Hierarchical Control

Laboratory

Exercise 2

Oil refinery

1 Linear programming

In mathematics, *linear programming* is a problem which can be expressed as follows:

$$\min_{\underline{x}} \underline{C}^T \, \underline{x}$$

with subject to

$$\underline{A}\,\underline{x} = \underline{B}$$

$$\underline{x} \geq \underline{0}$$

$\underline{x}$ is the vector of decision variables:

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

where $n \in N$, $x_i \in [0; \infty)$, $i \in \{1, 2, ..., n\}$.

$\underline{C}$ is the vector of the objective function coefficients:

$$\underline{C} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

where $c_i \in R$.

$\underline{A}$ and $\underline{B}$ are matrices of constraints coefficients:

$$\underline{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \quad \underline{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

where $m \in N$, $b_j \in R$, $a_{i,j} \in R$, $j \in \{1, 2, ..., m\}$.

Such a form of the linnear programming problem is called the standard form. The Matlab software is able to solve such a problems, but in the real cases problems rarely are formulated in this way.

2 A simple example

Let's consider a farmer, who has some limited resources: a piece of land, L square kilometers large, some amount of money M and fertilizer F. The farmer wants his land to be planted with either wheat or barley. All the data our farmer has is presented in the table 1. All these data are given for 1 square kilometer of the farmer's land. So, for example, in order to plant 1 square kilometer of wheat our farmer have to buy seed for P_w, then fertilize this land with F_w fertilizer, and then he can sell his crop (from this piece of his land) for S_w. The question is: what amounts of the farmer's land should be planted with wheat and barley in order to make the highest possible profit?

Our objective function can be expressed as follows:

	wheat	barley
selling price	S_w	S_b
seed price	P_w	P_b
fertilizer needed	F_w	F_b

Table 1: The farming data.

$$\text{profit} = S_w x_w + S_b x_b \tag{1}$$

where x_w is the amount of land planted with wheat, and x_b is the amount of land planted with barley.

The farmer cannot plant more than he has:

$$x_w + x_b \leq L \tag{2}$$

He cannot also spend more money than he has:

$$P_w x_w + P_b x_b \leq M \tag{3}$$

And finally he can't use more fertilizer than he has:

$$F_w x_w + F_b x_b \leq F \tag{4}$$

We have three constraints, two decision variables and the objective function to maximize. However, we cannot express our problem in the standard form mentioned in the point 1, because of the inequality symbols in our contraints. We have to modify our problem first, introducing slack variables.

Let's modify the constraint (2). The farmer cannot use more land than he has, hence the inequality symbol. Let's introduce the slack "land" variable x_u, which will stand for the amount of unused land. The constraint (2) can be expressed as follows:

$$x_w + x_b + x_u = L \tag{5}$$

In the same way we can modify the constraints (3) and (4), introducing slack variables x_m for the amount of unused money and x_f for the amount of unused fertilizer:

$$P_w x_w + P_b x_b + x_m = M \tag{6}$$

$$F_w x_w + F_b x_b + x_f = F \tag{7}$$

The equations (1) and (5)-(7) form a linear programming problem. The problem can be denoted with matrices:

$$\text{profit} = \underline{C}^T \underline{x}$$

where

$$\underline{C} = \begin{bmatrix} S_w \\ S_b \\ 0 \end{bmatrix} \quad \underline{x} = \begin{bmatrix} x_w \\ x_b \\ x_u \end{bmatrix}$$

and

$$\underline{A}\,\underline{x} = \underline{B}$$

where

$$\underline{A} = \begin{bmatrix} 1 & 1 & 1 \\ P_w & P_b & 1 \\ F_w & F_b & 1 \end{bmatrix} \quad \underline{B} = \begin{bmatrix} L \\ M \\ F \end{bmatrix}$$

3 Using the Matlab software to solve linear programming problems

The linear programming problems can be solved with the Matlab software. This is a `linprog` command in the optimization toolbox. Using this command we can solve the following problem:

$$\min_{\underline{x}} \underline{C}^T \underline{x} \quad \text{subject to:} \quad \underline{A}\,\underline{x} = \underline{B}$$

As you can see, this is one difference between the problem being solved with the `linprog` command and the problem formulated in the point 1: the $\underline{x}$ vector elements can have values lower than 0. The problem being solved with the `linprog` command is thus more general. (In fact, there are more differences; for more information please refer to the help for the `linprog` command).

Let's try to solve the farmer's problem using the Matlab software. First, let's create the $\underline{C}$ vector. It can be done with the following Matlab command:

```
>> C = [ Sw Sb 0 ]'
```

Of course, in order to use such a command it's needed to enter the values for Sw and Sb variables, but in this example the focus is rather on the commands syntax.

