PROBLEM SET

1. Find the coefficient of $x^5y^2z^2$ in $(x + y + z)^9$.

9!
5! 2! 2!

ANSWER

1. 756.

§ 5.30. RECURRENCE RELATIONS

Introduction. Suppose we ask from Mohan about his income. Mohan tells us that his income in Rs. 5,000 more than the income of Sohan. If we ask the income of Sohan, Mohan tells us that the income of Sohan is Rs. 2,000 more than the income of Ravi. If we ask the income of Ravi, he tells us that the income of Ravi is Rs. 1,000 more than the income of Sheela. If we ask the income of Sheela, he tells us that the income of Sheela is Rs. 12,000. Thus we have no difficulty to find the income of Mohan, that the income of Mohan is Rs. 20,000.

Our aim to give the above example is that we are trying to express a (general) term of an (unknown) sequence as a (known) function of its earlier terms. *The essential idea in a recurrence relation is that it expresses a general term of an unknown sequence as a known function of its earlier terms.*

Thus a recurrence relation can be defined as "Let $\{a_n\}$ be a sequence. A recurrence relation of $\{a_n\}$ is a relation that expresses a_n in terms of one or more of the earlier terms of $\{a_n\}$, namely $a_0, a_1, a_2, \ldots, a_{n-1}$ for all integers n with $n \geq n_0$ where n_0 is a non-negative integer. A sequence is said to be a solution of a recurrence relation if the terms of this sequence satisfy the recurrence relation."

Recurrence relations can be written in terms of the differences between the consecutive terms of a sequence, and hence *recurrence relations are also called* **difference equations.** Thus the terms *recurrence relations* and *difference equations* are used interchangeably. Now we may define a recurrence relation as follows :

Definition. A relation which involves an independent variable x, a dependent variable y and one or more than one differences $\Delta y, \Delta^2 y, \Delta^3 y, \ldots$ is called a recurrence relation (or difference equation).

Thus, a relation of the form

$$F(x, y, \Delta y, \Delta^2 y, \ldots, \Delta^n y) = 0$$

represents a recurrence relation (or difference equation).

From the calculus of finite differences, we know that $\Delta^n y_x$ is written as y_{x+n}.

Some examples, of recurrence relations (difference equations) are :

(i) $y_{x+2} - 5y_{x+1} + 6y_x = 0$, where x is independent variable.

(ii) $y_{x+1} - 3y_x = 3^x$

(iii) $(h+2)\, y_{h+2} - 2\,(h+1)\, y_{h+1} + hy_h = 0$, where h is independent variable.

(iv) $\Delta^2 y_x - 5\,\Delta y_x + 6y_x = 0$

Order of a Recurrence Relation (Difference Equation) :

The order of a recurrence relation (difference equation) is defined to be the difference between the highest and lowest subscripts (arguments) of dependent variable y occuring in the recurrence relation divided by the unit of difference interval.

For example, the orders of recurrence relations in (i) is $x + 2 - x$ *i.e.*, 2 in (ii) is $x + 1 - x$ *i.e.*, 1; in (iii) is $h + 2 - h$ *i.e.*, 2; and in (iv) is 2.

Degree of a Recurrence Relation (Difference Equation) :

The degree of a recurrence relation (difference equation) is defined to be the highest power of y_x.

For example (i) The recurrence relation $4y_x^3 - 6y_{x-1}^2 + 3y_{x-2} = 0$ is of degree 3 since the highest power of y_x is 3.

(ii) The recurrence relation $a_n - a_{n-1} = 2^n$ is of degree 1 since highest power of a_n is one.

§ 5.31. TRANSFORMATION OF RECURRENCE RELATION (DIFFERENCE EQUATION) IN SUBSCRIPT NOTATION

We know that

$$E \equiv \Delta + 1 \quad i.e., \quad \Delta \equiv E - 1.$$

$$\therefore \quad \Delta y_h = (E-1)\, y_h = Ey_h - y_h \quad \text{...(1)}$$

$$\Delta^2 y_h = E^2 y_h - 2Ey_h + y_h \quad \text{...(2)}$$

$$\Delta^3 y_h = E^3 y_h - 3E^2 y_h + 3Ey_h - y_h \quad \text{...(3)}$$

We denote $\quad y_x = y(x) = f(x)$.

Also $\quad E^n y_x = f(x+nk) = y(x+nk)$

$$= y_{x+n} \text{ for } k = 1.$$

Thus relations (1) to (3) are written as

$$\Delta y_h = y_{h+1} - y_h \quad \text{...(4)}$$

$$\Delta^2 y_h = y_{h+2} - 2y_{h+1} + y_h \quad \text{...(5)}$$

$$\Delta^3 y_h = y_{h+3} - 3y_{h+2} + 3y_{h+1} - y_h. \quad \text{...(6)}$$

In recurrence relations (difference equations) x belongs to the set of numbers and therefore we shall use the symbol r, n, h or k rather than x to denote a number in the domain of the functions related by the difference equation, and thus we shall write y_h rather than $y(h)$ for the value of y at h. Since integer h has consecutive values and therefore all difference operators are taken with a difference interval equal to 1. With these notations, the difference equations $\Delta^2 y_h + 4\Delta y_h + 2y_h = 4$ is written as

$$(y_{h+2} - 2y_{h+1} + y_h) + 4(y_{h+1} - y_h) + 2y_h = 4$$

$$\Rightarrow \quad y_{h+2} + 2y_{h+1} - y_h = 4. \quad ...(7)$$

§ 5.32. FORMATION OF RECURRENCE RELATIONS (DIFFERENCE EQUATIONS)

Consider a functional relation between the independent variable x, dependent variable y [*i.e.*, $f(x)$] and a constant c_1 given by

$$F[f(x), x, c_1] = 0 \quad ...(1)$$

operating Δ on (1), we have

$$\Delta F[f(x), x, c_1] = 0 \quad ...(2)$$

On eliminating c_1 between (1) and (2), we shall get a relation of the form

$$G[f(x), x, \Delta f(x)] = 0 \quad ...(3)$$

Since $\Delta f(x) = f(x+1) - f(x) = y_{x+1} - y_x$. So putting this value of $\Delta f(x)$ and also $f(x) = y$ in (3), we have a relation of the form

$$H[y_{x+1}, y_x, x] = 0 \quad ...(4)$$

which is the required recurrence relation (difference equation).

Similarly if the functional relation contains two independent constants c_1, c_2 (say) *i.e.*,

$$F[f(x), x, c_1, c_2] = 0 \quad ...(5)$$

Then operating Δ twice on (5) and then eliminating c_1, c_2 between the three relations as above, we have a relation of the form

$$H[y_{x+2}, y_{x+1}, y_x, x] = 0 \quad ...(6)$$

which is the required recurrence relation (difference equation).

In general, consider a functional relation containing n constants $c_1, c_2, ..., c_n$ (say)

$$F[f(x), x, c_1, c_2, ..., c_n] = 0. \quad ...(7)$$

Then operating Δ n times in succession on (7), we obtain following n relations :

$$\left.\begin{array}{l} \Delta F[f(x), x, c_1, c_2, ..., c_n] = 0 \\ \Delta^2 F[f(x), x, c_1, c_2, ..., c_n] = 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \Delta^n F[f(x), x, c_1, c_2, ..., c_n] = 0 \end{array}\right\} . \quad ...(8)$$

Now substituting $\Delta f(x) = y_{x+1} - y_x$, $\Delta^2 f(x) = y_{x+2} - 2y_{x+1} + y_x$ etc. in equations (8) and then eliminating $c_1, c_2, ..., c_n$, we get an equation of the form

$$H[y_{x+n}, y_{x+n-1}, ..., y_x, x] = 0 \quad ...(9)$$

which is the required recurrence relation (difference equation).

Note that the complete primitive of (9) will be of the form (7).

ILLUSTRATIVE EXAMPLES

Example 1. *Given* $y_h = A \cdot 2^h + B \cdot 3^h$, *find the corresponding recurrence relation (difference equation).*

Solution. We have $y_h = A \cdot 2^h + B \cdot 3^h$.

$$\therefore \quad y_{h+1} = A \,.\, 2^{h+1} + B \,.\, 3^{h+1}$$

and
$$y_{h+2} = A \,.\, 2^{h+2} + B \,.\, 3^{h+2}$$

Above three relations may be written as :

$$\begin{aligned} y_{h+2} - 4 \,.\, 2^h A - 9 \,.\, 3^h B &= 0 \\ y_{h+1} - 2 \,.\, 2^h A - 3 \,.\, 3^h B &= 0 \\ y_h - 2^h A - 3^h B &= 0. \end{aligned} \quad ...(1)$$

Eliminating $A \,.\, 2^h$ and $B \,.\, 3^h$ [in other words eliminating constants A and B] from above three relations given by (1), we get

$$\begin{vmatrix} y_{h+2} & -4 & -9 \\ y_{h+1} & -2 & -3 \\ y_h & -1 & -1 \end{vmatrix} = 0$$

or
$$y_{h+2}(2-3) - y_{h+1}(4-9) + y_h(12-18) = 0$$

or
$$y_{h+2} - 5y_{h+1} + 6y_h = 0$$

which is the required recurrence relation.

Second method. We have

$$y_h = A \,.\, 2^h + B \,.\, 3^h. \quad ...(1)$$

Here the equation contains two constants, so operating the operator Δ twice on (1), we get

$$\Delta y_h = \Delta (A \,.\, 2^h + B \,.\, 3^h)$$

$$\Rightarrow \quad y_{h+1} - y_h = (A \,.\, 2^{h+1} + B \,.\, 3^{h+1}) - (A \,.\, 2^h + B \,.\, 3^h)$$

$$\Rightarrow \quad y_{h+1} - y_h = A \,.\, 2^h + 2B \,.\, 3^h \quad ...(2)$$

And
$$\Delta^2 y_h = \Delta^2 (A \,.\, 2^h + B \,.\, 3^h)$$

$$\Rightarrow \quad \Delta [y_{h+1} - y_h] = \Delta [A \,.\, 2^h + 2B \,.\, 3^h], \quad \text{[Use (2)]}$$

$$\Rightarrow \quad y_{h+2} - 2y_{h+1} + y_h = (A \,.\, 2^{h+1} + 2B \,.\, 3^{h+1}) - (A \,.\, 2^h + 2B \,.\, 3^h)$$

$$\Rightarrow \quad y_{h+2} - 2y_{h+1} + y_h = A \,.\, 2^h + 4B \,.\, 3^h. \quad ...(3)$$

Solving (1) and (2) for A and B, we have

$$A \,.\, 2^h = -y_{h+1} + 3y_h, \; B \,.\, 3^h = y_{h+1} - 2y_h.$$

Substituting values in (3), we get

$$y_{h+2} - 2y_{h+1} + y_h = -y_{h+1} + 3y_h + 4y_{h+1} - 8y_h$$

$$\Rightarrow \quad y_{h+2} - 5y_{h+1} + 6y_h = 0$$

which is the required recurrence relation.

Example 2. *Eliminate the constant C from $y(x) = C \,.\, 3^x + x \,.\, 3^{x-1}$ and hence find the corresponding recurrence relation.*

Solution. $\quad y(x) = C \,.\, 3^x + x \,.\, 3^{x-1}. \quad ...(1)$

Here (1) contains only one constant and hence operating Δ on (1), we get

$$\Delta y_x = \Delta [C \,.\, 3^x + x \,.\, 3^{x-1}] \quad [\because \; y(x) = y_x]$$

$$\begin{aligned} \Rightarrow \quad y_{x+1} - y_x &= C \,.\, 3^{x+1} + (x+1)\, 3^{x+1-1} - C \,.\, 3^x - x \,.\, 3^{x-1} \\ &= 3C \,.\, 3^x + x \,.\, 3^x + 3^x - C \,.\, 3^x - x \,.\, 3^{x-1} \\ &= 2C \,.\, 3^x + 3x \,.\, 3^{x-1} + 3 \,.\, 3^{x-1} - x \,.\, 3^{x-1} \end{aligned}$$

$$y_{x+1} - y_x = 2\,[y_x - x \,.\, 3^{x-1}] + 2x \,.\, 3^{x-1} + 3^x$$

$\Rightarrow$

$$y_{x+1} - 3y_x = 3^x \qquad \text{[Putting for } C.3^x \text{ from (1)]}$$

$\Rightarrow$ which is the required recurence relation.

