Online Connectivity

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Online Connectivity

- Solve the following problem
  - Given a set of N objects
  - Union command
    - Connects two objects
  - Find query
    - Finds connected couples
- We do not want to know the path which connect two objects but only whether such a path exists or not
- N objects (whatever they are) can be mapped on N integers (from 0 to N-1)

Online Connectivity

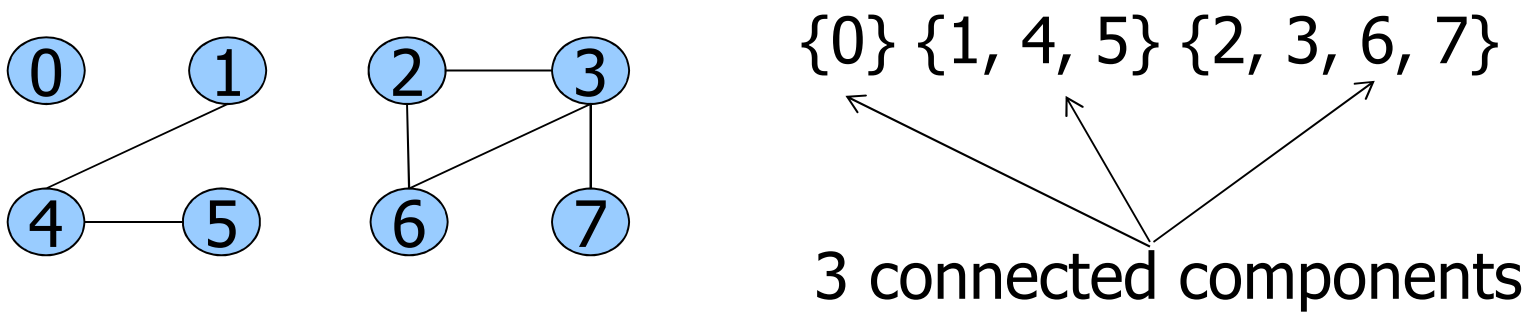
- Input: sequence of integer pairs (p, q)
  - Interpretation: p is connected to q
- Output
  - Null if p and q are already connected (directly or indirectly)
  - Else (p, q)

Online Connectivity

- Connectivity is an equivalence relation
  - Reflexive: *p is connected to p*
  - Symmetrical: *if p is connected to q, q is connected to p*
  - Transitive: *if p is connected to q and q is connected to r, then p is connected to r*

Online Connectivity

- Connected component
  - Maximal subset of mutually reachable nodes



  - There are no elements connecte to an element outside its connected component
    - Find is checking connected component
    - Union is replacing connected component with their union

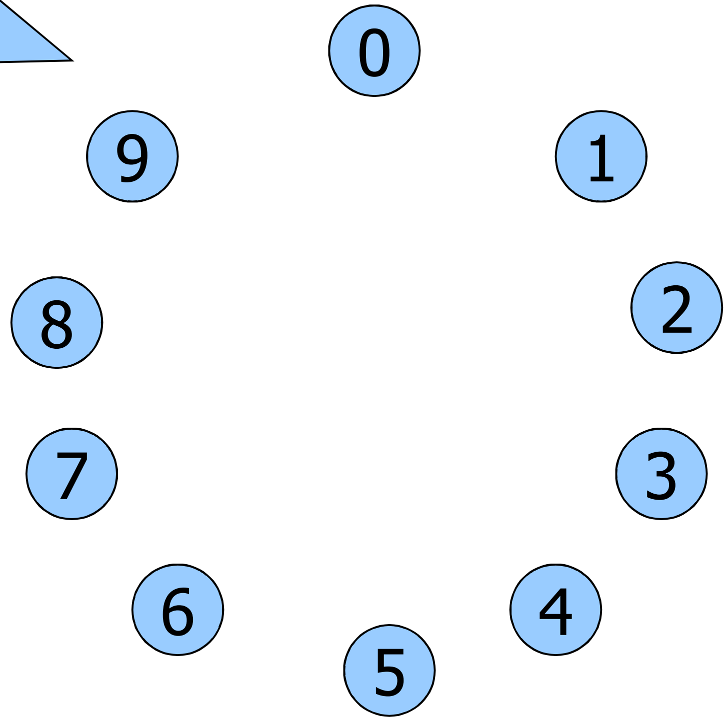
Applications

- Computer networks
  - Integers p and q represent computers
  - (p, q) connections between computers
- Electrical networks
  - Integers p and q represent contact points
  - (p, q) wires
- Programming environments
  - Integers p and q represent variables
  - (p, q) declarations of equivalent variables
- Social network
  - …

Example

- Pairs
  - 3-4, 4-9, 8-0, 2-3, 5-6, 2-9, 5-9, 7-3, 4-8, 5-6, 0-2, 6-1

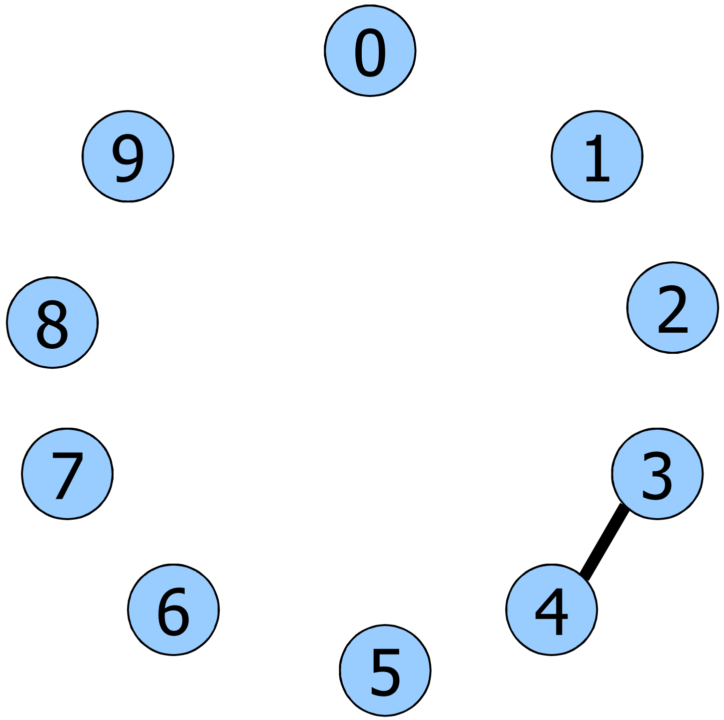
Graph:
structure representing
nodes (vertices)
and their connections
(edges)



