Hierarchical Control

Laboratory

Exercise 4

Price method of coordination

The aim of this exercise is to apply the price method of coordination to a complex static system composed of three cross-coupled subsystems. During the exercise students will learn about some practical aspects of applying the method.

1 Modification of the local performance indeces

During the last exercise, the direct method of coordination was used in order to find the optimal solution of the complex system problem. In this exercise, the system (presented in the fig. 1) is the same, although the problem is slightly modified.

In the first subsystem, the coordination variable v_{31} is replaced with a new variable v_{1in}, and the variable v_{12} with a new variable v_{1out}. The first performance index have to be changed from the old form:

$$\begin{aligned} p_1(\underline{v}, \underline{m_1}) &= p_1(v_{31}, v_{12}, m_{11}, m_{12}) = \\ &= -(v_{31}-2)^2 - 2(v_{12}-3)^2 - v_{31}m_{12} - (m_{11}-v_{12})^2 + m_{11} + 40 \end{aligned}$$

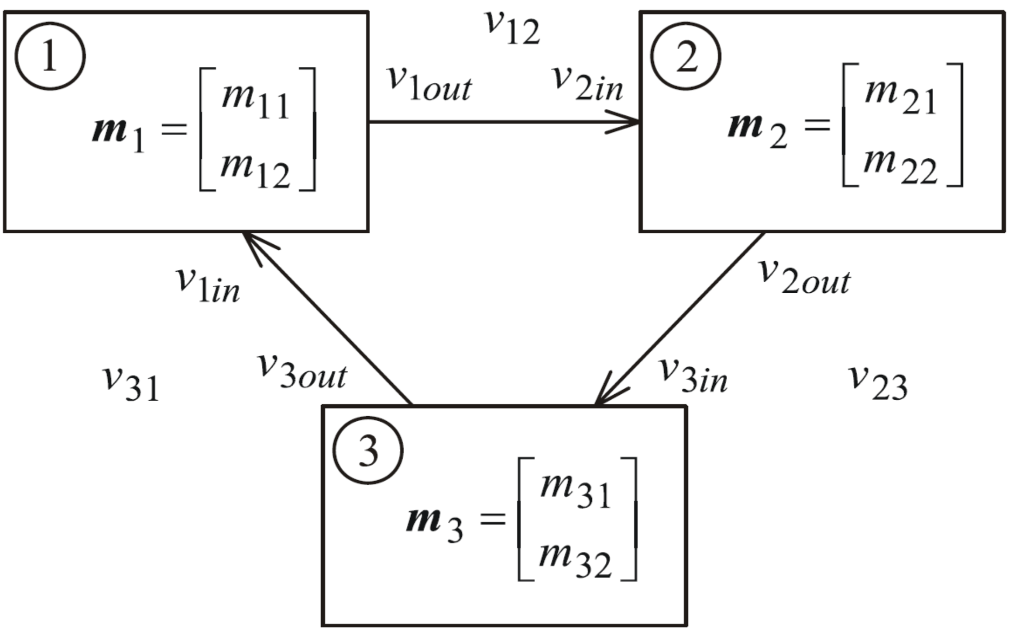

Figure 1: The complex system scheme.

to a new one:

$$\begin{aligned} p_1(\underline{v}, \underline{m_1}) &= p_1(v_{1in}, v_{1out}, m_{11}, m_{12}) = \\ &= -(v_{1in} - 2)^2 - 2(v_{1out} - 3)^2 - v_{1in}m_{12} - (m_{11} - v_{1out})^2 + \\ &\quad +m_{11} + 40 \end{aligned}$$

Accordingly, the new variables v_{2in}, v_{2out}, v_{3in} and v_{3out}, should replace the coordination variables v_{12}, v_{23} and v_{31} in performance indeces for subsystems 2 and 3. Many new variables are introduced, but they are also related to each other, since the system remains the same. A new constraints need to be introduced:

$$v_{1out} = v_{2in} \tag{1}$$
$$v_{2out} = v_{3in} \tag{2}$$
$$v_{3out} = v_{1in} \tag{3}$$

This is an optimisation problem with constraints. Such a problem can be solved by modifying the performance index to the form:

$$p'(\underline{v}, \underline{m}, \underline{\lambda}) = p(\underline{v}, \underline{m}) + \underline{\lambda}^T \underline{A}\, \underline{v} \tag{4}$$

where

$$\underline{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \quad \underline{v} = \begin{bmatrix} v_{1in} \\ v_{1out} \\ v_{2in} \\ v_{2out} \\ v_{3in} \\ v_{3out} \end{bmatrix}$$

