1}{2} \qquad \therefore \; 2c_1 + 3c_2 = -\frac{9}{4} \quad \text{——— (7)}$$

solve from (6) & (7)

$$c_1 = -3, \quad c_2 = \frac{5}{4}$$

Putting c_1 & c_2 in (5) required solⁿ is

$$a_r = -3 \cdot 2^r + \frac{5}{4} \cdot 3^r + \frac{11}{4} + \frac{r}{2}$$

GIRIRAJ

Q.4) Solve the recurrence relation ~~$a_r + 6a_{r-1}$~~

$a_r + 6a_{r-1} + 9a_{r-2} = 3$ given that

boundary condition $a_0 = 0$ & $a_1 = 1$

Sol^n ⇒

$$a_r + 6a_{r-1} + 9a_{r-2} = 3 \quad \text{——— (1)}$$

characteristic equation is

$$m^2 + 6m + 9 = 0$$

$$\therefore m = -3, -3$$

$$a_r^{(h)} = (c_1 + c_2 r)(-3)^r \quad \text{——— (2)}$$

Let particular sol^n corresponding to the term of RHS of (1)

$$A_0 r^2 + A_1 r + A_2$$

$$a_r^{(P)} = A_0 r^2 + A_1 r + A_2 \quad \text{——— (3)}$$

Now substituting (3) in (1)

$$(A_0 r^2 + A_1 r + A_2) + 6[A_0(r-1)^2 + A_1(r-1) + A_2]$$
$$+ 9[A_0(r-2)^2 + A_1(r-2) + A_2] = 3$$

$$16A_0 r^2 + (16A_1 - 48A_0)r + (16A_2 - 24A_1 + 42A_0) = 3$$

comparing side^s (4) ——— (4)

$$16A_0 = 0 \qquad \therefore A_0 = 0$$

$$16A_1 - 48A_0 = 0 \qquad \therefore A_1 = 0$$

$$16A_2 - 24A_1 + 42A_0 = 3 \qquad \therefore A_2 = 3/16$$

from (3)

$$a_r^{(P)} = \frac{3}{16}$$

DATE : / /

$\therefore$ Total sol^n of ①

$$a_r = a_r^{(h)} + a_r^{(p)}$$

$$a_r = (c_1 + c_2 r)(-3)^r + \frac{3}{16} \quad \text{———} ⑤$$

when putting $r = 0, 1$ in equation ⑤

$r=0 \Rightarrow \quad a_0 = 0 \qquad \therefore c_1 = -\frac{3}{16}$

$r=1 \Rightarrow \quad a_1 = 1 \qquad \therefore c_2 = \frac{1}{12}$

putting c_1 & c_2 in ⑤

$$a_r = \left[-\frac{3}{16} + \frac{r}{12}\right](-3)^r + \frac{3}{16}$$

Q.5)) ~~The given~~ solve $y_{n+2} - 4y_{n+1} + 4y_n = 3n + 2^n$

sol^n Recurrence relation

$$(E^2 - 4E + 4) y_n = 3n + 2^n \quad \text{———} ①$$

characteristic equation $m^2 - 4m + 4 = 0 \qquad \therefore m = 2, 2$

So Homogeneous sol^n $y_n^{(h)} = (c_1 + c_2 n) 2^n$

find particular sol^n

$$y_n^{(P)} = A_1 + A_2 n + B n^2 \cdot 2^n \quad \text{———} ②$$

$$(E^2 - 4E + 4)(A_1 + A_2 n + B n^2 \cdot 2^n) = 3n + 2^n$$

$$[A_1 + A_2(n+2) + B(n+2)^2 \cdot 2^{n+2}] - 4[A_1 + A_2(n+1) + B(n+1)^2 \cdot 2^{n+1}]$$
$$+ 4[A_1 + A_2 n + B n^2 \cdot 2^n] = 3n + 2^n$$

$$(A_1 - 2A_2) + A_2 n + 8B \cdot 2^n = 3n + 2^n$$

compair $\quad 8B = 1 \quad \therefore B = \frac{1}{8} \quad A_2 = 3, \quad A_1 - 2A_2 = 0 \quad \therefore A_1 = 6$

from ② $\quad y_n^{(P)} = 6 + 3n + \frac{n^2}{8} \cdot 2^n$

Total sol^n $y_n = (c_1 + c_2 n) 2^n + 6 + 3n + (\frac{1}{8}) n^2 \cdot 2^n \quad$ Ans