Example

- Pairs
  - 3-4, 4-9, 8-0, 2-3, 5-6, 2-9, 5-9, 7-3, 4-8, 5-6, 0-2, 6-1

Input
3-4

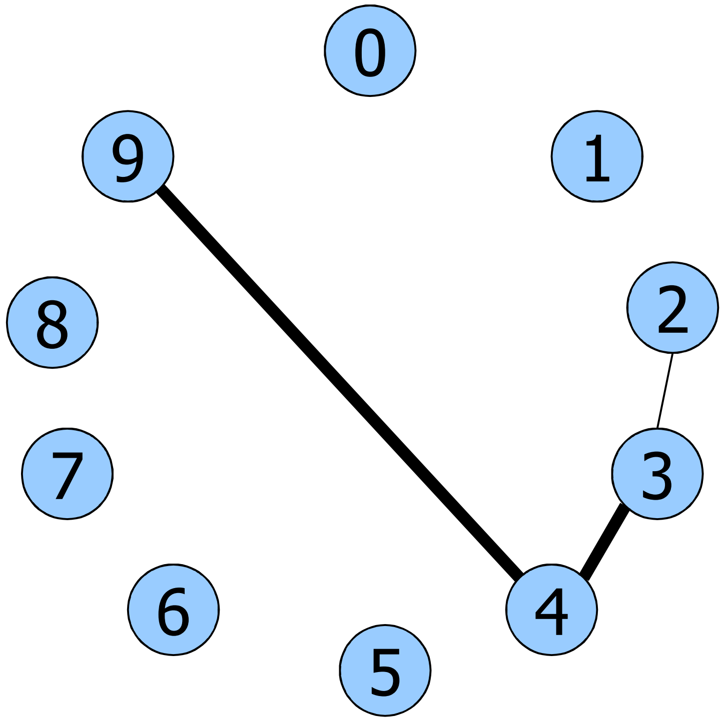


Output
3-4

Example

- Pairs
  - 3-4, 4-9, 8-0, 2-3, 5-6, 2-9, 5-9, 7-3, 4-8, 5-6, 0-2, 6-1

Input
4-9

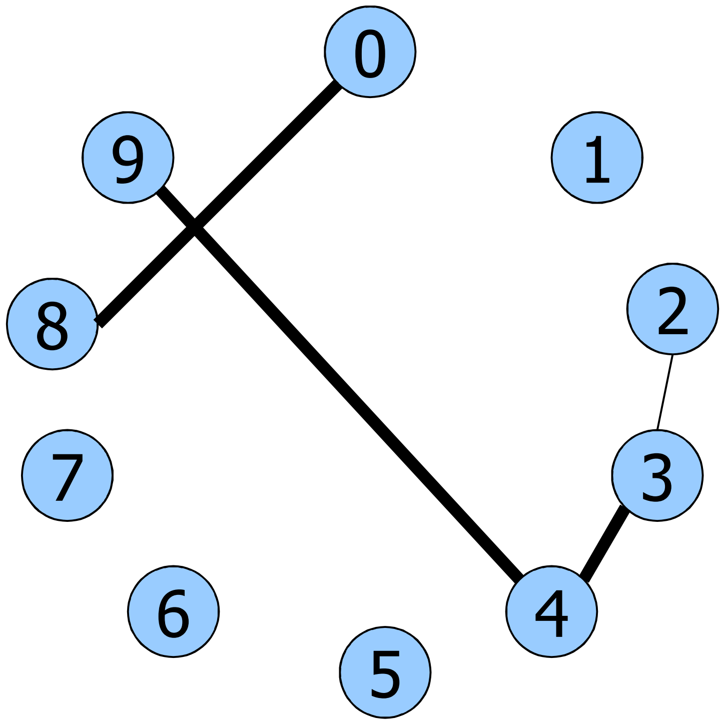


Output
4-9

Example

- Pairs
  - 3-4, 4-9, 8-0, 2-3, 5-6, 2-9, 5-9, 7-3, 4-8, 5-6, 0-2, 6-1

Input
8-0

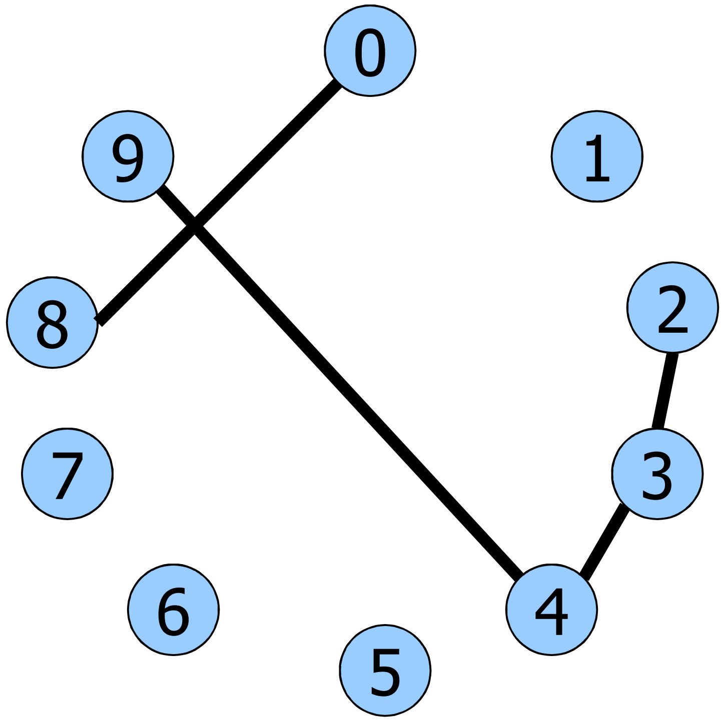


Output
8-0

Example

- Pairs
  - 3-4, 4-9, 8-0, 2-3, 5-6, 2-9, 5-9, 7-3, 4-8, 5-6, 0-2, 6-1

Input
2-3

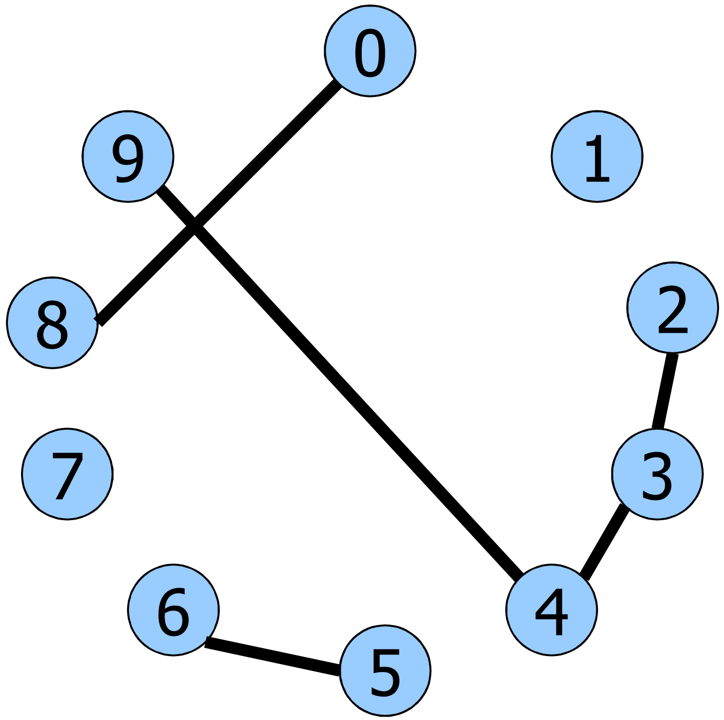


Output
2-3

Example

- Pairs
  - 3-4, 4-9, 8-0, 2-3, 5-6, 2-9, 5-9, 7-3, 4-8, 5-6, 0-2, 6-1

Input
5-6



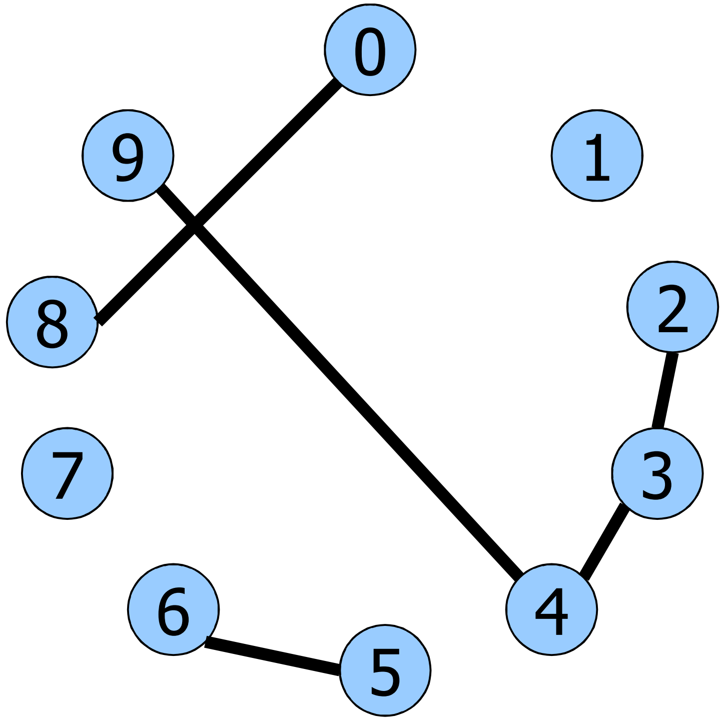
Output
5-6

Example

- Pairs
  - 3-4, 4-9, 8-0, 2-3, 5-6, 2-9, 5-9, 7-3, 4-8, 5-6, 0-2, 6-1

Input
2-9

Path
2-3-4-9
already exists

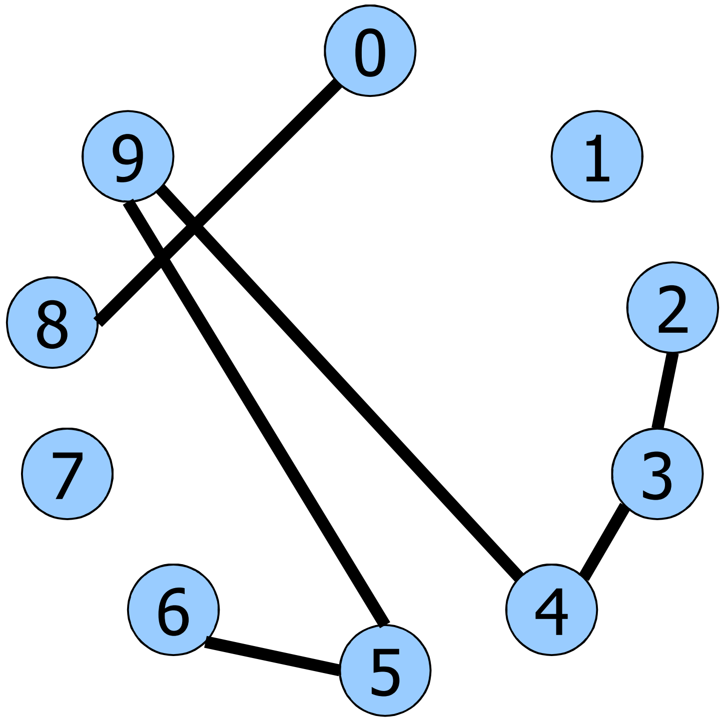


Output

Example

- Pairs
  - 3-4, 4-9, 8-0, 2-3, 5-6, 2-9, 5-9, 7-3, 4-8, 5-6, 0-2, 6-1

Input
5-9

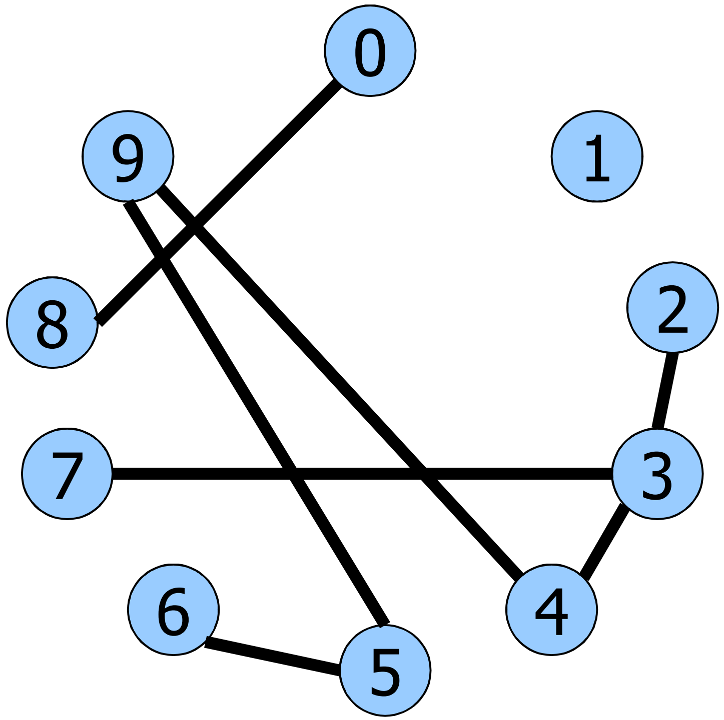


Output
5-9

Example

- Pairs
  - 3-4, 4-9, 8-0, 2-3, 5-6, 2-9, 5-9, 7-3, 4-8, 5-6, 0-2, 6-1

Input
7-3

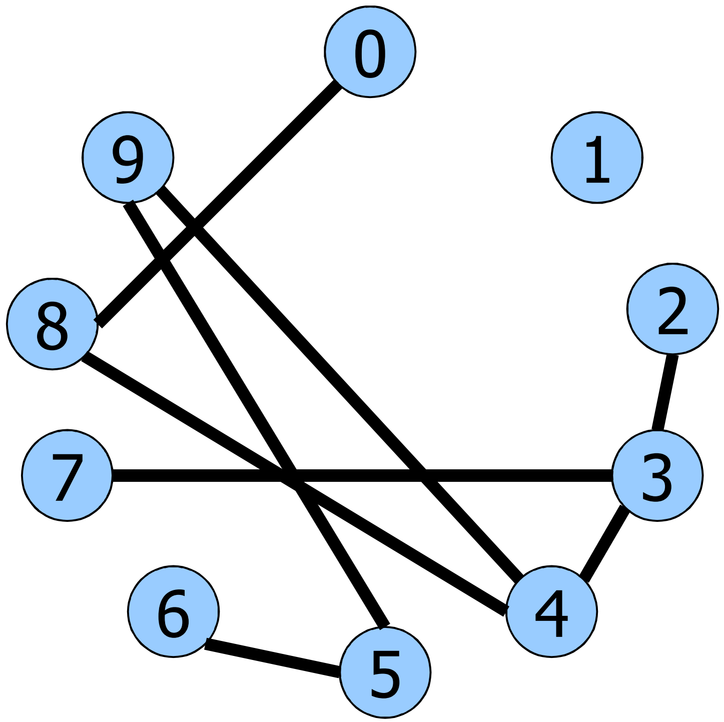


Output
7-3

Example

- Pairs
  - 3-4, 4-9, 8-0, 2-3, 5-6, 2-9, 5-9, 7-3, 4-8, 5-6, 0-2, 6-1

Input
4-8

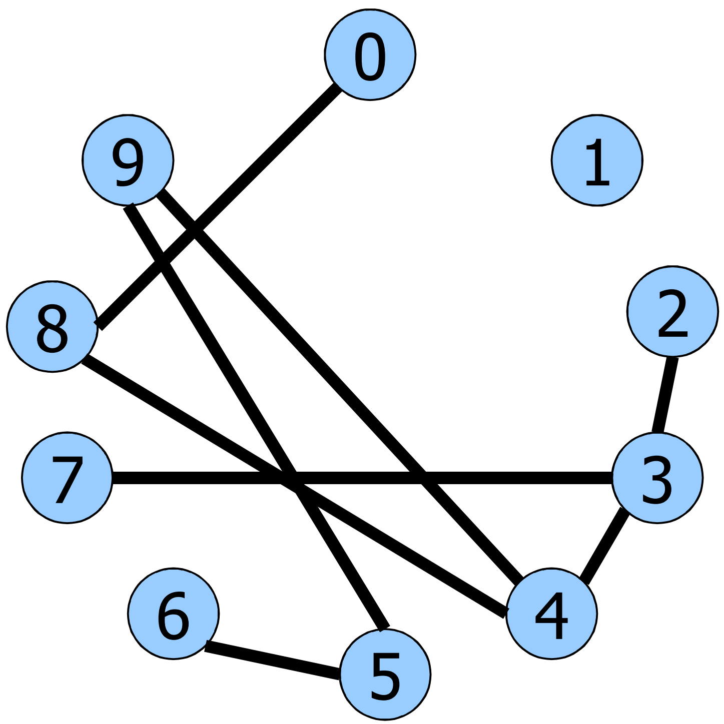


Output
4-8

Example

- Pairs
  - 3-4, 4-9, 8-0, 2-3, 5-6, 2-9, 5-9, 7-3, 4-8, 5-6, 0-2, 6-1

Input
5-6



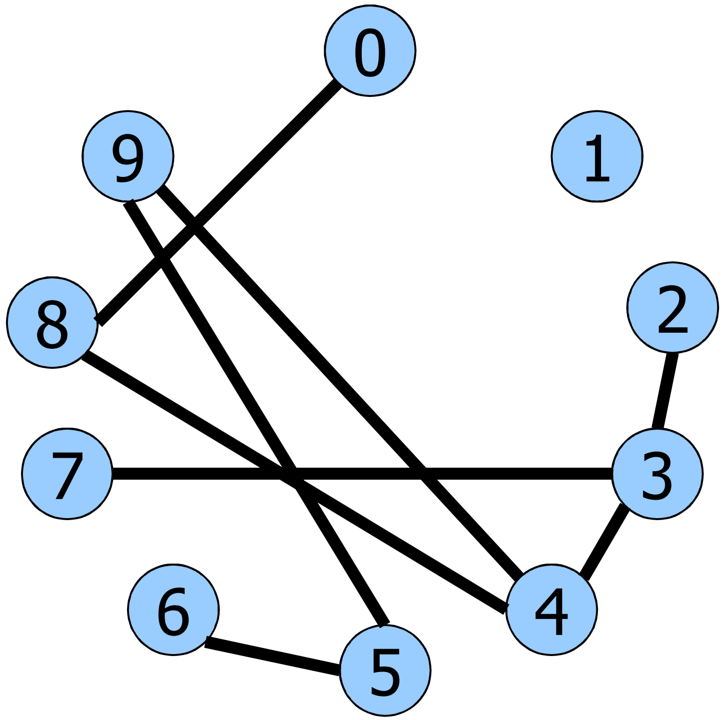
Output

Path
5-6
already exists

Example

- Pairs
  - 3-4, 4-9, 8-0, 2-3, 5-6, 2-9, 5-9, 7-3, 4-8, 5-6, 0-2, 6-1

Input
0-2



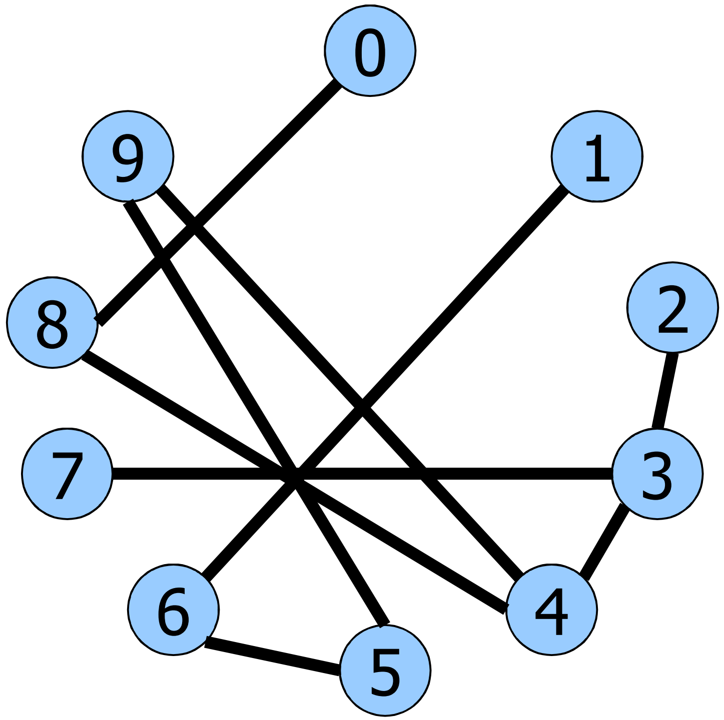
Output

Path
0-8-4-3-2
already exists

Example

- Pairs
  - 3-4, 4-9, 8-0, 2-3, 5-6, 2-9, 5-9, 7-3, 4-8, 5-6, 0-2, 6-1

Input
6-1



Output
6-1

Approach

Hypothesis

- We do not have the graph
- We work pair by pair
  - We keep and update information necessary to find out connectivity
  - Sets S_i of connected pairs, initially as many sets as nodes, each node being connected just with itself
- Abstract operations
  - find: find the set an object belongs to
  - union: merge two sets

- Algorithm: repeat for all pairs (p, q)
  - Read the pair (p, q)
  - Execute find on p: find an S_p such that $p \in S_p$
  - Execute find on q: find an S_q such that $q \in S_q$
  - If S_p and S_q coincide
    - Consider the next pair
    - Otherwise execute union on S_p and S_q

Quick find

- Represent sets S_i of connected pairs with array id
  - Initially id[i] = i (no connection)

id	0	1	2	3	4	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9

  - If p and q are connected, id[p] = id[q]

