स्तब्धम्
घमण्ड को नाश करना or
घमण्ड का नाश करने वाला

Unit-5
(RGPV Unit-4)

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Discrete Numeric functions ⇒

The functions whose domain is the set of natural numbers & whose Range is the set of real numbers these functions are called discrete numeric functions.
Numeric functions are denoted by bold lower case letters a, b, ---. If a is a numeric function then it's value at 0, 1, 2, --- r are denoted by $a_0, a_1, a_2, \ldots a_r \ldots$ written as

$$a = (a_0, a_1, a_2, \ldots\ldots a_r, \ldots\ldots)$$

Ex $(1, 2, 9, 28, \ldots\ldots r^3+1 \ldots)$ is the numeric function a whose r^{th} term a_r is r^3+1

or $a_r = r^3+1, \; r \geq 0$

Ex $(0, 3, 6, 7, 15, 31, \ldots\ldots)$

$$a_r = \begin{cases} 3r, & 0 \leq r \leq 2 \\ 2^r - 1, & r \geq 3 \end{cases}$$

Q.1) A person deposite 200/- in saving account at an interest of 8% per year compound annually. Find the numeric function a where a_r denotes the total amount in the account at the end of r^{th} year.

Sol^n I^{st} yr

$$SI = \frac{P \times R \times t}{100} = \frac{200 \times 8 \times 1}{100} = 16$$

∴ Amount = P + SI = 200 + 16 = 216/-

II^{nd} yr $\quad SI = \frac{216 \times 8 \times 1}{100} = \frac{1728}{100} = 17.28\%$

Amount $= P + SI = 216 + 17.28 = 233.28$

III^{rd} yr $\quad SI = \frac{233.28 \times 8 \times 1}{100} = 18.66$

Amount $= 233.28 + 18.66 = 251.94$

$\therefore a = (200, 216, 233.28, 251.94, \text{------})$

$a = 200\left(1 + \frac{8}{100}\right)^r$

$a = 200(1.08)^r$

$\therefore a_r = 200(1.08)^r, \quad r \geq 0$

Q.2) A ball is dropped to the floor from a height ~~of~~ of 80 m. Suppose that the ball always rebounds to each half of the height from it's falls. Find the numeric function a where a denotes the height it reaches in the r^{th} rebound.

Sol^n A ball is dropped from ~~the~~ a height ~~of~~ of 80 m above the floor.

So, the height after first rebound $= 80\left(\frac{1}{2}\right)$

$= 40$ m

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The height after second rebound $= 80\left(\frac{1}{2}\right)^2 = 20$

The height after third rebound $= 80\left(\frac{1}{2}\right)^3 = 10$

⋮

The height at the end of each rebound can be represented by numeric function

$$a = (80, 40, 20, 10, \ldots\ldots)$$

$$a = 80\left(\frac{1}{2}\right)^r \quad \therefore \quad a_r = 80\left(\frac{1}{2}\right)^r, \; r \geq 0$$

Manipulation of numeric functions ⇒

Sum of Numeric functions ⇒

Let a & b are two numeric functions. The sum of a & b is denoted by a+b & is a numeric function whose value at r is equal to the sum of value of a & b at r.

Ex ① Let a & b are two numeric functions

$$a_r = \begin{cases} 0 & 0 \leq r \leq 2 \\ 2^{-r} + 7 & r \geq 3 \end{cases}$$

$$b_r = \begin{cases} 5 - 2^r & 0 \leq r \leq 1 \\ r + 3 & r \geq 2 \end{cases}$$

find a+b (sum)

Sol^n Let $c = a + b$

r^{th} $\quad C_r = a_r + b_r$

$$\therefore C_r = \begin{cases} 5 - 2^r, & 0 \le r \le 1 \\ 5, & r = 2 \\ 2^{-r} + r + 10, & r \ge 3 \end{cases}$$

Multiplication of Numeric function ⇒

αa_r — real number

$$a_r = \begin{cases} 0, & 0 \le r \le 2 \\ 2^{-r} + 7, & r \ge 3 \end{cases}$$

then $5a$

$$\therefore 5a_r = \begin{cases} 0, & 0 \le r \le 2 \\ 5(2^{-r} + 7), & r \ge 3 \end{cases}$$

Product of numeric function ⇒

$C = ab \qquad \therefore C_r = a_r \cdot b_r$

$$C_r = \begin{cases} 0, & 0 \le r \le 2 \\ (2^{-r} + 7)(r + 3), & r \ge 3 \end{cases}$$

{ a_r & b_r [illegible] }

Accumulated sum of numeric function ⇒

$$b_r = \sum_{i=0}^{r} a_i$$

$y(t) = \sum_{n=0} y_n t = y_0 + y_1 t + y_2 t^2 + y_3 t^3 + \cdots$
$\cdots + y_n t^n + \cdots$

↑ for initial value problems

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GENERATING FUNCTIONS ⇒

Generating functions provide us an alternative way to represent discrete numeric functions & solving recurrence relations. Generating functions also provide us an alternative way to solve combinatorial problems.

Let $(a_0, a_1, a_2, \cdots a_r \cdots) = a$ be a discrete numeric function then an infinite series in terms of z parameter

$$A(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_r z^r + \cdots$$

EX ① $(1, 3^1, 3^2, 3^3, \cdots 3^r \cdots)$ is given by

$3^0 \to 1$

$3^0 z^0 + 3^1 z^1 + 3^2 z^2 + \cdots 3^r z^r$

$$A(z) = 1 + 3z + 3^2 z^2 + 3^3 z^3 + \cdots + 3^r z^r + \cdots$$

It is the G.P. (infinite G.P.) [RHS side]

Geometric progress

$$\therefore A(z) = \frac{1}{1-3z} \quad \left(Sum_\infty = \frac{a}{1-r}\right) \quad \left(r = \frac{T_2}{T_1}\right)$$