Example 3. *Given* $u_x = C_1 \,.\, 2^x + C_2 x$, *find the corresponding recurrence relation.*

Solution. Given $\qquad u_x = C_1 \,.\, 2^x + C_2 x \qquad$ ...(1)

Relation (1) has two constants, so operating twice by Δ on (1), we get

$$\Delta u_x = C_1 \,.\, 2^{x+1} + C_2(x+1) - C_1 \,.\, 2^x - C_2 x$$
$$= C_1 \,.\, 2^x + C_2 \qquad \text{...(2)}$$
$$\Delta^2 u_x = [C_1 \,.\, 2^{x+1} + C_2] - [C_1 \,.\, 2^x + C_2]$$
$$= C_1 \,.\, 2^x. \qquad \text{...(3)}$$

Solving (2) and (3) for C_1 and C_2, we have

$$C_1 \,.\, 2^x = \Delta^2 u_x, \;\; C_2 = \Delta u_x - \Delta^2 u_x.$$

Substituting values in (1), we get

$$u_x = \Delta^2 u_x + (\Delta u_x - \Delta^2 u_x) \,.\, x$$
$$= (u_{x+2} - 2u_{x+1} + u_x) + [(u_{x+1} - u_x) - (u_{x+2} - 2u_{x+1} + u_x)]\, x$$

or
$$u_{x+2} - 2u_{x+1} + [-u_{x+2} + 3u_{x+1} - 2u_x]\, x = 0$$

or
$$(1-x)\, u_{x+2} - (2-3x)\, u_{x+1} - 2x.u_x = 0$$

which is the required recurrence relation.

Example 4. *Given* $u_x = C_1.2^x + C_2.3^x + \frac{1}{2}$, *find the corresponding recurrence relation.*

Solution. Proceed as Ex. 1, above. The required recurrence relation is

$$u_{x+2} - 5u_{x+1} + 6u_x = 1.$$

Example 5. *Given* $y_x = ax^2 + bx$, *find the corresponding recurrence relation.*

Solution. We have $\quad y_x = ax^2 + bx \qquad$ ...(1)

$\therefore$
$$y_{x+1} = a\,(x+1)^2 + b\,(x+1) \qquad \text{...(2)}$$

and
$$y_{x+2} = a\,(x+2)^2 + b\,(x+2) \qquad \text{...(3)}$$

Writing relations (1), (2) and (3) as

$$y_{x+2} - a\,(x+2)^2 - b\,(x+2) = 0 \qquad \text{...(4)}$$
$$y_{x+1} - a\,(x+1)^2 - b\,(x+1) = 0 \qquad \text{...(5)}$$
$$y_x - ax^2 - bx = 0 \qquad \text{...(6)}$$

Eliminating a and b from (4), (5), (6), we get

$$\begin{vmatrix} y_{x+2} & (x+2)^2 & x+2 \\ y_{x+1} & (x+1)^2 & x+1 \\ y_x & x^2 & x \end{vmatrix} = 0$$

or $$y_{x+2}\,[x(x+1)^2 - x^2(x+1)] - y_{x+1}\,[x(x+2)^2 - x^2(x+2)] + y_x\,[(x+2)^2(x+1) - (x+1)^2(x+2)] = 0$$

or $$(x^2+x)\,y_{x+2} - 2x\,(x+2)\,y_{x+1} + (x+1)(x+2)\,y_x = 0.$$ Ans.

Example 6. *From the recurrence relation for $y_h = (a/h) + b$.*

Solution. We have $$y_h = \frac{a}{h} + b$$

$$y_{h+1} = \frac{a}{h+1} + b$$

$$y_{h+2} = \frac{a}{h+2} + b.$$

Eliminating a and b, we have

$$\begin{vmatrix} y_{h+2} & 1/(h+2) & 1 \\ y_{h+1} & 1/(h+1) & 1 \\ y_h & 1/h & 1 \end{vmatrix} = 0$$

or $$(h+2)\,y_{h+2} - 2\,(h+1)\,y_{h+1} + h y_h = 0.$$ Ans.

§ 5.33. SOLUTION OF A RECURRENCE RELATION

A relation between the independent variable and dependent variable is said to be a solution of a recurrence relation if this relation satisfies the recurrence relation.

General Solution. A general solution of a recurrence relation of order n is a solution which involves n arbitrary contants (or periodic functions of period 1). A general solution is also called complete primitive.

Particular Solution. A particular solution is a solution obtained from the general solution by assigning particular values to one or more arbitrary constants.

Linear Recurrence Relation. A recurrence relation of degree one is called a linear recurrence relation.

§ 5.34. LINEAR RECURRENCE RELATION WITH CONSTANT COEFFICIENTS

The general form of linear recurrence relation with constant coefficients is given by

$$c_0 a_r + c_1 a_{a-r} + c_2 a_{r-2} + \ldots + c_k a_{r-k} = f(r) \qquad \ldots(1)$$

where $c_0, c_1, c_2, \ldots, c_k$ are constants and $f(r)$ is a function of r (but not of a_r). If both c_0 and c_k are non-zero then the recurrence relation in (1) is said to be or **order** k.

If $f(r) = 0$ in (1), we obtain

$$c_0 a_r + c_1 a_{r-1} + c_2 a_{r-2} + \ldots + c_k a_{r-k} = 0. \qquad \ldots(2)$$

The recurrence relation (2) is called *homogeneous linear recurrence relation with constant coefficients.* If both c_0 and c_k are non-zero then recurrence relation (2) is of order k.

If $f(r) \neq 0$, then the recurrence relation (1) is called *non-homogeneous linear recurrence relation* of order k.

§ 5.35. GENERAL SOLUTION OF HOMOGENEOUS LINEAR RECURRENCE RELATION OF SECOND ORDER WITH CONSTANT COEFFICIENTS (HOMOGENEOUS SOLUTION)

The homogeneous linear recurrence relation of second order with constant coefficients is given by

$$a_r + b_1 a_{r-1} + b_2 a_{r-2} = 0$$

where b_1 and b_2 are constants, and $b_2 \neq 0$. ...(1)

We assume that the solution of (1) is of the form

$$a_r = m^r$$

where m is suitably chosen constant different from zero. Substituting from (2) in (1), we get ...(2)

$$m^r + b_1 m^{r-1} + b_2 m^{r-2} = 0$$

or

$$m^2 + b_1 m + b_2 = 0 \quad \text{...(3)}$$

[On dividing by m^{r-2}]

The quadratic equation (3) is called the **characteristic equation** (or *auxiliary equation*) of the recurrence relation (1). Thus if m is a root of (3), then (2) will be a solution of (1). The characteristic equation (3) being quadratic in m, has two non-zero roots m_1 and m_2 say. (Note that zero roots are excluded since $b_2 \neq 0$, since it is a requirement for (1) to be of second order).

Now corresponding to the roots m_1 and m_2 of (3) there exists solutions

$$a_r^{(1)} = m_1^r \text{ and } a_r^{(2)} = m_2^r. \quad \text{...(4)}$$

Now there arises following three cases :

(I) The roots m_1 and m_2 both are real numbers and unequal.

(II) The roots m_1 and m_2 are real and equal.

(III) The roots m_1 and m_2 are complex numbers.

Now we shall study the above cases :

Case (I). Roots are real and unequal (*i.e.*, $m_1 \neq m_2$).

In this case, the solution in (4) forms a fundamental set. In order to prove it, we consider the following determinant.

$$\begin{vmatrix} a_0^{(1)} & a_0^{(2)} \\ a_1^{(1)} & a_1^{(2)} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ m_1 & m_2 \end{vmatrix} = m_2 - m_1 \neq 0 \text{ since } m_1 \neq m_2.$$

$[\because\ a_0^{(1)} = m_1^0 = 1, a_1^{(1)} = m_1^1 = m_1 \text{ etc.}]$

Hence, when the roots of characteristic equation are real and unequal, the general solution (1) is given by

$$\boldsymbol{a_r = C_1 m_1^r + C_2 m_2^r}$$

where C_1 and C_2 are constants to be determined by boundary conditions.

Case (II). Roots are Real and Equal (*i.e.*, $m_1 = m_2$).

Equation (1) may be written as

$$(E^2 + b_1 E + b_2)\, a_{r-2} = 0 \quad [\because\ E^2 a_{r-2} = a_r,\ E a_{r-2} = a_{r-1}] \quad \text{...(5)}$$

or

$$(E - m_1)^2\, a_{r-2} = 0.$$

To solve (5), we put $a_r = m_1^r v_r$ ...(6), where v_r is a new dependent variable.

Now (5) becomes, $(E - m_1)^2 m_1^{r-2} v_{r-2} = 0$

$\Rightarrow \quad (E^2 - 2m_1 E + m_1^2) m_1^{r-2} v_{r-2} = 0$

$\Rightarrow \quad v_r - 2v_{r-1} + v_{r-2} = 0$

$\Rightarrow \quad \Delta^2 v_{r-2} = 0 \Rightarrow v_{r-2} = C_1 + C_2 (r - 2)$

$\Rightarrow \quad v_r = C_1 + C_2 r.$

Putting the value of v_r in (6), the solution of (1) is

$$a_r = (C_1 + C_2 r)\, m_1^r.$$

Case (III). Roots are Complex Numbers.

We know that the complex roots of a quadratic equation occur in conjugate pairs. Therefore, if m_1 and m_2 are complex roots of the characteristic equation, then $m_1 \neq m_2$.

Let $\quad m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$.

Then by case (I), the corresponding solution of (1) is given by

$$a_r = c_1 (\alpha + i\beta)^r + c_2 (\alpha - i\beta)^r$$

By use of De-Maivre's theorem, this can be written as

$$\boldsymbol{a_r = (C_1 \cos r\theta + C_2 \sin r\theta)\, R^r} \quad \text{...(7)}$$

where $\quad R = \sqrt{(\alpha^2 + \beta^2)}$, and $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$.

Hence in this case the required solution of recurrence relation (1) is given by (7).

In case $\alpha = 0, \beta = 1$ then $\theta = \frac{1}{2}\pi$ and the solution (7) becomes

$$a_r = C_1 \cos \frac{\pi r}{2} + C_2 \sin \frac{\pi r}{2}$$

Remark. Above result (7) can also be written as

$$a_r = C_1 \cos (r\theta + C_2).\, R^r$$

where $\quad R = \sqrt{(\alpha^2 + \beta^2)}$ and $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$.

Summary. Let the homogeneous linear recurrence relation of order 2 be

$$a_r + b_1 a_{r-1} + b_2 a_{r-2} = 0 \quad \text{...(1)}$$

where b_1 and b_2 are constants and $b_2 \neq 0$.

The characteristic equation (in short Ch. E.) of (1) is

$$m^2 + b_1 m + b_2 = 0. \quad \text{...(2)}$$

Suppose m_1 and m_2 are roots of (2). Then following three cases arises :

(i) If m_1 and m_2 are real and unequal, then the general solution (homogeneous solution) of (1) is given by

$$a_r = C_1 m_1^r + C_2 m_2^r.$$

(ii) If $m_1 = m_2$, then general solution (homogeneous solution) is

$$a_r = (C_1 + C_2 r)\, m_1^r.$$

(iii) If m_1 and m_2 are complex conjugate with polar form $R(\cos\theta \pm i\sin\theta)$, then general solution (homogeneous solution) is

$$a_r = (C_1 \cos r\theta + C_2 \sin r\theta) R^r$$

or

$$a_r = C_1 \cos (r\theta + C_2) R^r.$$

Remark. The general solution of homogeneous linear recurrence relation is also called **homogeneous solution.**

§ 5.35. (A) HOMOGENEOUS SOLUTION (*i.e.*, GENERAL SOLUTION) OF THE HOMOGENEOUS LINEAR RECURRENCE RELATION OF ORDER k WITH CONSTANT COEFFICIENTS

The results of § 5.35 above can be generalised and summarised as follows :

Let the kth order homogeneous linear recurrence relation with constant coefficients $b_1, b_2, ..., b_k$ be given by

$$a_r + b_1 a_{r-1} + b_2 a_{r-2} + ... + b_k a_{r-k} = 0, \; b_k \neq 0. \qquad ...(1)$$

The Ch. E of (1) is given by

$$m^r + b_1 m^{r-1} + ... + b_k = 0. \qquad ...(2)$$

Let $m_1, m_2, ..., m_k$ be its k roots. The following cases arise :

(i) **Roots are real and distinct.** In this the homogeneous solution is given by

$$a_r = C_1 m_1^r + C_2 m_2^r + ... + C_k m_k^r.$$

(ii) **Some of the roots are equal.** Let the roots m_1 be repeated p times *i.e.*, $m_1 = m_2 = ...: = m_p, p \leq k$, then the term in homogeneous solution corresponding to this root is $(C_1 + C_2 r + C_3 r^2 + ... + C_p r^{p-1}) m_1^r$.