Bool's

Find

Operator method [particular sol^n]

Linear recurrence relation of n^{th} order is given by

$$y_{h+n} - b_1 y_{h+n-1} + \cdots + b_n y_h = F(h), \quad b_n \neq 0$$

$$(E^n + b_1 E^{n-1} + \cdots + b_n) y_h = F(h)$$

$$f(E) \cdot y_h = F(h) \quad \text{--- (1)}$$

$$f(E) = E^n + b_1 E^{n-1} + \cdots + b_n$$

so, Particular sol^n of (1) is $= \dfrac{1}{f(E)} F(h)$

If $F(h) = b^h$ then Particular sol^n is

$$\frac{1}{f(E)} b^h = \frac{b^n}{f(b)} \quad \text{but } f(b) \neq 0$$

$$f(E) \cdot b^h = (E^n + b_1 E^{n-1} + \cdots + b_n) b^h$$

$$= E^n b^n + b_1 E^{n-1} b^h + \cdots + b_n b^h$$

$$= b^{h+n} + b_1 b^{h+n-1} + \cdots + b_n b^h$$

$$= (b^n + b_1 b^{n-1} + \cdots + b_n) b^h$$

$$= f(b) \cdot b^h$$

Thus by operator inversion we get

$$\frac{1}{f(E)} \cdot b^h = \frac{b^h}{f(b)} \quad \text{provided } f(b) \neq 0$$

Note This method is fails when b is a root of auxiliary equation,

DATE: / /

Q.1) Solve $y_{h+1} - 3y_h = 2^h$

Sol^n Given difference equation may be written as

$$(E-3)\, y_h = 2^h \quad \text{--- (1)}$$

$\therefore$ Characteristic equation $m-3=0$ $\therefore m=3$

$\therefore$ homogenous sol^n $= C\cdot 3^h$

Particular sol^n $= \frac{1}{E-3}\cdot 2^h$

$= \frac{1}{2-3}\cdot 2^h$ when $E=2$

$= -2^h$

$\therefore$ $y_h = C\cdot 3^h - 2^h$ (Total sol^n)

Q.2) Solve $y_{x+2} - 4y_x = 10\cdot 3^x$

Sol^n Given difference $(E^2-4)\, y_x = 10\cdot 3^x$

characteristic equation $m^2-4=0$ $\therefore m=-2, 2$

Homogenous sol^n $= C_1\cdot(-2)^x + C_2(2)^x$ --- (1)

$\therefore$ Particular sol^n $= \frac{1}{E^2-4}\cdot 10\cdot 3^x$

$= \frac{10\cdot 3^x}{(3)^2-4} = 2\cdot 3^x$ --- (2)

Total (general) sol^n equation (1) + (2)

$$y_x = C_1(-2)^x + C_2\cdot 2^x + 2\cdot 3^x$$

Recurrence relation or difference Equation

PAGE NO.
DATE: / /

Q.3) Solve $y_{n+2} + y_{n+1} + y_n = (-1)^n$

$\underline{\text{sol}^n}$

$$(E^2+E+1)\, y_n = (-1)^n$$

Character. Equation $m^2+m+1=0$

$$\therefore m = -\frac{1}{2} \pm \frac{\sqrt{3}\, i}{2}$$

In Polar forms

$$-\frac{1}{2} \pm \frac{\sqrt{3}\, i}{2} = r\cos\theta \pm i r \sin\theta$$

$$r\cos\theta = -\frac{1}{2} \quad , \quad r\sin\theta = \frac{\sqrt{3}}{2}$$

$$r = 1 \quad , \quad \theta = \tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}$$