EX ② $a_r = 1$

$$A(z) = 1 + z + z^2 + z^3 + \cdots + z^r + \cdots$$

$$\therefore A(z) = \frac{1}{1-z}$$

EX ③ $a_r = 2^r$

$$A(z) = 1 + 2z + 2^2 z^2 + \cdots + 2^r z^r + \cdots$$

$$\therefore A(z) = \frac{1}{1-2z}$$

EX ④ ~~A(z) =~~ $a_r = \alpha^r$

$$A(z) = 1 + \alpha z + \alpha^2 z^2 + \cdots + \alpha^r z^r + \cdots \quad \therefore A(z) = \frac{1}{1-\alpha z}$$

Ex ⑤ $a_r = r$

$A(z) = z + 2z^2 + 3z^3 + \cdots + rz^r + \cdots$

$\therefore A(z) = (1 + 2z + 3z^2 + 4z^3 + \cdots) - (1 + z + z^2 + z^3 + \cdots)$

$= \frac{1}{(1-z)^2} - \frac{1}{(1-z)} = \frac{z}{(1-z)^2}$

Ex ⑥ $a_r = r(r+1)$

$A(z) = 1\cdot 2z + 2\cdot 3z^2 + 3\cdot 4z^3 + \cdots + r(r+1)z^r + \cdots$

$\therefore A(z) = \frac{2z}{(1-z)^3}$

(*)

(Q.1) Let $A(z)$, $B(z)$ & $C(z)$ represent the generating functions of the numeric functions a, b, & c respectively show that

(i) If $b_r = \alpha a_r$ where α is a real constant then $B(z) = \alpha A(z)$

(ii) If $c_r = a_r + b_r$ then $C(z) = A(z) + B(z)$

(iii) If $c = a * b$ then $C(z) = A(z)\,B(z)$

Solⁿ Let $a = (a_0, a_1, a_2, a_3, \cdots a_r, \cdots)$

$b = (b_0, b_1, b_2, b_3, \cdots b_r, \cdots)$

$c = (c_0, c_1, c_2, c_3, \cdots c_r, \cdots)$

Then $A(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_r z^r + \cdots$

$B(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_r z^r + \cdots$

$C(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_r z^r + \cdots$

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(I) If $b_r = \alpha a_r$ then

$$B(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_r z^r + \cdots$$

$$B(z) = \alpha a_r$$

$$B(z) = \alpha a_0 + \alpha a_1 z + \alpha a_2 z^2 + \cdots + \alpha a_r z^r + \cdots$$

$$B(z) = \alpha (a_0 + a_1 z + a_2 z^2 + \cdots + a_r z^r + \cdots)$$

$$B(z) = \alpha A(z)$$

$\because b_r = \alpha a_r$ then $B(z) = \alpha A(z)$

(II) If

$C_r = \underline{a_r + b_r}$ then

$$C(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots + c_r z^r + \cdots$$

Similarly

$$C(z) = (a_0 + b_0) + (a_1 + b_1) z + (a_2 + b_2) z^2 + \cdots + (a_r + b_r) z^r + \cdots$$

$$C(z) = (a_0 + a_1 z + a_2 z^2 + \cdots + a_r z^r + \cdots) + (b_0 + b_1 z + b_2 z^2 + \cdots + b_r z^r + \cdots)$$

$$C(z) = A(z) + B(z)$$

$\because C_r = a_r + b_r \quad \therefore C(z) = A(z) + B(z)$

(III) If $c = a * b$ then

$$C(z) = A(z) * B(z)$$

Q.2) Let $A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots + a_r z^r + \dots$ be generating function of the numeric function $a = (a_0, a_1, a_2, \dots a_r, \dots)$ then to prove that $\frac{1}{1-z} A(z)$ is the generating function of the numeric function b which is the accumulated sum of a

$$b_r = \sum_{k=0}^{r} a_k$$

Sol^n ⟹

$$\frac{1}{1-z} = (1-z)^{-1} = 1 + z + z^2 + \dots + z^r + \dots$$

there fore $\frac{1}{1-z}$ is the generating function of the numeric function $(1, 1, 1, 1, \dots)$

Again

$$A(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_r z^r + \dots$$

multiply by $\frac{1}{1-z}$ with $A(z)$

$$\left(\frac{1}{1-z}\right) A(z) = a_0 + (a_0 + a_1) z + (a_0 + a_1 + a_2) z^2 + \dots$$
$$\dots + (a_0 + a_1 + a_2 + \dots + a_r) z^r + \dots$$

b is the accumulated sum of a

$$\therefore b_r = \sum_{k=0}^{r} a_k$$

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Q.3) Find the generating function of the given numeric function.

(i) $5 \cdot 2^r$, $r \geq 0$

Sol^n ⇒ Here $a_r = 5 \cdot 2^r$

we know that generating function of 2^r

$\therefore \frac{1}{1-2z}$

$A(z) = \frac{5}{1-2z}$

(ii) $a_r = 5 \cdot 2^{r+2}$

$\therefore a_r = 5 \cdot 2^r \cdot 2^2 = 5 \cdot 2^r \cdot 4 = 20 \cdot 2^r$

$\therefore A(z) = \frac{20}{1-2z}$

(iii) $a_r = 2^r + 3^r$

$A(z) = \left(\frac{1}{1-2z}\right) + \left(\frac{1}{1-3z}\right) = \frac{1-3z + 1-2z}{(1-2z)(1-3z)}$

$= \frac{2-5z}{(1-2z)(1-3z)}$

Q.4) Determine the generating function of the numeric function a_r

$$a_r = \begin{cases} 2^r, & \text{if } r \text{ is even} \\ -2^r, & \text{if } r \text{ is odd} \end{cases}$$

Sol^n) Let generating function of a_r

$$A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$$

Where $r = 0, 1, 2, 3, 4, 5, \cdots$

We are given

$$a_r = \begin{cases} 2^r, & r \text{ is even} \\ -2^r, & r \text{ is odd} \end{cases}$$