6 and 8 are connected

id	0	1	1	3	4	5	6	6	6	9
	0	1	2	3	4	5	6	7	8	9

Quick find

Repeat for all pairs (p, q)

- Read pair (p, q)
- Find
  - Check if (id[p] = id[q])
  - Do nothing and move to the next pair
- Else Union
  - Scan the array, replacing id[p] values with id[q] values

Union 3 and 4

id	0	1	2	4	4	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9

Tree representation

- Some objects represent the set they belong to
- Other objects point to the object that represents the set they belong to

0 1 2 3 4

5 6 7 8 9

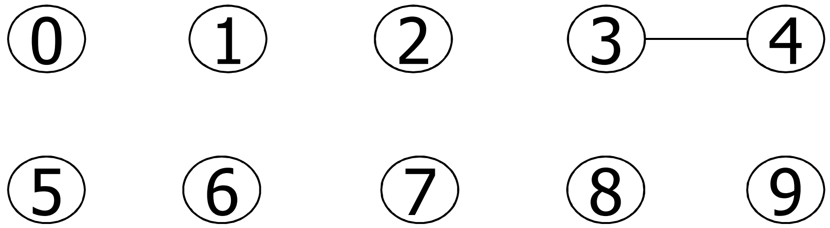
Example

Initially

id	0	1	2	3	4	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9

$S_0 = \{0\}$, $S_1 = \{1\}$, $S_2 = \{2\}$, $S_3 = \{3\}$, $S_4 = \{4\}$
$S_5 = \{5\}$, $S_6 = \{6\}$, $S_7 = \{7\}$, $S_8 = \{8\}$, $S_9 = \{9\}$

0 1 2 3 4 5 6 7 8 9



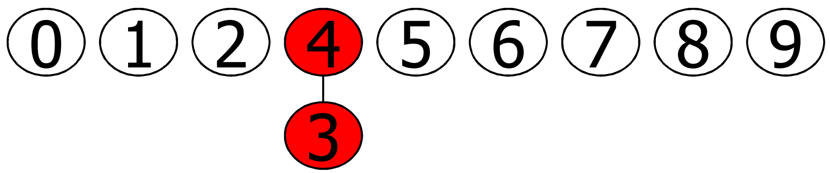
p q = 3 4

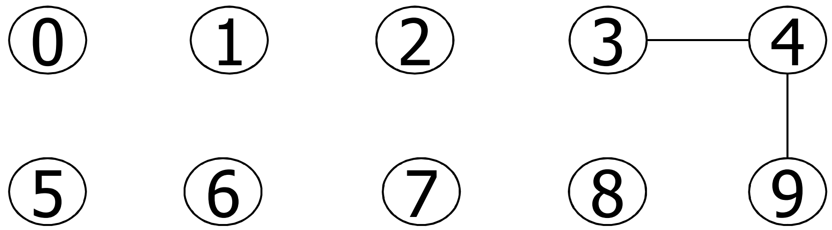
id	0	1	2	3	4	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9

id[p]=3 $\neq$ id[q]=4
replace all id[p] values with id[q] values

id	0	1	2	4	4	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9

$S_0 = \{0\}$, $S_1 = \{1\}$, $S_2 = \{2\}$, $S_{3\text{-}4} = \{3,4\}$,
$S_5 = \{5\}$, $S_6 = \{6\}$, $S_7 = \{7\}$, $S_8 = \{8\}$, $S_9 = \{9\}$





p q = 4 9

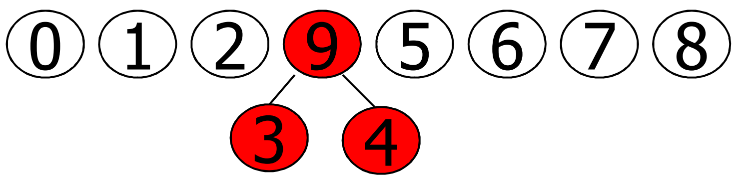
id	0	1	2	4	4	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9

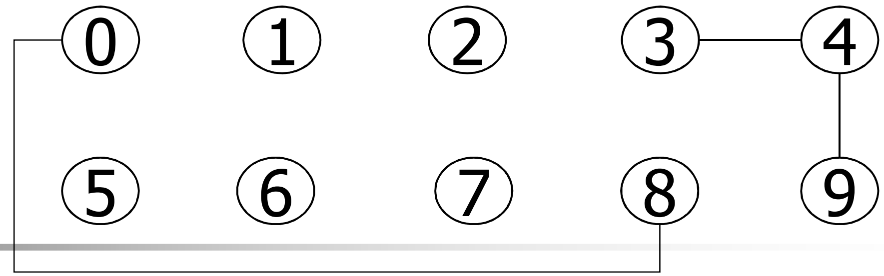
id[p]=4 $\neq$ id[q]=9

replace all id[p] values with id[q] values

id	0	1	2	9	9	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9

S_0 = {0}, S_1 = {1}, S_2 = {2}, $S_{3\text{-}4\text{-}9}$ = {3,4,9},
S_5 = {5}, S_6 = {6}, S_7 = {7}, S_8 = {8}





p q = 8 0

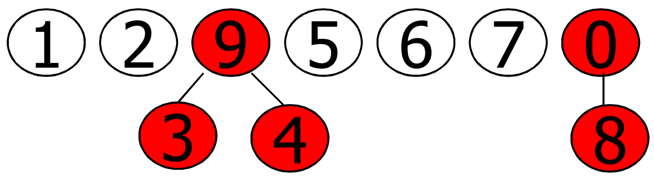
id	0	1	2	9	9	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9

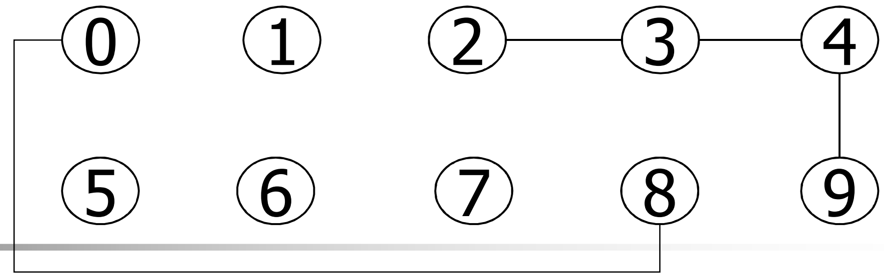
id[p]=8 $\neq$ id[q]=0

replace all id[p] values with id[q] values

id	0	1	2	9	9	5	6	7	0	9
	0	1	2	3	4	5	6	7	8	9

$S_{0\text{-}8} = \{0,8\}$, $S_1 = \{1\}$, $S_2 = \{2\}$, $S_{3\text{-}4\text{-}9} = \{3,4,9\}$,
$S_5 = \{5\}$, $S_6 = \{6\}$, $S_7 = \{7\}$





p q = 2 3

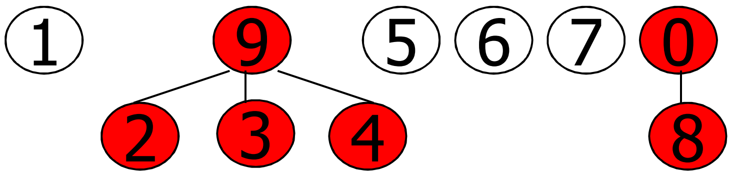
id	0	1	2	9	9	5	6	7	0	9
	0	1	2	3	4	5	6	7	8	9

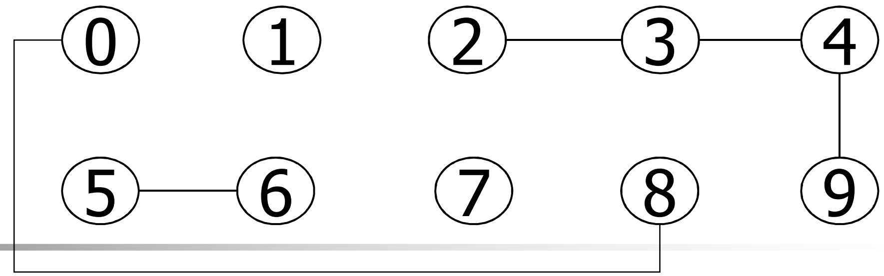
id[p]=2 $\neq$ id[q]=9

replace all id[p] values with id[q] values

id	0	1	9	9	9	5	6	7	0	9
	0	1	2	3	4	5	6	7	8	9

$S_{0\text{-}8} = \{0,8\}$, $S_1 = \{1\}$, $S_{2\text{-}3\text{-}4\text{-}9} = \{2,3,4,9\}$,
$S_5 = \{5\}$, $S_6 = \{6\}$, $S_7 = \{7\}$





p q = 5 6

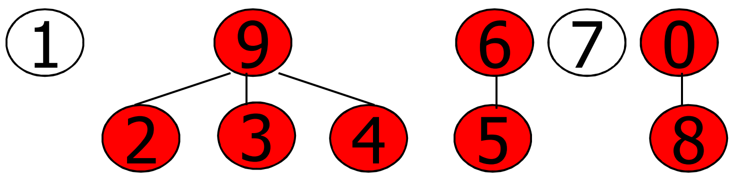
id										
	0	1	9	9	9	5	6	7	0	9
	0	1	2	3	4	5	6	7	8	9

