Date : Page :

Mathematics

Discrete ~~Structure~~

CS3BS03

UNIT - 1

SET THEORY

Collection of well - defined objects is called set.
The object of the set is called the elements of the set.
Generally, set is ~~done~~ denoted by capital letters A, B, C, ... and the elements of set are denoted by small letters a, b, c...

Symbol	Meaning
$\Rightarrow$ (Implies)	$a \Rightarrow b$
$\Leftrightarrow$ (Both implies)	$a \Leftrightarrow b$
$\forall$ (for all)	
$\in$ (belongs to)	
$\notin$ (does not belongs to)	
$\because$ (Since)	
$\therefore$ (Therefore)	
$\exists$ (there exist)	
... (Show on)	
$\mid$ or : (such that)	
$\subseteq$ (subset)	
$\subsetneq$ (propere subset)	

1 If A having three elements namely x_1, x_2, x_3 then we write...

$A = \{x_1, x_2, x_3\}$ / $\forall x \in A$

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Date : Page :

> Types of Set – (a) Null set
(b) Singleton set
(c) finite set
(d) Infinite set

(e) Subset ⇒ Let A and B be two sets, if every element of A is in B then A is called a subset of B.
It is represented as $A \subseteq B$.

Ex:- $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5, 6\}$
⇒ $A \subseteq B$ but $B \not\subseteq A$ or $B \supseteq A$

* Each set is a subset of itself.
i.e., $A \subseteq A$ and $B \subseteq B$
* Null set is a subset of all sets.

* Proper subset → (1) Let A and B be two sets. Then A is called a proper subset of B, if
(a) $A \subseteq B$
(b) $\exists\, x \in B$ such that $x \in A$, it is denoted by $A \subsetneq B$.
* Each set is not a proper subset of itself.

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Date : Page :

2) $\mathbb{N}$ → The set of all natural numbers
$= \{1, 2, 3, \ldots\}$ → Set builder form or Tabular form
$= \{x \mid x \text{ is a natural no.}\}$

3) $\mathbb{Z}$ → The set of all integers
$= \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$
$= \{0, \pm 1, \pm 2, \pm 3, \ldots\}$
$= \{x \mid x \text{ is an integer}\}$

4) $\mathbb{W}$ → The set of all whole no.
$= \{0, 1, 2, 3, \ldots\}$

5) $\mathbb{N}$ –
6) $\mathbb{R}$ –
7) $\mathbb{C}$ –

γ → gamma
Γ → (capital) gamma

> Representation of Set –

1) Set builder form. [Tabular form]
2) Property form.

Ex:- If $A = \{x \mid x^2 - 9 = 0, x \in N\}$, then write tabular form of A.

Sol.) $x^2 - 9 = 0$
$x = 3, -3 \Rightarrow A = \{3\}$

Ex:- $A = \{x \mid x^2 + 9 = 0, x \in \mathbb{Z}\}$
$= \{\} = \phi$

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Discrete Mathematics

UNIT - 1

1) Equal set :- Two set A and B are called equal set
(i) Every element of A is an element of B
i.e., $A \subseteq B$
(ii) Every element of B is also an element of A.
i.e., $B \subseteq A$

$$A = B \Leftrightarrow A \subseteq B \ \& \ B \subseteq A$$

2) Universal set
3) Power set :- Let A be a set. The set of all subsets of A is called a power set of A. It is denoted by P(A)

Ex:- If $A = \{1, 2, 3\}$, then
$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

Here, $|A| = 3$, $|P(A)| = 2^3 = 8$.

Q.) If $A = \{a, b, c, d\}$, then find $P(A) = ?$

Theorem :- If $|A| = n$, then show $|P(A)| = 2^n$.

Proof → Suppose $A = \{a_1, a_2, a_3, \ldots\ldots a_n\}$
The no. of all subsets having 1-1 elements $= {}^nC_1$
The no. of all subsets having r elements $= {}^nC_r$.

The no. of all subsets having n elements $= {}^nC_n$
And, we know that null set is a subset of all sets

PAGE NO.:
DATE:

$\Rightarrow$ Total no. of subsets of $A = 1 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \dots {}^nC_n$

We know that,

$(x+a)^n = x^n + {}^nC_1 x^{n-1} \cdot a + {}^nC_2 x^{n-2} \cdot a^2 + \dots + {}^nC_n \cdot a^n$

$x = 1, a = 1$

$2^n = 1 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n$

$\Rightarrow \boxed{2^n = P(A)}$

Q.E.D

> Operation on sets — $A \cup B$, $A \cap B$, $A - B$

(1) Union of sets $\Rightarrow$ Let A and B be two sets, the union of A and B is the set of all those elements which are in A or in B or in A and B.

$A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ or } x \in A \,\&\, B\}$

If $A = \{1, 2, 3, 4\}$ & $B = \{3, 4, 5, 6\}$, then

$A \cup B = \{1, 2, 3, 4, 5, 6\}$

(2) $A \cap B = \{3, 4\}$

(3) $A - B \rightarrow$ Elements present in A not in B.

$= \{1, 2\}$

Symmetric difference of A and B

$A \Delta B = (A - B) \cup (B - A)$

) If $A = \{a, b, c, d\}$ and $B = \{a, d, e, f, g\}$ then find $A \cup B$, $B \cup A$, $A \cap B$, $B \cap A$, $A - B$, $B - A$, $A \Delta B$.

A.) $A \cup B = B \cup A = \{a, b, c, d, e, f, g\}$

$A \cap B = \{a, d\} = B \cap A$

$A - B = \{b, c\}$, $B - A = \{e, f, g\}$

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$A \Delta B = \{b, c, e, f, g\}$

Show that $\rightarrow$ $A - (A - B) = A \cap B$

▸ Symmetric Difference –

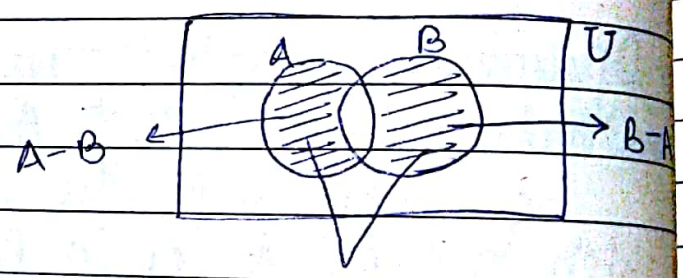
Let A and B be two sets. The symm. diff. of A & B is denoted by $A \Delta B$ and defined as,

$$A \Delta B = (A - B) \cup (B - A)$$

Remark: 1 $\Rightarrow$

If $x \in A \cup B$

$\Rightarrow$ $x \in A$ or $x \in B$ or $x \in A$ ~~and~~ & B both.



$A \Delta B$

or $\rightarrow$ union

& $\rightarrow$ intersection

2) If $x \notin A \cup B$

$\Rightarrow$ $x \notin A$ and $x \notin B$

3) If $x \in A \cap B$

$\Rightarrow$ $x \in A$ and $x \in B$

4) If $x \notin A \cap B$

$\Rightarrow$ $x \notin A$ or $x \notin B$ or $x \notin A$ & B both.

5) If $x \notin A$ $\Rightarrow$ $x \in A^c$

6) If $x \in A - B$ $\Rightarrow$ $x \in A$ but $x \notin B$.

7) $x \notin A - B$ $\Rightarrow$ $x \in A$ & $x \in B$ or $x \notin A$

($x \in A$, $x \in B$ or $x \notin A$)

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PAGE NO.:
DATE:

Algebra on Set →

Q.) If A and B are two sets, show that
(i) commutative law
(a) $A \cup B = B \cup A$
(b) $A \cap B = B \cap A$

Sol^n (a) → Let $x \in A \cup B$.
$\Rightarrow$ $x \in A$ or $x \in B$ or $x \in A$ & $\cancel{x \in}$ B.
$\Rightarrow$ $x \in B$ or $x \in A$ or $x \in B$ & A.
$\Rightarrow$ $x \in B \cup A$
$\Rightarrow$ $A \cup B \subseteq B \cup A$ — (1)

Let $x \in B \cup A$
$\Rightarrow$ $x \in B$ or $x \in A$ or $x \in B$ & A.
$\Rightarrow$ $x \in A$ or $x \in B$ or $x \in A$ & B.
$\Rightarrow$ $x \in A \cup B$
$\Rightarrow$ $B \cup A \subseteq A \cup B$ — (2)
from (1) and (2)
$$\boxed{A \cup B = B \cup A}$$

$$\boxed{(B \cup C)^c = U - (B - C)}$$

Q) If A, B, C are three sets, then show that
(a) $A - (B \cup C) = (A - B) \cap (A - C)$
(b) $A - (B \cap C) = (A - B) \cup (A - C)$

Sol.^{n} (a) Let $x \in A - (B \cup C)$
$\Leftrightarrow$ $x \in A$ but $x \notin B \cup C$
$\Leftrightarrow$ $x \in A$ but $x \notin B$ and $x \notin C$

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$\Leftrightarrow$ $x \in A$ but $x \notin B$ and $x \in A$ but $x \notin C$

$\Leftrightarrow$ $x \in A - B$ and $x \in A - C$

$\Leftrightarrow$ $x \in (A-B) \cap (A-C)$

Q.) If A and B are two sets, then show that

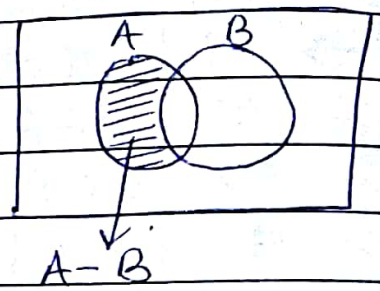
$$A - (A-B) = A \cap B$$

Sol^n) Let $x \in A - (A-B)$

$\Rightarrow$ $x \in A$ but $x \notin A - B$

$\Leftrightarrow$ $x \in A$ but $x \in A$ ~~and~~ & $x \in B$

or $x \notin A$



$\Leftrightarrow$ $x \in A$ but $x \in A$, $x \in B$.