$\therefore$ $a_0 = 1$, $a_1 = -2$, $a_2 = 2^2 = 4$

$a_3 = -2^3 = -8$, $a_4 = 2^4 = 16$, $a_5 = -2^5 = -32$

$\cdots\cdots\cdots$

$\therefore$ $A(z) = 1 - 2z + 4z^2 - 8z^3 + 16z^4 - 32z^5 + \cdots$

$= 1 - 2z + (2z)^2 - (2z)^3 + (2z)^4 - (2z)^5 + \cdots$

$= (1+2z)^{-1}$ by Binomial Theorem

$\therefore$ $A(z) = \dfrac{1}{1+2z}$

(II) $(1, -2, 3, -4, 5, \cdots\cdots (-1)^r (r+1), \cdots)$

Sol^n generatin function of given numeric function

$A(z) = 1 - 2z + 3z^2 - 4z^3 + 5z^4 - \cdots + (-1)^r (r+1) z^r + \cdots$

$A(z) = (1+z)^{-2}$ By Binomial Theorem

$\therefore$ $A(z) = \dfrac{1}{(1+z)^2}$

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or
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(III) $a_r = \left(1, \frac{2}{4}, \frac{3}{16}, \ldots\right)$

$$A(z) = 1 + \frac{2}{4}z + \frac{3}{16}z^2 + \frac{4}{64}z^3 + \cdots + \frac{(r+1)}{4^r}z^r + \cdots$$

$$= 1 + 2\left(\frac{z}{4}\right) + 3\left(\frac{z}{4}\right)^2 + 4\left(\frac{z}{4}\right)^3 + \cdots + (r+1)\left(\frac{z}{4}\right)^r + \cdots$$

$$= \left(1 - \frac{z}{4}\right)^{-2}$$

$$A(z) = \frac{1}{\left(1 - \frac{z}{4}\right)^2} = \frac{16}{(4-z)^2}$$

(IV) $A(z) = \frac{2}{1-4z^2}$ Determine Discrete numeric function

Sol^n

$$A(z) = \frac{2}{1-4z^2} = \frac{1}{(1-2z)} + \frac{1}{(1+2z)}$$

$$= (1-2z)^{-1} + (1+2z)^{-1}$$

$$= \left(1 + 2z + 2^2z^2 + \cdots + 2^r z^r + \cdots\right) + \left(1 - 2z + 2^2z^2 + \cdots + (-1)^r 2^r z^r + \cdots\right)$$

$$= 2 + 2^3z^2 + 2^5z^4 + \cdots + \{2^r + (-2)^r\}z^r + \cdots$$

$$\therefore a_r = 2^r + (-2)^r$$

$$a_r = \begin{cases} 0, & \text{if } r \text{ is odd} \\ 2 \cdot 2^r, & \text{if } r \text{ is even} \end{cases}$$

or

$$a_r = \begin{cases} 0, & r \text{ is odd} \\ 2^{r+1}, & r \text{ is even} \end{cases}$$

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Recurrence Relations ⇒

Recurrence relation in terms of difference b/w the consecutive terms of a sequence & hense recurrence relations are also called difference equation.

⇒ A relation which involves an independent variable x, a dependent variable y & one or more than one differences Δy, $\Delta^2 y$, $\Delta^3 y$ .... is called a recurrence relation.

Degree of Recurrence Relation ⇒

The degree of a recurrence relation is defined to be the highest power of y_x

Q.1) Given $y_n = A \cdot 2^n + B \cdot 3^n$ find the corresponding recurrence relation.

Sol^n

$$y_n = A \cdot 2^n + B \cdot 3^n$$

$$y_{n+1} = A \cdot 2^{n+1} + B \cdot 3^{n+1}$$

$$y_{n+2} = A \cdot 2^{n+2} + B \cdot 3^{n+2}$$

Above three relation may be written as

Same 2

$y_{n+2} - 4 \cdot 2^n A - 9 \cdot 3^n B = 0$

$y_{n+1} - 2 \cdot 2^n A - 3 \cdot 3^n B = 0$

$y_n - 2^n A - 3^n B = 0$

Eliminating $A, 2^n, 3^n$

$$\begin{vmatrix} y_{n+2} & -4 & -9 \\ y_{n+1} & -2 & -3 \\ y_n & -1 & -1 \end{vmatrix} = 0$$

$y_{n+2}(2-3) - y_{n+1}(4-9) + y_n(12-18) = 0$

$y_{n+2} - 5y_{n+1} + 6y_n = 0$

Q.2) Given $y_x = ax^2 + bx$ find the corresponding recurrence relation

Sol:

$y_x = ax^2 + bx$ — Same

$y_{x+1} = a(x+1)^2 + b(x+1)$

$y_{x+2} = a(x+2)^2 + b(x+2)$

we may written as

$y_{x+2} - a(x+2)^2 - b(x+2) = 0$

$y_{x+1} - a(x+1)^2 - b(x+1) = 0$

$y_x - ax^2 - bx = 0$

L.R.R. (linearly recurrence relation)

$$\begin{vmatrix} y_{x+2} & (x+2)^2 & (x+2) \\ y_{x+1} & (x+1)^2 & (x+1) \\ y_x & x^2 & x \end{vmatrix} = 0$$

$$y_{x+2}\left[x(x+1)^2 - x^2(x+1)\right] - y_{x+1}\left[x(x+2)^2 - x^2(x+2)\right]$$
$$+ y_x\left[(x+2)^2(x+1) - (x+1)^2(x+2)\right] = 0$$

$$(x^2+x)\,y_{x+2} - 2x(x+2)\,y_{x+1} + (x+1)(x+2)\,y_x = 0$$

Q.) from the recurrence relation $y_n = \frac{a}{n} + b$

Sol^n

$$y_n = \frac{a}{n} + b$$

same

$$y_{n+1} = \frac{a}{n+1} + b$$

$$y_{n+2} = \frac{a}{n+2} + b$$

$$\begin{vmatrix} y_{n+2} & \frac{1}{n+2} & 1 \\ y_{n+1} & \frac{1}{n+1} & 1 \\ y_n & \frac{1}{n} & 1 \end{vmatrix} = 0$$

$$(n+2)\,y_{n+2} - 2(n+1)\,y_{n+1} + n y_n = 0$$

Linear Recurrence solⁿ with constant coefficients

general form of L. R. R.