(iii) **Some of the roots are Complex Numbers.** If the roots m_1 and m_2 of Ch. E are complex conjugates with the polar form $R(\cos\theta \pm i\sin\theta)$, then the term in homogeneous solution corresponding to these roots is $R^r (C_1 \cos r\theta + C_2 \sin r\theta)$ or $C_1 R^r \cos (r\theta + C_2)$.

(iv) **Some of the Roots are repeated Complex Roots.** If $R(\cos\theta \pm i\sin\theta)$ be a pair of complex conjugate roots (repeated two times) in polar form, then the term in homogeneous solution corresponding to these roots is

$$R^r [(C_1 + C_2 r) \cos r\theta + (C_3 + C_4 r) \sin r\theta].$$

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the following recurrence relations :*

(*i*) $a_r - 6a_{r-1} + 8a_{r-2} = 0$ *given* $a_0 = 3$ *and* $a_1 = 2$.

(*ii*) $2a_r - 5a_{r-1} + 2a_{r-2} = 0$, *given* $a_0 = 0, a_1 = 1$.

Solution. (i) The given recurrence relation is

$$a_r - 6a_{r-1} + 8a_{r-2} = 0. \qquad ...(1)$$

The Characteristic equation is

$$m^2 - 6m + 8 = 0 \Rightarrow (m-2)(m-4) = 0 \Rightarrow m = 2, 4.$$

∴ The general solution of (1) is given by

$$a_r = C_1 . 2^r + C_2 . 4^r. \qquad ...(2)$$

Putting $r = 0$ in (2), we get

$$a_0 = C_1 + C_2 \Rightarrow 3 = C_1 + C_2.$$

Again putting $r = 1$ in (2), we get

$$a_1 = 2C_1 + 4C_2 \Rightarrow 2 = 2C_1 + 4C_2 \Rightarrow 1 = C_1 + 2C_2.$$

Solving (3) and (4), $C_1 = 5$, $C_2 = -2$.

Putting for C_1 and C_2 in (2), the required solution of (1) is

$$a_r = 5 \,.\, 2^r - 2 \,.\, 4^r.$$

(ii) The given recurrence relation is

$$2a_r - 5a_{r-1} + 2a_{r-2} = 0.$$

The Characteristic equation is

$$2m^2 - 5m + 2 = 0 \Rightarrow m = \frac{1}{2}, 2.$$

The general solution of (1) is given by

$$a_r = C_1\left(\frac{1}{2}\right)^r + C_2 .\, 2^r.$$

Putting $r = 0, 1$ in (2), we get

$$a_0 = C_1 + C_2 \text{ and } a_1 = \frac{1}{2}C + 2C_2$$

$$\Rightarrow \quad 0 = C_1 + C_2,\ 1 = \frac{1}{2}C_1 + 2C_2 \Rightarrow C_1 = -\frac{2}{3}, C_2 = \frac{2}{3}.$$

Putting values of C_1 and C_2 in (2), the required solution of (1) is given by

$$a_r = -\frac{2}{3}\cdot\left(\frac{1}{2}\right)^r + \frac{2}{3} .\, 2^r.$$

Ans.

Example 2. *Solve the following recurrence relations :*

(*i*) $9a_r - 6a_{r-1} + a_{r-2} = 0$, *given* $a_0 = 0, a_1 = 1$

(*ii*) $a_r + a_{r-2} = 0$, *given* $a_0 = 0, a_1 = 1$

(*iii*) $a_r + 2a_{r-1} + 2a_{r-2} = 0$, *given* $a_0 = 0, a_1 = -1$

(*iv*) $a_r + 6a_{r-1} + 25a_{r-2} = 0$.

Solution. (i) The Characteristic equation is

$$9m^2 - 6m + 1 = 0 \Rightarrow (3m-1)^2 = 0 \Rightarrow m = \frac{1}{3}, \frac{1}{3}.$$

The general solution of given recurrence formula is

$$a_r = (C_1 + C_2 r)\left(\frac{1}{3}\right)^r \qquad \text{[by Case II]} \quad ...(1)$$

Putting $r = 0, 1$ in (1), we get

$$a_0 = C_1 \Rightarrow 0 = C_1$$

and

$$a_1 = (C_1 + C_2)\left(\frac{1}{3}\right) \Rightarrow 1 = \frac{1}{3}C_2 \Rightarrow C_2 = 3.$$

Putting values of C_1 and C_2 in (1), the required solution is

$$a_r = 3r\left(\frac{1}{3}\right)^r \text{ i.e., } a_r = \frac{r}{3^{r-1}}.$$

Ans.

(ii) The given recurrence relation is $a_r + a_{r-2} = 0$. ...(1)

The Ch. E. is $m^2 + 1 = 0 \Rightarrow m = \pm i \Rightarrow m = \left(\cos\frac{\pi}{2} \pm i \sin\frac{\pi}{2}\right)$

$$\theta = \frac{\pi}{2},\ R = 1.$$

$\therefore$ The general solution of (1) is

$$a_r = C_1 R^r \cos(r\theta + C_2) \qquad \text{[By Case III]}$$

or
$$a_r = C_1 \cos\left(r\frac{\pi}{2} + C_2\right). \qquad ...(2)$$

Putting $r = 0, 1$ in (2), we have

$$a_0 = C_1 \cos C_2 \Rightarrow 0 = C_1 \cos C_2 \Rightarrow \cos C_2 = 0 \Rightarrow C_2 = \frac{\pi}{2}$$

and
$$a_1 = C_1 \cos\left(\frac{\pi}{2} + C_2\right) \Rightarrow 1 = C_1 \cos\left(\frac{\pi}{2} + \frac{\pi}{2}\right) \Rightarrow C_1 = -1.$$

Putting values in (2), the required solution is

$$a_r = -\cos\left(r\frac{\pi}{2} + \frac{\pi}{2}\right) \text{ i.e., } a_r = \sin\left(r\frac{\pi}{2}\right). \qquad \textbf{Ans.}$$

(iii) The given recurrence formula is

$$a_r + 2a_{r-1} + 2a_{r-2} = 0. \qquad ...(1)$$

The Ch. E. is $\quad m^2 + 2m + 2 = 0 \Rightarrow m = -1 \pm i$

$$\Rightarrow \qquad m = \sqrt{2}\left(-\frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{3\pi}{4} \pm i \sin\frac{3\pi}{4}\right)$$

$$\therefore \qquad R = \sqrt{2},\ \theta = \frac{3\pi}{4}.$$

The general solution of (1) is

$$a_r = (\sqrt{2})^r \left(C_1 \cos\frac{3\pi}{4} r + C_2 \sin\frac{3\pi}{4} r\right). \qquad ...(2)$$

Putting $r = 0, 1$ in (2), we have

$$a_0 = C_1 \Rightarrow 0 = C_1$$

and
$$a_1 = \sqrt{2}\left[0 + C_2 \sin\frac{3\pi}{4}\right] \Rightarrow -1 = \sqrt{2}\,.\,C_2\left(\frac{1}{\sqrt{2}}\right) \Rightarrow C_2 = -1.$$

Putting values of C_1 and C_2 in (2), the required solution is

$$a_r = -(\sqrt{2})^r \sin\left(\frac{3\pi r}{4}\right). \qquad \textbf{Ans.}$$

(iv) The given recurrence formula is

$$a_r + 6a_{r-1} + 25a_{r-2} = 0. \qquad ...(1)$$

Its Ch. E. is

$$m^2 + 6m + 25 = 0$$

$$\therefore \qquad m = \frac{-6 \pm \sqrt{36 - 100}}{2} = -3 \pm 4i.$$

Putting $\quad -3 \pm 4i = R(\cos\theta \pm i \sin\theta)$

$$\Rightarrow \qquad R\cos\theta = -3,\ R\sin\theta = 4$$

$$\Rightarrow \qquad R = 5,\ \theta = \tan^{-1}\left(-\frac{4}{3}\right).$$

$\therefore$ The general solution of (1) is

$$a_r = (5)^r \,[C_1 \cos r\theta + C_2 \sin r\theta]$$

where $$\theta = \tan^{-1}\left(-\frac{4}{3}\right).$$ **Ans.**

Example 3. *Solve the recurrence relation* $a_r = a_{r-1} + a_{r-2}$, *given* $a_0 = 1, a_1 = 1$.

Solution. The Characteristic equation is

$$m^2 - m - 1 = 0$$

$$\Rightarrow \quad m = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$\therefore$ The homogeneous (general) solution is

$$a_r = C_1\left(\frac{1+\sqrt{5}}{2}\right)^r + C_2\left(\frac{1-\sqrt{5}}{2}\right)^r. \quad \text{...(1)}$$

Putting $r = 0, 1$ in (1), we have

$$a_0 = C_1 + C_2 \Rightarrow C_1 + C_2 = 1 \quad \text{...(2)}$$

and $$a_1 = C_1\left(\frac{1+\sqrt{5}}{2}\right) + C_2\left(\frac{1-\sqrt{5}}{2}\right)$$

i.e., $$C_1\left(\frac{1+\sqrt{5}}{2}\right) + C_2\left(\frac{1-\sqrt{5}}{2}\right) = 1. \quad \text{...(3)}$$

Solving (2) and (3)

$$C_1 = \frac{\sqrt{5}+1}{2\sqrt{5}}, \; C_2 = \frac{\sqrt{5}-1}{2\sqrt{5}}.$$

Putting values of C_1 and C_2 in (1), the required solution is

$$a_r = \left(\frac{\sqrt{5}+1}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^r + \left(\frac{\sqrt{5}-1}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^r$$

$$= \frac{1}{2^{r+1}.\sqrt{5}}\,[(\sqrt{5}+1)^{r+1} + (-1)^r(\sqrt{5}-1)^{r+1}].$$ **Ans.**

Example 4. *Solve the recurrence relation :*

$a_r - 7a_{r-1} + 10a_{r-2} = 0$, *given* $a_0 = 0, a_1 = 3$. **[R.G.P.V. Dec. 2002]**

Solution. The Characteristic equation is

$$m^2 - 7m + 10 = 0 \Rightarrow (m-2)(m-5) = 0 \Rightarrow m = 2, 5.$$

The homogeneous (general) solution of given recurrence formula is

$$a_r = C_1 \,.\, 2^r + C_2 \,.\, 5^r. \quad \text{...(1)}$$

Putting $r = 0, 1$ in (1), we obtain

$$a_0 = C_1 + C_2 \Rightarrow 0 = C_1 + C_2 \quad \text{...(2)}$$

and $$a_1 = 2C_1 + 5C_2 \Rightarrow 3 = 2C_1 + 5C_2, \quad \text{...(3)}$$

Solving (2) and (3), $C_1 = -1, C_2 = 1$.

Putting values of C_1 and C_2 in (1), the required solution is

$$a_r = 5^r - 2^r.$$ **Ans.**

Example 5. *Solve* $a_r - 7a_{r-2} - 6a_{r-3} = 0$ *with initial conditions* $a_0 = 9$, $a_1 = 10, a_2 = 32$.