The $\underline{A}$ and $\underline{B}$ matrices have to be entered also:

```
>> A = [ 1 1 1; Pw Pb 1; Fw Fb 1 ]
>> B = [ L M F ]'
```

In the farmer's problem, all the decision variables are non-negative. In Matlab this constraint can be expressed in a form of lower bounds for these variables:

```
>> lb = zeros(3,1)'
```

The `linprog` command needs also the upper bounds for devision variables, but since there are no such bounds in the farmer's problem, they might be declared as a vector of infinite numbers:

```
>> ub = inf(3,1)'
```

Now, the result can be obtained using the Matlab software:

```
>> [X,Fvalneg] = linprog(-C, [], [], A, B, lb, ub)
```

The `linprog` command solves a problem of minimization of the given performance index, and in the farmer's problem the objective is to maximize the profit. Therefore the minus sign in the `linprog` command above. The `X` vector will contain the values of all devision variables, and the `Fvalneg` will contain the negated value of the optimal profit. Thus, the profit can be obtained as follows:

```
>> Fval = - Fvalneg
```

4 The oil refinery problem

Let's suppose that we have to optimise production in the oil refinery presented in the figure 1. The oil refinery consists of many subsystems:

1. evaporator
2. heater
3. distillation tower
4. cracking furnace
5. blending installation

The refinery buys oil from three different delivers and sales nine different products shown in the figure 1. The main technological processes are:

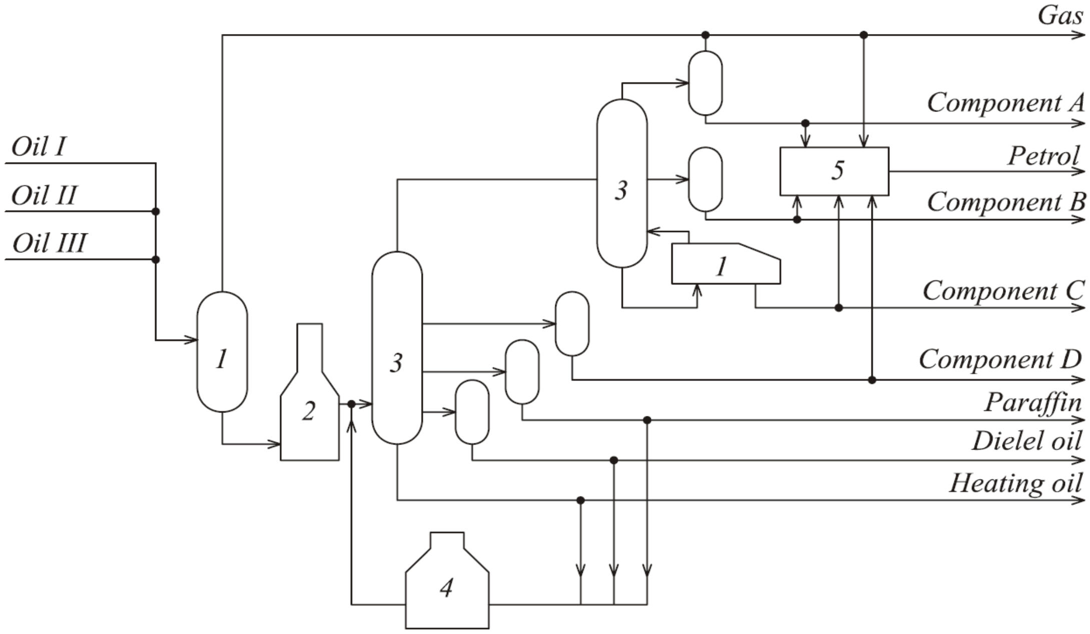

Figure 1: Oil refinery scheme.

- distillation
- cracking
- blending of the petrol

The goal of the refinery owner is to maximize daily profit. The profit can be expressed as follows:

$$Q = Q_{sale} - Q_{prod} - Q_{buy} \tag{8}$$

where Q_{sale} is the sale income, Q_{prod} is the production cost, and Q_{buy} is the cost of mineral oil purchase. All these variables are dependant on material flows:

$$\begin{aligned} Q_{sale} &= 1.8s_{po} + 4.0s_{do} + 4.2s_{ho} + 5.5s_p + \\ &\quad + 4.0s_{cD} + 4.1s_{cC} + 4.2s_{cB} + 4.3s_{cA} + 3.3s_g \quad (9) \\ Q_{prod} &= 0.4(d_{oI} + d_{oII} + d_{oIII}) + 0.16c_{po} + 0.21c_{do} + 0.21c_{ho} \quad (10) \\ Q_{buy} &= 1.4d_{oI} + 1.5d_{oII} + 1.6d_{oIII} \quad (11) \end{aligned}$$