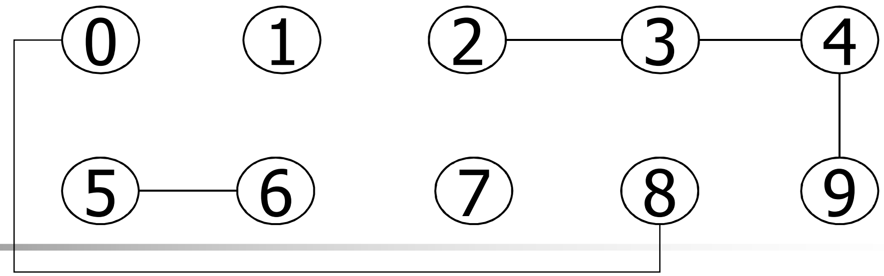
id[p]=5 $\neq$ id[q]=6

replace all id[p] values with id[q] values

id										
	0	1	9	9	9	6	6	7	0	9
	0	1	2	3	4	5	6	7	8	9

$S_{0\text{-}8}$ = {0,8}, S_1 = {1}, $S_{2\text{-}3\text{-}4\text{-}9}$ = {2,3,4,9},
$S_{5\text{-}6}$ = {5,6}, S_7 = {7}





p q = 2 9

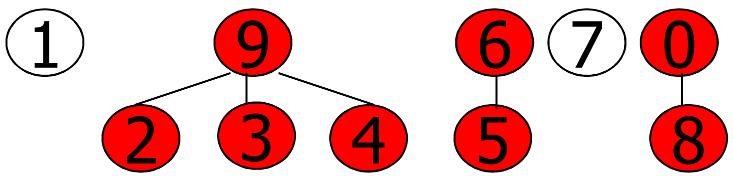
id	0	1	9	9	9	6	6	7	0	9
	0	1	2	3	4	5	6	7	8	9

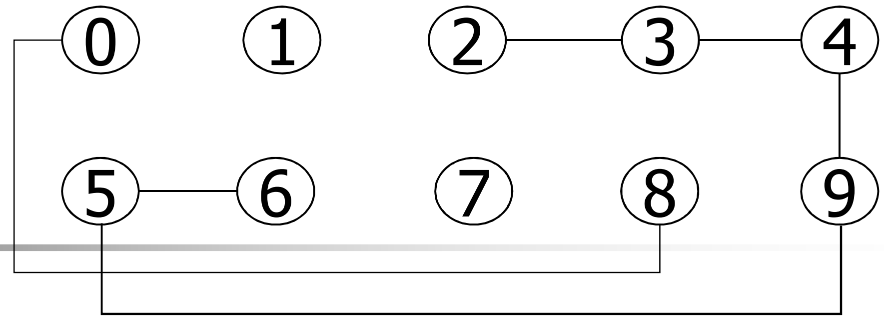
id[p]=9 = id[q]=9

no change

id	0	1	9	9	9	6	6	7	0	9
	0	1	2	3	4	5	6	7	8	9

$S_{0\text{-}8} = \{0,8\}$, $S_1 = \{1\}$, $S_{2\text{-}3\text{-}4\text{-}9} = \{2,3,4,9\}$,
$S_{5\text{-}6} = \{5,6\}$, $S_7 = \{7\}$





p q = 5 9

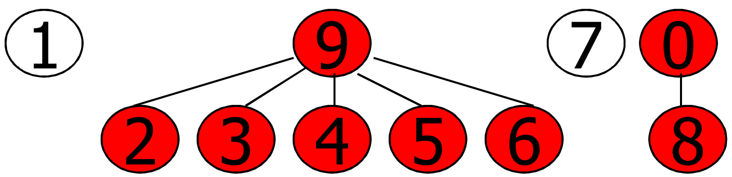
id	0	1	9	9	9	6	6	7	0	9
	0	1	2	3	4	5	6	7	8	9

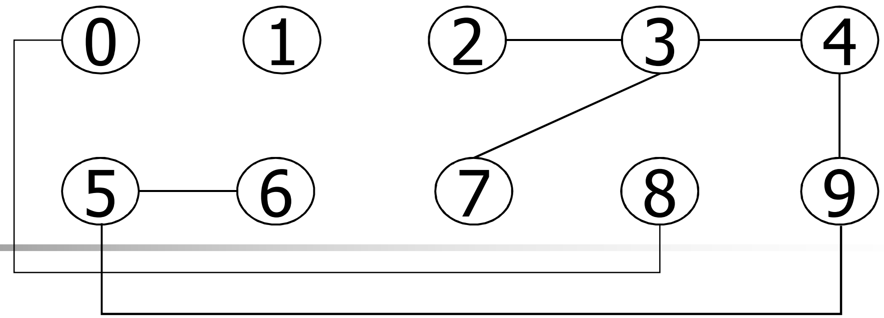
id[p]=6 $\neq$ id[q]=9

replace all id[p] values with id[q] values

id	0	1	9	9	9	9	9	7	0	9
	0	1	2	3	4	5	6	7	8	9

$S_{0\text{-}8}$ = {0,8}, S_1 = {1}, $S_{2\text{-}3\text{-}4\text{-}5\text{-}6\text{-}9}$ = {2,3,4,5,6,9}, S_7 = {7}





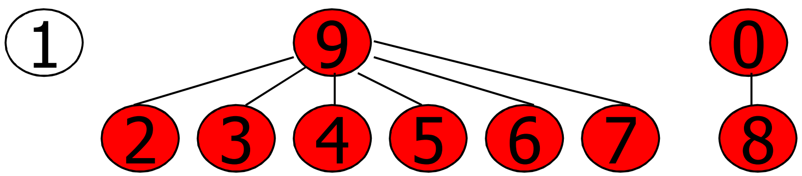
p q = 7 3

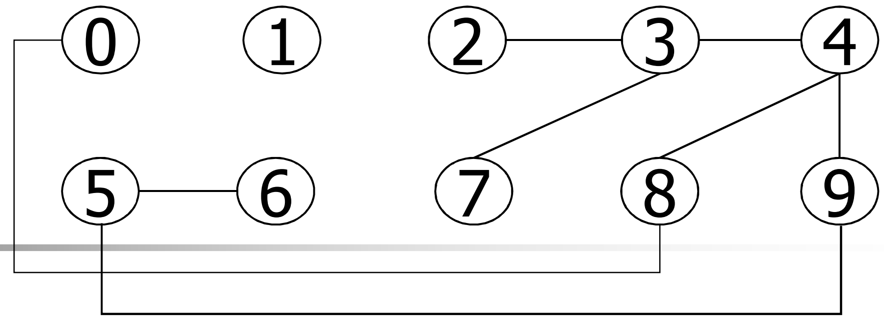
id	0	1	9	9	9	9	9	7	0	9
	0	1	2	3	4	5	6	7	8	9

id[p]=7 $\neq$ id[q]=9
replace all id[p] values with id[q] values

id	0	1	9	9	9	9	9	9	0	9
	0	1	2	3	4	5	6	7	8	9

$S_{0\text{-}8} = \{0,8\}$, $S_1 = \{1\}$, $S_{2\text{-}3\text{-}4\text{-}5\text{-}6\text{-}7\text{-}9} = \{2,3,4,5,6,7,9\}$





p q = 4 8

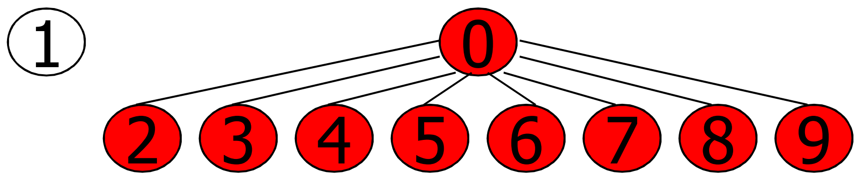
id										
	0	1	9	9	9	9	9	9	0	9
	0	1	2	3	4	5	6	7	8	9

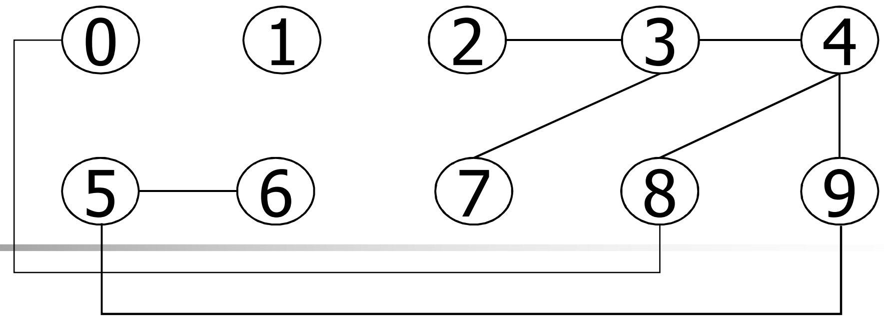
id[p]=9 $\neq$ id[q]=0

replace all id[p] values with id[q] values

id										
	0	1	0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6	7	8	9

$S_1 = \{1\}$, $S_{0\text{-}2\text{-}3\text{-}4\text{-}5\text{-}6\text{-}7\text{-}8\text{-}9} = \{0,2,3,4,5,6,7,8,9\}$





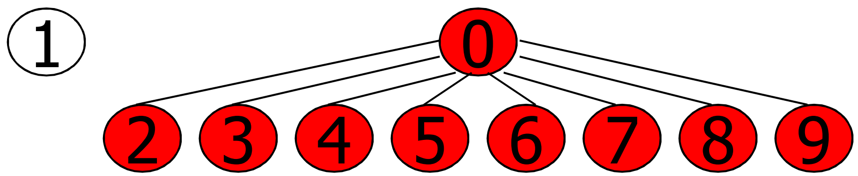
p q = 5 6

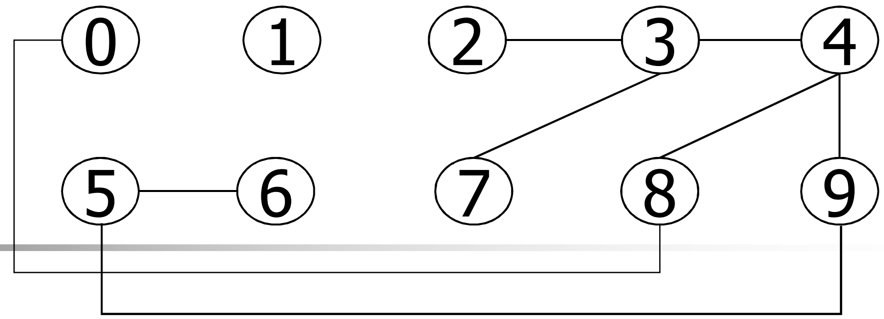
id										
value	0	1	0	0	0	0	0	0	0	0
index	0	1	2	3	4	5	6	7	8	9

id[p]=0 = id[q]=0
no change

id										
value	0	1	0	0	0	0	0	0	0	0
index	0	1	2	3	4	5	6	7	8	9

$S_1 = \{1\}, S_{0\text{-}2\text{-}3\text{-}4\text{-}5\text{-}6\text{-}7\text{-}8\text{-}9} = \{0,2,3,4,5,6,7,8,9\}$





p q = 0 2

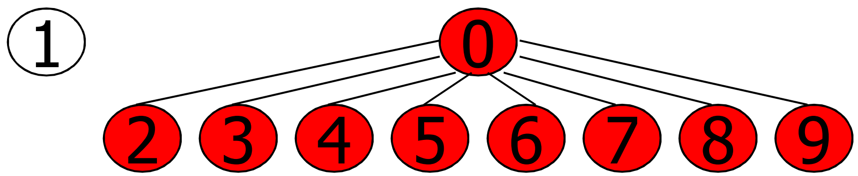
id									
0	1	0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	9

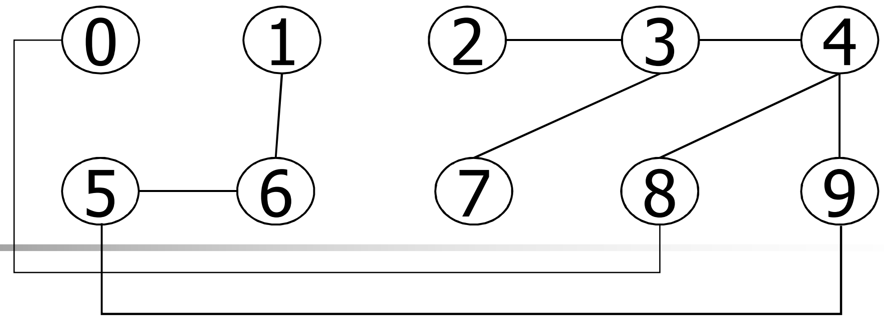
id[p]=0 = id[q]=0

no change

id									
0	1	0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	9

$S_1 = \{1\}$, $S_{0\text{-}2\text{-}3\text{-}4\text{-}5\text{-}6\text{-}7\text{-}8\text{-}9} = \{0,2,3,4,5,6,7,8,9\}$





p q = 6 1

id

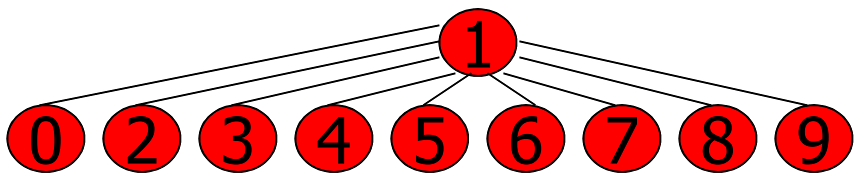
0	1	0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	9

id[p]=0 = id[q]=1
replace all id[p] values with id[q] values

id

1	1	1	1	1	1	1	1	1	1
0	1	2	3	4	5	6	7	8	9

$S_{0\text{-}1\text{-}2\text{-}3\text{-}4\text{-}5\text{-}6\text{-}7\text{-}8\text{-}9} = \{0,1,2,3,4,5,6,7,8,9\}$



```
#include <stdio.h>
#define N 10000
main() {
  int i, t, p, q, id[N];
  for(i=0; i<N; i++)
    id[i] = i;
  printf("Input pair p q:  ");
  while (scanf("%d %d", &p, &q) ==2) {
    if (id[p] == id[q])
      printf("%d %d already connected\n", p,q);
    else {
      for (t = id[p], i = 0; i < N; i++)
        if (id[i] == t)
          id[i] = id[q];
        printf("pair %d %d not yet connected\n", p, q);
      }
      printf("Input pair p q:  ");
  }
}
```