Solution. The Characteristic equation is

$$m^3 - 7m - 6 = 0$$

$$(m+1)(m+2)(m-3) = 0 \Rightarrow m = -1, -2, 3$$

The general solution is

$$a_r = C_1(-1)^r + C_2(-2)^r + C_3(3)^r \quad \ldots(1)$$

Putting $r = 0, 1, 2$ and using initial conditions in (1), we have

$$C_1 + C_2 + C_3 = 9 \quad \ldots(2)$$

$$-C_1 - 2C_2 + 3C_3 = 10 \quad \ldots(3)$$

$$C_1 + 4C_2 + 9C_3 = 32 \quad \ldots(4)$$

Solving (2), (3) and (4) $\Rightarrow C_1 = 8, C_2 = -3, C_3 = 4$.

Putting values in (1), the required solution is

$$a_r = 8(-1)^r - 3(-2)^r + 4(3)^r.$$

Ans.

Example 6. *Solve* $a_r + 3a_{r-1} + 3a_{r-2} + a_{r-3} = 0$ *with* $a_0 = 1, a_1 = -2$ *and* $a_2 = -1$.

Solution. The Characteristic equation is

$$m^3 + 3m^2 + 3m + 1 = 0 \Rightarrow (m+1)^3 = 0 \Rightarrow m = -1, -1, -1.$$

The homogeneous solution is

$$a_r = (C_1r^2 + C_2r + C_3)(-1)^r. \quad \ldots(1)$$

Applying initial conditions in (1), we get

$$C_3 = 1, -(C_1 + C_2 + C_3) = -2, 4C_1 + 2C_2 + C_3 = -1.$$

Solving,

$$C_1 = -2, C_2 = 3, C_3 = 1.$$

Putting values in (1), the required solution is

$$a_r = (1 + 3r - 2r^2)(-1)^r.$$

Ans.

PROBLEM SET

Solve the following recurrence relations :

1. $a_r - 7a_{r-1} + 12a_{r-2} = 0$.
2. $a_r + 3a_{r-1} + 2a_{r-2} = 0$ with $a_0 = 1, a_1 = 2$.
3. $a_r - 5a_{r-1} + 6a_{r-2} = 0$ with $a_0 = 1, a_1 = 2$.
4. $3a_r - 7a_{r-1} - 6a_{r-2} = 0$ with $a_0 = 0, a_1 = 1$.
5. $a_r - 4a_{r-1} + 4a_{r-2} = 0$ with $a_0 = 1, a_1 = 2$.

ANSWERS

1. $a_r = C_1 3^r + C_2 4^r$ **2.** $a_r = 4(-1)^r - 3(-2)^r$ **3.** $a_r = 2^r$

4. $a_r = \frac{3}{11}\, 3^r - \frac{3}{11}\left(-\frac{2}{3}\right)^r$ **5.** $a_r = 2^r$.

5.36. TOTAL SOLUTION

A non-homogeneous linear recurrence relation (*i.e.*, difference equation) of order k with constant coefficients (see § 5.34) is given by

$$c_0a_r + c_1a_{r-1} + c_2a_{r-2} + \ldots + c_ka_{r-k} = f(r) \quad \ldots(1)$$

where $c_0, c_1, \ldots, c_k$ are constants and both ... of r.

The total (general) solution of (1) consists of two parts, namely

(i) homogeneous solution,

and (ii) particular solution.

Then the total (*i.e.*, general or complete) solution is given by

$$a_n = \text{homogeneous solution} + \text{particular solution}.$$

The homogeneous solution has been discussed in § 5.35 and § 5.35 (A). Now we shall discuss the methods of determining **particular solution** of (1).

§ 5.37. PARTICULAR SOLUTION

There is no general procedure for determining the particular solution of a recurrence relation (difference equation). However, there are a number of special techniques for finding particular solution of (1) in § 5.36. Here we shall discuss following two important methods for obtaining particular solution :

(i) Method of inspection or method of undetermined coefficients,

(ii) Operator Method.

§ 5.37. (A) METHOD OF INSPECTION (OR METHOD OF UNDETERMINED COEFFICIENTS)

This method is useful in finding the particular solution when $f(r)$ consists of terms having certain special forms. In this method, we consider a trial solution consisting of unknown constant coefficients corresponding to each term which is present in $f(r)$. The unknown coefficients are determined by substituting the trial solution in the recurrence relation (*i.e.*, difference equation).

The trial solutions (a_r, say) to be used in different cases are given in the following table :

S. No.	Terms is $f(r)$	Trial solution a_r
(i)	b^r	$A \cdot b^r$
(ii)	a polynomial of degree k in r	$A_0 + A_1 r + A_2 r^2 + \ldots + A_k r^k$
(iii)	b^r.(a polynomial of degree of k in r)	$b^r \cdot (A_0 + A_1 r + A_2 r^2 + \ldots + A_k r^k)$
(iv)	$\sin br$ or $\cos br$	$A \sin br + B \cos br$
(v)	$a^r \sin br$	$a^r (A \sin br + B \cos br)$
(vi)	$a^r \cos br$	$a^r (A \sin br + B \cos br)$

where $a, b, A, B, A_0, A_1, A_2, \ldots, A_k$ are unknown constants coefficient to be determined.

In case $f(r)$ is a linear combination of some functions then trial solution is taken as the sum of the corresponding trial functions with different unknown constant coefficients.

Important Note. It is necessary that no term of the trial solution should appear in the homogeneous solution.

For clear understanding of the method see the following examples.

ILLUSTRATIVE EXAMPLES

Example 1. (*a*) *Solve* $a_r - 5a_{r-1} + 6a_{r-2} = 5^r$.

Solution. We have $a_r - 5a_{r-1} + 6a_{r-2} = 5^r$. ...(1)

The Ch. E. is $m^2 - 5m + 6 = 0 \Rightarrow (m-2)(m-3) = 0 \Rightarrow m = 2, 3$

$\therefore$ Homogeneous solution $(a_r^{(h)})$ is given by

$$a_r^{(h)} = C_1 . 2^r + C_2 . 3^r. \quad ...(2)$$

The particular solution (trial solution) corresponding to the term on R.H.S. of (1) is $A . 5^r$.

$\therefore$
$$a_r^{(p)} = A . 5^r. \quad ...(3)$$

Substituting (3) in (1), we get

$$A . 5^r - 5A . 5^{r-1} + 6 . A5^{r-2} = 5^r$$

$\Rightarrow$
$$A\,[6 . 5^{r-2}] = 5^r \Rightarrow A = \frac{25}{6}.$$

Putting for A in (3), the particular solution is

$$a_r^{(p)} = \frac{25}{6} \cdot 5^r \Rightarrow a_r^{(p)} = \frac{1}{6} \cdot 5^{r+2}.$$

$\therefore$ Total solution of (1) is given by

$$a_r = a_r^{(h)} + a_r^{(p)}$$

i.e.,
$$a_r = C_1.2^r + C_2.3^r + \frac{1}{6} \cdot 5^{r+2}.$$ **Ans.**

Example 1. (*b*) *Determine the particular solution for the diference equation* $a_r - 2a_{r-1} = f(r)$ *where* $f(r) = 7r$. **[R.G.P.V. June 2007]**

Solution. The given equation is

$$a_r - 2a_{r-1} = 7r. \quad ...(1)$$

The particular solution (trial solution) corresponding to the term $7r$ on R.H.S. of (1) is $A_0 + A_1 r$.

$\therefore$
$$a_r^{(p)} = A_0 + A_1 r. \quad ...(2)$$

Substituting (2) in (1), we get

$$(A_0 + A_1 r) - 2\,[A_0 + A_1(r-1)] = 7r \quad ...(3)$$

or
$$(-A_0 + 2A_1) + (-A_1 r) = 7r.$$

Comparing two sides of (3), we get

$$-A_1 = 7 \text{ and } -A_0 + 2A_1 = 0 \Rightarrow A_1 = -7,\ A_0 = -14.$$

Putting for A_0 and A_1 in (2), we get

$$a_r^{(p)} = -14 - 7r.$$

$\therefore$ The required particular solution is $-14 - 7r$.

Example 2. *Solve the recurrence relation*

$$a_r - 5a_{r-1} + 6a_{r-2} = 2 + r, \quad r \geq 2$$

with boundary conditions $a_0 = 1$ *and* $a_1 = 1$. **[R.G.P.V. June 2001]**

Solution. The given equation is

$$a_r - 5a_{r-1} + 6a_{r-2} = 2 + r \quad \ldots(1)$$

The Ch. E. is $m^2 - 5m + 6 = 0 \Rightarrow m = 2, 3$.

$$\therefore \quad a_r^{(h)} = C_1 \cdot 2^r + C_2 \cdot 3^r. \quad \ldots(2)$$

The particular solution (trial solution) corresponding to the term $2 + r$ on R.H.S. of (1) is $A_0 + A_1 r$

$$\therefore \quad a_r^{(p)} = A_0 + A_1 r. \quad \ldots(3)$$

Substituting (3) in (1), we get

$$(A_0 + A_1 r) - 5\{A_0 + A_1(r-1)\} + 6\{A_0 + A_1(r-2)\} = 2 + r$$

or

$$(2A_0 - 7A_1) + 2A_1 r = 2 + r. \quad \ldots(4)$$

Comparing two sides of (4), we get

$$2A_0 - 7A_1 = 2 \text{ and } 2A_1 = 1.$$

Solving, $\quad A_0 = \frac{11}{4}, A_1 = \frac{1}{2}.$

Putting for A_0 and A_1 in (3), we get

$$a_r^{(p)} = \frac{11}{4} + \frac{1}{2} r.$$

$\therefore$ Total solution of (1) is given by

$$a_r = a_r^{(h)} + a_r^{(p)}$$

i.e.,

$$a_r = C_1 . 2^r + C_2 . 3^r + \frac{11}{4} + \frac{1}{2} r. \quad \ldots(5)$$

Now putting $r = 0$, 1 and using boundary conditions in (5), we get

$$1 = C_1 + C_2 + \frac{11}{4} \Rightarrow C_1 + C_2 = -\frac{7}{4} \quad \ldots(6)$$

and

$$1 = 2C_1 + 3C_2 + \frac{11}{4} + \frac{1}{2} \Rightarrow 2C_1 + 3C_2 = -\frac{9}{4}. \quad \ldots(7)$$

Solving (6) and (7), $C_1 = -3$, $C_2 = \frac{5}{4}$.