$\Leftrightarrow$ $x \in A$ and $x \in B$

$\Leftrightarrow$ $x \in A \cap B$ $\Leftrightarrow$ $\boxed{A - (A-B) = A \cap B}$

> Principle of Inclusion and Exclusion (App^n of set the

Theorem. 1 If A and B are two finite sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Proof $\Rightarrow$ We know that $A \cup B$ is the set of all those elements which are in A or in B or in A and both.

$\Rightarrow$ The no. of distinct elements in $A \cup B$ = The no. of distinct elements in A + The no. of distin elements in B − the no. of distinct elements in A

$\Rightarrow$ $|A \cup B| = |A| + |B| - |A \cap B|$

Q.E.D

PAGE NO.:
DATE:

Theorem: 2 If A, B and C are three finite sets, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Proof :- let $P = B \cup C$

LHS :- $|A \cup P| = |A| + |P| - |A \cap P|$

substitute the values of P,

$$\Rightarrow |A \cup B \cup C| = |A| + |B \cup C| - |A \cap (B \cup C)|$$

$$= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)|$$

(by distributive law)

$$= |A| + |B| + |C| - |B \cap C| - [|A \cap B| + |A \cap C| - |A \cap B \cap A \cap C|]$$

$$= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$$

$$\Rightarrow |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Q.E.D

Theorem: 3 If A_1, A_2, A_3 and A_4 are four finite sets, then

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| + |A_4| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4| + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_4|$$

Q.) Among 100 students, 32 study maths, 30 stud physics, 45 study ~~pt~~ biology, 15 study maths & biology, 7 study maths & physics, 10 study

physics & biology and 30 do not study any of the 3 subjects.

(i) Find the no. of students studying all the 3 subjects.

(ii) Find the no. of students studying exactly 1 of the 3 subjects.

Sol^n) Let X denote the set of all students, & then

$|X| = 100$

Let M, P and B be the set of all students studying maths, physics and bio respectively, then

$|M| = 32$, $|P| = 20$, $|B| = 45$

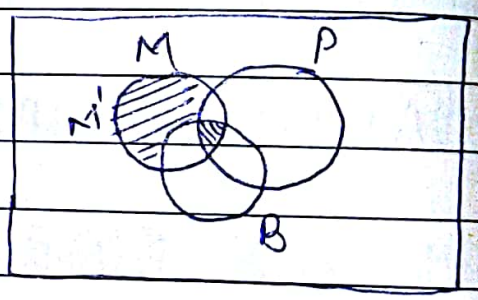
$|M \cap B| = 15$, $|M \cap P| = 7$, $|P \cap B| = 10$,

$|(M \cup P \cup B)^c| = 30$

Here,

$|M \cup P \cup B| = |X| - |(M \cup P \cup B)^c|$

$= 100 - 30 = 70$



(i) We know that

$|M \cup P \cup B| = |M| + |P| + |B| - |M \cap P| - |M \cap B| - |P \cap B| + |M \cap P \cap B|$

$\Rightarrow |M \cap P \cap B| = 70 - 32 - 20 - 45 + 7 + 15 + 10$

$= 5$ Ans.

(ii) Let M', P' and B' be the set of all students studying only maths, physics and bio resp., then

$|M'| = |M| - |M \cap P| - |M \cap B| + |M \cap P \cap B|$

$= 32 - 7 - 15 + 5$

$= 15$ Ans.

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Similarly, $|P'| = |P| - |M \cap P| - |P \cap B| + |M \cap P \cap B|$

$= 20 - 7 - 10 + 5$

$= 8$ Ans

And,

$|B| = |B| - |M \cap B| - |P \cap B| + |M \cap P \cap B|$

$= 45 - 15 - 10 + 5$

$= 25$ Ans

$\therefore$ Total no. of students studying exactly 1 subject

$= 15 + 8 + 25 = 48$ Ans.

Imp

Q) Find the no. of integers b/w 1 and 250 that are divisible by any of the integers, 2, 3, 5 and 7.

Sol^n) Let A_1, A_2, A_3, A_4 denote the set of all integers b/w 1 and 250 that are divisible by 2, 3, 5 and 7 resp., then

$|A_1| = \left\lfloor \frac{250}{2} \right\rfloor = 125$

GIF

$|A_2| = \left\lfloor \frac{250}{3} \right\rfloor = \lfloor 83.3 \rfloor = 83$

$|A_3| = \left\lfloor \frac{250}{5} \right\rfloor = 50$

$|A_4| = \left\lfloor \frac{250}{7} \right\rfloor = \lfloor 35.7 \rfloor = 35$

$|A_1 \cap A_2| = \left\lfloor \frac{250}{6} \right\rfloor = \lfloor 41.6 \rfloor = 41$

$|A_1 \cap A_3| = \left\lfloor \frac{250}{10} \right\rfloor = 25$

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PAGE NO.:
DATE:

Cartesian product of two sets.

Let A and B be two sets. The cartesian product of A and B is the set of all ordered pair (a, b) where first component belongs to A and the second component belongs to B. It is denoted $A \times B$.

Mathematically,

$$A \times B = \{ (a, b) \mid a \in A, b \in B \}$$

Q.) If $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, then $A \times B = ?$

Sol^n) $A \times B = \{ (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,2), (3,3), (3,4) \}$

* Remark : (i) If $|A| = m$, $|B| = n$ then $|A \times B| = |A||B|$
$= mn$

$B \times A = \{ (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (4,1), (4,2), (4,3) \}$

(ii) $A \times B \neq B \times A$

(iii) $A \times B = B \times A$, if $A = B$

(iv) $|A \times B| = |B \times A|$

> Relation –

If $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$

R = An element of A is less than or equal to the elements of B ($\leq$)

(statement)

$= (1 \leq 2), (1 \leq 3), (1 \leq 4), (2 \leq 2), (2 \leq 3), (2 \leq 4), (3 \leq 3), (3 \leq 4)$

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or

$1R2,\ 1R3,\ 1R4,\ 2R2,\ 2R3,\ 2R4,\ 3R3,\ 3R4$

or

$R = \{(1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4)\}$

Total no. of Relations from A to B = Subsets of $R = 2^{9} = 2^{mn}$
or if $|A| = m$ and $|B| = n$.

$R = \{(x,y) \mid x \in A, y \in B, xRy\}$

Let A and B be two sets. A subset of $A \times B$ is called a relation from A to B. It is denoted by R.
Mathematically,

$$R = \{(x,y) \mid x \in A, y \in B, xRy\}$$

- Domain of a relation - Let R be a relation from A to B i.e., $R \subseteq A \times B$.

The domain of R is the set of all first component of the ordered pair (x,y) which belongs to R.

$$D(R) = \{x \mid (x,y) \in R, \text{ for some } y \in B\}$$

Ex: If $A = \{1,2,3,4,5\}$ & $B = \{a,b,c,d\}$
Total no. of relations, $R = 2^{5\times4} = 2^{20}$

$R_1 = \{(1,a), (1,b), (2,c), (3,c)\}$

$D(R_1) = \{1,2,3\}$

Range, $R(R_1) = \{a,b,c\}$

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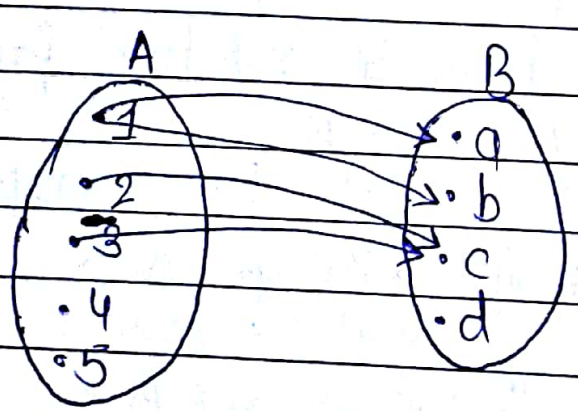
PAGE NO.:
DATE:

> Representation of a relation –
(i) Graphical Rep^n (Pictorial)
(ii) Matrix Rep^n
(iii) Disc Rep^n

Q.) Write the matrix rep^n of and disc rep^n
$R_1 = \{(1,a), (1,b), (2,c), (3,c)\}$ if
$A = \{1,2,3,4,5\}$ & $B = \{a,b,c,d\}$

R_1	a	b	c	d
1	1	1	0	0
2	0	0	1	0
3	0	0	1	0
4	0	0	0	0
5	0	0	0	0

⇒ Matrix rep^n



⇒ Disc rep^n

Q.) Give the pictorial rep^n of the relation,
prop. ⇒ $R = \{(x,y) \mid x,y \in \mathbb{R},\ x^2+y^2 = 1\}$

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Inverse of a Relation –

Let R be a relation from A to B i.e., $R = A \times B$.
The inverse of R is a relation from B to A and it is defined as $R^{-1} = \{ (y,x) \mid (x,y) \in R, x \in A, y \in B \}$

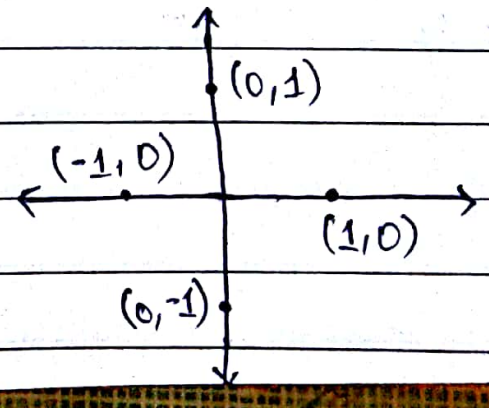
Ex:- If $A = \{1,2,3\}$
$B = \{a,b,c,d\}$ and
$R_1 = \{ (1,a) \ (1,b) \ (2,c) \}$ → dir^n from A to B
Total no. of relations $= 2^{3 \times 4} = 2^{12}$
find R_1^{-1}.

Sol^n)
$R_1^{-1} = \{ (a,1), (b,1), (c,2) \}$
dir^n from B to A.