homogenous $\text{sol}^n = A\cos\frac{2\pi n}{3} + B\sin\frac{2\pi n}{3}$ —— (1)

particular $\text{sol}^n = \frac{1}{E^2+E+1}(-1)^n$

$$= \frac{(-1)^n}{(-1)^2+(-1)+1} = (-1)^n$$

Total sol^n (general sol^n) = homogenous + Particular sol^n

$$y_n = A\cos\frac{2\pi n}{3} + B\sin\frac{2\pi n}{3} + (-1)^n$$

Particular sol^n when $f(n) = \sin bn$ or $\cos bn$

Particular $\text{sol}^n = \frac{1}{f(E)}\sin bn$ or $\frac{1}{f(E)}\cos bn$

$$\sin bn = \frac{e^{ibn} - e^{-ibn}}{2i} \qquad \cos bn = \frac{e^{ibn} + e^{-ibn}}{2}$$

We know that $e^{ibn} = \cos bn + i\sin bn$

Particular sol^n $= \frac{1}{f(E)} \sin bh$

Imaginary part in $\frac{1}{f(E)} e^{ibh}$

(ii) Particular sol^n $= \frac{1}{f(E)} \cos bh$

Real part $\frac{1}{f(E)} e^{ibh}$

x

Q.1) Solve $y_{n+2} + a^2 y_n = \sin ah$

sol^n

Difference equation may be written as

$(E^2 + a^2) y_n = \sin ah$

characteristic Equation (C.E.) $m^2 + a^2 = 0$

$\therefore m = \pm ia = a\left(\cos\frac{\pi}{2} \pm i\sin\frac{\pi}{2}\right)$

$\therefore$ homogenous sol^n $= a^h \left(A\cos h\frac{\pi}{2} + B\sin h\frac{\pi}{2}\right)$

Particular sol^n $= \frac{1}{E^2+a^2} \cdot \sin ah$

$=$ Imaginary part $\frac{1}{E^2+a^2} e^{iah}$

$=$ Imaginary $\frac{1}{E^2+a^2} \cdot (e^{ia})^h$

$=$ Imaginary $\frac{e^{iah}}{(e^{ia})^2 + a^2}$

$=$ Imaginary $\frac{e^{iah}}{(e^{2ai} + a^2)} \times \frac{e^{-2ai} + a^2}{e^{-2ai} + a^2}$

$$= \text{Imaginary} \frac{e^{ia(h-2)} + a^2 e^{iah}}{1+a^2(e^{2ai}+e^{-2ai})+a^4}$$

$$= \text{Imaginary} \frac{\cos[a(h-2)] + i\sin[a(h-2)] + a^2[\cos ah + i\sin ah]}{1+2a^2\cos 2a + a^4}$$

$$= \frac{\sin[a(h-2)] + a^2\sin ah}{1+2a^2\cos 2a + a^4}$$

$\therefore$ General sol^n (Total Sol^n)

$$y_h = a^h\left[A\cos\frac{h\pi}{2} + B\sin\frac{h\pi}{2}\right] + \frac{a^2\sin ah + \sin[a(h-2)]}{1+2a^2\cos 2a + a^4}$$

Q.2) Solve $y_{h+2} + a^2 y_h = \cos ah$

Sol^n similarly Q.1

Homogenous $sol^n = a^h\left[A\cos\frac{h\pi}{2} + B\sin\frac{h\pi}{2}\right]$

Particular $sol^n = \frac{1}{E^2+a^2}\cos ah$

$$= \text{Real Part } \frac{1}{E^2+a^2} e^{iah}$$

$$= \text{Real } \frac{e^{ia(h-2)} + a^2 e^{iah}}{1+2a^2\cos 2a + a^4}$$

$$= \frac{\cos[a(h-2)] + a^2\cos ah}{1+2a^2\cos 2a + a^4}$$

General sol^n

$$y_h = a^h\left[A\cos\frac{h\pi}{2} + B\sin\frac{h\pi}{2}\right] + \frac{a^2\cos ah + \cos[a(h-2)]}{1+2a^2\cos 2a + a^4}$$