$c_0 a_r + c_1 a_{r-1} + c_2 a_{r-2} + \dots + c_k a_{r-k} = f(r)$ order of K

If $f(r) = 0$ then called homogeneous linear ~~recurrent~~ relation with ~~constant coefficients~~

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solⁿ of Recurrence Relation ⇒

A relation b/w the independent & dependent variable is said to be solⁿ of recurrence relation. There are three types of solⁿ

(Total solⁿ)
① General or solⁿ ② Particular solⁿ ③ Linear Recurrence solⁿ.

⟶ $c_0 a_r + c_1 a_{r-1} + c_2 a_{r-2} + c_3 a_{r-3} + \dots + c_k a_{r-k} = f(r)$

Homogeneous solⁿ [Linear recurrence solⁿ]

(i) Roots are real & distinct

$a_r = c_1 m_1^r + c_2 m_2^r + \dots + c_k m_k^r$

(ii) Some roots are Equal

$(c_1 + c_2 r + c_3 r^2 + \dots + c_p r^{p-1}) m_1^r$

(iii) Some roots are complex number

$R^r (c_1 \cos r\theta + c_2 \sin r\theta)$

or $c_1 R^r \cos(r\theta + c_2)$

(iv) Some roots are repeated complex roots

$R^r [(c_1 + c_2 r) \cos r\theta + (c_3 + c_4 r) \sin r\theta]$

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Q.1) Solve the following Recurrence relations

(1) $a_r - 6a_{r-1} + 8a_{r-2} = 0$ given $a_0 = 3$ & $a_1 = 2$

sol^n Given recurrence relations

$$a_r - 6a_{r-1} + 8a_{r-2} = 0$$

Characteristic Equation is

$$m^2 - 6m + 8 = 0$$

$$(m-2)(m-4) = 0$$

$$\therefore \quad m = 2, 4$$

$$a_r = C_1 2^r + C_2 \cdot 4^r \quad \text{(A)}$$

Putting $r = 0$ we get

$$a_0 = C_1 + C_2 = 3 \quad \text{or} \quad C_1 + C_2 = 3 \quad (1)$$

Again $r = 1$

$$a_1 = 2C_1 + 4C_2 = 2 \quad \text{or} \quad 2C_1 + 4C_2 = 2$$

$$\text{or} \quad C_1 + 2C_2 = 1 \quad (2)$$

Solve Equation (1) & (2)

$$C_1 = 5, \quad C_2 = -2$$

Putting C_1 & C_2 in equation (A)

$$a_r = 5 \cdot 2^r - 2 \cdot 4^r$$

(II) $2a_r - 5a_{r-1} + 2a_{r-2} = 0$ given $a_0 = 0$, $a_1 = 1$

sol^n we given recurrence relation is

$$2a_r - 5a_{r-1} + 2a_{r-2} = 0$$

characteristic Equation is

$$2m^2 - 5m + 2 = 0 \quad \text{--- (1)}$$

$$\therefore m = \frac{1}{2}, 2$$

general sol^n of equation (1) is given by

$$a_r = c_1\left(\frac{1}{2}\right)^r + c_2 \cdot 2^r \quad \text{--- (2)}$$

put $r = 0$

$$a_0 = c_1 + c_2 \qquad \therefore c_1 + c_2 = 0 \quad \text{--- (3)}$$

put $r = 1$ $\quad a_1 = c_1\left(\frac{1}{2}\right)^1 + c_2(2)^1 \quad \therefore \frac{c_1}{2} + 2c_2 = 1$

or $\quad c_1 + 4c_2 = 2$ --- (4)

solve (3) & (4) equation

$$c_1 = \frac{-2}{3} \qquad c_2 = \frac{2}{3}$$

$$\therefore \boxed{a_r = -\frac{2}{3}\left(\frac{1}{2}\right)^r + \frac{2}{3}(2)^r}$$

(III) $a_r + a_{r-2} = 0$ given $a_0 = 0$, $a_1 = 1$ --- (1)

sol^n characteristic equation

$$m^2 + 1 = 0$$

$$m = \pm i \qquad \text{or} \qquad m = \left(\cos\frac{\pi}{2} \pm i\sin\frac{\pi}{2}\right)$$

$$\theta = \frac{\pi}{2} \qquad R = 1$$

$$a_r = c_1 R^r \cos(r\theta + c_2)$$

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$$a_r = c_1 \cos\left(r\frac{\pi}{2} + c_2\right) \longrightarrow (2)$$

put $r=0$

$$a_0 = c_1 \cos c_2$$

$$0 = c_1 \cos c_2$$

$\therefore \cos c_2 = 0$

$\cos c_2 = \cos \pi/2$

$\boxed{c_2 = \pi/2}$

$r=1$ put

$$a_1 = c_1 \cos\left(\frac{\pi}{2} + c_2\right)$$

$$c_1 \cos\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 1$$

$$c_1 \cos 180 = 1 \qquad c_1(-1) = 1 \qquad \boxed{\therefore c_1 = -1}$$

$$\therefore a_r = -\cos\left(r\frac{\pi}{2} + \frac{\pi}{2}\right)$$

or $\quad a_r = \sin\left(r\frac{\pi}{2}\right)$

(IV) $\quad a_r + 2a_{r-1} + 2a_{r-2} = 0 \longrightarrow (1) \qquad$ given $a_0 = 0$, $a_1 = -1$

sol^n

characteristic equation

$$m^2 + 2m + 2 = 0$$

$$m = -1 \pm i$$

$$m = \sqrt{2}\left[-\frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}\right] = \sqrt{2}\left[\cos\frac{3\pi}{4} \pm i \sin\frac{3\pi}{4}\right]$$

$$\therefore R = \sqrt{2} \qquad \theta = \frac{3\pi}{4}$$

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$$a_r = (\sqrt{2})^r \left[C_1 \cos \frac{3\pi}{4} r \pm C_2 \sin \frac{3\pi}{4} r \right]$$