Imp
Relation in a set A – (Binary rel^n)
A subset of $A \times A$ is called a relation in a set A
i.e.,
$R = \{ (x,y) \mid x,y \in A \ \& \ xRy \}$

Q.) If I denote the set of all integers, defined a relation R in a set
(prop. form) $R = \{ (x,y) \mid x^2 + y^2 = 1, x,y \in I \}$
[* Not a circle, only if, $x,y \in R$]
OR
(Tabular form) $R = \{ (1,0), (0,1), (-1,0), (0,-1) \}$

(Pictorial form)



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PAGE NO.:
DATE:

▸ Properties of a relation in set A -

(i) Reflexive
(ii) Symmetric
(iii) Transitive
(iv) Anti-symmetric

(i) Let R be a relation in a set A and every element of A related to itself, ~~where~~ under R then is called a reflexive relation.

i.e., $xRx, \forall x \in A$

or $(x, x) \in R, \forall x \in A$.

Ex:- If $A = \{1, 2, 3\}$, then $R_1 = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$ is a reflexive but

$R_2 = \{(1,1), (2,2), (1,2), (1,3)\}$

is not reflexive relation b/s $(3,3) \notin R_2$.

(ii) Let R be a relation in a set A, then R is called a symmetric rel^n if

$xRy \Rightarrow yRx$, where $x, y \in A$.

or

$(x, y) \in R \Rightarrow (y, x) \in R$, where $x, y \in A$

Ex:- R_1 is symmetric.

R_2 is non-symmetric.

• Theorem :- If R is a symmetric relation, then show that $R^{-1} = R$.

Proof :-

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(iii) A relation R in a set A is called a transitive relation if xRy and $yRz \Rightarrow xRz$, where $x, y, z \in A$.

or

$(x,y) \in R$ and $(y,z) \in R \rightarrow p$

$\Rightarrow (x,z) \in R$, where $x, y, z \in A$.

$\downarrow$

q

or

$(x,z) \notin R \Rightarrow (x,y) \notin R$ or $(y,z) \notin R$, where $x, y, z \in A$.

Tautology,

$p \Rightarrow q$

i.e., $\sim p \Rightarrow \sim q$

Demorgan's law,

$\sim(t \wedge s) \equiv \sim t \vee \sim s$

Ex:- If $A = \{1, 2, 3\}$, then and

$R_1 = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$

$\Rightarrow R_1$ is transitive.

$R_2 = \{(2,3), (3,2)\}$

$\Rightarrow R_2$ is non-transitive.

$(1,1)$ & $(1,2)$

$\downarrow$

$(1,2)$

$(2,2)$ & $(2,1)$

$\downarrow$

$(2,1)$

* $R_3 = \{(3,4)\}$

Applying contrapositive statement,

$(3,z) \in R_3 \Rightarrow (3,4) \in R_3$ but $(4,z) \notin R_3$

$\therefore R_3$ is transitive.

* $R_4 = \{(1,2), (3,4)\}$

R_4 is transitive.

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(iv) A relation R in a set A is called an anti-symmetric relation if xRy and yRx where $x, y \in A$.
$\Rightarrow x = y$,

If $\underbrace{(x,y) \in R \ \& \ (y,x) \in R}_{p} \Rightarrow \underbrace{x = y}_{q}$

If (Contrapositive statement)
$x \neq y \Rightarrow (x,y) \notin R$ or $(y,x) \notin R$

$p \Rightarrow q$
$\sim q \Rightarrow \sim p$

Ex:- $A = \{1, 2, 3\}$

(i) $R_1 = \{(1,2), (2,1), (1,3)\}$ Total relations $= 2^{3\times3}$
Not an anti-symmetric relation.

(ii) $R_2 = \{(1,2)\}$
Anti-symmetric relation by contrapositive statem

(iii) $R_3 = \{(1,2), (1,3), (3,1)\}$
Not an " —— "

(iv) $R_4 = \{(1,2), (1,1), (2,2), (3,3)\}$
Anti-symmetric relation.

$(x,y) \in R_4$, $(y,z) \in R_4$
$\Rightarrow (x,z) \in R_4$

or

If $(x,z) \notin R_4 \Rightarrow (x,y) \notin R_4$ or $(y,z) \notin R$

> Equivalence relation – A relation R in a set is called an equivalence relation, if

(i) R is reflexive i.e., $xRx \ \forall x \in A$ or $(x,x) \in R, \ \forall x \in A$

(ii) R is symmetric i.e.,

$xRy \Rightarrow yRx$, where $x, y \in A$.

or $(x,y) \in R \Rightarrow (y,x) \in R$, $\forall\ x, y \in A$.

(iii) R is transitive i.e.,

If xRy & $yRz \Rightarrow xRz$, where $x, y, z \in A$

or $(x,y) \in R$ & $(y,z) \in R \Rightarrow (x,z) \in R$, where $x, y, z \in A$.

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Ex:- If I denote the set of all integers, defined a relation R in a set I as,

$R = \{ (x,y) \mid x-y \text{ is divisible by } 3 \}$

Show that R is an equivalence relation.

Sol^n) (i) R is reflexive :

Let $x \in I$ be an arbitrary element.

To show :- $(x,x) \in R$

Now, $x - x = 0 = 3 \cdot 0$ which is divisible by 3.

$\Rightarrow (x,x) \in R$, $\forall\ x \in I$.

$\Rightarrow$ R is reflexive.

(ii) R is symmetric :

Let $(x,y) \in R$, where $x, y \in I$

To show :- $(y,x) \in R$

Since, $(x,y) \in R$

$\Rightarrow (x-y)$ is divisible by 3.

$\Rightarrow (x-y) = 3k$, $k \in I$

$\Rightarrow y - x = -3k = 3(-k)$

$\Rightarrow (y-x)$ is divisible by 3.

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$\Rightarrow$ $(y, x) \in R$, $x, y \in I$

$\Rightarrow$ R is symmetric.

(iii) R is transitive:

Let $(x, y) \in R$ & $(y, z) \in R$, where $x, y, z \in$

To show :- $(x, z) \in R$

Since, $(x, y) \in R$ & $(y, z) \in R$

$\Rightarrow$ $x - y = 3K_1$ — (a) $K_1 \in I$

$\Rightarrow$ $y - z = 3K_2$ — (b) $K_2 \in I$

Add (a) and (b)

$\Rightarrow$ $x - z = 3(K_1 + K_2)$

$\Rightarrow$ $x - z = 3K$, $K = K_1 + K_2 \in I$

$\Rightarrow$ $(x - z)$ is divisible by 3

$\Rightarrow$ $(x, z) \in R$

$\Rightarrow$ R is transitive.

Hence, R is an equivalence relation.

IMP

Ex :- If R_1 and R_2 are two equivalence relation in a set A, then show that $R_1 \cap R_2$ is also an equivalence relation in a set A.

Sol^n) Given, $R_1 \subseteq A \times A$, $R_2 \subseteq A \times A$

$R_1 \cap R_2 \subseteq A \times A$

$\Rightarrow$ * $R_1 \cap R_2$ is a relation in a set A.

(i) $R_1 \cap R_2$ is reflexive :

To show :- $(x, x) \in R_1 \cap R_2$, $\forall x \in A$.

Since, R_1 & R_2 are eq^n relation,

$\Rightarrow$ R_1 and R_2 are reflexive

$\Rightarrow (x,x) \in R_1, \quad \forall x \in A$ &

$(x,x) \in R_2, \quad \forall x \in A$

$\Rightarrow (x,x) \in R_1 \cap R_2, \quad \forall x \in A$

$\Rightarrow R_1 \cap R_2$ is reflexive.

(ii) $R_1 \cap R_2$ is symmetric:

Let $(x,y) \in R_1 \cap R_2$, where $x, y \in A$.

To show :- $(y,x) \in R_1 \cap R_2$

Since, $(x,y) \in R_1 \cap R_2$

$\Rightarrow (x,y) \in R_1$ & $(x,y) \in R_2$

$\Rightarrow (y,x) \in R_1$ & $(y,x) \in R_2$

$\because R_1$ and R_2 are eq^n rel^n.

$\Rightarrow R_1$ and R_2 are symmetric.

$\Rightarrow (y,x) \in R_1 \cap R_2$

$\Rightarrow R_1 \cap R_2$ is symmetric.

(iii) $R_1 \cap R_2$ is transitive:

Let $(x,y) \in R_1 \cap R_2$, & $(y,z) \in R_1 \cap R_2$,

where $x, y, z \in A$.

To show :- $(x,z) \in R_1 \cap R_2$

Since $(x,y) \in R_1 \cap R_2$, $(y,z) \in R_1 \cap R_2$

$\Rightarrow (x,y) \in R_1$ & $(x,y) \in R_2$ and

$(y,z) \in R_1$ & $(y,z) \in R_2$

$\Rightarrow (x,z) \in R_1$ & $(x,z) \in R_2$

$\because R_1$ and R_2 are equivalence rel^n.

$\Rightarrow R_1$ and R_2 are transitive

$\Rightarrow (x,z) \in R_1 \cap R_2$

$\Rightarrow R_1 \cap R_2$ is transitive.

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Hence, $R_1 \cap R_2$ is an equivalence relation.
Q.E.D

Q) If R is an eq^n rel^n in a set A, then show that R^{-1} is also an eq^n rel^n.

Q) If I denote the set of all integers other than 0 defined a relation R in a set I as,

$$xRy \Rightarrow x^y = y^x$$

Is R an equivalence rel^n?

FUNCTION

Let A and B be two sets. If every element of A is related to an unique element of B by some definite rule 'f'. This rule f is called a fu

Mathematically,

$$f : A \to B$$

Where, A is called a domain of f and B is called co-domain of f.

Range of f: If $f : A \to B$ is a f^n

$$R(f) = \{ y \in B \mid \exists\, x \in A,\ f(x) = y \}$$

is called the range of f.

Q.) If $A = \{1, 2, 3\}$ and $B = \{a, b, c, d\}$

Total no. of relations $= 2^{3 \times 4} = 12$

(i) $R_1 = \{(1, a), (1, b), (2, c)\}$

It is not a f^n b/s element 3 is not present and or element d has no pre-image in A.