where:

s_{po} sold paraffin oil

s_{do} sold disel oil

s_{ho} sold heating oil

s_p sold petrol

s_{cA} sold component A

s_{cB} sold component B

s_{cC} sold component C

s_{cD} sold component D

s_g sold gas

d_{oI} purchased and distilled oil I

d_{oII} purchased and distilled oil II

d_{oIII} purchased and distilled oil III

c_{po} cracked paraffin oil

c_{do} cracked diesel oil

c_{ho} cracked heating oil

Maximization of the given performance index is constrained by various factors:

- The amounts of purchased oils are limited. The maximal amounts for the given oil delivers are: 9500 [m^3] for the 1st oil deliver, 8500 [m^3] for the 2nd one, and 8500 for the 3rd one. Hence the following constraints:

$$\begin{aligned} d_{oI} + u_{oI} &= 9500 && (12) \\ d_{oII} + u_{oII} &= 8500 && (13) \\ d_{oIII} + u_{oIII} &= 8500 && (14) \end{aligned}$$

  where:

  u_{oI} unused amount of oil I

  u_{oII} unused amount of oil II

  u_{oIII} unused amount of oil III

u_{oI}, u_{oII} and u_{oIII} are non-negative slack variables.

- Maximal daily distillation capacity is 14000 [m^3].

$$d_{oI} + d_{oII} + d_{oIII} + u_{dc} = 14000 \tag{15}$$

where u_{dc} is unused distillation capacity.

- Maximal daily cracking ability is 3500 [m^3]:

$$c_{po} + c_{do} + c_{ho} + u_{cc} = 3500 \tag{16}$$

where u_{cc} is unused cracking capacity.

- The mass balance of gas:

$$\begin{aligned} 0.02d_{oI} + 0.03d_{oII} + 0.04d_{oIII} + \\ + 0.12c_{po} + 0.16c_{do} + 0.14c_{ho} - s_g - 0.15s_p &= 0 \end{aligned} \tag{17}$$

- The mass balance of component A:

$$\begin{aligned} 0.02d_{oI} + 0.02d_{oII} + 0.03d_{oIII} + \\ + 0.06c_{po} + 0.12c_{do} + 0.10c_{ho} - s_{cA} - 0.1s_p &= 0 \end{aligned} \tag{18}$$

- The mass balance of component B:

$$\begin{aligned} 0.03d_{oI} + 0.04d_{oII} + 0.05d_{oIII} + \\ + 0.04c_{po} + 0.04c_{do} + 0.04c_{ho} - s_{cB} - 0.25s_p &= 0 \end{aligned} \tag{19}$$

- The mass balance of component C:

$$\begin{aligned} 0.04d_{oI} + 0.05d_{oII} + 0.08d_{oIII} + \\ + 0.4c_{po} + 0.2c_{do} + 0.3c_{ho} - s_{cC} - 0.3s_p &= 0 \end{aligned} \tag{20}$$

- The mass balance of component D:

$$\begin{aligned} 0.04d_{oI} + 0.06d_{oII} + 0.05d_{oIII} + \\ + 0.06c_{po} + 0.28c_{do} + 0.3c_{ho} - s_{cD} - 0.2s_p &= 0 \end{aligned} \tag{21}$$

- The mass balance of the heating oil:

$$s_{ho} + c_{ho} - 0.10d_{oI} - 0.07d_{oII} - 0.14d_{oIII} - 0.6c_{po} - 0.2c_{do} = 0 \tag{22}$$

- The mass balance of the diesel oil:

$$s_{do} + c_{do} - 0.20d_{oI} - 0.12d_{oII} - 0.11d_{oIII} = 0 \tag{23}$$

- The mass balance of the paraffin oil:

$$s_{po} + c_{po} - 0.55d_{oI} - 0.61d_{oII} - 0.50d_{oIII} = 0 \tag{24}$$

Of course, all those variables should be non-negative.

So, the problem of oil refinery optimisation is the LP problem of maximization of the performance index (8) with subject to the constraints (9)-(24).

5 The exercise program

1. Write a Matlab script which solves the oil refinery problem. The script should provide the following information:
   - how much oil should be purchased from each oil deliver
   - how much more oil could've been purchased from each oil deliver
   - how much more oil could've been cracked
   - how much more oil could've been distilled
2. Check how changes of market oil prices affect those values.
3. In the oil refinery optimization problem the amount of oil refinery owner money is "unlimited". What additional contraints would have to be introduced, if this amount of money was limited?