Put $r = 0$

$a_0 = C_1$ $\therefore \boxed{C_1 = 0}$

Put $r = 1$

$a_1 = \sqrt{2} \left[0 + C_2 \sin \frac{3\pi}{4} \right] = -1$ $\therefore \boxed{C_2 = -1}$

$$\therefore a_r = -(\sqrt{2})^r \sin\left(\frac{3\pi r}{4}\right)$$

(V) $a_r = a_{r-1} + a_{r-2}$ given $a_0 = 1$, $a_1 = 1$

sol^n $a_r - a_{r-1} - a_{r-2} = 0$ —— (1)

C.E. $m^2 - m - 1 = 0$

$$m = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$\therefore$ Homogenous sol^n (general) is

$$a_r = C_1 \left[\frac{1+\sqrt{5}}{2}\right]^r + C_2 \left[\frac{1-\sqrt{5}}{2}\right]^r$$

Put $r = 0$ & 1

$a_0 = C_1 + C_2$ $\therefore C_1 + C_2 = 1$ —— (2)

$a_1 = C_1\left(\frac{1+\sqrt{5}}{2}\right) + C_2\left(\frac{1-\sqrt{5}}{2}\right)$ $\therefore \left(\frac{1+\sqrt{5}}{2}\right)C_1 + \left(\frac{1-\sqrt{5}}{2}\right)C_2$ —— (3)

Sol^n (2) & (3)

$$C_1 = \frac{\sqrt{5}+1}{2\sqrt{5}} \qquad C_2 = \frac{\sqrt{5}-1}{2\sqrt{5}}$$

$$\therefore a_r = \left(\frac{\sqrt{5}+1}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^r + \left(\frac{\sqrt{5}-1}{2\sqrt{2}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^r$$

or

$$a_r = \frac{1}{2^{r+1} \cdot \sqrt{5}} \left[(\sqrt{5}+1)^{r+1} + (-1)^r (\sqrt{5}-1)^{r+1} \right]$$

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(General solution)
or
Total sol^n ⇒

A non-homogenous linear recurrence relation of order k with constant coefficients

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \cdots + C_k a_{r-k} = f(r)$$

where $C_0, C_1, C_2 \cdots C_k$ are constants & both C_0 & C_k are non zero. $f(r)$ is a function of r.

Total sol^n consists of two parts

(i) homogenous sol^n

(ii) Particular sol^n

Total sol^n

$$\boxed{\begin{aligned} a_n &= \text{homogenous} + \text{Particular } sol^n \\ a_n &= a_n^{(h)} + a_n^{(P)} \end{aligned}}$$

Particular sol^n

There is no general procedure for determining the particular sol^n of a recurrence relation. There are two general method for particular sol^n.

① Method of inspection or Undetermined coefficients

② Operator method.

	$f(r)$ Terms	Trial sol^n a_r
1	b^r	$A \cdot b^r$
2	degree k in r	$A_0 + A_1 r + A_2 r^2 + \cdots + A_k r^k$
3	b^r & degree k in r	$b^r \cdot (A_0 + A_1 r + A_2 r^2 + \cdots + A_k r^k)$
4	$\sin br$ or $\cos br$	$A \sin br + B \cos br$
5	$a^r \sin br$	$a^r (A \sin br + B \cos br)$
6	$a^r \cos br$	— " —

where $a, b, A, B, A_0, A_1, A_2 \cdots A_k$ are unknown coefficients

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Q.1) solve $a_r - 5a_{r-1} + 6a_{r-2} = 5^r$

Sol^n

$$a_r - 5a_{r-1} + 6a_{r-2} = 5^r \quad \text{——— (1)}$$

C.E. $m^2 - 5m + 6 = 0$

$$(m-2)(m-3) = 0$$

$$m = 2, 3$$

Homogenous sol^n $(a_r^{(h)})$ is given by

$$a_r^{(h)} = C_1 \cdot 2^r + C_2 \cdot 3^r \quad \text{——— (2)}$$

Particular sol^n corresponding to the term on RHS of (1) is $A \cdot 5^r$

$$a_r^{(p)} = A \cdot 5^r \quad \text{——— (3)}$$

substiting in (1) of (3)

$$A \cdot 5^r - 5 A \cdot 5^{r-1} + 6 \cdot A 5^{r-2} = 5^r$$

$$A[6 \cdot 5^{r-2}] = 5^r \qquad A = \frac{25}{6}$$

$\frac{\cancel{5} A 5^r}{\cancel{5}}$

$A5^r$

$\frac{5^r \cdot 5^2}{5^2} = \frac{5^r}{25}$

Putting A in (3)

$$a_r^{(p)} = \frac{25}{6} 5^r = \frac{1}{6} \cdot 5^{r+2}$$

Total sol^n

$$\boxed{a_r^{(T)} = a_r^{(h)} + a_r^{(p)}}$$

$$\boxed{a_r = C_1 \cdot 2^r + C_2 \cdot 3^r + \frac{1}{6} \cdot 5^{r+2}}$$

Ans

Q.2) Determine the Particular solⁿ for the difference equation

$$a_r - 2a_{r-1} = f(r) \quad \text{where } f(r) = 7r$$

Solⁿ

$$a_r - 2a_{r-1} = 7r \quad \text{——— (1)}$$

Particular solⁿ

$$a_r^{(P)} = A_0 + A_1 r \quad \text{——— (2)}$$

Substituting (2) in (1) we get

$$(A_0 + A_1 r) - 2[A_0 + A_1(r-1)] = 7r$$

$$(-A_0 + 2A_1) + (-A_1 r) = 7r \quad \text{——— (3)}$$

comparing two sides of (3)

$$-A_1 = 7 \quad , \quad -A_0 + 2A_1 = 0 \quad \therefore$$

$$\therefore \quad A_1 = -7 \qquad A_0 = -14$$

Putting for A_0 & A_1 in (2)

$$a_r^{(P)} = -14 - 7r$$

Q.3) Solve the recurrence relation

$$a_r - 5a_{r-1} + 6a_{r-2} = 2 + r \quad , \quad r \geq 2$$

with boundary conditions $a_0 = 1$, $a_1 = 1$

Solⁿ The given equation is

$$a_r - 5a_{r-1} + 6a_{r-2} = 2 + r \quad \text{——— (I)}$$

Characteristic equation is

$$m^2 - 5m + 6 = 0$$

$$\therefore \quad m = 2, 3$$

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$$a_r^{(h)} = C_1 \cdot 2^r + C_2 \cdot 3^r \quad \text{---} \quad (2)$$