~~given relation~~

(b) $R_2 = \{(1,a), (2,b), (3,c)\}$ → 1-1, into

(c) $R_3 = \{(1,d), (2,c)$

(d) $R_4 = \{(1,c), (2,c), (3,d)\}$ → many one, into

(e) $R_5 = \{(2,d)\}$

(f) $R_6 = \{(1,d), (2,d), (3,c)\}$ → many one, into

(g) $R_7 = \phi$

(h) $R_8 = \{(1,b), (2,c)\}$

One-one function → A f^n $f: A \to B$ is called a one-one function, if different elements of A have different images in B. i.e.,

1st $x_1 \neq x_2$

$\Rightarrow f(x_1) \neq f(x_2)$ or

2nd $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Onto function → A f^n $f: A \to B$ is called an onto f^n, if $\forall y \in B \ \exists x \in A$

$f(x) = y$ → (Image of x is y / Pre-image of y is x)

i.e., $\boxed{R(f) = B}$

Ex:- Defined a f^n $f: R - \{0\} \to R$ as $f(x) = 1/x$. Is f one-one, onto?

Ans) for 1-1:

1st Let $x_1, x_2 \in R - \{0\}$ such that

$x_1 \neq x_2$

$1/x_1 \neq 1/x_2$

$f(x_1) \neq f(x_2) \Rightarrow f$ is one-one.

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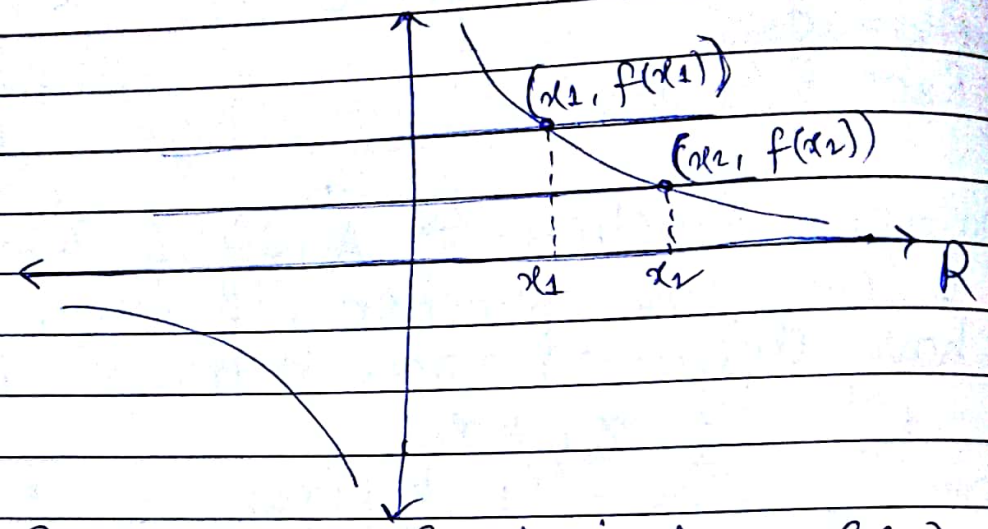
PAGE NO.:
DATE:

2nd Let $f(x_1) = f(x_2)$, where $x_1, x_2 \in R$

$$\frac{1}{x_1} = \frac{1}{x_2}$$

$\Rightarrow$ $x_1 = x_2$ $\Rightarrow$ f is one-one.

Graphically,



Q.) If $f: R \to R$ is a f^n defined as $f(x) = e^x$, where R is the set of all real numbers, then show that f is one-one, but not onto.

Sol.) For one-one,

1st Let $x_1, x_2 \in R$ such that

$$x_1 \neq x_2$$

$$\Rightarrow e^{x_1} \neq e^{x_2}$$

$$\Rightarrow f(x_1) \neq f(x_2)$$

$\Rightarrow$ f is one-one.

2nd Let $f(x_1) = f(x_2)$, where $x_1, x_2 \in R$

$$e^{x_1} = e^{x_2}$$

Taking log both sides,

$$\Rightarrow \log e^{x_1} = \log e^{x_2}$$

$$\Rightarrow x_1 = x_2$$

$\Rightarrow$ f is one-one.

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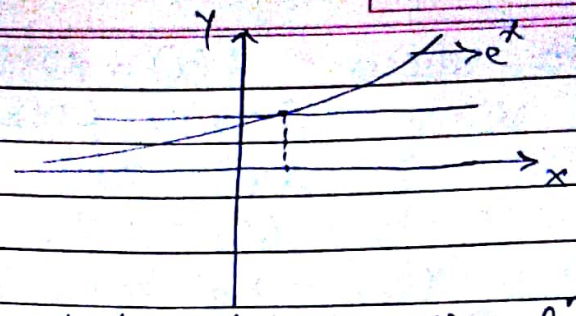
3rd Graphically,
A line ∥ to x-axis cuts the graph of $f(x)$ at only one points OR



There is only one pt. of intersection of the f^n $f(x)$ and ~~a~~ line ∥ to x-axis.
Therefore, f is one-one.

for onto,
Let $y \in R$ $\exists$ $x \in R$ (above and below)
Graphically, each line ∥ to x-axis must intersect the graph of $f(x)$ at one point.

$$f(x) = y$$
$$\Rightarrow e^x = y$$
$$\Rightarrow x = \ln y$$

If $y > 0$ $\exists$ $x = \ln y$ such that $f(x) = f(\ln y)$

If $f : R \to R^+$, then
Let $y \in R^+$ $\exists$ $x \in R$
such that $f(x) = y$
$$\Rightarrow e^x = y$$
$$\Rightarrow x = \ln y$$
for each $y \in R^+$ $\exists$ $x = \ln y$ ~~satisfy~~ such that f.
$$\Rightarrow f(x) = y$$
$\Rightarrow$ f is onto.

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PAGE NO.:
DATE:

Q.) To show $f(x) = x^2$ many one & onto.

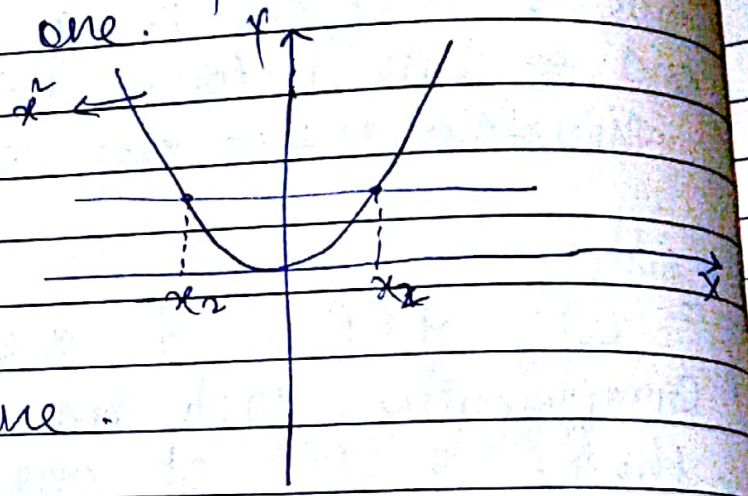
Sol.) Since, $f(x_1) = f(x_2)$, such that $x_1, x_2 \in$

$x_1^2 = x_2^2$

$x_1 = \pm x_2$

$\Rightarrow$ $x_2, -x_2$ have same pre-image i.e., x_1

$\Rightarrow$ f is many one.



Graphically,
Graph of $f(x)$ and a line $\parallel$ to x-axis intersect at 2 points.
Therefore, f is many one.

for onto,

Let $y \in R^+$ $\exists$ $x \in R$

such that $f(x) = y$

$\Rightarrow x^2 = y$

$\Rightarrow x = \sqrt{y}$

~~for each~~ $y \in R^+$ $\exists$ $x = \sqrt{y}$ such that $f(x) = y$

$\Rightarrow$ f is onto.

Imp.

Q.) If X, Y are two sets and A, B are subsets and $f: X \to Y$ is a f^n. Then show that,

(i) $f(A \cup B) = f(A) \cup f(B)$

(ii) $f(A \cap B) = f(A) \cap f(B)$

Sol^n) Let $y \in f(A \cup B)$ be an arbitrary element.

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PAGE NO.:
DATE:

To show :- $y \in f(A) \cup f(B)$

Since, $y \in f(A \cup B)$

$\Leftrightarrow$ $y = f(x)$ for some $x \in A \cup B$.

$\Leftrightarrow$ $y = f(x)$ for some $x \in A$ or $x \in B$ or $x \in A$ & B

$\Leftrightarrow$ $y \in f(A)$ or $y \in f(B)$ or $y \in f(A)$ & $f(B)$

$\Leftrightarrow$ $y \in f(A) \cup f(B)$

$\Leftrightarrow$ $f(A \cup B) = f(A) \cup f(B)$

Q.) If $f : X \to Y$ is a one-one and onto f^n, then show that $f^{-1} : Y \to X$ is also one-one and onto.

Ex:- If $X = \{1, 2, 3, 4, 5, 6\}$, $Y = \{a, b, c, d, e, f\}$

Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5\}$

defined $f : X \to Y$ as,

$f(1) = a$, $f(2) = a$, $f(3) = c$,

$f(4) = c$, $f(5) = d$, $f(6) = 6$

$f(A) = \{a, c\}$, $f(B) = \{c, d\}$

$A \cup B = \{1, 2, 3, 4, 5\}$

$f(A \cup B) = \{a, c, d\} = f(A) \cup f(B)$

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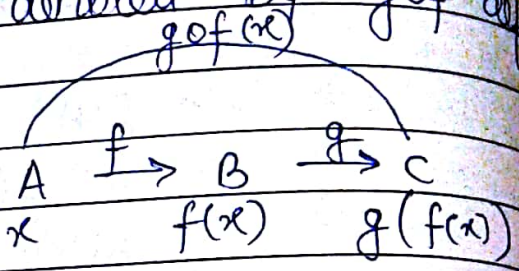
Composition of two functions -

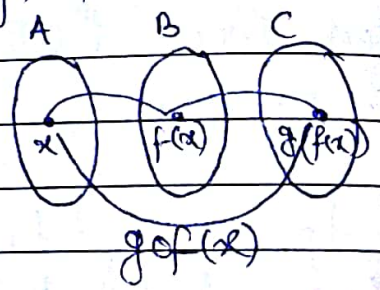
Let f and g be two functions from A to B, and B to C resp. The composition f^n of f and g is the f^n from A to C and is denoted by gof defined as $gof(x) = g(f(x))$



• Condition :-

(i) $R(f) \subseteq D(g)$

* If $R(f)$ is not a subset of $D(g)$, then we cannot define $gof(x)$.