λ_1, λ_2 and λ_3 are Lagrange multipliers for constraints. The $\underline{A}$ matrix consists of three rows and six columns:

$$\underline{A} = \begin{bmatrix} a_{1,v_{1in}} & a_{1,v_{1out}} & a_{1,v_{2in}} & a_{1,v_{2out}} & a_{1,v_{3in}} & a_{1,v_{3out}} \\ a_{2,v_{1in}} & a_{2,v_{1out}} & a_{2,v_{2in}} & a_{2,v_{2out}} & a_{2,v_{3in}} & a_{2,v_{3out}} \\ a_{3,v_{1in}} & a_{3,v_{1out}} & a_{3,v_{2in}} & a_{3,v_{2out}} & a_{3,v_{3in}} & a_{3,v_{3out}} \end{bmatrix}$$

Each row corresponds to one constraint, and each column corresponds to one variable v. The first constraint can be expressed in the form $v_{1out} - v_{2in} = 0$, so in the first row of the $\underline{A}$ matrix the element $a_{1,v_{1out}} = 1$, the element $a_{1,v_{2in}} = -1$ and all the remaining elements are equal to 0.

From the matrix $\underline{A}$ three submatrices can be derived:

$$\underline{A_1} = \begin{bmatrix} a_{1,v_{1in}} & a_{1,v_{1out}} \\ a_{2,v_{1in}} & a_{2,v_{1out}} \\ a_{3,v_{1in}} & a_{3,v_{1out}} \end{bmatrix} \quad \underline{A_2} = \begin{bmatrix} a_{1,v_{2in}} & a_{1,v_{2out}} \\ a_{2,v_{2in}} & a_{2,v_{2out}} \\ a_{3,v_{2in}} & a_{3,v_{2out}} \end{bmatrix} \quad \underline{A_3} = \begin{bmatrix} a_{1,v_{3in}} & a_{1,v_{3out}} \\ a_{2,v_{3in}} & a_{2,v_{3out}} \\ a_{3,v_{3in}} & a_{3,v_{3out}} \end{bmatrix}$$

These matrices correspond to subsystems in the complex system. In order to apply the price method of coordination to optimize these subsystems, the local performance indeces have to be modified:

$$\begin{aligned} p_1(\underline{v_1}, \underline{m_1}, \underline{\lambda}) &= p_1(v_{1in}, v_{1out}, m_{11}, m_{12}) + \underline{\lambda}^T \, \underline{A_1} \, \underline{v_1} \\ p_2(\underline{v_2}, \underline{m_2}, \underline{\lambda}) &= p_2(v_{2in}, v_{2out}, m_{21}, m_{22}) + \underline{\lambda}^T \, \underline{A_2} \, \underline{v_2} \\ p_3(\underline{v_3}, \underline{m_3}, \underline{\lambda}) &= p_3(v_{3in}, v_{3out}, m_{31}, m_{32}) + \underline{\lambda}^T \, \underline{A_3} \, \underline{v_3} \end{aligned}$$

Of course, the sum of the local performance indeces presented above is the global performance index (4).

2 Optimization

The optimization should be performed in iterative process. In each iteration, all the local optimizations should be done:

$$\max_{\underline{m_1},\underline{v_1}} p_1(\underline{v_1}, \underline{m_1}, \underline{\lambda}) \Longrightarrow \underline{m_{1(i)}},\ \underline{v_{1(i)}}$$

$$\max_{\underline{m_2},\underline{v_2}} p_2(\underline{v_2}, \underline{m_2}, \underline{\lambda}) \Longrightarrow \underline{m_{2(i)}},\ \underline{v_{2(i)}}$$

$$\max_{\underline{m_3},\underline{v_3}} p_3(\underline{v_3}, \underline{m_3}, \underline{\lambda}) \Longrightarrow \underline{m_{3(i)}},\ \underline{v_{3(i)}}$$

where $\underline{m_{j(i)}}$ and $\underline{v_{j(i)}}$ are vectors $\underline{m_j}$ and $\underline{v_j}$ obtained in the i-th iteration.

The main advantage of the price method of coordination is the fact that the optimization on the global level is performed without any constraints. During the exercise, a gradient method will be used. The $\underline{\lambda}$ vector for the $(i+1)$-th iteration will be calculated using the following formula:

$$\underline{\lambda_{i+1}} = \underline{\lambda_i} - c \begin{bmatrix} v_{1out} - v_{2in} \\ v_{2out} - v_{3in} \\ v_{3out} - v_{1in} \end{bmatrix}$$

3 The exercise program

1. Compose the $\underline{A}$ matrix.

2. Write a Matlab script which solves the complex system problem using the price method of coordination. The optimization should be repeated until the difference between two subsequent values of performance index is small enough. The script should provide the following information:

   - the values of the variables v_{1in}, v_{1out}, v_{2in}, v_{2out}, v_{3in} and v_{3out}
   - the values of the variables m_{11}, m_{12}, m_{21}, m_{22}, m_{31} and m_{32}
   - the values of the Lagrange multipliers λ_1, λ_2 and λ_3
   - the value of the performance index

3. Check the influence of the initial variables values on the obtained results and the number of performed iterations.

4. Compare the the values obtained using the price method of coordination with the ones obtained using direct method of coordination and without any coordination method.