The particular solⁿ corresponding to the term $2+r$ on RHS of (1) is $A_0 + A_1 r$

$$a_r^{(p)} = A_0 + A_1 r \quad \text{---} \quad (3)$$

substituting (3) in (1) we get

$$(A_0 + A_1 r) - 5[A_0 + A_1(r-1)] + 6[A_0 + A_1(r-2)] = 2+r$$

$$(2A_0 - 7A_1) + 2A_1 r = 2+r \quad \text{---} \quad (4)$$

comparing two sides

$$2A_0 - 7A_1 = 2 \qquad \therefore A_0 = \frac{11}{4}, \quad A_1 = \frac{1}{2}$$

$$2A_1 = 1$$

Putting for A_0 & A_1 in (3) we get

$$a_r^{(p)} = \frac{11}{4} + \frac{r}{2}$$

$\therefore$ Total solⁿ $\quad a_r = a_r^{(h)} + a_r^{(p)}$

$$a_r = C_1 \cdot 2^r + C_2 \cdot 3^r + \frac{11}{4} + \frac{r}{2} \quad \text{---} \quad (5)$$

Now putting $r=0$ & 1 using boundary conditions in (5) we get

$$r=0 \Rightarrow 1 = C_1 + C_2 + \frac{11}{4} \qquad \therefore C_1 + C_2 = -\frac{7}{4} \quad \text{---} \quad (6)$$

$$r=1 \Rightarrow 1 = 2C_1 + 3C_2 + \frac{11}{4} + \frac{1}{2} \qquad \therefore 2C_1 + 3C_2 = -\frac{9}{4} \quad \text{---} \quad (7)$$

solve from (6) & (7)

$$C_1 = -3, \quad C_2 = \frac{5}{4}$$

Putting C_1 & C_2 in (5) required solⁿ is

$$a_r = -3 \cdot 2^r + \frac{5}{4} \cdot 3^r + \frac{11}{4} + \frac{r}{2}$$

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Q.4) Solve the recurrence relation ~~$a_r + 6a_{r-1}$~~
$a_r + 6a_{r-1} + 9a_{r-2} = 3$ given that boundary condition $a_0 = 0$ & $a_1 = 1$

Solⁿ ⇒

$$a_r + 6a_{r-1} + 9a_{r-2} = 3 \quad \text{——— (1)}$$

Characteristic equation is

$$m^2 + 6m + 9 = 0$$

$$\therefore m = -3, -3$$

$$a_r^{(h)} = (C_1 + C_2 r)(-3)^r \quad \text{——— (2)}$$

Let particular solⁿ corresponding to the term of RHS of (1)

$$A_0 r^2 + A_1 r + A_2$$

$$a_r^{(P)} = A_0 r^2 + A_1 r + A_2 \quad \text{——— (3)}$$

Now substituting (3) in (1)

$$(A_0 r^2 + A_1 r + A_2) + 6[A_0(r-1)^2 + A_1(r-1) + A_2]$$
$$+ 9[A_0(r-2)^2 + A_1(r-2) + A_2] = 3$$

$$16A_0 r^2 + (16A_1 - 48A_0)r + (16A_2 - 24A_1 + 42A_0) = 3$$

comparing side of (4) ——— (4)

$$16A_0 = 0 \qquad \therefore A_0 = 0$$

$$16A_1 - 48A_0 = 0 \qquad \therefore A_1 = 0$$

$$16A_2 - 24A_1 + 42A_0 = 3 \qquad \therefore A_2 = 3/16$$

from (3)

$$a_r^{(P)} = \frac{3}{16}$$

$\therefore$ Total sol^n of (1)

$$a_r = a_r^{(h)} + a_r^{(p)}$$

$$a_r = (c_1 + c_2 r)(-3)^r + \frac{3}{16} \quad \text{——— (5)}$$

when putting $r = 0, 1$ in equation (5)

$r=0 \Rightarrow \quad a_0 = 0 \qquad \therefore \; c_1 = -\frac{3}{16}$

$r=1 \Rightarrow \quad a_1 = 1 \qquad \therefore \; c_2 = \frac{1}{12}$

putting c_1 & c_2 in (5)

$$a_r = \left[-\frac{3}{16} + \frac{r}{12}\right](-3)^r + \frac{3}{16}$$

Q.5) ~~The given~~ solve $y_{n+2} - 4y_{n+1} + 4y_n = 3n + 2^n$

$\underline{sol^n}$ Recurrence relation

$$(E^2 - 4E + 4)\,y_n = 3n + 2^n \quad \text{——— (1)}$$

characteristic equation $m^2 - 4m + 4 = 0 \qquad \therefore m = 2, 2$

So Homogeneous sol^n $y_n^{(h)} = (c_1 + c_2 n)\,2^n$

find Particular sol^n

$$y_n^{(P)} = A_1 + A_2 n + Bn^2 \cdot 2^n \quad \text{——— (2)}$$

$$(E^2 - 4E + 4)(A_1 + A_2 n + Bn^2 \cdot 2^n) = 3n + 2^n$$

$$[A_1 + A_2(n+2) + B(n+2)^2 \cdot 2^{n+2}] - 4[A_1 + A_2(n+1) + B(n+1)^2 \cdot 2^{n+1}]$$
$$+ 4[A_1 + A_2 n + Bn^2 \cdot 2^n] = 3n + 2^n$$

$$(A_1 - 2A_2) + A_2 n + 8B \cdot 2^n = 3n + 2^n$$

Compair $\quad 8B = 1 \quad \therefore B = \frac{1}{8} \quad A_2 = 3, \quad A_1 - 2A_2 = 0 \quad \therefore A_1 = 6$

from (2) $\quad y_n^{(P)} = 6 + 3n + \frac{n^2}{8} \cdot 2^n$

Total sol^n $\quad y_n = (c_1 + c_2 n)\,2^n + 6 + 3n + \left(\frac{1}{8}\right)n^2 \cdot 2^n \quad$ Ans