Imp

Q.) Let $f : R \to R$ be given by $f(x) = -x^2$ and $g : R^+ \to R^+$ be given by $g(x) = \sqrt{x}$, where R^+ is the set of all non-negative real numbers and R is the set of all real numbers. Compute fog. Can gof be defined?

Sol.) Here,

$D(f) = R$ —(i) $C(f) = R$,

$R(f) = (-\infty, 0]$

$D(g) = R^+$, $C(g) = R^+$,

$R(g) = [0, \infty)$ —(ii)

To compute fog, this cond. must be satisfied,

* $R(g) \subseteq D(f)$

from (i) and (ii)

$\Rightarrow R(g) \subseteq D(f)$

So, fog exists

$\Rightarrow fog : R^+ \to R$

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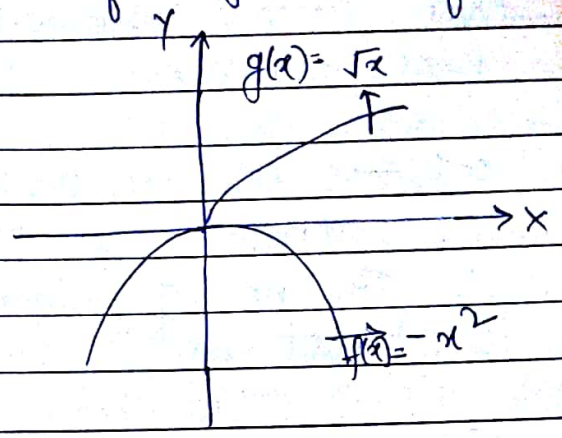
Now,

$$fog(x) = f(g(x)) = f(\sqrt{x}) = -(\sqrt{x})^2 = -x$$

Since, $R(f)$ is not a subset of $D(g)$ therefore we cannot define gof.



Imp

Q.) Let $f : R \to R$ be given by $f(x) = x^2$ and $g : R \to R$ be given by $g(x) = \sin x$, where R is the set of all real numbers. Compute fog. Can gof be defined?

Sol.) Here,

$D(f) = R$, $C(f) = R$ $R(f) = [0, \infty)$

$D(g) = R$, $C(g) = R$ $R(g) = [-1, 1]$

Here, for fog,

$R(g) \subseteq D(f)$

$\therefore$ fog exists

$\Rightarrow$ $fog : R \to R$

$$\Rightarrow fog(x) = f(g(x)) = f(\sin x) = (\sin x)^2$$

for gof,

Here, $R(f) \subseteq D(g)$

$\therefore$ gof exists

$$\Rightarrow gof(x) = g(f(x)) = g(x^2) = \sin x^2$$ Ans

Also, $fog(x) \neq gof(x)$

State and prove Pigeonhole principle :-

- Statement → If n pigeons (object) are assigned m pigeonhole (box), such that $n > m$, then e that atleast one pigeonhole (box) contains two or more pigeons.

- Proof → let us assume, each pigeonhole contains atmost one pigeon.

∴ The no. of pigeons assigned to m pigeonhole is atmost m.

$n = 5$ (1, 2, 3, 4, 5)
$m = 0 \quad 0 \quad 0 \quad 0$

Since, $m < n$, therefore $(n - m)$ or more pigeons are left without having assigned pigeonhole.

Hence, if all pigeons are to be assigned pigeonh then atleast one pigeon hole contains two or mo pigeons.

Generalised Pigeonhole principle :-

If n pigeons (object) are assigned to m pigeonho (box), such that $n > m$ then some pigeonho must contain atleast $\left[\frac{n-1}{m}\right] + 1$ pigeons.

- Proof → let us assume each pigeonhole contai atmost $\left[\frac{n-1}{m}\right]$ pigeons.

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Therefore, the no. of pigeons assigned to m pigeonhole is atleast $m \times \left[\frac{n-1}{m}\right] \leq (n-1)$.

Since, $m<n$, therefore $n-(n-1)$ or more pigeons are left without having assigned pigeonhole.

Hence, if all pigeons are to be assigned pigeonhole then atleast one pigeonhole contains two or more pigeons.

P-1) A company has 13 employees then show that atleast two of the employees were born in the same month.

Sol^n) Taking 13 employees as pigeons and 12 months as pigeonhole.

Then by pigeonhole principle, there are atleast two of the employees who were born in the same month.

P-2) Prove that among 1 lakh people, there are two people who ~~were~~ are born on same time in a day.

Sol^n) Here, $n = 1,00,000$

$m = 24 \times 60 \times 60 = 86,400$ sec.

By generalised pigeonhole principle,

$$= \left[\frac{n-1}{m}\right] + 1$$

$$= \left[\frac{99,999}{86,400}\right] + 1 = 1 + 1 = \underline{\underline{2}}$$

Hence, there are two people who are born on same time in a day.

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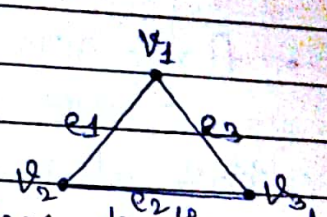
UNIT - 4
Graph Theory

- Graph (G) is a set of vertices and edges denoted by
  $G = \{V, E\}$

where, $V = \{v_1, v_2, v_3, \ldots, v_n\}$
$E = \{e_1, e_2, e_3, \ldots, e_n\}$



Graph is a mapping from the set of edges to the set of vertices such that every edge is associated with pair of two vertices (calling end pt.).

$$G : E \rightarrow V$$

- Selfloop – If an edge start and finish on the same vertex, then it is called selfloop.

- Parallel edges (multiple edges) –
  If two or more edges have same pair of vertices (same end pt.) then they are called parallel edges.

- Simple graph – A graph which has neither a selfloop nor parallel edges.

~~Max. no. of vertex~~ =
Max. degree of n vertices = $n-1$

- Multi graph – A graph which include multiple edg (no selfloop).

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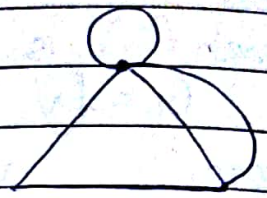
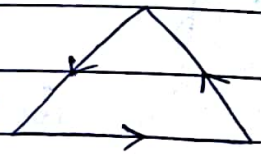
- General graph (pseudo) – A graph which contains self-loops and multiple edges



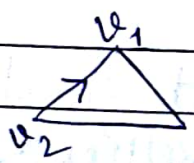
- finite and Infinite graph –
  If no. of vertices and edges are limited (countable), then it is called finite graph otherwise infinite.

- Directed graph – A corelation b/w ordered vertices express directed graph.

(v_1, v_2)



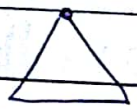
(v_2, v_1)



- Order of graph – Total no. of vertices is called order of that graph. (No. of points) $o(G)$

- Degree of vertex – Total no. of passing edges.

*In self loop, degree = 2



$d = 2$

- Isolated vertex – If degree of vertex is zero, then it is called isolated vertex.

  Symbol → •

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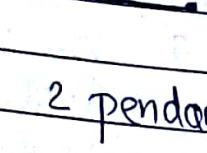
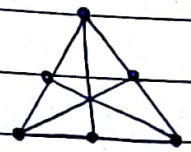
PAGE NO.:
DATE:

- Pendant vertex – If degree of the vertex is 1, then it is called pendant vertex.
  Symbol →



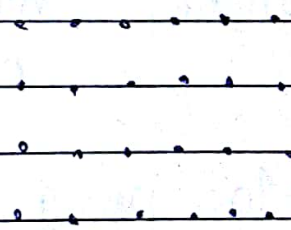
2 pendant vertex →

- Regular Graph – If in the graph all vertices have equal degree, then it is called regular graph.



- Trivial Graph – A graph which contains only one isolated vertex.

- Null graph – A graph which contains multiple isolated vertices.



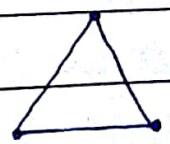
- Complete graph (Universal graph) –

If every vertex is connected with remaining vertices.

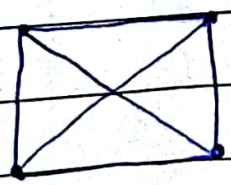
* Degree = $n-1$

It is denoted by K_n.

Ex:- K_3 →



K_4 →



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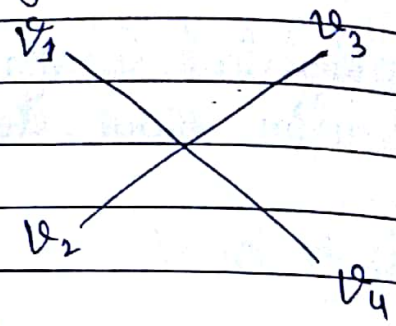
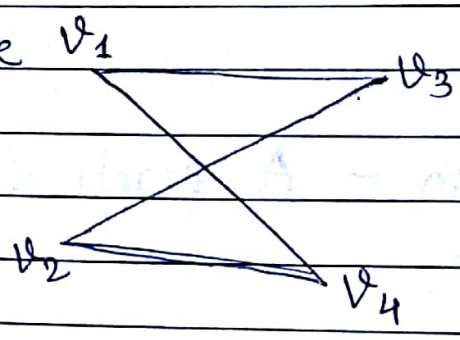
PAGE NO.:
DATE:

▷ Bipartite graph – If the vertex (V) of any graph can be divided into two non-empty disjoint subsets V_1 and V_2, such that the vertex of V_1 connected with the vertex of V_2 with the help of any edge, then the graph is called bipartite graph.