DATE: / /

Particular soln when $F(h) = P(h) \Rightarrow$ $P(h)$ being Polynomial of degree n in h

Particular soln $= \frac{1}{f(E)} \cdot P(h)$

$$= \frac{1}{f(1+\Delta)} \cdot P(h) \qquad \because E = 1+\Delta$$

$$= (a_0 + a_1\Delta + a_2\Delta^2 + \cdots + a_n\Delta^n + \cdots)\, P(h)$$

$$= (a_0 + a_1\Delta + a_2\Delta^2 + \cdots + a_n\Delta^n)\, P(h)$$

Q.1) solve $y_{h+2} - 2y_{h+1} + y_h = 3h+4$

soln

$$(E^2 - 2E + 1)\, y_h = 3h+4$$

$$(E-1)^2 y_h = 3h+4$$

$\therefore$ characteristic equation $(m-1)^2 = 0$ $\therefore m = 1, 1$

$\therefore$ Homogenous soln $= (c_1 + c_2 h)\cdot 1^h = c_1 + c_2 h$

Particular soln $= \frac{1}{(E-1)^2}\cdot(3h+4) = \frac{1}{\Delta^2}\cdot 3h+4$

$$= \Delta^{-2}\left[3h^{(1)} + 4h^{(0)}\right]$$

$$= \frac{3h^{(3)}}{2\cdot 3} + \frac{4h^{(2)}}{1\cdot 2} = \frac{1}{2}(h)^3 + 2h^{(2)}$$

$$= \frac{h(h-1)(h-2) + 2h(h-1)}{2}$$

$$= \frac{h(h-1)(h+2)}{2}$$

$\therefore$ general soln $y_h = c_1 + c_2 h + \frac{1}{2} h(h-1)(h+2)$

Particular sol^n when $F(h) = b^h \cdot P(h) \Rightarrow$

Q.1) Solve $y_{x+1} - 3y_x = 3^x \cdot x^2$

Sol^n

$$y_{x+1} - 3y_x = 3^x \cdot x^2$$

$$(E-3)\, y_x = 3^x \cdot x^2$$

C.E. $\quad m - 3 = 0 \quad \therefore m = 3$

$\therefore$ Homogenous $sol^n = C \cdot 3^x$

Particular $sol^n = \frac{1}{E-3} \cdot 3^x \cdot x^2$

$$= 3^x \cdot \frac{1}{3E-3} \cdot x^2 = \frac{3^x}{3} \cdot \frac{1}{(E-1)} \cdot x^2$$

$$= 3^{x-1} \cdot \frac{1}{\Delta}\left[x^{(2)} + x^{(1)}\right] = 3^{x-1}\left[\frac{x^{(3)}}{3} + \frac{x^{(2)}}{2}\right]$$

$$= \frac{3^{x-1}}{6}\left[2x(x-1)(x-2) + 3x(x-1)\right]$$

$$= \frac{3^x}{18}\left[x(x-1)(2x-1)\right]$$

General sol^n is given by

$$y_x = C \cdot 3^x + \frac{3^x}{18} \cdot x(x-1)(2x-1)$$

Q.2) Solve $y_{h+2} + 4y_{h+1} - 12y_h = 2^h (h-1)$

Ans

Generating functions ⟶

The method of generating functions is a powerful method to solve the difference equations (recurrence relations)