Generating functions ~~for~~ →

The method of generating functions is a powerful method to solve the difference equations (recurrence relations)

If $y_0, y_1, y_2 \dots$ is a sequence of a real numbers the function Y defined for some interval of real numbers containing zero whose value at t is given by the series

$$y_0 t^0 + y_1 t^1 + y_2 t^2$$

$$Y(t) = \sum_{n=0}^{\infty} y_n t^n = y_0 + y_1 t + y_2 t^2 + \dots + y_n t^n + \dots$$

is called the generating function of the sequence $\{y_t\}$

Q.1) Apply the generating function technique to solve the initial value problem

$y_{n+1} - 2y_n = 0$ with $y_0 = 1$

Sol^n

Given $y_{n+1} - 2y_n = 0$ ——— (1)

Consider the generating function $Y(t)$ given by

$$Y(t) = \sum_{n=0}^{\infty} y_n t^n = y_0 + y_1 t + y_2 t^2 + \cdots \quad \text{——— (2)}$$

multiply the given difference equation (1) by t^n & summing from $n=0$ to $n=\infty$ we get

$$\sum_{n=0}^{\infty} y_{n+1} t^n - 2\sum_{n=0}^{\infty} y_n t^n = 0$$

$n \to 0$ $y_{n+0} t^n$ $y_n t$

$$(y_1 + y_2 t + y_3 t^2 + \cdots) - 2Y(t) = 0$$

$$\frac{1}{t}(y_1 t + y_2 t^2 + y_3 t^3 + \cdots) - 2Y(t) = 0$$

$$\frac{(Y(t) - y_0)}{t} - 2Y(t) = 0$$

$$Y(t) - y_0 - 2tY(t) = 0$$

$$(1-2t)Y(t) = y_0$$

$(\because y_0 = 1$ put$)$

$$Y(t) = \frac{y_0}{(1-2t)} = (1-2t)^{-1}$$

$$\therefore \sum_{n=0}^{\infty} y_n t^n = 1 + 2t + (2t)^2 + \cdots + (2t)^n + \cdots$$

$\therefore$ equating coefficient of t^n

$y_n = 2^n$

Q.2) Solve $y_{n+2} - 7y_{n+1} + 10y_n = 0$ with $y_0 = 0$, $y_1 = 3$ by the method of generating function.

Sol^n we have

$$y_{n+2} - 7y_{n+1} + 10y_n = 0 \quad \text{---} (1)$$

with $y_0 = 0$, $y_1 = 3$

consider the generating function

$$Y(t) = \sum_{n=0}^{\infty} y_n t^n = y_0 + y_1 t + y_2 t^2 + \cdots\cdots \quad \text{---} (2)$$

multiplying (1) by t^n & summing from $n=0$ to $n=\infty$

we have

$$\sum_{n=0}^{\infty} y_{n+2} t^n - 7\sum_{n=0}^{\infty} y_{n+1} t^n + 10 \sum_{n=1}^{\infty} y_n t^n = 0$$

$$(y_2 + y_3 t + \cdots\cdots) - 7(y_1 + y_2 t + \cdots\cdots) + 10\,Y(t) = 0$$

$$\frac{Y(t) - y_0 - y_1 t}{t^2} - 7\,\frac{Y(t) - y_0}{t} + 10\,Y(t) = 0$$

put $y_0 = 0$, $y_1 = 3$

$$\frac{Y(t) - 3t}{t^2} - \frac{7\,Y(t)}{t} + 10\,Y(t) = 0.$$

$$Y(t) - 3t - 7t\,Y(t) + 10t^2 Y(t) = 0$$

$$(1 - 7t + 10t^2)\,Y(t) = 3t$$

$$Y(t) = \frac{3t}{1 - 7t + 10t^2}$$

P.T.O

$1-2t - 1+5t = 3t$

$\overline{\cancel{2t}}$

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$$y(t) = \frac{3t}{(1-2t)(1-5t)}$$

Breaking into partial fraction

$$y(t) = \frac{1}{(1-5t)} - \frac{1}{(1-2t)}$$

$$\sum_{n=0}^{\infty} y_n t^n = (1-5t)^{-1} - (1-2t)^{-1}$$

$$= \left[1 + 5t + (5t)^2 + \cdots + (5t)^n + \cdots\right] - \left[1 + 2t + (2t)^2 + \cdots + (2t)^n + \cdots\right]$$

Equating coefficients of t^n

$$y_n = 5^n - 2^n$$

Q.) Solve $y_{n+2} - y_{n+1} - y_n = 0$ with $y_0 = 0$, $y_1 = 1$ by the method of generating function.

* **Partitions of Integers** ⇒ A Partition of a positive integer n is a multiset of positive integers that sum of n. It is denoted the number of partitions of n by P_n.

P_n = No. of ways of representing an integer as addition of positive integers

Example
$5 \Rightarrow \{5+0\}$
$5 \Rightarrow \{4+1\}$
$5 \Rightarrow \{3+2\}$
$5 \Rightarrow \{3+1+1\}$
$5 \Rightarrow \{2+2+1\}$
$5 \Rightarrow \{2+1+1+1\}$
$5 \Rightarrow \{1+1+1+1+1\}$

$\therefore P_5 = 7$ or $P_5(n) = 7$

increasing order से लिखें

$$\{\prod_{K=1}^{\infty}\}$$

$$(1+x+x^2+x^3+\cdots)(1+x^2+x^4+x^6+\cdots)(1+x^3+x^6+\cdots)$$
$$\cdots(1+x^K+x^{2K}+x^{3K}+\cdots)\cdots = \sum_{i=0}^{\infty} x^{iK}$$

Q.) Find the Partition P_8?