$V = \{v_1, v_2, v_3, v_4\}$
$V_1 = \{v_1, v_2\}$
$V_2 = \{v_3, v_4\}$



* Complete bipartite



It is denoted by $K_{m,n}$.
Where, m = total no. of vertices of subset V_1.
n = total no. of vertices of subset V_2.

Max. no. of vertices of whole graph (complete bipartite) $= m+n$

Max. no. of edges of whole graph $= mn$

(1) ▷ Hand shaking theorem :-
The sum of decrease of all vertices in a graph is equal to the twice of the no. of edges.

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$$\boxed{\sum_{i=1}^{n} d(v_i) = 2e}$$

$d \rightarrow$ degree
$v \rightarrow$ vertex

Proof :- Let a graph $G\{V, E\}$
where, order of vertices $o(V) = n$ and
$o(E) = e$

This theorem is based on mathematical induction concept.

Case 1 ⇒ When $e = 0$, i.e., no edges and degree of each vertex is 0.

$$\Rightarrow \sum_{i=1}^{n} d(v_i) = 0 = 0 \times 2 = 2e$$

Hence, theo. is true in this case.

Case 2 ⇒ When $e = 1$, i.e., one edge

$$\Rightarrow \sum_{i=1}^{n} d(v_i) = 1 + 1 = 2 \times 1 = 2e$$

Hence, theo. is true in this case.

Case 3 ⇒ When $e = e - 1$.
Assume that theo. is true for all graphs having $e-1$ edges.
Let G' is a new graph such that $G' = G - e$.
By hypothesis,

$$\sum_{i=1}^{n} d(v_i) = 2(e-1)$$

Now, we replace the edge in G' and as a result degree of each vertex will be increased by 1.

$$\Rightarrow \sum_{i=1}^{n} d(v_i) = 2(e-1) + 2 = 2e$$

Hence, it is proved that theo. is completely established

(2) In any graph, the no. of vertices of odd degree is always even.

$G(V, E)$

$V = V_o \cap V_e$

where, V_o = Vertices of odd degrees

V_e = Vertices of even degrees

$$\sum_{i=1}^{n} d(v_i) = \sum_{\text{odd deg}} d(v_i) + \sum_{\text{even deg}} d(v_i)$$

Acc. to the hand shaking theo.,

$$\Rightarrow 2e - 2k = \sum_{\text{odd deg}} d(v_i)$$

$$\Rightarrow \text{even quantity} = \sum_{\text{odd deg}} d(v_i)$$

To make the sum of odd no. an even quantity, it is compulsory that the no. of terms is even

(3) The max. no. of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$

Given that, graph is simple graph, i.e., no selfloops and no parallel edges.

There are n vertices, it means

$V = \{v_1, v_2, v_3, \ldots, v_n\}$

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To obtain max. no. of edges, we have to connect every vertex with remaining vertices.

⇒ v_1 will be connected with $v_2, v_3, v_4, \ldots, v_n$ and we obtain $(n-1)$ no. of edges.

⇒ v_2 will be connected with $v_3, v_4, \ldots, v_n$ and we obtain $(n-2)$ no. of edges.

⇒ v_n will not be connected with any ~~edge~~ vertex and we obtain (0) no. of edges.

⇒ v_{n-1} will be connected with v_n and we obtain (1) no. of edge.

To obtain max. no. of edges, we have to collect all results.

$$\text{Max. no. of edges} = (n-1) + (n-2) + \ldots + 2 + 1 + 0$$
$$= 1 + 2 + 3 + \ldots + (n-1) \cancel{+(n-2)}$$

Above series is divergent series of natural no.

We know that,

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

⇒ Here, $n = n - 1$

$$\Rightarrow \quad = \frac{n(n-1)}{2}$$

Theo:

(4) The total no. of edges in a complete graph with n vertices is $\frac{n(n-1)}{2}$

given that,

The graph is complete, it means every vertex is created with remaining vertices.

We can say that degree of each vertex is $n-1$.

Acc. to the hand shaking theo.,

$$\sum_{i=1}^{n} d(v_i) = 2e$$

$$\Rightarrow d(v_1) + d(v_2) + d(v_3) + \cdots + d(v_n) = 2e$$

$$\Rightarrow (n-1) + (n-1) + (n-1) + \cdots + (n-1) = 2e$$

$$n(n-1) = 2e$$

$$\boxed{\frac{n(n-1)}{2} = e}$$

Theo:

(5) Let G be a simple graph with n vertices. If G has k components then max. no. of edges that ~~we~~ G can have is $\frac{(n-k)(n-k+1)}{2}$.

Given that, graph is simple and having n vertices. It is also given that there exists k components.

$G_1, G_2, G_3, \ldots, G_k$ with

$n_1, n_2, n_3, \ldots, n_k$ vertices.

Now,

$$n_1 + n_2 + n_3 + \cdots + n_k = \sum_{i=1}^{k} n_i = n \quad \text{—— (i)}$$

We know that,

Max. no. of edges in a simple graph is $\frac{n(n-1)}{2}$.

$$\Rightarrow \frac{n_1(n_1-1)}{2} + \frac{n_2(n_2-1)}{2} + \cdots + \frac{n_k(n_k-1)}{2}$$

$$= \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2}$$

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$$= \frac{1}{2} \sum_{i=1}^{k} n_i (n_i - 1)$$ or

$$= \frac{1}{2} \sum_{i=1}^{k} n_i^2 - n_i \quad \text{—— (ii)}$$

$$\Rightarrow \sum_{i=1}^{k} (n_i - 1) = (n_1 - 1) + (n_2 - 1) + (n_3 - 1) + \dots + (n_k - 1)$$
$$= n - k$$

On squaring both sides,

$$\Rightarrow \sum_{i=1}^{k} n_i^2 + \sum_{i=1}^{k} 1 - \sum_{i=1}^{k} 2n_i = (n-k)^2$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 = (n-k)^2 - k + 2n$$

Subtract $\sum_{i=1}^{k} n_i$ both the sides;

$$\Rightarrow \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i\right] = \left[(n-k)^2 - k + 2n - \sum_{i=1}^{k} n_i\right] \frac{1}{2}$$

$$\Rightarrow \frac{1}{2} \sum_{i=1}^{k} n_i^2 - n_i = \frac{1}{2}\left[(n-k)^2 - k + n\right]$$

$$= \frac{1}{2}(n-k)\left[(n-k) + 1\right]$$

Hence proved

Isomorphic Graphs –

Two graphs are called isomorphic if their graph theo. prop. are same. It means there exist one to one correspondence b/w vertices & edges.

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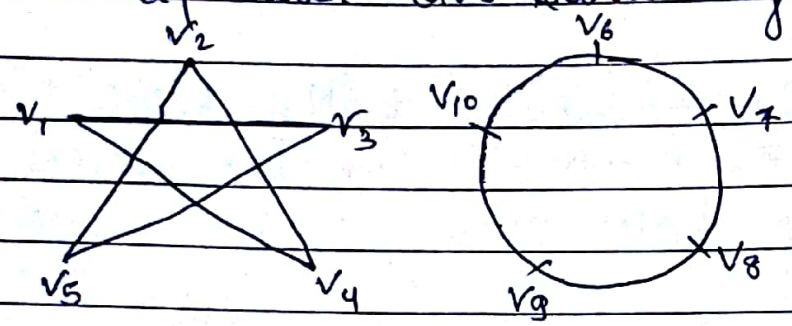
for isomorphism following prop. should be satisfied,

(i)

(ii) They have same no. of vertices with a given degree.

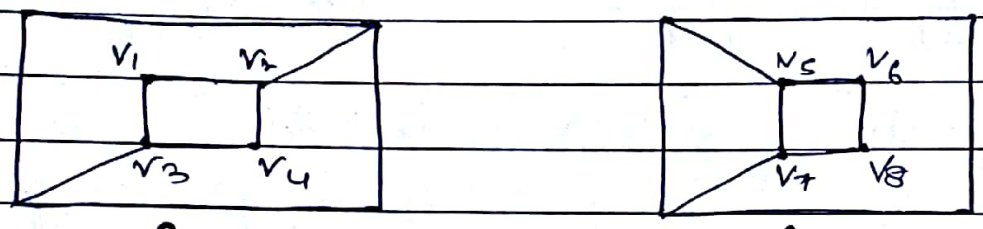
*(iii) They must have same degree sequences.

(I)



⇒ Isomorphic graphs

(II)

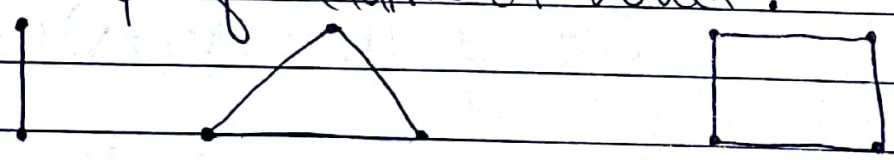


G_1 G_2

(They are not isomorphic graphs)

Homeomorphic Graphs -

Two graphs are said to be homeomorphic, if one can be obtained from another by dividing any edge with the help of additional vertex.