Putting for C_1 and C_2 in (5), the required solution is

$$a_r = -3 \cdot 2^r + \frac{5}{4} \cdot 3^r + \frac{11}{4} + \frac{1}{2} r. \quad \textbf{Ans.}$$

Example 3. *Solve the recurrence relation*

$$a_r - 5a_{r-1} + 6a_{r-2} = 2^r + r, r \geq 2$$

with boundary conditions $a_0 = 1$ *and* $a_1 = 1$. **[R.G.P.V. June 2002]**

Solution. We have

$$a_r - 5a_{r-1} + 6a_{r-2} = 2^r + r \quad \ldots(1)$$

$$\therefore \quad a_r^{(h)} = C_1 \cdot 2^r + C_2 \cdot 3^r \quad \text{[See Ex. 2]} \quad \ldots(2)$$

The particular solution corresponding to the term $2^r + r$ on R.H.S. of (1) is

$$A_0 r \cdot 2^r + A_1 + A_2 r.$$

[Note that , here we have chosen $Ar.2^r$ corresponding to the term 2^r and not $A_0.2^r$ since 2^r is a term in homogeneous solution.]

$$\therefore \quad a_r^{(p)} = A_0 r \cdot 2^r + A_1 + A_2 r. \quad \ldots(3)$$

Substituting (3) in (1), we get

$$(A_0 r \cdot 2^r + A_1 + A_2 r) - 5\,[A_0 (r-1)\, 2^{r-1} + A_1 + A_2 (r-1)]$$
$$+ 6\,[A_0 (r-2)\, 2^{r-2} + A_1 + A_2 (r-2)] = 2^r + r$$

or
$$\left[A_0 - \frac{5}{2} A_0 + \frac{3}{2} A_0\right] r \cdot 2^r + \left[\frac{5}{2} A_0 - 3A_0\right] 2^r$$
$$+ [A_1 - 5A_1 + 5A_2 + 6A_1 - 12A_2] + [A_2 - 5A_2 + 6A_2]\, r = 2^r + r$$

or
$$-\frac{1}{2} A_0 2^r + (2A_1 - 7A_2) + 2A_2 r = 2^r + r. \qquad \text{...(4)}$$

Comparing two sides of (4), we get

$$-\frac{1}{2} A_0 = 1,\ 2A_1 - 7A_2 = 0,\ 2A_2 = 1$$

$$\Rightarrow \qquad A_0 = -2,\ A_1 = \frac{7}{4},\ A_2 = \frac{1}{2}.$$

Putting values of A_0, A_1 and A_2 in (3), the particular solution is given by

$$a_r^{(p)} = -2r.2^r + \frac{7}{4} + \frac{1}{2} r.$$

Hence the total solution of (1) is

$$a_r = a_r^{(h)} + a_r^{(p)}$$

i.e.,
$$a_r = C_1 \cdot 2^r + C_2 \cdot 3^r - 2r \cdot 2^r + \frac{7}{4} + \frac{1}{2} r. \qquad \text{...(5)}$$

Putting $r = 0, 1$ and using boundary conditions in (5), we have

$$C_1 + C_2 + \frac{7}{4} = 1 \Rightarrow C_1 + C_2 = -\frac{3}{4} \qquad \text{...(6)}$$

and
$$2C_1 + 3C_2 - 4 + \frac{7}{4} + \frac{1}{2} = 1 \Rightarrow 2C_1 + 3C_2 = \frac{11}{4}. \qquad \text{...(7)}$$

Solving (6) and (7)

$$C_1 = -5,\ C_2 = \frac{17}{4}$$

Putting values of C_1 and C_2 in (5), the required solution of (1) is

$$a_r = -5.2^r + \frac{17}{4} \cdot 3^r - 2r\, 2^r + \frac{1}{2} r + \frac{7}{4}. \qquad \textbf{Ans.}$$

Example 4. *Solve the recurrence relation*

$$a_r - 4a_{r-1} + 4a_{r-2} = (r+1)^2,\ r \geq 2. \qquad \textbf{[R.G.P.V. June 2003]}$$

Solution. The given equation is

$$a_r - 4a_{r-1} + 4a_{r-2} = (r+1)^2. \qquad \text{...(1)}$$

The Ch. E. is

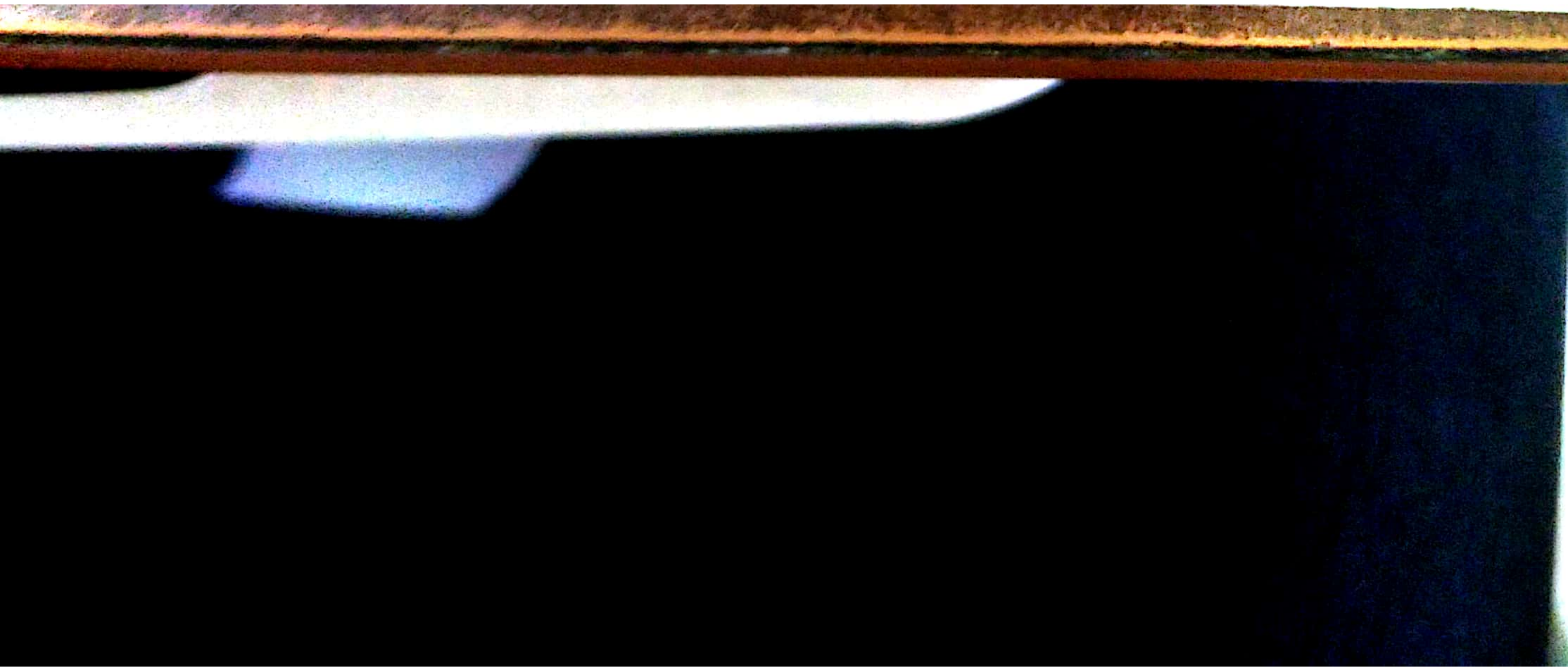
$$m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0 \Rightarrow m = 2, 2.$$

$\therefore$ The homogeneous solution is

$$a_r^{(h)} = (C_1 + C_2 r)\, 2^r. \qquad \text{...(2)}$$

The particular solution corresponding to the term $1 + 2r + r^2$ on the R.H.S. of (1) is $A_0 + A_1 r + A_2 r^2$. ...(3)

$$\therefore \qquad a_r^{(p)} = A_0 + A_1 r + A_2 r^2.$$



Putting values in (3), we get

$$a_r^{(p)} = \frac{1}{4} r^2 + \frac{13}{24} r + \frac{71}{288}.$$

Hence the total solution of (1) is

$$a_r = a_r^{(h)} + a_r^{(p)}$$

i.e.,
$$a_r = C_1 (-2)^r + C_2 (-3)^r + \frac{1}{4} r^2 + \frac{13}{24} r + \frac{71}{288}.$$
Ans.

Example 7. *Solve* $y_{h+2} - 4y_{h+1} + 4y_h = 3h + 2^h$.

Solution. The given recurrence relation may be written as

$$(E^2 - 4E + 4) y_h = 3h + 2^h. \qquad ...(1)$$

$\therefore$ Ch. E. is $\quad m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0 \Rightarrow m = 2, 2.$

So homogeneous solution, $y_h^{(h)} = (C_1 + C_2 h) 2^h$.

To find the particular solution of (1), let the trial solution be

$$y_h^{(p)} = A_1 + A_2 h + Bh^2 . 2^h. \qquad ...(2)$$

The question arises that why $Bh^2 . 2^h$ has been corresponding to 2^h and not $B . 2^h$ or $Bh . 2^h$. The reason is simple. Since 2^h is a term in homogeneous solution, so we do not take $B . 2^h$ as our trial function. Therefore we multiply by h and try $Bh . 2^h$, but this too is a term in homogeneous solution. We again multiply by h and so arrive at the try function $Bh^2 . 2^h$.

Now putting (2) in (1), we get

$$(E^2 - 4E + 4)(A_1 + A_2 h + Bh^2 . 2^h) = 3h + 2^h$$

$$\Rightarrow \quad [A_1 + A_2 (h+2) + B(h+2)^2 . 2^{h+2}] - 4 [A_1 + A_2 (h+1) + B(h+1)]^2 . 2^{h+1}]$$
$$+ 4 [A_1 + A_2 h + Bh^2 . 2^h] = 3h + 2^h$$

$$\Rightarrow \quad (A_1 - 2A_2) + A_2 h + 8B . 2^h = 3h + 2^h.$$

Comparing coefficients on both sides, we have

$$8B = 1 \Rightarrow B = 1/8, A_2 = 3, A_1 - 2A_2 = 0 \Rightarrow A_1 = 6.$$

Putting values of A_1, A_2 and B in (2), the particular solution

$$y_h^{(p)} = 6 + 3h + \frac{1}{8} h^2 . 2^h.$$

$\therefore$ Total (General) solution, $y_h = (C_1 + C_2 h) 2^h + 6 + 3h + (1/8) h^2 . 2^h$. **Ans.**

§ 5.37. (B) SPECIAL OPERATOR METHOD TO FIND THE PARTICULAR SOLUTION (OR BOOL'S OPERATIONAL METHOD)

In § 5.37 (A) above, we have discussed a general method (method of undetermined coefficients) to obtain the particular solution. This method generally leads to laborious calculations. Here in this article, we are giving short methods which will avoid the long calculations.

The linear recurrence relation (difference equation) of nth order is given by

$$y_{h+n} - b_1 y_{h+n-1} + \ldots + b_n y_h = F(h), \; b_n \neq 0$$

$$(E^n + b_1E^{n-1} + \dots + b_n)\, y_h = F(h)$$

or
$$f(E) \cdot y_h = F(h)$$

or
$$f(E) = E^n + b_1E^{n-1} + \dots + b_n. \quad \dots(1)$$

where

So particular solution of (1) $= \dfrac{1}{f(E)} F(h)$.

Now we shall discuss particular solution when $F(h)$ is of the following special forms :

(I) b^h, where b is some constant,

(II) $\sin bh$ or $\cos bh$, where b is some constant,

(III) $P(h)$, where $P(h)$ is a polynomial of degree n,

(IV) $b^h \cdot P(h)$, where $P(h)$ is polynomial of degree n and b is some constant.

Now we shall discuss these forms one by one.

§ 5.37. (C) IF $F(h) = b^h$, THEN PARTICULAR SOLUTION $= \dfrac{1}{f(E)} \cdot b^h = \dfrac{b^h}{f(b)}$ PROVIDED $f(b) \neq 0$

Proof. We have

$$\begin{aligned} f(E) \cdot b^h &= (E^n + b_1E^{n-1} + \dots + b_n)\, b^h \\ &= E^n b^h + b_1E^{n-1}b^h + \dots + b_n \cdot b^h \\ &= b^{h+n} + b_1b^{h+n-1} + \dots + b_n \cdot b^h \\ &= (b^n + b_1b^{n-1} + \dots + b_n)\, b^h \\ &= f(b) \cdot b^h. \end{aligned}$$

Thus by operator inversion, we get

$$\frac{1}{f(E)} \cdot b^h = \frac{b^h}{f(b)}, \text{ provided } f(b) \neq 0.$$

Note. This method fails when b is a root of auxiliary equation.

ILLUSTRATIVE EXAMPLES

Example. 1. *Solve* $y_{h+1} - 3y_h = 2^h$.

Solution. The given difference equation may be written as

$$(E - 3)\, y_h = 2^h. \quad \dots(1)$$

$\therefore$ Ch. E. $m - 3 = 0 \Rightarrow m = 3$.

$\therefore$ homogeneous solution $= C \cdot 3^h$.

Particular solution $= \dfrac{1}{E-3} \cdot 2^h$

$= \dfrac{1}{2-3} \cdot 2^h$ [Replacing E by 2]

$= -2^h$.

$\therefore$ General solution, $y_h = C \cdot 3^h - 2^h$. **Ans.**

Example 2. *Solve* $y_{x+2} - 4y_x = 10 \cdot 3^x$.

Solution. The given difference equation is

$$(E^2 - 4)\, y_x = 10 \cdot 3^x$$

Putting $y_0 = A + 1 \Rightarrow 1 = A + 1 \Rightarrow A = 0$ ...(2)

and $y_1 = B \sin \frac{\pi}{2} + 2 \Rightarrow 0 = B + 2 \Rightarrow B = -2.$ Ans.