Sol^n
$$(1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8)(1+x^2+x^4+x^6+x^8)(1+x^3+x^6)(1+x^4+x^8)(1+x^5)(1+x^6)(1+x^7)(1+x^8)$$

$$= 1+x+2x^2+3x^3+5x^4+7x^5+11x^6+15x^7+22x^8+\cdots$$
$$\cdots + x^{56}$$

$P_8 = 22$

$8 \Rightarrow \{8+0\}$
$8 \Rightarrow \{7+1\}$
$8 \Rightarrow \{6+1+1\}$
$8 = \{5+1+1+1\}$
$8 \Rightarrow \{4+1+1+1+1\}$
$8 = \{3+1+1+1+1+1\}$
$8 = \{2+1+1+1+1+1+1\}$
$8 = \{1+1+1+1+1+1+1+1\}$
$8 = \{6+2\}$
$8 = \{4+2+2\}$

$8 = \{4+2+1+1\}$
$8 = \{3+1+2+1+1\}$
$8 = \{2+2+1+1+1+1\}$
$8 = \{3+3+2\}$
$8 = \{2+2+2+1+1\}$
$8 = \{2+2+2+2\}$
$8 = \{2+2+3+1\}$
$8 = \{4+3+1\}$
$8 = \{5+3\}$
$8 = \{5+2+1\}$

$8 = \{4+4\}$
$8 = \{3+3+1+1\}$

$\boxed{\therefore P_8 = 22}$

(OR)

An integer partitions is a way of writting a given Positive integer (X) as a sum of other Positive integers

(OR)

It is the number ~~tho~~ of ways that a positive integer can be written using positive integers, such that the sum adds up to the origional number (X).

* Euler's theorem ⇒ The no. of ways to partion an integer as a sum of unique integers is Equal to the no. of ways to partition an integer as a sum of odd integers

Example

Partitions of 5:-	Partions of 5 as a sum of unique integer	Partitions of 5 as sum of odd integers
5	5	5
4+1	4+1	3+1+1
3+2	3+2	1+1+1+1+1
3+1+1		
2+2+1		
2+1+1+1		
1+1+1+1+1		

Polynomial Representations:-

$$(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\cdots$$

$$(1+x)(1+x) = x^2+2x+1$$

Exponential Generating function ⇒

A normal Generating function for a sequence (a_r) is

$$\sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_r x^r + \cdots$$

$$= A(x) \quad \text{or} \quad f(x)$$

$(a_r) = (1)$ A sequence of $(1, 1, 1, \ldots)$

$= (a_0, a_1, a_2, a_3, \ldots)$ $(\because |x|<1)$

$$\therefore \sum_{r=0}^{\infty} a_r x^r = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x} = f(x)$$

(EGF) $\Rightarrow$ Exponential generating function:- The exponential generating function for the sequence (a_r) is

$$a_0 + a_1 \frac{x}{\lfloor 1} + a_2 \frac{x^2}{\lfloor 2} + \ldots + a_r \frac{x^r}{\lfloor r} + \ldots$$

$$= \sum_{r=0}^{\infty} a_r \frac{x^r}{\lfloor r}$$ Here a_r is cofficient of $\frac{x^r}{\lfloor r}$

Example $\Rightarrow$ Consider $(1, 1, 1, 1, \ldots) \Rightarrow (a_0, a_1, a_2, a_3, \ldots)$

it's Expontial generating function

$$1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \ldots + \frac{x^r}{\lfloor r} \ldots = \sum_{r=0}^{\infty} \frac{x^r}{\lfloor r} = e^x$$

it's EGF

Example $\Rightarrow$ consider $(\lfloor 0, \lfloor 1, \lfloor 2, \lfloor 3, \ldots \lfloor r, \ldots)$

$$\lfloor 0 + \lfloor 1 \frac{x}{\lfloor 1} + \lfloor 2 \frac{x^2}{\lfloor 2} + \ldots \lfloor r \frac{x^r}{\lfloor r} + \ldots = \sum_{r=0} \frac{x^r}{\cancel{\lfloor r}} = \frac{1}{1-x}$$

$\therefore$ Exponential generating function for $(a_r) = \lfloor r$ is $\frac{1}{1-x}$

Example $\Rightarrow$ consider $(1, k, k^2, k^3, \ldots k^r, \ldots)$ $k \neq 0$

then exponential generating function is e^{kx}

$$1 + k\frac{x}{\lfloor 1} + k^2 \frac{x^2}{\lfloor 2} + k^3 \frac{x^3}{\lfloor 3} + \ldots + \frac{k^r x^r}{\lfloor r} + \ldots$$

$$1 + \frac{kx}{\lfloor 1} + \frac{(kx)^2}{\lfloor 2} + \frac{(kx)^3}{\lfloor 3} + \ldots + \frac{(kx)^r}{\lfloor r} + \ldots$$

$$= \sum_{r=0}^{\infty} \frac{(kx)^r}{\lfloor r} = e^{kx}$$

Q.1) Show that the exponential generating function for the sequence $(1, 1\cdot3, 1\cdot3\cdot5, 1\cdot3\cdot5\cdot7, \ldots)$ is $(1-2x)^{-3/2}$

Sol^n To Show the coefficient of

$\frac{x^r}{\lfloor r}$ is $(1\cdot3\cdot5\cdot7\cdots(2r+1))$

or coefficient of x^r in $(1-2x)^{-3/2}$ is

$$\left(\frac{1\cdot3\cdot5\cdots(2r+1)}{\lfloor r}\right)$$

$\because$ coefficient of x^r in $(1-2x)^{-3/2}$

$$(1-2x)^{-3/2} = \sum_{i=0}^{\infty} \binom{-3/2}{i} (-2x)^i$$

$\therefore$ coefficient of x^r is $\binom{-3/2}{r}(-2)^r$

$$= \frac{(-2)^r\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)\left(-\frac{3}{2}-2\right)\cdots\left(-\frac{3}{2}-r+1\right)}{\lfloor r} = (-2)^r\left(-\frac{1}{2}\right)^r$$

$$= \cancel{(-2)^r} \times \cancel{\left(-\frac{1}{2}\right)} \frac{3\cdot5\cdot7\cdots(2r+1)}{\lfloor r}$$

$\therefore$ coefficient of $\frac{x^r}{\lfloor r}$ in $(1-2x)^{-3/2}$ is $3\cdot5\cdot7\cdots(2r+1)$

Q.2