If $y_0, y_1, y_2 \ldots$ is a sequence of a real numbers the function Y defined for some interval of real numbers containing zero whose value at t is given by the series

$$y_0 t^0 + y_1 t^1 + y_2 t^2$$

$$Y(t) = \sum_{n=0}^{\infty} y_n t^n = y_0 + y_1 t + y_2 t^2 + \cdots + y_n t^n + \cdots$$

is called the generating function of the sequence $\{y_t\}$

Q.1) Apply the generating function technique to solve the initial value problem

$y_{n+1} - 2y_n = 0$ with $y_0 = 1$

Sol^n

Given $y_{n+1} - 2y_n = 0$ ——— (1)

Consider the generating function $Y(t)$ given by

$$Y(t) = \sum_{n=0}^{\infty} y_n t^n = y_0 + y_1 t + y_2 t^2 + \cdots \text{ ——— (2)}$$

multiply the given difference equation (1) by t^n & summing from $n=0$ to $n=\infty$ we get

$$\sum_{n=0}^{\infty} y_{n+1} t^n - 2\sum_{n=0}^{\infty} y_n t^n = 0$$

$y_{n+0} t^n$

$y_n t$

$$(y_1 + y_2 t + y_3 t^2 + \cdots) - 2Y(t) = 0$$

$$\frac{1}{t}(y_1 t + y_2 t^2 + y_3 t^3 + \cdots) - 2Y(t) = 0$$

$$\frac{(Y(t) - y_0)}{t} - 2Y(t) = 0$$

$$Y(t) - y_0 - 2tY(t) = 0$$

$$(1-2t)Y(t) = y_0$$

$(\because y_0 = 1$ put$)$

$$Y(t) = \frac{y_0}{(1-2t)} = (1-2t)^{-1}$$

$$\therefore \sum_{n=0}^{\infty} y_n t^n = 1 + 2t + (2t)^2 + \cdots + (2t)^n + \cdots$$

$\therefore$ equating coefficient of t^n

$y_n = 2^n$

DATE : / /

Q.2) Solve $y_{n+2} - 7y_{n+1} + 10y_n = 0$ with $y_0 = 0$, $y_1 = 3$ by the method of generating function.

Sol^n we have

$$y_{n+2} - 7y_{n+1} + 10y_n = 0 \quad \text{—— (1)}$$

with $y_0 = 0$, $y_1 = 3$

consider the generating function

$$Y(t) = \sum_{n=0}^{\infty} y_n t^n = y_0 + y_1 t + y_2 t^2 + \cdots \quad \text{—— (2)}$$

multiplying (1) by t^n & summing from $n=0$ to $n=\infty$

we have

$$\sum_{n=0}^{\infty} y_{n+2} t^n - 7\sum_{n=0}^{\infty} y_{n+1} t^n + 10\sum_{n=1}^{\infty} y_n t^n = 0$$

$$(y_2 + y_3 t + \cdots) - 7(y_1 + y_2 t + \cdots) + 10Y(t) = 0$$

$$\frac{Y(t) - y_0 - y_1 t}{t^2} - 7\,\frac{Y(t) - Y_0}{t} + 10Y(t) = 0$$

put $y_0 = 0$, $y_1 = 3$

$$\frac{Y(t) - 3t}{t^2} - \frac{7Y(t)}{t} + 10Y(t) = 0$$

$$Y(t) - 3t - 7tY(t) + 10t^2Y(t) = 0$$

$$(1 - 7t + 10t^2)\,Y(t) = 3t$$

$$Y(t) = \frac{3t}{1 - 7t + 10t^2}$$

P.T.O

$$\frac{1-2t-1+5t}{2t}=3t$$

$$y(t)=\frac{3t}{(1-2t)(1-5t)}$$

Breaking into partial fraction

$$y(t)=\frac{1}{(1-5t)}-\frac{1}{(1-2t)}$$

$$\sum_{n=0}^{\infty} y_n t^n = (1-5t)^{-1}-(1-2t)^{-1}$$

$$=\left[1+5t+(5t)^2+\cdots+(5t)^n+\cdots\right]-\left[1+2t+(2t)^2+\cdots+(2t)^n+\cdots\right]$$

Equating coefficients of t^n

$$y_n = 5^n - 2^n$$

Q.) Solve $y_{n+2}-y_{n+1}-y_n=0$ with $y_0=0$, $y_1=1$ by the method of generating function.