Performance

- Find
  - Simple reference to cell in array id[index], unit cost
- Union
  - Scan array to replace p values with q values, cost linear in array size
- overall number of operations related to

  # pairs · array size

Quadratic
…
Too slow

Quick union

- Represent sets S_i of connected pairs with an array id
  - Initially all the objects point to themselves

    id[i] = i (no connection)
  - Each object points either to an object to which it is connected or to itself (no loops)

    Writing (id[i])* for id[id[id[… id[i]]]]

    if objects i are j connected

    (id[i])* = (id[j])*

Quick union

- Each object points either to an object to which it is connected or to itself (no loops)

  Writing (id[i])* for id[id[id[… id[i]]]]

  if objects i are j connected

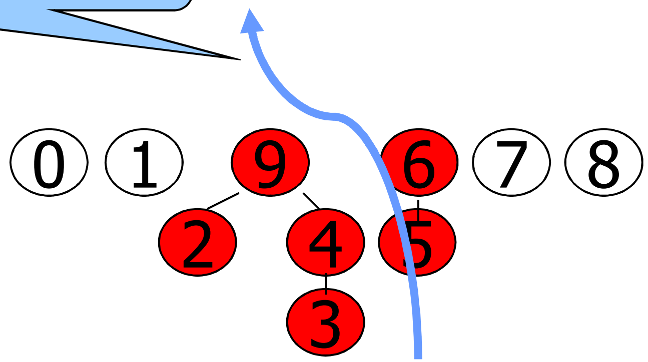
  $$(id[i])^* = (id[j])^*$$

- Example

Keep going until it doesn't change

id

0	1	9	4	9	6	6	7	8	9
0	1	2	3	4	5	6	7	8	9



Quick union

Algorithm

- Repeat for all the pairs (p, q)
  - Read pair (p, q)
  - If(id[p])* = (id[q])*
    - Do nothing (the pair is already connected) and move on to the next pair
    - Else id[(id[p])*] = (id[q])* (connect the pair)

Example

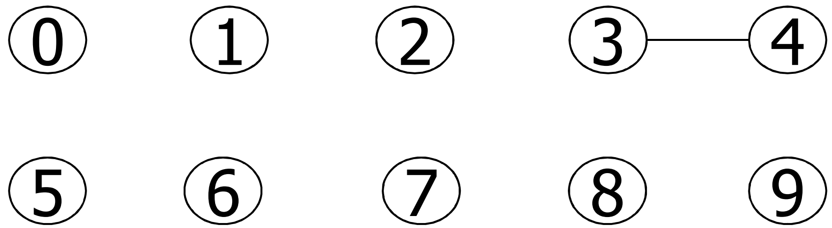
0 1 2 3 4

5 6 7 8 9

Initially

id	0	1	2	3	4	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9

0 1 2 3 4 5 6 7 8 9



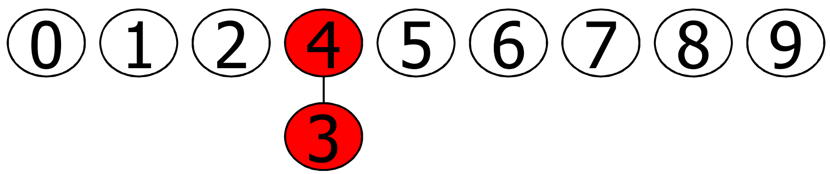
p q = 3 4

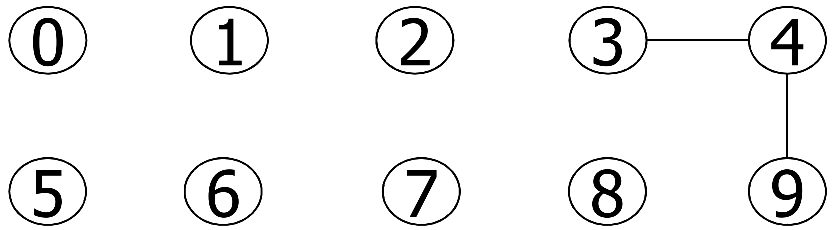
id	0	1	2	3	4	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9

id[p]=3 $\neq$ id[q]=4

p points to q: id[p]=4

id	0	1	2	4	4	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9





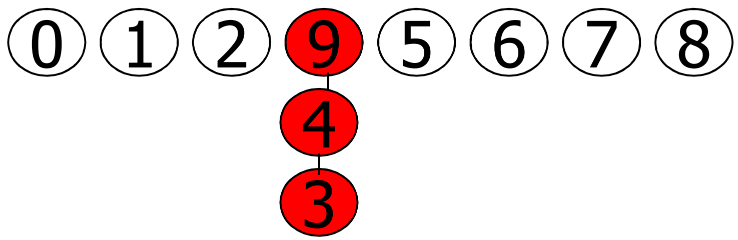
p q = 4 9

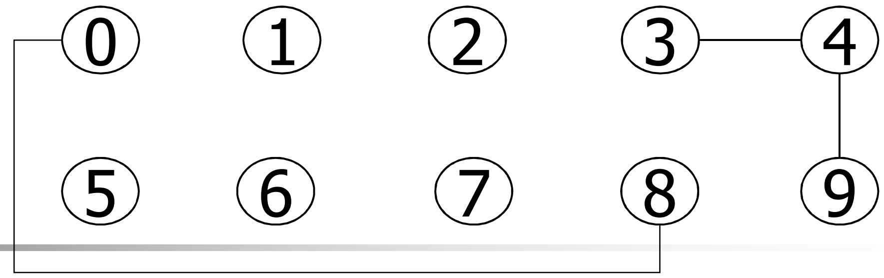
id	0	1	2	4	4	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9

id[p]=4 $\neq$ id[q]=9

p points to q: id[p]=9

id	0	1	2	4	9	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9





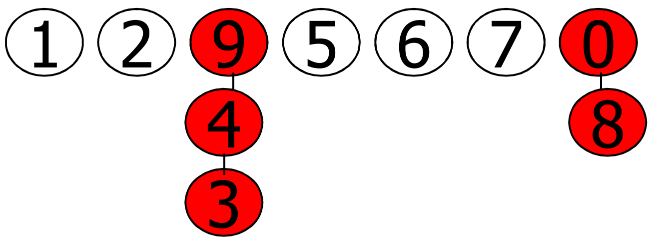
p q = 8 0

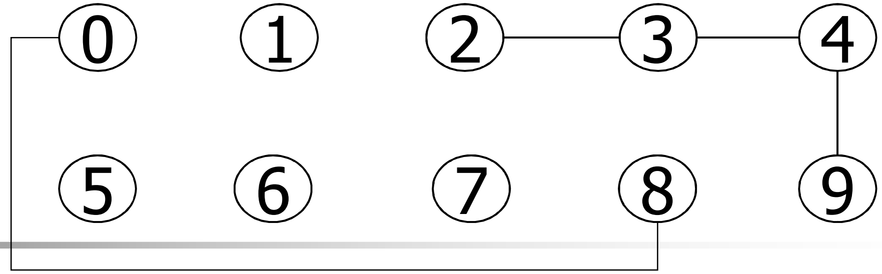
id	0	1	2	4	9	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9

id[p]=8 $\neq$ id[q]=0

p points to q: id[p]=0

id	0	1	2	4	9	5	6	7	0	9
	0	1	2	3	4	5	6	7	8	9





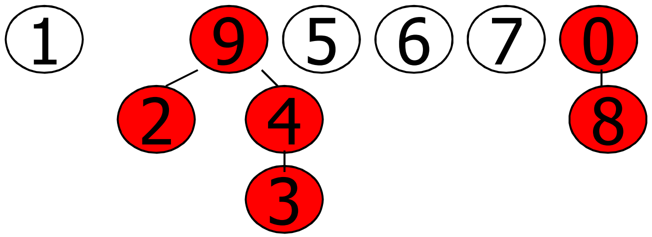
p q = 2 3

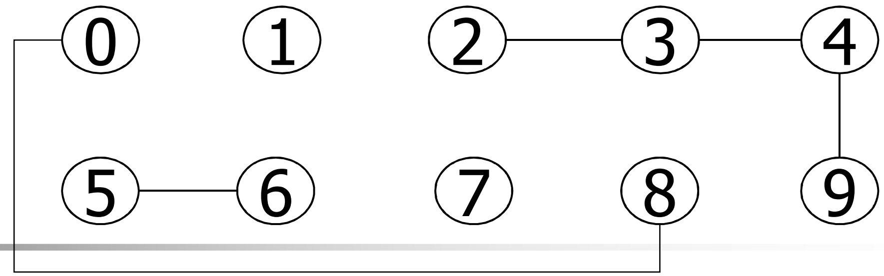
id	0	1	2	4	9	5	6	7	0	9
	0	1	2	3	4	5	6	7	8	9

id[p]=2 $\neq$ id[id[id[q]]]=9

p points to q: id[p]=9

id	0	1	9	4	9	5	6	7	0	9
	0	1	2	3	4	5	6	7	8	9





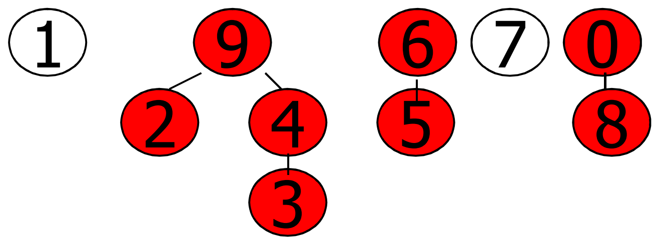
p q = 5 6

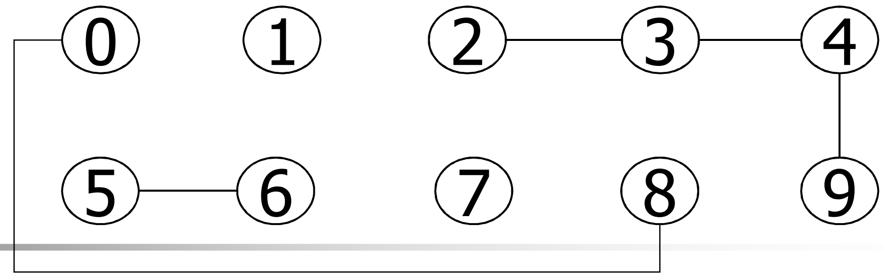
id	0	1	9	4	9	5	6	7	0	9
	0	1	2	3	4	5	6	7	8	9

id[p]=5 $\neq$ id[q]=6

p points to q: id[p]=6

id	0	1	9	4	9	6	6	7	0	9
	0	1	2	3	4	5	6	7	8	9





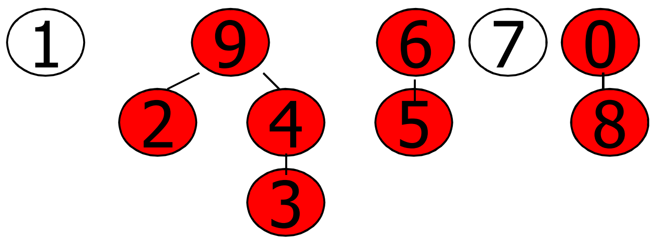
p q = 2 9

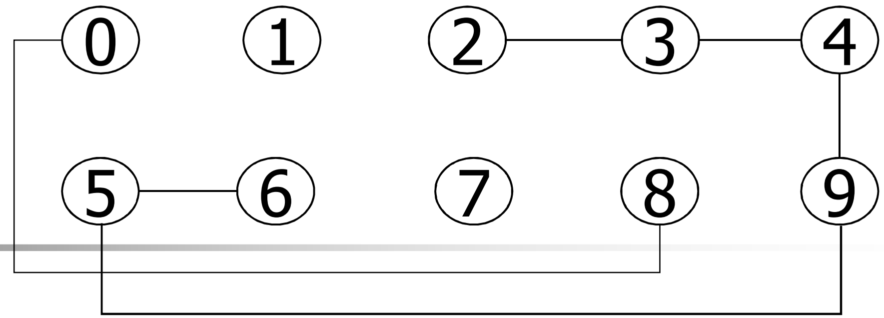
id	0	1	9	4	9	6	6	7	0	9
	0	1	2	3	4	5	6	7	8	9