Subgraph -

Let $G\{V, E\}$ be a graph and $G' = \{V', E'\}$ is called subgraph of G if following prop. are satisfied

(i) $V' \subseteq V$

(ii) $E' \subseteq E$

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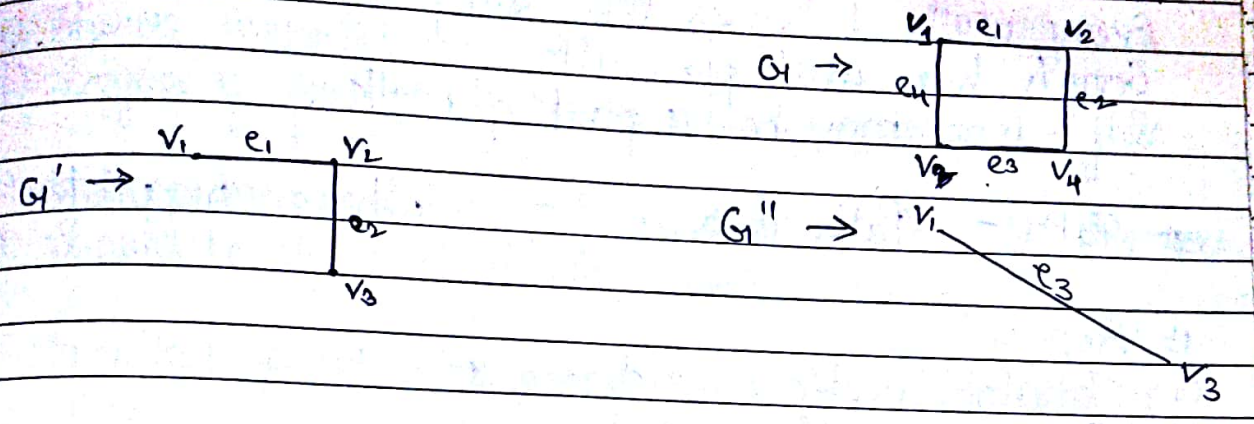
Each edge of G_1' has the same terminals like G_1



Rank of graph $\rightarrow$ $r = n - k$

Nullity, $\mu = e - r$

$= e - n + k$

Q.) $K_4 \rightarrow$ graph

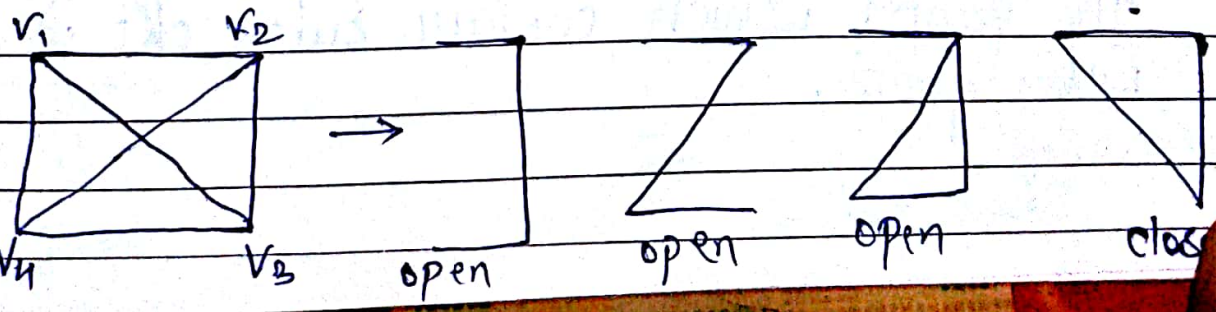
$n = 4$, $r = 4 - 1 = 3$

$e = \frac{n(n-1)}{2} = \frac{4 * 3}{2} = 12/2 = 6$

$\mu = 6 - 4 + 4 = 6$

Walk / Edge Chain $\rightarrow$

Walk is an alternative sequence of vertices and edges, beginning & ending with same or different vertices, such that no edge can appear more than once.



PAGE NO.:
DATE:

~~Their are~~ There are two walks ~~and~~ → open and close.
Every walk behaves like subgraph
Graph has multiple walk.
Self-loop may be a part of walk.

Path - Path is a where no vertex

Prop. :-
(1) Tabular vertices = degree 1
(2) Rest of vertices = degree 2

(3) Self loop cannot be a part of path
(4) Length of path is always one less than the total no. of vertices in that path.

Circuit → It is a closed walk where repetition of vertices is not allowed (except terminal vertices).

Any graph where path is

Euler circuit / Euler graph / Euler path -
A close walk (circuit which contains exactly one time is called Euler circuit.

The graph which contain Euler ckt is called Euler graph.

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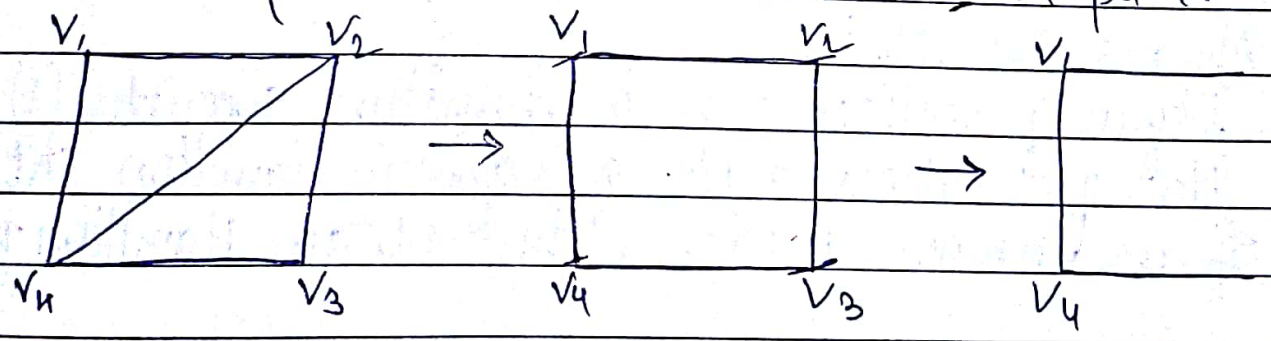
Prop. of Euler's graph –

(i) Always connected.
(ii) Degree of vertices is even in Euler graph.
$d(v_i) = \text{even}$.
(iii) Not contain isolated and pendant vertex into Euler graph.
(iv) Should be decomposed into circuit.

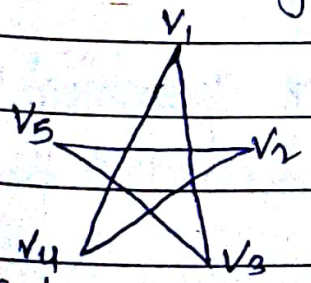
Hamilton graph / Hamilton circuit / Hamilton path –

- Hamilton ckt is a closed ~~box~~ walk which included all vertices of the graph exactly one time (except initial and final terminals).

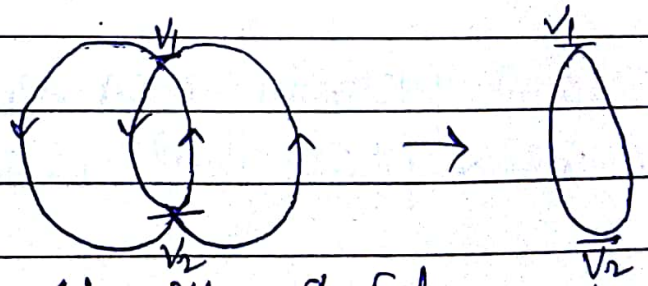
- The graph which include Hamilton ckt known as Hamilton graph.
- A path (open walk) in Hamilton graph that cover every vertex exactly one time known as Hamilton path.



Hamilton graph Hamilton ckt Hamilton path



Euler graph



Hamilton & Euler graph

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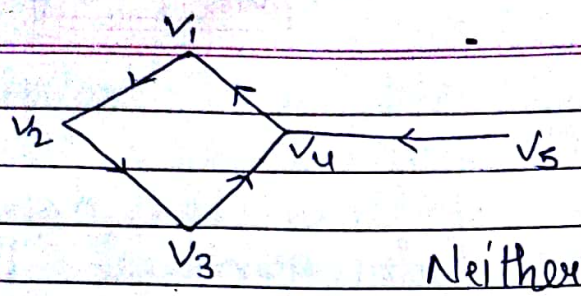
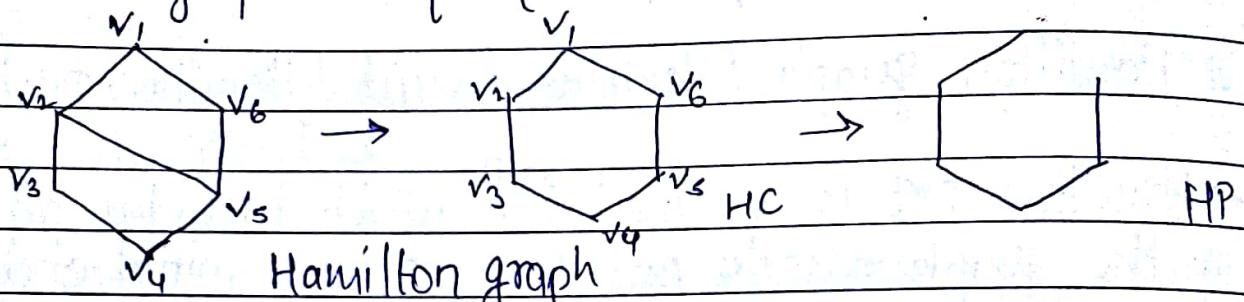


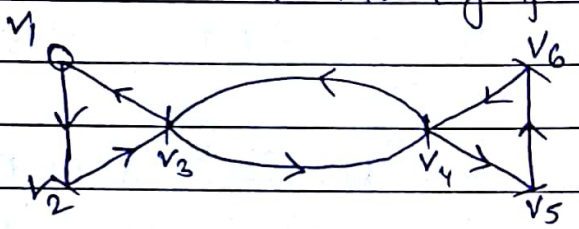
Neither Hamilton nor Euler graph.

Q.) With the help of 6 vertices make Euler graph and Hamilton graph uniquely.

Sol.)



Hamilton graph



EC and EG

▷ Prop. of Hamilton graph –

(i) Always connected.
(ii) Degree of vertices = 2 in Hamilton circuit.
(iii) No. of vertices = No. of edges in Hamilton ckt.
(iv) If we remove 1 edge, then obtain Hamilton path.

Matrix representation of graph –

It is a pictorial representation of any graphical pro
where we suppose that vertices are in ordered form

PAGE NO.:
DATE:

* Types :-

(1) Incidence Matrix. $v\overset{e}{[\ ]}$

~~In~~ It is $m \times n$ matrix which is defined by

$A = [a_{ij}]_{m \times n}$

$$a_{ij} = \begin{cases} 1 & \text{vertex } v_i \text{ connected with } e_j. \\ 0 & \text{Otherwise.} \end{cases}$$

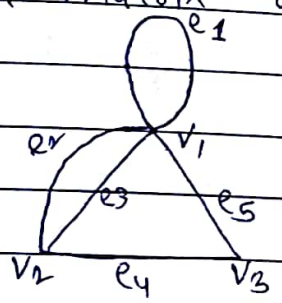
(2) Adjacency Matrix. $v\overset{v}{[\ ]}$

It is $n \times n$ matrix defined by

$A = [a_{ij}]_{n \times n}$

$$a_{ij} = \begin{cases} 1 & \text{vertex } v_i \text{ connected with } v_j \\ 0 & \text{otherwise.} \end{cases}$$

Q.) Obtain matrix representation of given graph.