Hence the solution of (1) satisfying the initial conditions is

$$y_h = -2 \sin \frac{\pi h}{2} + 2^h.$$

Ans.

Example 5. *Solve* $y_{x+2} - 2ay_{x+1} + (a^2 + b^2)\, y_x = a^x.$

Solution. The given difference equation may be written as

$$[E^2 - 2aE + (a^2 + b^2)]\, y_x = a^x.$$

$\therefore$ Ch. E. $m^2 - 2am + a^2 + b^2 = 0$ ...(1)

$\Rightarrow \quad (m - a)^2 + b^2 = 0 \Rightarrow (m - a)^2 = -b^2$

$\Rightarrow \quad m - a = \pm ib \Rightarrow m = a \pm ib.$

Let $\quad a \pm ib = r \cos \theta \pm ir \sin \theta.$

$\therefore \quad a = r \cos \theta$ and $b = r \sin \theta$

so that $\quad r^2 = a^2 + b^2$ *i.e.*, $r = \sqrt{(a^2 + b^2)}$

and $\quad \tan \theta = b/a$ *i.e.*, $\theta = \tan^{-1}(b/a).$

$\therefore$ homogeneous solution $= r^x [A \cos x\theta + B \sin x\theta]$

$$= [\sqrt{(a^2 + b^2)}]^x [A \cos \{x \tan^{-1}(b/a)\} + B \sin \{x \tan^{-1}(b/a)\}].$$

Particular solution $= \dfrac{1}{E^2 - 2aE + a^2 + b^2} \cdot a^x$

$$= \frac{a^x}{a^2 - 2a \,.\, a + a^2 + b^2} = \frac{a^x}{b^2}.$$

$\therefore$ General solution is given by

$$y_x = [\sqrt{a^2 + b^2}]^x [A \cos \{x \tan^{-1}(b/a)\} + B \sin \{x \tan^{-1}(b/a)\}] + \frac{a^x}{b^2}.$$ Ans.

PROBLEM SET

Solve the following difference equations :

1. $y_{h+2} - y_h = 3^h.$
2. $y_{x+2} - 3y_{x+1} + 2y_x = 5^x.$
3. $u_{h+2} - 7u$ ... 3^h

5. $y_h = (C_1 + C_2 h) \cdot a^h + \dfrac{1}{(1-a)^2}$

6. $y_h = \dfrac{4}{5}(-3)^h - \dfrac{17}{20}(-2)^h + \dfrac{2^h}{20}$.

§ 5.37. (D) TO FIND PARTICULAR SOLUTION WHEN F(h) = sin bh OR cos bh

In this case, particular solution $= \dfrac{1}{f(E)} \sin bh$ or $\dfrac{1}{f(E)} \cos bh$.

Write
$$\sin bh = \frac{e^{ibh} - e^{-ibh}}{2i},$$
$$\cos bh = \frac{e^{ibh} + e^{-ibh}}{2}$$
and then apply method of § 5.37 (C).

We know that $e^{ibh} = \cos bh + i \sin bh$.

Therefore, to evaluate particular solution, the following procedure is useful:

(i) Particular solution $= \dfrac{1}{f(E)} \sin bh$

$= \text{Imaginary part in } \dfrac{1}{f(E)} e^{ibh}$.

Now apply § 5.37 (C)

(ii) Particular solution $= \dfrac{1}{f(E)} \cos bh$

$= \text{Real part } \dfrac{1}{f(E)} e^{ibh}$.

Now apply § 5.37 (C).

ILLUSTRATIVE EXAMPLES

Example 1. *Solve* $y_{h+2} + a^2 y_h = \sin ah$.

Solution. The given difference equation may be written as
$$(E^2 + a^2)\, y_h = \cos ah. \qquad \text{...(1)}$$

$\therefore$ Ch. E. $m^2 + a^2 = 0 \Rightarrow m = \pm ia = a\left(\cos \dfrac{\pi}{2} \pm i \sin \dfrac{\pi}{2}\right)$.

$\therefore$ homogeneous solution $= a^h \left(A \cos \dfrac{h\pi}{2} + B \sin \dfrac{h\pi}{2}\right)$.

Particular solution $= \dfrac{1}{E^2 + a^2} \cdot \sin ah$

$= \text{Imaginary part in } \dfrac{1}{E^2 + a^2} e^{iah}$

$= \text{Imaginary } \dfrac{1}{(E^2 + a^2)} \cdot (e^{ia})^h$

$= \text{Imaginary } \dfrac{e^{iah}}{(e^{ia})^2 + a^2}$

$$= \text{Imaginary } \frac{e^{iah}}{(e^{2ai}+a^2)} \times \frac{e^{-2ai}+a^2}{e^{-2ai}+a^2}$$

$$= \text{Imaginary } \frac{e^{ia(h-2)}+a^2e^{iah}}{1+a^2(e^{2ai}+e^{-2ai})+a^4}$$

$$= \text{Imaginary } \frac{\cos[a(h-2)]+i\sin[a(h-2)]+a^2[\cos ah+i\sin ah]}{1+2a^2\cos 2a+a^4}$$

$$= \frac{\sin[a(h-2)]+a^2\sin ah}{1+2a^2\cos 2a+a^4}$$

$\therefore$ General solution of (1) is given by

$$y_h = a^h\left[A\cos\frac{h\pi}{2}+B\sin\frac{h\pi}{2}\right]+\frac{a^2\sin ah+\sin[a(h-2)]}{1+2a^2\cos 2a+a^4}. \quad \textbf{Ans.}$$

Example 2. *Solve* $y_{h+2}+a^2y_h = \cos ah.$

Solution. Proceeding as Ex. 1, above.

Homogeneous solution $= a^h\left(A\cos\dfrac{h\pi}{2}+B\sin\dfrac{h\pi}{2}\right).$

Particular solution $= \dfrac{1}{E^2+a^2}\cos ah$

$$= \text{Real part in } \frac{1}{E^2+a^2}e^{iah}$$

$$= \text{Real } \frac{e^{ia(h-2)}+a^2e^{iah}}{1+2a^2\cos 2a+a^4} \quad \text{[See Ex. 1]}$$

$$= \frac{\cos[a(h-2)]+a^2\cos ah}{1+2a^2\cos 2a+a^4}.$$

$\therefore$ General solution is given by

$$y_h = a^h\left(A\cos\frac{h\pi}{2}+B\sin\frac{h\pi}{2}\right)+\frac{a^2\cos ah+\cos[a(h-2)]}{1+2a^2\cos 2a+a^4}.$$

Example 3. *Solve* $y_{h+2}+a^2y_h = \cos bh.$

Solution. Proceeding as Ex. 1 above.

Homogeneous solution $= a^h\left[A\cos\dfrac{\pi h}{2}+B\sin\dfrac{\pi h}{2}\right].$

Particular solution $= \dfrac{1}{E^2+a^2}\cos bh$

$$= \text{Real part in } \frac{1}{E^2+a^2}e^{ibh}$$

$$= \text{Real part in } \frac{e^{ibh}}{e^{2ib}+a^2}$$

$$= \text{Real part in } \frac{e^{ibh}}{e^{2ib}+a^2}\times\frac{e^{-2ib}+a^2}{e^{-2ib}+a^2}$$

$$= \text{Real part in } \frac{e^{ib(h-2)}+a^2e^{ibh}}{1+a^2\cdot(e^{2ib}+e^{-2ib})+a^4}$$

$$= \frac{\cos\{b(h-2)\} + a^2 \cos bh}{1 + 2a^2 \cos 2b + a^4}.$$

$\therefore$ General solution is given by

$$y_h = a^h \left[A \cos \frac{\pi h}{2} + B \sin \frac{\pi h}{2} \right] + \frac{a^2 \cos bh + \cos\{b(h-2)\}}{1 + 2a^2 \cos 2b + a^4}$$

Example 4. *Solve* $y_{x+2} - 3y_{x+1} + 2y_x = \cos 2x$.

Solution. The given difference equation is given by

$$(E^2 - 3E + 2)\, y_x = \cos 2x. \quad \text{...(1)}$$

Ch. E. is $m^2 - 3m + 2 = 0 \Rightarrow m = 1, 2$.

$\therefore$ Homogeneous solution $= C_1 \,.\, 1^x + C_2 \,.\, 2^x = C_1 + C_2 \,.\, 2^x$.

$$\text{Particular solution} = \frac{1}{E^2 - 3E + 2} \,.\, \cos 2x$$

$$= \text{Real part in } \frac{1}{(E-1)(E-2)} \cdot e^{2xi}$$

$$= \text{Real part in } \frac{e^{2xi}}{(e^{2i}-1)(e^{2i}-2)}$$

$$= \text{Real part in } \frac{e^{2xi}(e^{-2i}-1)(e^{-2i}-2)}{\{(e^{2i}-1)(e^{-2i}-1)\}\{(e^{2i}-2)(e^{-2i}-2)\}} \quad \text{...(2)}$$

Nr. of (2) $= e^{2xi}[-e^{-4i} - 3e^{-2i} + 2]$

$= e^{2(x-2)i} - 3e^{2(x-1)i} + 2e^{2xi}$,

Dr. of (2) $= \{(e^{2i}-1)(e^{-2i}-1)\}\{(e^{2i}-2)(e^{-2i}-2)\}$

$= \{1 - (e^{2i} + e^{-2i}) + 1\}\{1 - 2(e^{2i} + e^{-2i}) + 4\}$

$= (2 - 2\cos 2) + (5 - 4\cos 2)$

$= 10 - 18\cos 2 + 8\cos^2 2$

$= 10 - 18\cos 2 + 4(1 + \cos 4)$

$= 14 - 18\cos 2 + 4\cos 4$.

Thus (2) becomes,

$$\text{Particular solution} = \text{Real part in } \frac{e^{2(x-2)i} - 3e^{2(x-1)i} + 2e^{2xi}}{14 - 18\cos 2 + 4\cos 4}$$

$$= \frac{\cos\{2(x-2)\} - 3\cos\{2(x-1)\} + 2\cos 2x}{14 - 18\cos 2 + 4\cos 4}.$$

$\therefore$ General solution of (1) is given by

$$y_x = C_1 + C_2 \,.\, 2^x + \frac{\cos\{2(x-2)\} - 3\cos\{2(x-1)\} + 2\cos 2x}{14 - 18\cos 2 + 4\cos 4}. \quad \textbf{Ans.}$$

PROBLEM SET

$\therefore y_h = A\cos\frac{...}{2}$ ... $2\left(1+\cos\frac{2}{3}\right)$

5.57. (E) TO FIND PARTICULAR SOLUTION WHEN F(h) = P(h), P(H) BEING POLYNOMIAL OF DEGREE n IN h

In this case,

The particular solution $= \frac{1}{f(E)} \cdot P(h)$

$$= \frac{1}{f(1+\Delta)} \cdot P(h) \qquad [\text{Since } E \equiv 1+\Delta]$$

$$= (a_0 + a_1\Delta + a_2\Delta^2 + \ldots + a_n\Delta^n + \ldots)\, P(h)$$

$$= (a_0 + a_1\Delta + \ldots + a_n\Delta^n)\, P(h)$$

where the expansion is carried out only upto the term containing Δ^n since $\Delta^m P(h) = 0$ if $m \geq n+1$.