* **Partitions of Integers** ⇒ A Partition of a positive integer n is a multiset of positive integers that sum of n. It is denoted the number of partitions of n by P_n.

P_n = No. of ways of representing an integer as addition of positive integer

Example $5 \Rightarrow \{5+0\}$
$5 \Rightarrow \{4+1\}$
$5 \Rightarrow \{3+2\}$
$5 \Rightarrow \{3+1+1\}$
$5 \Rightarrow \{2+2+1\}$
$5 \Rightarrow \{2+1+1+1\}$
$5 \Rightarrow \{1+1+1+1+1\}$

← increasing order में लिखें

$\therefore P_5 = 7$ or $P_5(n) = 7$

$\{\prod_{K=1}^{\infty}\}$

$$(1+x+x^2+x^3+\cdots)(1+x^2+x^4+x^6+\cdots)(1+x^3+x^6+\cdots$$
$$\cdots(1+x^K+x^{2K}+x^{3K}+\cdots)\cdots = \sum_{i=0}^{\infty} x^{iK}$$

Q.) Find the Partition P_8?

Sol^n

$$(1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8)(1+x^2+x^4+x^6+x^8)(1+x^3+x^6)(1+x^4+x^8)(1+x^5)(1+x^6)(1+x^7)(1+x^8)$$

$$= 1+x+2x^2+3x^3+5x^4+7x^5+11x^6+15x^7+22x^8+\cdots$$
$$\cdots + x^{56}$$

$P_8 = 22$

$8 \Rightarrow \{8+0\}$
$8 \Rightarrow \{7+1\}$
$8 \Rightarrow \{6+1+1\}$
$8 = \{5+1+1+1\}$
$8 \Rightarrow \{4+1+1+1+1\}$
$8 = \{3+1+1+1+1+1\}$
$8 = \{2+1+1+1+1+1+1\}$
$8 = \{1+1+1+1+1+1+1+1\}$
$8 = \{6+2\}$
$8 = \{4+2+2\}$

$8 = \{4+2+1+1\}$
$8 = \{3+1+2+1+1\}$
$8 = \{2+2+1+1+1+1\}$
$8 = \{3+3+2\}$
$8 = \{2+2+2+1+1\}$
$8 = \{2+2+2+2\}$
$8 = \{2+2+3+1\}$
$8 = \{4+3+1\}$
$8 = \{5+3\}$
$8 = \{5+2+1\}$

$8 = \{4+4\}$
$8 = \{3+3+1+1\}$

$\boxed{\therefore P_8 = 22}$

(OR)

An integer partitions is a way of writting a given Positive integer (x) as a sum of other positive integers

(OR)

It is the number ~~th~~ of ways that a positive integer can be written using positive integers, such that the sum adds up to the origional number (x).

* **Euler's theorem** ⇒ The no. of ways to partion an integer as a sum of unique integers is equal to the no. of ways to partition an integer as a sum of odd integers

Example

Partitions of 5:-	Partions of 5 as a sum of unique integer	Partitions of 5 as sum of odd integers
5	5	5
4+1	4+1	3+1+1
3+2	3+2	1+1+1+1+1
3+1+1		
2+2+1		
2+1+1+1		
1+1+1+1+1		

Polynomial Representations -

$$(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\cdots$$

$$(1+x)(1+x) = x^2+2x+1$$

Exponential Generating function ⇒

A normal Generating function for a sequence (a_r) is

$$\sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_r x^r + \cdots$$

$$= A(x) \text{ or } f(x)$$

$(a_r) = (1)$ A sequence of $(1, 1, 1, \ldots)$

$= (a_0, a_1, a_2, a_3, \ldots)$ $(\because |x|<1)$

$\therefore \sum_{r=0}^{\infty} a_r x^r = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x} = f(x)$