Sol.) (1) Incidence matrix.

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}_{3 \times 5} \end{array}$$

(2) Adjacency Matrix.

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{3 \times 3} \end{array}$$

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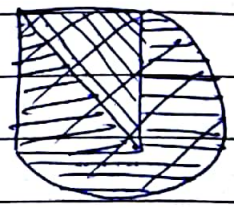
Planar graph → A graph is said to be planar if it can be drawn on a plane without overlapping edges. The process to make planar graph is called Embedding.

- Region ⇒ Region is an area of the graph which may be bounded or not.
- Degree of Region ⇒ No. of edges required.

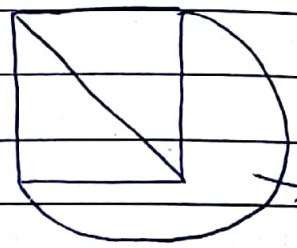


Euler's theorem →

If G is a connected planar graph with n vertices and e edges, then regions are can be given by,

$r = e - n + 2$ or

$n - e + r = 2$



Proof :- By using PMI.

Step 1) If $e = 0$, $n = 1$

then $r = 1$

$\Rightarrow$ $n - e + r = 1 - 0 + 1 = 2$

Hence, theo. is valid in this case.

Step 2) If $e = 1$, $n = 2$ (for a line) and $n = 1$ (for self lo

then $r = 1 - 2 + 2 = 1$ and $r = 2$

$\Rightarrow$ $n - e + r = 2 - 1 + 1 = 2$ and $n - e + r = 1 - 1 + 2 = 2$

Hence, theo. is valid in this case.

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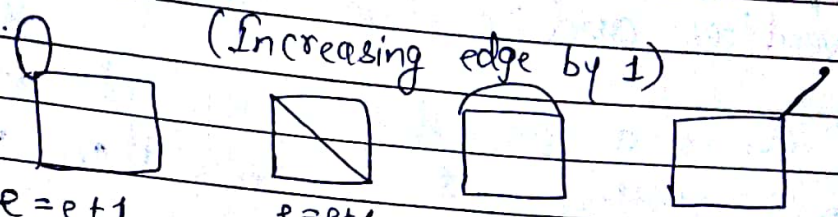
Step 3) $e = e - 1$

By induction hypothesis, we suppose that theo. is valid in this case also.

Step 4) $e = e + 1$. (Increasing edge by 1)

Ex :-



$e = e+1$, $n = n$, $r = r+1$

$e = e+1$, $n = n$, $r = r+1$

$e = e+1$, $n = n$, $r = r+1$

$e = e+1$, $n = n+1$, $r = r$

(1) $n - e + r = 2$

$n - (e+1) + (r+1) = 2$

$n - e + r = 2$

(2) $(n+1) - (e+1) + r = 2$

$n - e + r = 2$

Step 5) When graph is a tree (ckt less).

$n - e + r = n - (n-1) + 1$

$= 2$

$e = n - 1$

$n = n$

$r = 1$

Step 6) When graph is not a tree. (close walk)

$n - e + r = \cancel{n} - \cancel{n} + 2$

$= 2$

$n = n$

$e = n$

$r = 2$

After analysis, Euler's theo. is proved.

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PAGE NO.:
DATE:

Properties of planar graph →

(1) If G is a connected simple planar graph with $n \geq 3$ vertices and e edges then $3n-6 \geq e$.

(2) If G is a connected simple planar graph with $n \geq 3$ vertices and e edges then $2n-4 \geq e$.

Q.) Check that K_4, K_5 planar or not.

Sol.) K_4 means simple complete connected graph where $n=4$ and $e=?$

But in simple graph, total no. of edges $= \frac{n(n-1)}{2}$

Therefore, $e=6$.

Now, use the following formula,

$$3n-6 \geq e$$

$$3(4)-6 \geq 6 \quad \Rightarrow \quad \underline{6 \geq 6}$$

K_5 means simple complete connected graph where $n=5$ and $e=?$

But in simple graph, total no. of edges $= \frac{n(n-1)}{2}$

Therefore, $e=10$

Now, $3n-6 \geq e$

$$3(5)-6 \geq 10 \quad \Rightarrow \quad 9 \geq 10 \text{ (not valid)}$$

$\because$ this is mathematically invalid, we can say that K_5 is not planar.

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and $K_{2,2}$

Q.) Check that $K_{3,3}$ planar or not.

Sol.) $K_{3,3}$ means complete bipartite graph where

total no. of vertices $= 3+3 = 6$

total no. of edges $= 3\times 3 = 9$

Now,

$$2n - 4 \geq e$$

$$2(6) - 4 \geq 9 \Rightarrow 8 \geq 9 \text{ (invalid)}$$

$\because$ this is mathematically invalid, we can say that $K_{3,3}$ is not planar.

Total no. of vertices $= 2+2 = 4$

Total no. of edges $= 2\times 2 = 4$

Now,

$$2n - 4 \geq e$$

$$2(4) - 4 \geq 4 \Rightarrow \underline{4 \geq 4}$$

$\Rightarrow K_{2,2}$ is planar.

Q.) A connected planar graph has 10 vertices each of degree 3. Into how many regions does a presentation of this planar graph split the plane?

Sol.) Total no. of vertices $= 10$

Total no. of degrees $= 10\times 3 = 30$

But

$$\sum_{i=1}^{n} d(v_i) = 2e$$

$$30 = 2e$$

$$\Rightarrow \boxed{e = 15}$$

Now, by Euler's theo.,

$$r = e - n + 2$$

$$= 15 - 10 + 2 = 7$$

Ans. Into 7 regions.

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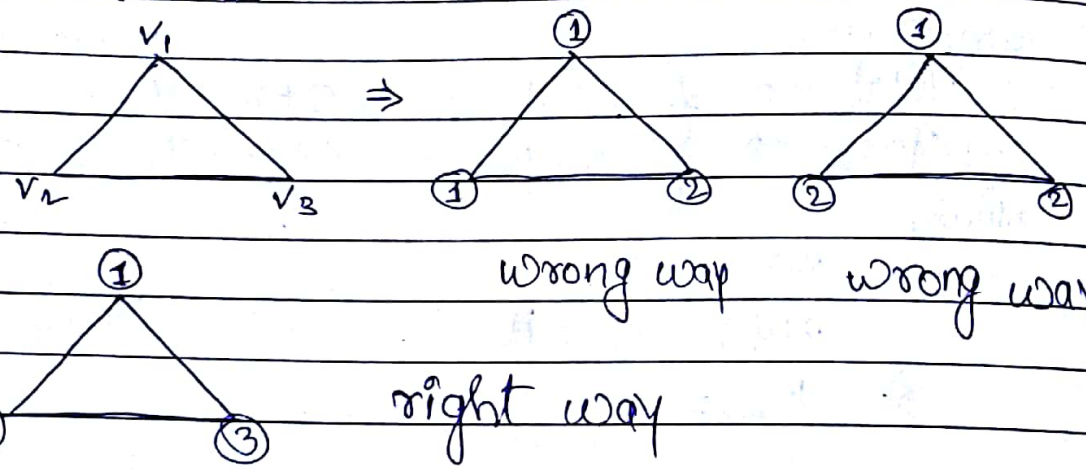
* Def^n of Tree – It is a graph without any cycle (close walk), self loop and multiedges. There always exist only one path b/w every pair of vertices.

* Coloring of graph – It is a concept where we assign a colour to every vertex of graph such that two adjacent vertex not assigned with same colour code.

for eg:-



* Chromatic No :- Minimum no. of req. colours for proper colouring. Symbol $\chi(G)$ or $C(G)$.

* Prop. of colouring →

(1) If no. of vertices $= n$ then $\chi(G) \leq n$.

(2) If degree of vertex $= d$ then atmost d colours are used to colour the all adjacent vertices.

(3) If gp has only one circuit with $n \geq 3$ vertices, then

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$\chi(G) = 2$ if n = even
$\chi(G) = 3$ if n = odd.

(4) Null graph = 1 - chromatic graph.
(5) Isolated vertex = " — "
(6) Tree = 2 - chromatic graph
(7) Bipartite gp = " — "
(8) Complete gp with n vertices always n - chromatic gp.

Theorem :- If Δ is the min. degree of any vertex of graph G, then $\chi(G) \leq \Delta + 1$.

Proof ⇒ Given that degree = Δ i.e., from any vertex atmost Δ adjacent vertices are connected. No. of vertices = n. We can also say that $n = \Delta + 1$.

By PMI,

Step (I) → $n = 1$ then $\Delta = 0$ and $\chi(G) = 1$

i.e., $\chi(G) \leq \Delta + 1$

$1 \leq 0 + 1 \Rightarrow \underline{1 \leq 1}$

$n = 2$ then $\Delta = 1$ and $\chi(G) = 2$

i.e., $\chi(G) \leq \Delta + 1$

$2 \leq 1 + 1 \Rightarrow \underline{2 \leq 2}$

Step (II) → for $n = n - 1$

By induction hypothesis suppose that theo. is true for all graphs for less than n vertices,

i.e., $\chi\{G - V\} \leq \Delta + 1$

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or $\chi(G') \leq \Delta + 1$

Step (III) → for $n = n$

for this add one vertex in G.

But we know that degree of each vertex is atmost Δ means no. of adjacent vertex $= \Delta$.

By the prop. we req. Δ no. of colours for colouring and one more colour for additional vertex.

Hence, it is proved that

$$\chi(G) \leq \Delta + 1$$

Theorem :- Every tree with two or more vertices is 2-chromatic but converse is not true.

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