id[id[p]]=9 = id[q]=9

no change

id	0	1	9	4	9	6	6	7	0	9
	0	1	2	3	4	5	6	7	8	9





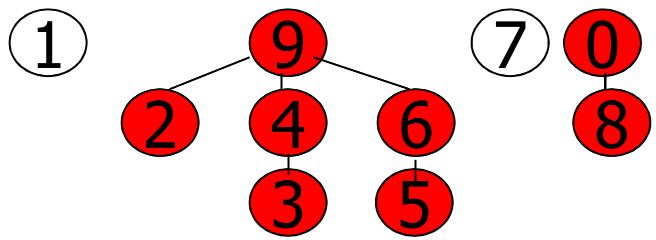
p q = 5 9

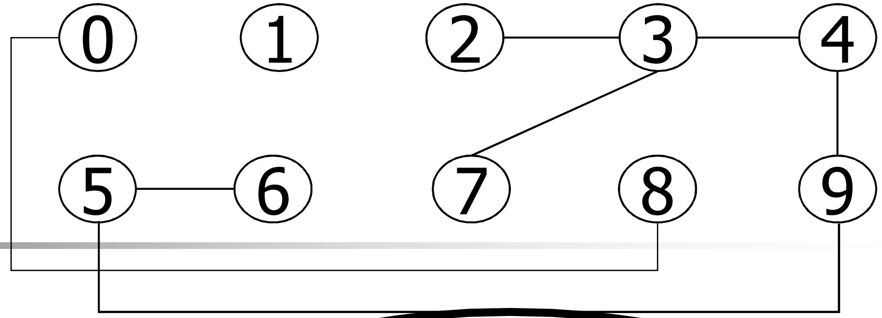
id	0	1	9	4	9	6	6	7	0	9
	0	1	2	3	4	5	6	7	8	9

id[id[p]]=6 ≠ id[q]=9

p points to q: id[id[p]]=9

id	0	1	9	4	9	6	9	7	0	9
	0	1	2	3	4	5	6	7	8	9





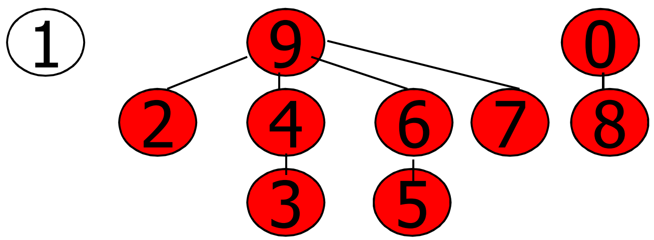
p q = 7 3

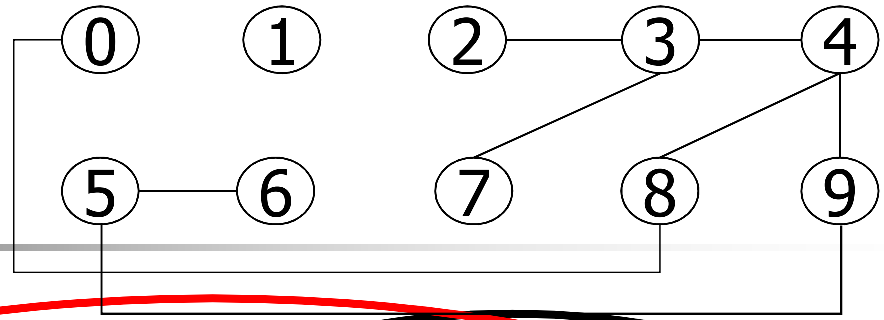
id	0	1	9	4	9	6	9	7	0	9
	0	1	2	3	4	5	6	7	8	9

id[p]=7 $\neq$ id[id[id[q]]]=9

p points to q: id[p]=9

id	0	1	9	4	9	6	9	9	0	9
	0	1	2	3	4	5	6	7	8	9





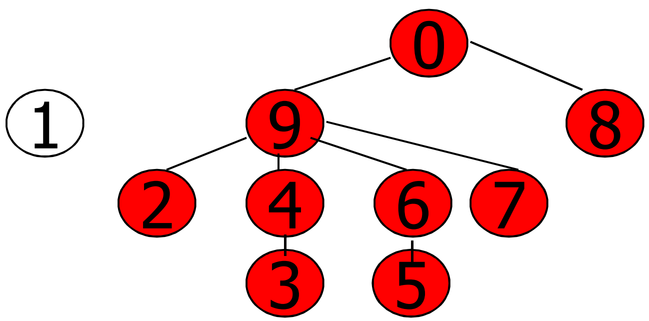
p q = 4 8

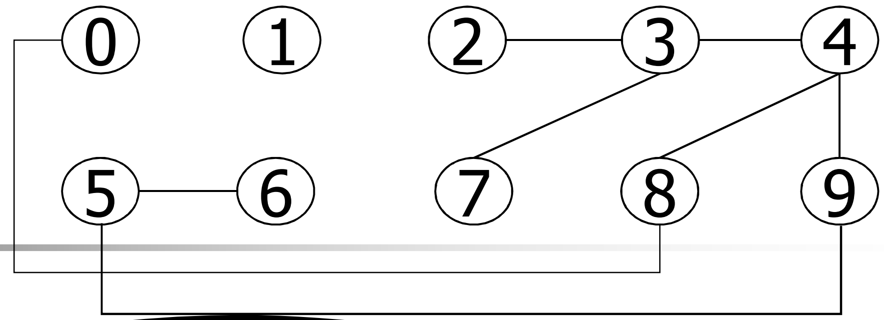
id										
	0	1	9	4	9	6	9	9	0	9
	0	1	2	3	4	5	6	7	8	9

id[id[p]]=9 $\neq$ id[id[q]]=0

p points to q: id[id[p]]=0

id										
	0	1	9	4	9	6	9	9	0	0
	0	1	2	3	4	5	6	7	8	9





p q = 5 6

id

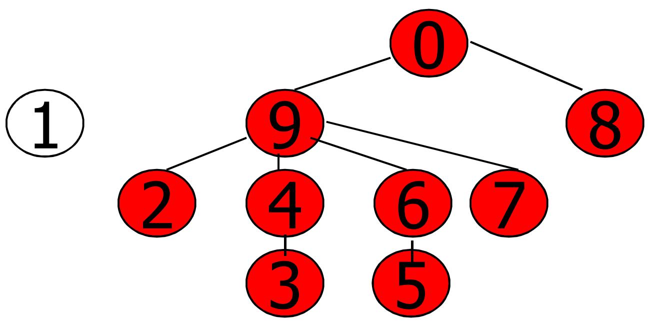
0	1	9	4	9	6	9	9	0	0
0	1	2	3	4	5	6	7	8	9

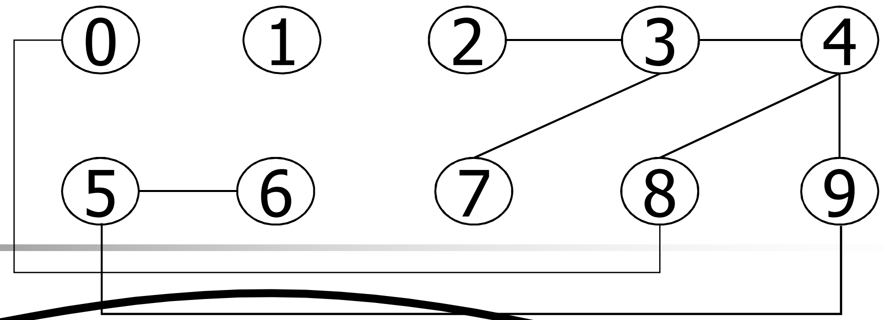
id[id[id[id[p]]]]=0 = id[id[q]]=0

no change

id

0	1	9	4	9	6	9	9	0	0
0	1	2	3	4	5	6	7	8	9





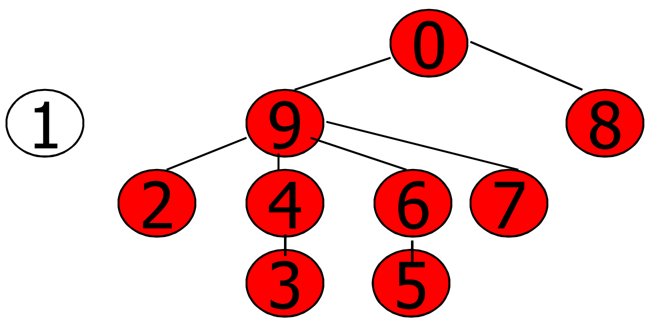
p q = 0 2

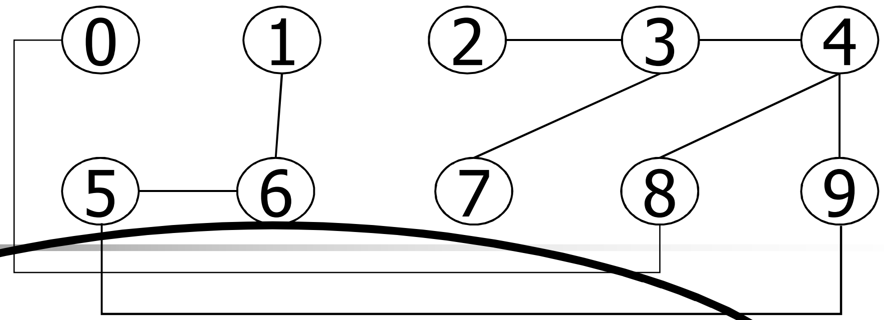
id	0	1	9	4	9	6	9	9	0	0
	0	1	2	3	4	5	6	7	8	9

id[p]=0 = id[id[id[q]]]=0

no change

id	0	1	9	4	9	6	9	9	0	0
	0	1	2	3	4	5	6	7	8	9





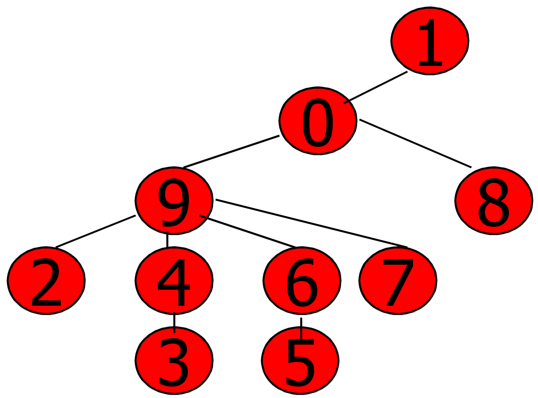
p q = 6 1

id	0	1	9	4	9	6	9	9	0	0
	0	1	2	3	4	5	6	7	8	9

id[id[id[p]]]=0 $\neq$ id[q]=1

p points to q: id[id[id[p]]]=1

id	1	1	9	4	9	6	9	9	0	0
	0	1	2	3	4	5	6	7	8	9



```c
#include <stdio.h>
#define N 10000
main() {
  int i, j, p, q, id[N];
  for(i=0; i<N; i++)
    id[i] = i;
  printf("Input pair p q:  ");
  while (scanf("%d %d", &p, &q) == 2) {
    for (i = p; i!= id[i]; i = id[i]);
    for (j = q; j!= id[j]; j = id[j]);
    if (i == j) {
      printf("pair %d %d already connected\n", p,q);
    } else {
      id[i] = j;
       printf("pair %d %d not yet connected\n", p, q);
    }
    printf("Input pair p q:  ");
  }
}
```