(EGF) $\Rightarrow$ Exponential generating function:- The exponential generating function for the sequence (a_r) is

$$a_0 + a_1 \frac{x}{\lfloor 1} + a_2 \frac{x^2}{\lfloor 2} + \ldots + a_r \frac{x^r}{\lfloor r} + \ldots$$

$$= \sum_{r=0}^{\infty} a_r \frac{x^r}{\lfloor r}$$ Here a_r is cofficient of $\frac{x^r}{\lfloor r}$

Example $\Rightarrow$ Consider $(1, 1, 1, 1, \ldots) \Rightarrow (a_0, a_1, a_2, a_3, \ldots)$

it's expontial generating function

$$1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \ldots + \frac{x^r}{\lfloor r} \ldots = \sum_{r=0}^{\infty} \frac{x^r}{\lfloor r} = e^x$$

it's EGF

Example $\Rightarrow$ consider $(\lfloor 0, \lfloor 1, \lfloor 2, \lfloor 3, \ldots \lfloor r, \ldots)$

$$\lfloor 0 + \lfloor 1 \frac{x}{\lfloor 1} + \lfloor 2 \frac{x^2}{\lfloor 2} + \ldots \lfloor r \frac{x^r}{\lfloor r} + \ldots = \sum_{r=0}^{\infty} \frac{x^r}{\lfloor r} = \frac{1}{1-x}$$

$\therefore$ Exponential generating function for $(a_r) = \lfloor r$ is $\frac{1}{1-x}$

Example $\Rightarrow$ consider $(1, k, k^2, k^3, \ldots k^r, \ldots)$ $k \neq 0$

then exponential generating function is e^{kx}

$$1 + k \frac{x}{\lfloor 1} + k^2 \frac{x^2}{\lfloor 2} + k^3 \frac{x^3}{\lfloor 3} + \ldots + k^r \frac{x^r}{\lfloor r} + \ldots$$

$$1 + \frac{kx}{\lfloor 1} + \frac{(kx)^2}{\lfloor 2} + \frac{(kx)^3}{\lfloor 3} + \ldots + \frac{(kx)^r}{\lfloor r} + \ldots$$

$$= \sum_{r=0}^{\infty} \frac{(kx)^r}{\lfloor r} = e^{kx}$$

Q.1) Show that the exponential generating function for the sequence $(1, 1\cdot3, 1\cdot3\cdot5, 1\cdot3\cdot5\cdot7, \ldots)$ is $(1-2x)^{-3/2}$

Sol^n To show the coefficient of

$\frac{x^r}{\lfloor r}$ is $(1\cdot3\cdot5\cdot7\cdot \ldots (2r+1))$

or coefficient of x^r in $(1-2x)^{-3/2}$ is

$$\left(\frac{1\cdot3\cdot5 \ldots (2r+1)}{\lfloor r}\right)$$

$\because$ coefficient of x^r in $(1-2x)^{-3/2}$

$$(1-2x)^{-3/2} = \sum_{i=0}^{\infty} \binom{-3/2}{i} (-2x)^i$$

$\therefore$ coefficient of x^r is $\binom{-3/2}{r}(-2)^r$

$$= \frac{(-2)^r\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)\left(-\frac{3}{2}-2\right)\cdots\left(-\frac{3}{2}-r+1\right)}{\lfloor r} = (-2)^r\left(-\frac{1}{2}\right)^r$$

$$= \cancel{(-2)^r} \times \cancel{\left(-\frac{1}{2}\right)} \frac{3\cdot5\cdot7 \cdots (2r+1)}{\lfloor r}$$

$\therefore$ coefficient of $\frac{x^r}{\lfloor r}$ in $(1-2x)^{-3/2}$ is $3\cdot5\cdot7 \cdots (2r+1)$

Q.7(a)