ILUSTRATIVE EXAMPLES

Example 1. *Solve* $y_{h+2} - 2y_{h+1} + y_h = 3h + 4.$

Solution. Given $y_{h+2} - 2y_{h+1} + y_h = 3h + 4$

$$\Rightarrow \quad (E^2 - 2E + 1)\, y_h = 3h + 4$$

$$\Rightarrow \quad (E-1)^2\, y_h = 3h + 4.$$

$\therefore$ Ch. E. $(m-1)^2 = 0 \Rightarrow m = 1, 1.$

Homogeneous solution $= (C_1 + C_2 h) \,.\, 1^h = C_1 + C_2 h.$

Particular solution $= \frac{1}{(E-1)^2} \cdot (3h+4) = \frac{1}{\Delta^2}(3h+4)$

$$= \Delta^{-2}\,[3h^{(1)} + 4h^{(0)}]$$

[Since in factorial notation, $3h + 4 = 3h^{(1)} = 4h^{(0)}$]

$y_h = a$ [

$$y_t = A\cos\frac{x\pi}{2} + B\sin\frac{x\pi}{2} + \frac{\sin\left(\frac{x}{3}\right) + \sin\left(\frac{x-2}{3}\right)}{2\left(1+\cos\frac{2}{3}\right)}$$

37. (E) TO FIND PARTICULAR SOLUTION WHEN F(h) = P(h), P(H) BEING POLYNOMIAL OF DEGREE n IN h

In this case,

$$\text{The particular solution} = \frac{1}{f(E)} \cdot P(h)$$

$$= \frac{1}{f(1+\Delta)} \cdot P(h) \qquad [\text{Since } E \equiv 1 + \Delta]$$

$$= (a_0 + a_1\Delta + a_2\Delta^2 + \ldots + a_n\Delta^n + \ldots)\, P(h)$$

$$= (a_0 + a_1\Delta + \ldots + a_n\Delta^n)\, P(h)$$

ere the expansion is carried out only upto the term containing Δ^n since $P(h) = 0$ if $m \geq n + 1$.

ILUSTRATIVE EXAMPLES

Example 1. *Solve* $y_{h+2} - 2y_{h+1} + y_h = 3h + 4.$

Solution. Given $y_{h+2} - 2y_{h+1} + y_h = 3h + 4$

$\Rightarrow$ $$(E^2 - 2E + 1)\, y_h = 3h + 4$$

$\Rightarrow$ $$(E-1)^2\, y_h = 3h + 4.$$

$\therefore$ Ch. E. $(m-1)^2 = 0 \Rightarrow m = 1, 1.$

omogeneous solution $= (C_1 + C_2 h) \,.\, 1^h = C_1 + C_2 h.$

$$\text{Particular solution} = \frac{1}{(E-1)^2} \cdot (3h+4) = \frac{1}{\Delta^2}(3h+4)$$

$$= \Delta^{-2}\, [3h^{(1)} + 4h^{(0)}]$$

[Since in factorial notation, $3h + 4 = 3h^{(1)} = 4h^{(0)}$]

$$= \frac{3h^{(3)}}{2.3} + \frac{4h^{(2)}}{1.2} = \frac{1}{2}h^{(3)} + 2h^{(2)}$$

$$= \frac{1}{2}h(h-1)(h-2) + 2h(h-1)$$

$$= \frac{1}{2}h(h-1)(h+2).$$

Hence general solution is

$$y_h = C_1 + C_2 h + \frac{1}{2}h(h-1)(h+2).$$

Example 2. *Solve* $y_{x+2} + y_{x+1} + y_x = x^2 + x + 1$.

Solution. The given equation may be written as

$$(E^2 + E + 1)\, y_x = x^2 + x + 1.$$

$\therefore$ Ch. E. is $m^2 + m + 1 = 0$

$$\Rightarrow \qquad m = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}.$$

Let $\qquad -\frac{1}{2} \pm \frac{i\sqrt{3}}{2} = r(\cos\theta \pm i\sin\theta).$

$\therefore \qquad r\cos\theta = -\frac{1}{2}$ and $r\sin\theta = \frac{\sqrt{3}}{2}.$

So that $\qquad r = 1, \theta = \tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}.$

$\therefore$ Homogeneous solution $= A\cos\frac{2x\pi}{3} + B\sin\frac{2x\pi}{3}.$

Particular Solution (P.S.) $= \dfrac{1}{E^2 + E + 1}\cdot(x^2 + x + 1).$ ...(1)

Now, $\qquad E^2 + E + 1 = (1+\Delta)^2 + (1+\Delta) + 1 = 3 + 3\Delta + \Delta^2$

$$= 3\left[1 + \frac{3\Delta + \Delta^2}{3}\right].$$

Also factorial notations of $x^2 + x + 1$ is

$$x^2 + x + 1 = x^2 - x + 2x + 1 = x(x-1) + 2(x) + 1$$

$$= x^{(2)} + 2x^{(1)} + x^{(0)}.$$

Therefore from (1), we have

$$\text{P. S.} = \frac{1}{3\left(1 + \frac{3\Delta + \Delta^2}{3}\right)}\cdot[x^{(2)} + 2x^{(1)} + x^{(0)}]$$

$$= \frac{1}{3}\left[1 + \left(\frac{3\Delta + \Delta^2}{3}\right)\right]^{-1}\cdot[x^{(2)} + 2x^{(1)} + x^{(0)}]$$

$$= \frac{1}{3}\left[1 - \frac{3\Delta + \Delta^2}{3} + \left(\frac{3\Delta + \Delta^2}{3}\right)^2 - \ldots\right][x^{(2)} + 2x^{(1)} + x^{(0)}]$$

$$= \frac{1}{3}\left[1 - \Delta - \frac{\Delta^2}{3} + \Delta^2 + \ldots\right][x^{(2)} + 2x^{(1)} + x^{(0)}]$$

$$= \frac{1}{3}\left[1 - \Delta + \frac{2}{3}\Delta^2 + \ldots\right][x^{(2)} + 2x^{(1)} + x^{(0)}]$$

$$= \frac{1}{3}[x^{(2)} + 2x^{(1)} + x^{(0)} - \Delta(x^{(2)} + 2x^{(1)} + x^{(0)}) + \frac{2}{3}\Delta^2(x^{(2)} + 2x^{(1)} + x^{(0)})]$$

$$= \frac{1}{3}[x^{(2)} + 2x^{(1)} + 1 - (2x^{(1)} + 2 + 0) + \frac{2}{3}(2! + 0 + 0)]$$

$$= \frac{1}{3}[x(x-1) + 2x + 1 - 2x - 2 + \frac{4}{3}] = \frac{x^2}{3} - \frac{x}{3} + \frac{1}{9}.$$

$\therefore$ General solution is given by

$$y_x = A\cos\frac{2x\pi}{3} + B\sin\frac{2x\pi}{3} + \frac{x^2}{3} - \frac{x}{3} + \frac{1}{9}.$$

Ans.

Example 3. *Solve* $y_{h+2} - y_{h+1} - 2y_h = h^2$.

Solution. The given difference equation is

$$(E^2 - E - 2)\,y_h = h^2.$$

$\therefore$ Ch. E. $m^2 - m - 2 = 0 \Rightarrow (m+1)(m-2) = 0 \Rightarrow m = -1, 2.$

$\therefore$ Homogeneous solution $= C_1 . (-1)^h + C_2 . 2^h$.

$$\text{P.S.} = \frac{1}{E^2 - E - 2} . h^2. \qquad \text{...(1)}$$

Now $\quad E^2 - E - 2 = (1+\Delta)^2 - (1+\Delta) - 2 = -2 + \Delta + \Delta^2$

$$= -2\left[1 - \frac{\Delta + \Delta^2}{2}\right].$$

Factorial notation of h^2 is

$$h^2 = h(h-1) + h = h^{(2)} + h^{(1)}.$$

$$\therefore \quad \text{P. S.} = \frac{1}{-2\left[1 - \frac{\Delta+\Delta^2}{2}\right]} . \{h^{(2)} + h^{(1)}\} \qquad \text{[From (1)]}$$

$$= -\frac{1}{2}\left[1 - \frac{\Delta+\Delta^2}{2}\right]^{-1} . \{h^{(2)} + h^{(1)}\}$$

$$= -\frac{1}{2}\left[1 + \frac{\Delta}{2} + \frac{\Delta^2}{2} + \frac{\Delta^2}{4} + \ldots\right]\{h^{(2)} + h^{(1)}\}$$

$$= -\frac{1}{2}\left[h^{(2)} + h^{(1)} + \frac{1}{2}\Delta\{h^{(2)} + h^{(1)}\} + \frac{3}{4}\Delta^2\{h^{(2)} + h^{(1)}\}\right]$$

$$= -\frac{1}{2}\left[h^{(2)} + h^{(1)} + \frac{1}{2}\{2h^{(1)} + 1\} + \frac{3}{4}\{2 + 0\}\right]$$

$$= -\frac{1}{2}\left[h(h-1) + h + h + \frac{1}{2} + \frac{3}{2}\right]$$

$$= -\frac{1}{2}h^2 - \frac{1}{2}h - 1.$$

Ans.

$\therefore$ General solution is given by

$$y_h = C_1 . (-1)^h + C_2 . 2^h - 1 - \frac{1}{2}h - \frac{1}{2}h^2.$$

Example 4. *Solve* $y_{x+1} - y_x = x^2$ *when* $y_0 = 1$.

Solution. The given difference equation may be written as

$$(E-1)\,y_x = x^2.$$

Ch. E. $m - 1 = 0 \Rightarrow m = 1$.

$\therefore$ Homogeneous solution $= C_1 \,.\, 1^x = C_1$.

$$\text{P.S.} = \frac{1}{E-1} \,.\, x^2 = \frac{1}{\Delta}\,[x^{(2)} + x^{(1)}]$$

$$= \Delta^{-1}\,[x^{(2)} + x^{(1)}] = \frac{x^{(3)}}{3} + \frac{x^{(2)}}{2}$$

$$= \frac{1}{6}\,[2x^{(3)} + 3x^{(2)}] = \frac{1}{6}\,[2x\,(x-1)\,(x-2) + 3x\,(x-1)]$$

$$= \frac{1}{6}\,x\,(x-1)\,(2x-1).$$

$\therefore$ The general solution of the given equation is

$$y_x = C_1 + \frac{1}{6}\,x\,(x-1)\,(2x-1).$$

Now putting $x = 0$ in (1), we get

$$y_0 = C_1 + 0 \Rightarrow 1 = C_1.$$

Putting this value of C_1 in (1), the required solution of the given equation is

$$y_x = 1 + \frac{1}{6}\,x\,(x-1)\,(2x-1).$$

Ans.

Example 5. *Solve* $y_{h+2} + 2y_{h+1} + y_h = 2^h + h^2 + h$.

Solution. The given difference equation may be written as

$$(E^2 + 2E + 1)\,y_h = 2^h + h^2 + h$$

Ch. E. $m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0$

$$\Rightarrow m = -1, -1.$$

Homogeneous solution $= (C_1 + C_2 h)\,.\,(-1)^h$.

$$\text{P. S.} = \frac{1}{E^2 + 2E + 1}\,[2^h + h^2 + h]$$

$$= \frac{1}{(E+1)^2} \cdot 2^h + \frac{1}{(E+1)^2} \cdot (h^2 + h) = I_1 + I_2 \text{ (say)}$$

$$I_1 = \frac{1}{(E+1)^2} \cdot 2^h = \frac{2^h}{(2+1)^2}$$ [Replacing E by 2]

$$= \frac{1}{9}\,.\,2^h$$

$$I_2 = \frac{1}{(E+1)^2}\,(h^2 + h) = \frac{1}{(2+\Delta)^2}\,[h^{(2)} + 2h^{(1)}]$$

$$= \frac{1}{4}\left[1 + \frac{\Delta}{2}\right]^{-2}\,[h^2 + 2h^{(1)}]$$

$$= \frac{1}{4}\left[1 - 2\,.\,\frac{\Delta}{2} + \frac{\Delta^2}{4} \ldots\right]\,[h^{(2)} + 2h^{(1)}]$$

Scanned with CamScanner

$$= \frac{1}{4}\left[h^{(2)} + 2h^{(1)} - \Delta\{h^{(2)} + 2h^{(1)}\} + \frac{1}{4}\Delta^2\{h^{(2)} + 2h^{(1)}\}\right]$$

$$= \frac{1}{4}\left[h^{(2)} + 2h^{(1)} - \{2h^{(1)} + 2\} + \frac{1}{4}\{2 + 0\}\right]$$

$$= \frac{1}{4}\left[h(h-1) + 2h - 2h - 2 + \frac{1}{2}\right] = \frac{h^2}{4} - \frac{h}{4} - \frac{3}{8}$$

$$= \frac{1}{4}\left(h^2 - h - \frac{3}{2}\right).$$

$\therefore$ General solution is given by

$$y_h = (C_1 + C_2 h)(-1)^h + \frac{2^h}{9} + \frac{1}{4}\left(h^2 - h - \frac{3}{2}\right).$$

Example 6. *Solve* $y_{h+3} + 2y_{h+2} + y_{h+1} = 2^h + h^2 + h.$

Solution. The given equation may be written as

$$(E^2 + 2E + 1)\, y_{h+1} = 2^h + h^2 + h. \quad \text{...(1)}$$

Now proceeding exactly as Ex. 5, above, the general solution of (1) is given by

$$y_{h+1} = (C_1 + C_2 h)(-1)^h + \frac{2^h}{9} + \frac{1}{4}\left(h^2 - h - \frac{3}{2}\right).$$

Now replacing h by $h-1$, we get

$$y_h = \{C_1 + C_2(h-1)\}(-1)^{h-1} + \frac{2^{h-1}}{9} + \frac{1}{4}\left[(h-1)^2 - (h-1) - \frac{3}{2}\right]$$

$$= (A + Bh)(-1)^h + \frac{2^h}{18} + \frac{1}{4}\left(h^2 - 3h + \frac{1}{2}\right)$$

[where $A = -C_1 + C_2$ and $B = -C_2$]

Ans.