Performance

- Find
  - Scan a "chain" of objects, upper bound linear cost in the number of objects, in general well below upper bound
- Union
  - Simple, as it is enough that an object points to another object, unit cost
- Overall number of operations related to

$$\# \text{ pairs} \cdot \text{chain length}$$

Still
too slow

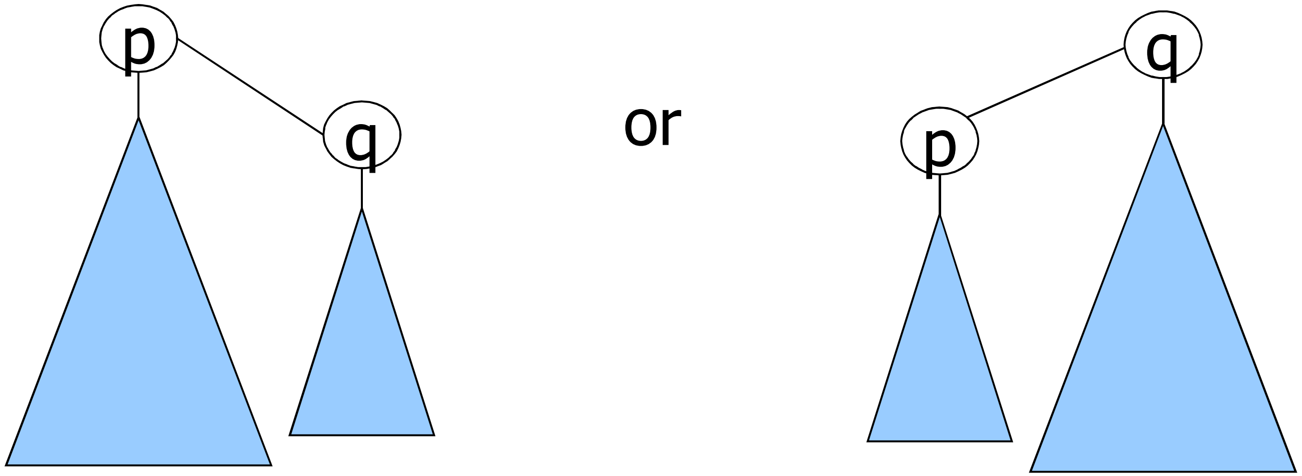
Quick union optimizations

Weighted quick union

- To shorten the chain length, keep track of the number of elements in each tree (array `sz`) and connect the smaller tree to the larger one

Union by height or "rank",
i.e., always link the
root of smaller tree
to root of lager tree

- According to which one is the larger, there might be 2 solutions



- It is irrelevant if p appears at the right or at the left of q

Example

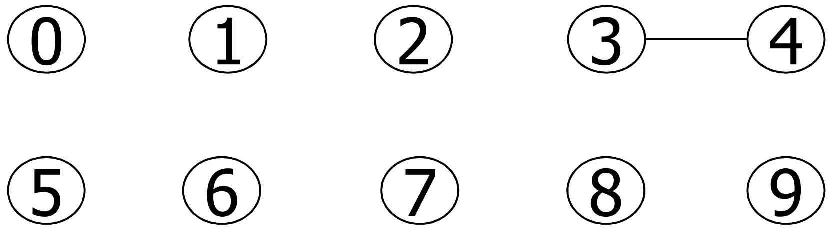
0 1 2 3 4

5 6 7 8 9

Initially

id										
	0	1	2	3	4	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9

0 1 2 3 4 5 6 7 8 9



p q = 3 4

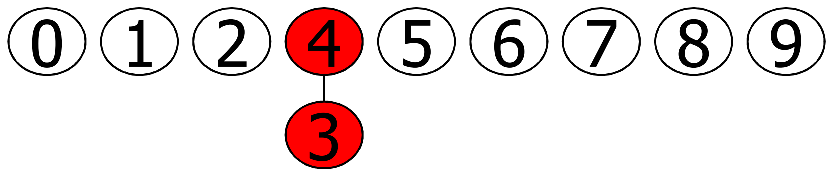
id

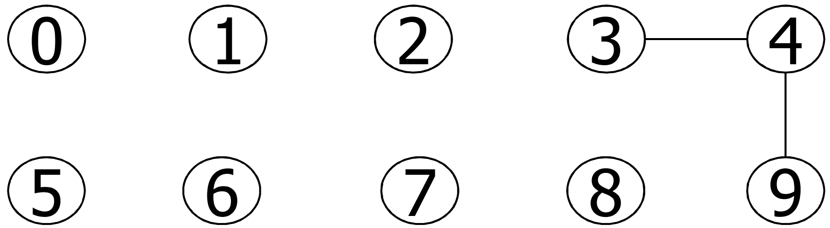
0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9

id[p]=3 $\neq$ id[q]=4
p points to q: id[p]=4

id

0	1	2	4	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9





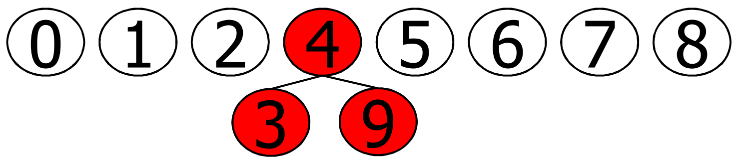
p q = 4 9

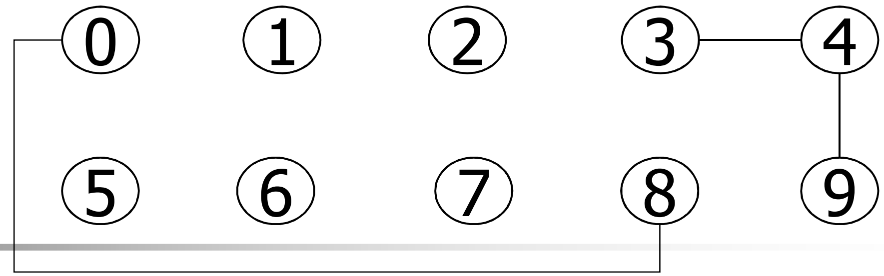
id	0	1	2	4	4	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9

id[p]=4 $\neq$ id[q]=9

the smaller tree q points to the larger one p: id[q]=4

id	0	1	2	4	4	5	6	7	8	4
	0	1	2	3	4	5	6	7	8	9





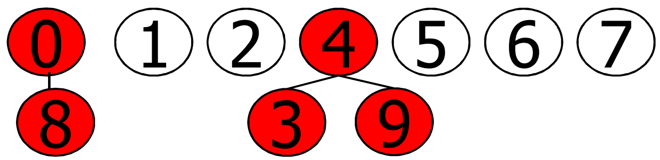
p q = 8 0

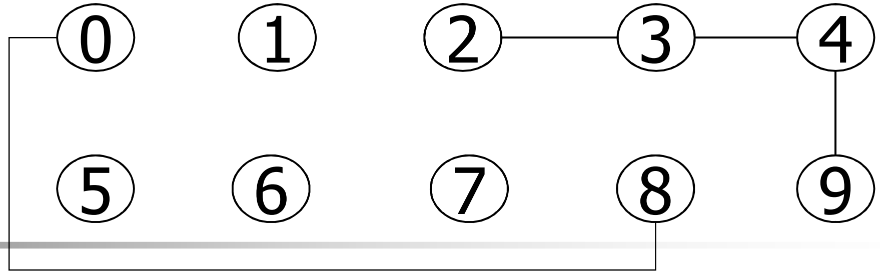
id	0	1	2	4	4	5	6	7	8	4
	0	1	2	3	4	5	6	7	8	9

id[p]=8 $\neq$ id[q]=0

p points to q: id[p]=0

id	0	1	2	4	4	5	6	7	0	4
	0	1	2	3	4	5	6	7	8	9





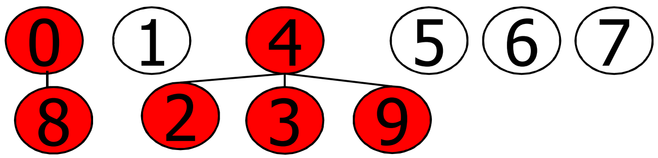
p q = 2 3

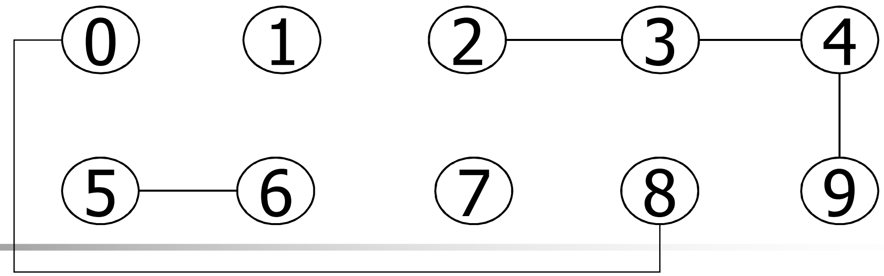
id	0	1	2	4	4	5	6	7	0	4
	0	1	2	3	4	5	6	7	8	9

id[p]=2 $\neq$ id[id[q]]=4

the smaller tree p points to the larger one q: id[p]=4

id	0	1	4	4	4	5	6	7	0	4
	0	1	2	3	4	5	6	7	8	9





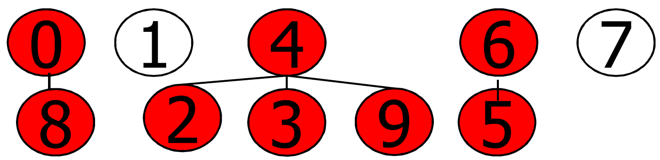
p q = 5 6

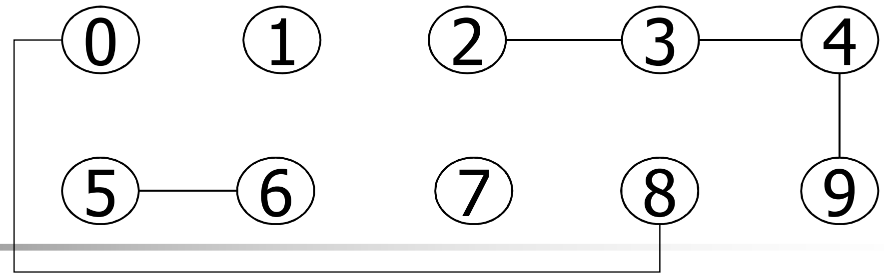
id	0	1	4	4	4	5	6	7	0	4
	0	1	2	3	4	5	6	7	8	9

id[p]=5 $\neq$ id[q]=6

p points to q: id[p]=6

id	0	1	4	4	4	6	6	7	0	4
	0	1	2	3	4	5	6	7	8	9





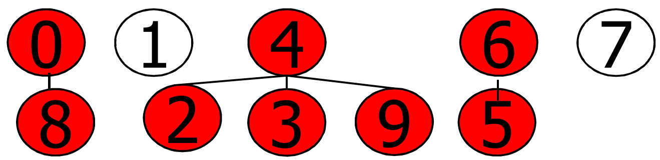
p q = 2 9

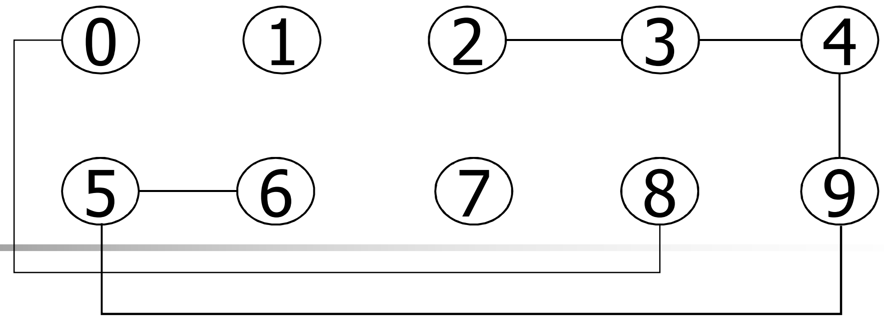
id	0	1	4	4	4	6	6	7	0	4
	0	1	2	3	4	5	6	7	8	9

id[id[p]]=4 = id[q]=4

no change

id	0	1	4	4	4	6	6	7	0	4
	0	1	2	3	4	5	6	7	8	9





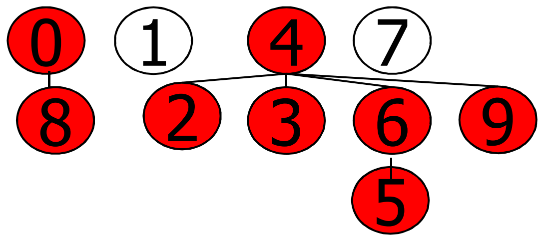
p q = 5 9

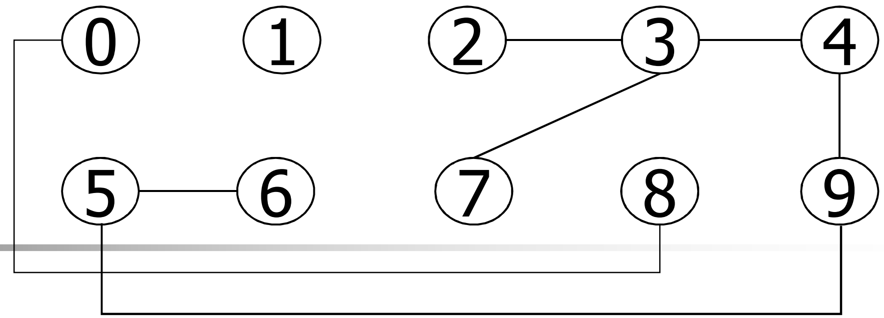
id										
	0	1	4	4	4	6	6	7	0	4
	0	1	2	3	4	5	6	7	8	9

id[id[p]]=6 ≠ id[id[q]]=4

the smaller tree p points to the larger one q: id[id[p]]=4

id										
	0	1	4	4	4	6	4	7	0	4
	0	1	2	3	4	5	6	7	8	9





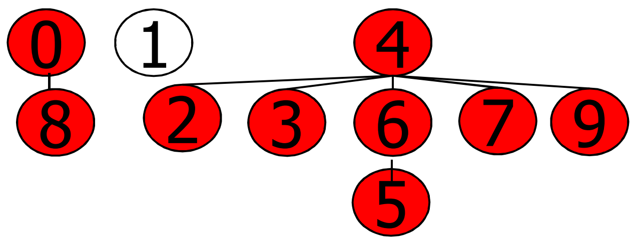
p q = 7 3

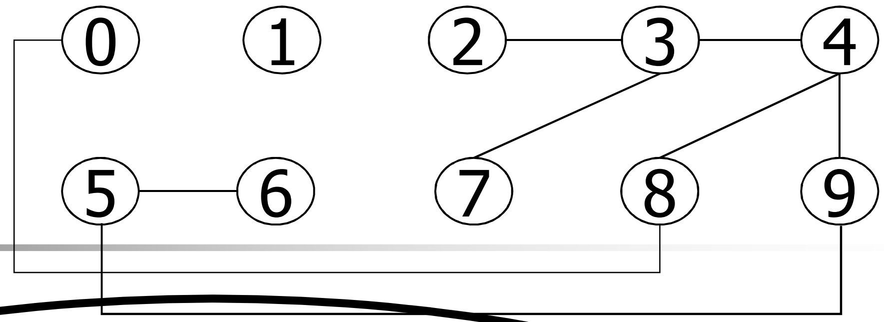
id	0	1	4	4	4	6	4	7	0	4
	0	1	2	3	4	5	6	7	8	9

id[p]=7 $\neq$ id[id[q]]=4

the smaller tree p points to the larger one q: id[p]=4

id	0	1	4	4	4	6	4	4	0	4
	0	1	2	3	4	5	6	7	8	9





p q = 4 8

id

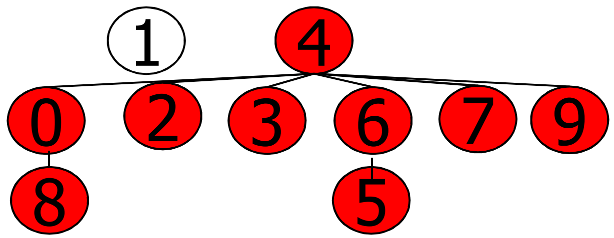
0	1	4	4	4	6	4	4	0	4
0	1	2	3	4	5	6	7	8	9

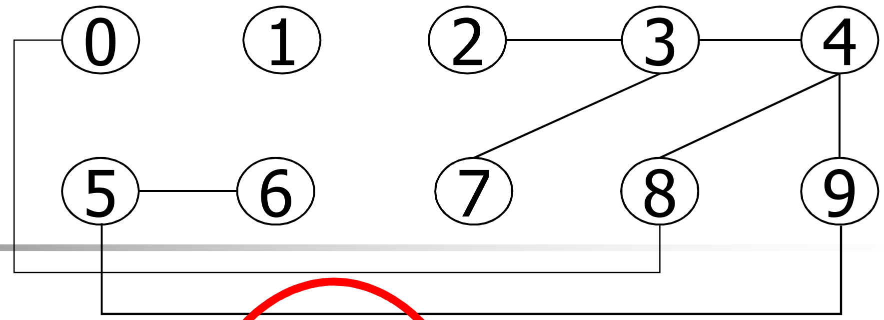
id[p]=4 $\neq$ id[id[q]]=0

the smaller tree q points to the larger one p: id[id[q]]=4

id

4	1	4	4	4	6	4	4	0	4
0	1	2	3	4	5	6	7	8	9





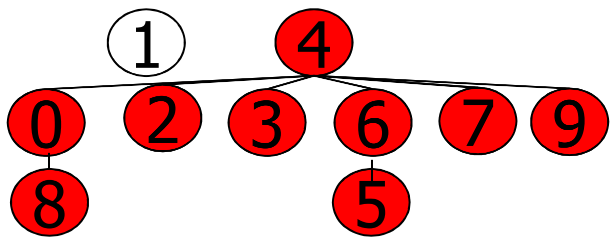
p q = 5 6

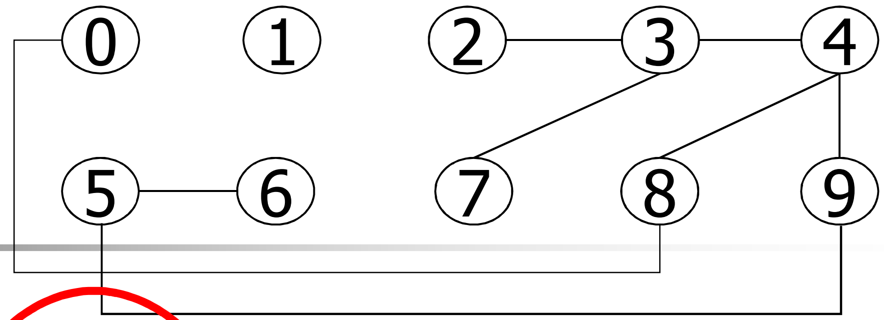
id	4	1	4	4	4	6	4	4	0	4
	0	1	2	3	4	5	6	7	8	9

id[id[id[p]]]=4 = id[id[q]]=4

no change

id	4	1	4	4	4	6	4	4	0	4
	0	1	2	3	4	5	6	7	8	9



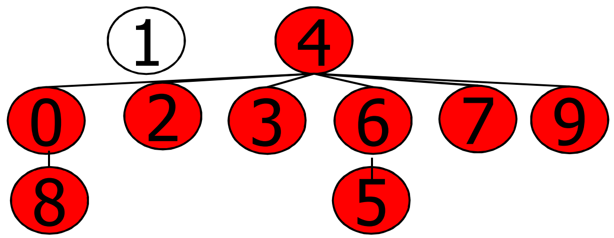


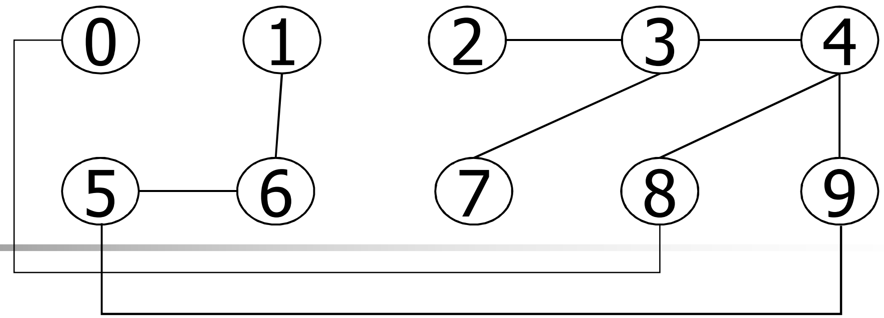
p q = 0 2

id	4	1	4	4	4	6	4	4	0	4
	0	1	2	3	4	5	6	7	8	9

id[id[p]]=4 = id[id[q]]=4
no change

id	4	1	4	4	4	6	4	4	0	4
	0	1	2	3	4	5	6	7	8	9





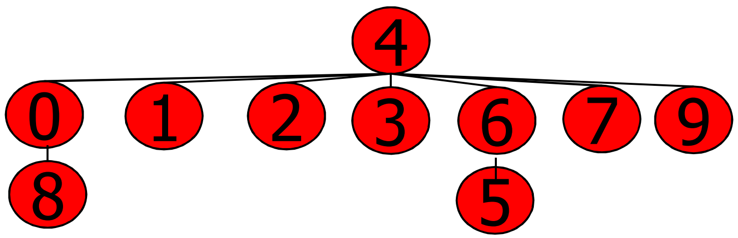
p q = 6 1

id	4	1	4	4	4	6	4	4	0	4
	0	1	2	3	4	5	6	7	8	9

id[id[p]]=4 ≠ id[q]=1

the smaller tree q points to the larger one p: id[q]=4

id	4	4	4	4	4	6	4	4	0	4
	0	1	2	3	4	5	6	7	8	9



```
...
int i, j, p, q, id[N], sz[N];
  for(i=0; i<N; i++) {
    id[i] = i; sz[i] =1;
  }
  printf("Input pair p q:  ");
  while (scanf("%d %d", &p, &q) ==2) {
    for (i = p; i!= id[i]; i = id[i]);
    for (j = q; j!= id[j]; j = id[j]);
    if (i == j)
      printf("pair %d %d already connected\n", p,q);
    else {
       printf("pair %d %d not yet connected\n", p, q);
       if (sz[i] < sz[j]) {
         id[i] = j; sz[j] += sz[i];
        }
        else {
          id[j] = i; sz[i] += sz[j];
        }
    }
...
```

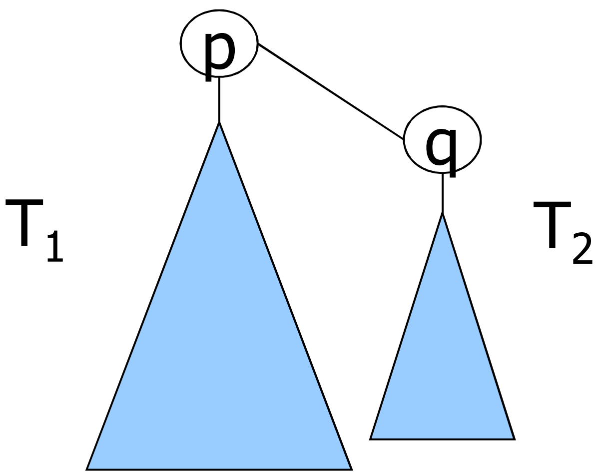
- Find
  - Scanning a "chain" of objects, cost at most logarithmic in the number of objects
- Union
  - Simple, because it is enough that an object points to another object, unit cost
- Globally the number of operations is bounded by

numb. of pairs * "chain" length

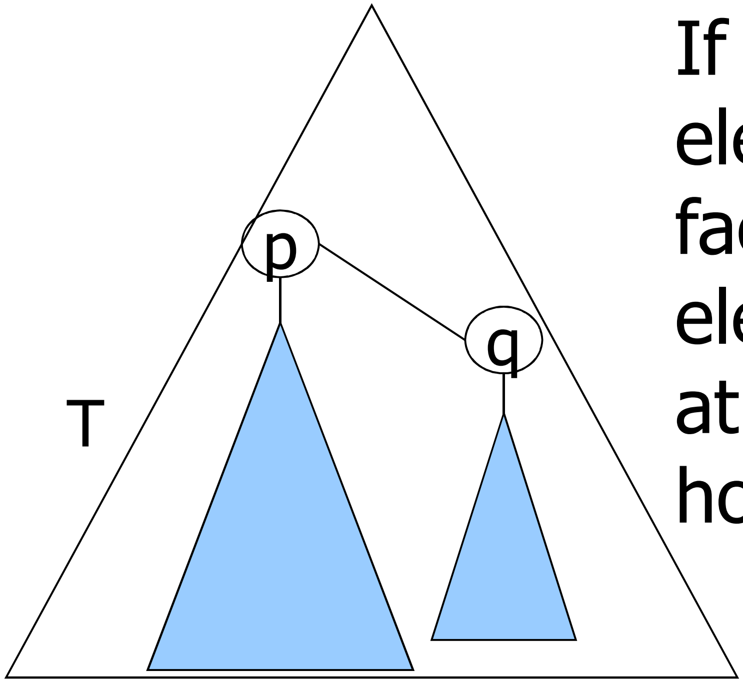
but chain length grows logarithmically !

Why logarithmically?

What matters is the maximum distance between a node and the root. The distance increases by 1 when we connect a smaller tree (whose size is T_2) to a larger tree (whose size is T_1).



But if $T_1 \geq T_2$ each time we connect a smaller tree to a larger one we generate a tree whose size T is at least twice as big as T_2.



If at each step the number of elements increases by at least a factor 2 and if there are N elements, after i steps there will be at least 2^i elements. $2^i \leq N$ must, hold, thus $i \leq \log_2 N$