PROBLEM SET

Solve the following difference equations :

1. $3y_{h+1} - y_h = h.$
2. $y_{h+1} - 3y_h = h, y_0 = 1.$
3. $y_{n+2} - 4y_n = n^2.$
4. $y_{n+2} - 4y_n = n^2 + n - 1.$
5. $y_{h+2} + 2y_{h+1} + y_h = 3h^2 + 4^h.$
6. $y_{x+2} - 6y_{x+1} + 8y_x = x^2 + 3 . 5^x.$

ANSWERS

2. $y_h = \frac{5}{4} . 3^h - \frac{1}{4}(2h+1)$

... $(2^n) - \frac{n^2}{3} - \frac{7n}{9} - \frac{17}{27}$

§ 5.37. (F) TO FIND PARTICULAR SOLUTION WHEN F(h) = b^h . P(h), P(H) BEING POLYNOMIAL OF DEGREE n IN h

$$\textit{Particular solution} = \frac{1}{f(E)} \cdot b^h P(h)$$

$$= b^h \cdot \frac{1}{f(bE)} \cdot P(h).$$

The method of § 5.37 (E) *is used.*

Proof. We have

$$F(h) = b^h \,.\, P(h)$$

$$\therefore \quad f(E)\, F(h) = f(E)\, b^h \,.\, P(h)$$

or $$f(E)\,[b^h \,.\, P(h)] = (E^n + g_1 E^{n-1} + \ldots + g_n)\, b^h \,.\, P(h)$$

$$= E^n\,[b^h P(h)] + g_1 E^{n-1}\,[b^h P(h)] + \ldots + g_n\, b^n P(h). \quad \ldots(1)$$

Now, $$E^n\,[b^n P(h)] = b^{h+n} P(h+n)$$

$$= b^h \,.\, b^n \,.\, E^n P(h)$$

$$= b^h (bE)^n P(h). \quad \ldots(2)$$

Now replacing n by $n-1, n-2, \ldots, 2, 1$ in (2), we get

$$E^{n-1}\,[b^h P(h)] = b^h (bE)^{n-1} P(h)$$

$$E^{n-2}\,[b^h P(h)] = b^h (bE)^{n-2} P(h)$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$E\,[b^h P(h)] = b^h (bE)\, P(h).$$

Substituting these values in (1), we obtain

$$f(E)\,[b^h P(h)] = b^h\,[(bE)^n + g_1 (bE)^{n-1} + \ldots + g_n]\, P(h)$$

$$= b^h f(bE)\, P(h).$$

The operator inversion gives

$$\frac{1}{f(E)}\,[b^h P(h)] = b^h \frac{1}{f(bE)} \cdot P(h).$$

Particular case. To find $\frac{1}{f(E)} \cdot b^h$ when $f(b) = 0$.

In this case, we have

$$\text{Particular solution} = \frac{1}{f(E)}\, b^h = \frac{1}{f(E)}\, b^h \,.\, 1 = \frac{1}{f(E)}\, b^h \,.\, h^{(0)}$$

$$= b^h \cdot \frac{1}{f(bE)} \cdot h^{(0)}.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Solve* $y_{x+1} - 3y_x = 3^x \,.\, x^2$.

$$= b^{x} \cdot \frac{}{f(bE)} \cdot h^{\cdot}$$

ILLUSTRATIVE EXAMP

Example 1. *Solve* $y_{x+1} - 3y_x = 3^x \cdot x^2$.

Solution. $$y_{x+1} - 3y_x = 3^x \cdot x^2$$

$$\Rightarrow \quad (E - 3)\, y_x = 3^x \cdot x^2.$$

Ch. E. is $m - 3 = 0 \Rightarrow m = 3$.

Homogeneous solution $= C \cdot 3^x$.

$$= 3^x \frac{1}{3E - 3} x^{(2)} + \frac{3^x}{3} \frac{1}{(E - 1)} x^{(1)}$$

$$= 3^{x-1} \frac{1}{\Delta} [x^{(2)} + x^{(1)}] = 3^{x-1} \left[\frac{x^{(3)}}{3} + \frac{x^{(2)}}{2}\right]$$

$$= \frac{3^{x-1}}{6} [2x(x - 1)(x - 2) + 3x(x - 1)]$$

$$= \frac{3^x}{18} [x(x - 1)(2x - 1)]$$

The general solution is given by

$$y_x = C \cdot 3^x + \frac{3^x}{18} x(x - 1)(2x - 1)$$ **Ans.**

Example 2. *Solve* $y_{k+1} - a y_k = a^k (2k + 1)$ *with* $y_0 = \frac{1}{a}$

Solution. We have $y_{k+1} - a y_k = a^k (2k + 1)$

$$(E - a) y_k = a^k (2k + 1)$$

Ch. E. $m - a = 0 \Rightarrow m = a$.

Homogeneous solution $= C \cdot a^k$.

$$\text{P. S.} = \frac{1}{E - a} a^k (2k + 1) = a^k \frac{1}{aE - a} (2k + 1)$$

$$= \frac{a^k}{a} \frac{1}{E - 1} [2k^{(1)} + k^{(0)}] = a^{k-1} \frac{1}{\Delta} [2k^{(1)} + k^{(0)}]$$

$$= a^{k-1} \left[\frac{2k^{(2)}}{2} + \frac{k^{(1)}}{1}\right] = a^{k-1} [k(k - 1) + k] = k^2 a^{k-1}$$

General solution, $y_k = C \cdot a^k + k^2 a^{k-1}$ ...(1)

Putting $k = 0$ in (1), we get

$$y_0 = C \Rightarrow \frac{1}{a} = C$$

Hence from (1), **Ans.**

$$y_k = \frac{1}{a} \cdot a^k + k^2 a^{k-1} \quad \text{or} \quad y_k = (1 + k^2) a^{k-1}$$

Example 3. *Solve* $y_{k+2} + 4y_{k+1} - 12y_k = 2^k \cdot (k - 1)$.

Solution. The given equation may be written as

$$(E^2 + 4E - 12) y_k = 2^k \cdot (k - 1)$$

$$(E - 2)(E + 6) y_k = 2^k \cdot (k - 1)$$

Ch. E. is $(m - 2)(m + 6) = 0 \Rightarrow m = 2, -6$.

Homogeneous solution $= C_1 \cdot 2^k + C_2 \cdot (-6)^k$

$$\text{P. S.} = \frac{1}{(E - 2)(E + 6)} \cdot 2^k (k - 1)$$ (Note)

$$= 2^k \cdot \frac{1}{(2E - 2)(2E + 6)} (k - 1)$$

$\therefore$ General solution is given by

$$y_h = C_1 \cdot 2^h + C_2 \cdot (-6)^h + \frac{2^h}{64}(2h^2 - 7h).$$

Example 4. *Solve* $y_{h+2} - 5y_{h+1} + 6y_h = 3^h$.

Solution. We have $y_{h+2} - 5y_{h+1} + 6y_h = 3^h$

or $$(E^2 - 5E + 6)\, y_h = 3^h$$

or $$(E-2)(E-3)\, y_h = 3^h.$$

Ch. E. is $(m-2)(m-3) = 0 \Rightarrow m = 2, 3.$

Homogeneous solution $= C_1 \cdot 2^h + C_2 \cdot 3^h.$

$$\text{P.S.} = \frac{1}{(E-2)(E-3)} \cdot 3^h$$

$$= \frac{1}{E-3} \cdot \left[\frac{1}{(E-2)} 3^h\right] = \frac{1}{E-3} \cdot \left[\frac{3^h}{3-2}\right]$$

$$= \frac{1}{E-3} \cdot 3^h = 3^h \cdot \frac{1}{3E-3} \cdot 1$$

$$= \frac{3h}{3} \cdot \frac{1}{E-1} \cdot 1 = \frac{3^h}{3} \cdot \frac{1}{\Delta}(1) = 3^{h-1} \cdot h.$$

$\therefore$ General solution is given by

$$y_h = C_1 \cdot 2^h + C_2 \cdot 3^h + h \cdot 3^{h-1}.$$

Example 5. *Solve* $u_{x+3} - 5u_{x+2} + 8u_{x+1} - 4u_x = 2^x \cdot x.$

Solution. The given difference equation may be written as

$$(E^3 - 5E^2 + 8E - 4)\, u_x = 2^x \cdot x.$$

$\therefore$ Ch. E. is $m^3 - 5m^2 + 8m - 4 = 0$

or $$m^2(m-1) - 4m(m-1) + 4(m-1) = 0$$

or $$(m-1)(m^2 - 4m + 4) = 0$$

or $$(m-1)(m-2)^2 = 0 \Rightarrow m = 1, 2, 2.$$

$\therefore$ Homogeneous function $= C_1 + (C_2 + C_3 x) \cdot 2^x$

$$\text{P. S.} = \frac{1}{(E^3 - 5E^2 + 8E - 4)} \cdot 2^x \cdot x$$

$$= \frac{1}{(E-1)(E-2)^2} 2^x \cdot x = 2^x \cdot \frac{1}{(2E-1)(2E-2)^2} \cdot x$$

$$= \frac{2^x}{4} \cdot \frac{1}{\{2(1+\Delta)-1)\} \cdot \Delta^2} \cdot x$$

$$= 2^{x-2} \cdot \frac{1}{\Delta^2 (1 + 2\Delta)} \cdot x = 2^{x-2} \cdot \frac{1}{\Delta^2} (1 + 2\Delta)^{-1} \cdot x$$

$$= 2^{x-2} \cdot \frac{1}{\Delta^2} [1 - 2\Delta + \ldots] \, x^{(1)}$$

$$= 2^{x-2} \cdot \frac{1}{\Delta^2} [x^{(1)} - 2x^{(0)}]$$

$$= 2^{x-2} \left[\frac{x^{(3)}}{2 \times 3} - 2 \cdot \frac{x^{(2)}}{1 \times 2} \right]$$

$$= \frac{2^{x-2}}{6} [x(x-1)(x-2) - 6x(x-1)]$$

$$= \frac{2^{x-2}}{6} x(x-1)(x-8).$$

eneral solution is

$$u_x = C_1 + (C_2 + C_3 x) \cdot 2^x + \frac{2^{x-2}}{6} \cdot x(x-1)(x-8)$$

$$u_x = C_1 + (C_2 + C_3 x)\, 2^x + \frac{2^x}{24} (x^3 - 9x^2 + 8x)$$

$$= C_1 + (C_2 + C_3 x)\, 2^x + 2^x \left(\frac{x^3}{24} - \frac{3}{8} x^2 \right).$$ **Ans.**

[The term $\frac{1}{3} x \cdot 2^x$ is included in $C_3 x \cdot 2^x$]

ple 6. *Solve* $y_{x+2} - 2 \cos \alpha \, y_{x+1} + y_x = \cos x\alpha.$ ...(1)

be written as