Graphs

Single Source Shortest Paths

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Problem definition

- Example
  - Given a road map on which the distance between each pair of adjacent intersactions is marked
  - How is it possible to determine the shortest route?
  - One possibility is to
    - Enumerate all routes, add the distance on each route, disallowing routes with cycles
    - Select the shortes routes
  - This implies examining an enourmous number of possibilities
- A better solution implies solving the so called Single-Source Shortest Path problem

Shortest Paths

- Given a graph G = (V, E)
  - Directed
  - Weighted
    - With a positive real-value weight function $w: E \rightarrow R$
  - With a weight w(p) over a path
    - $p = <v_0, v_1, \dots, v_k>$

    is equal to
    - $w(p) = \Sigma_{i=1}^{k} w(v_{i-1}, v_i)$

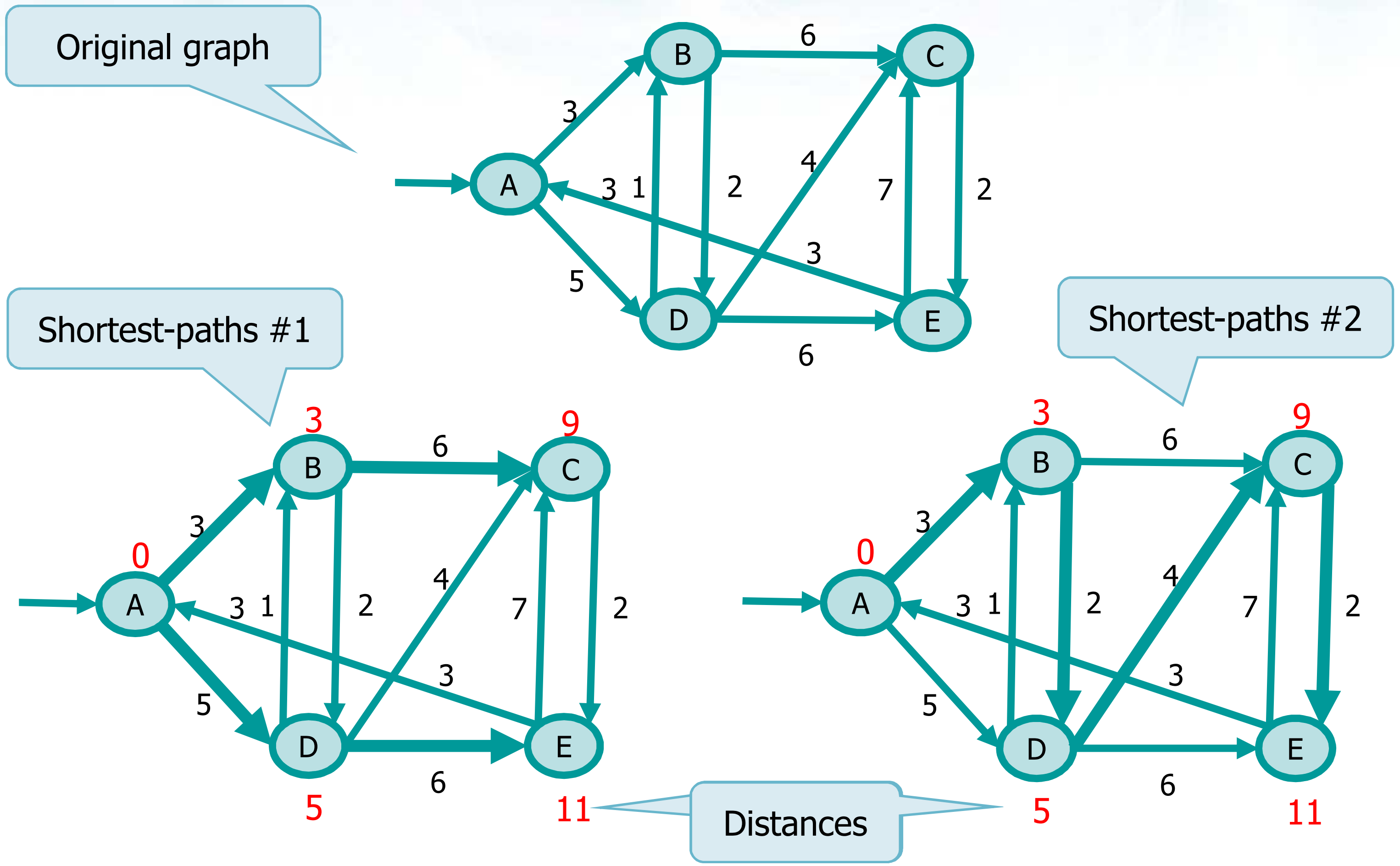
Shortest Paths

- We define the shortest path weight $\delta(u,v)$ from u to v as
  - $$\delta(u,v) = \begin{cases} \min\{w(p)\} & \text{if } \exists\ u \rightarrow_p v \\ \infty & \text{otherwise} \end{cases}$$
- A shortest path from u to v is any path p with weigth
  - $w(p) = \delta(u,v)$

Variants

- Shortest path problems
  - Single-source shortest-paths
    - Minimum path and its weight from s to all other vertices v
      - **Dijkstra**'s algorithm
      - **Bellman-Ford**'s algorithm
  - Notice that with **unweighted** graph a simple **BFS** (Breadth-First Search) solves the problem

Example


Original graph
Shortest-paths #1
Shortest-paths #2
Distances

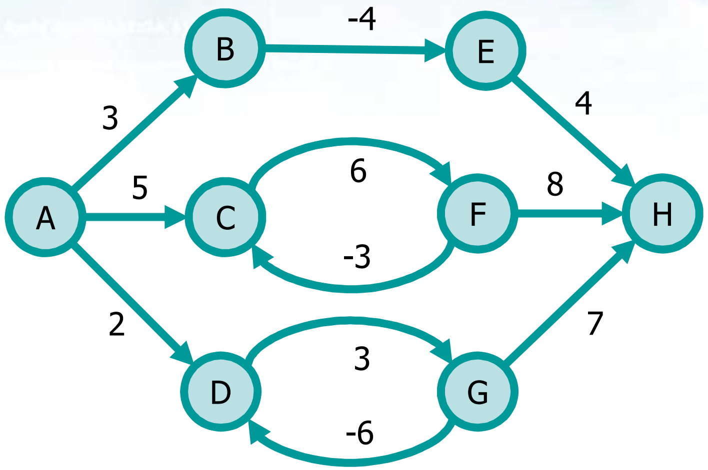
Variants

- Single-destination shortest-paths
  - Find the shortest path to a given destination
  - Use the reverse graph
- Single-pair shortest-paths
  - Find a shortest path from v_1 to v_2 given vertices v_1 to v_2
  - Soved when the SSSP is solved
  - All alternative solutions have the same worst-case asymptotic running time
- All-pairs shortest-path
  - Find a shortest-path for every vertex pair
  - Can be solved running SSSP from each vertex
  - Can be solved faster

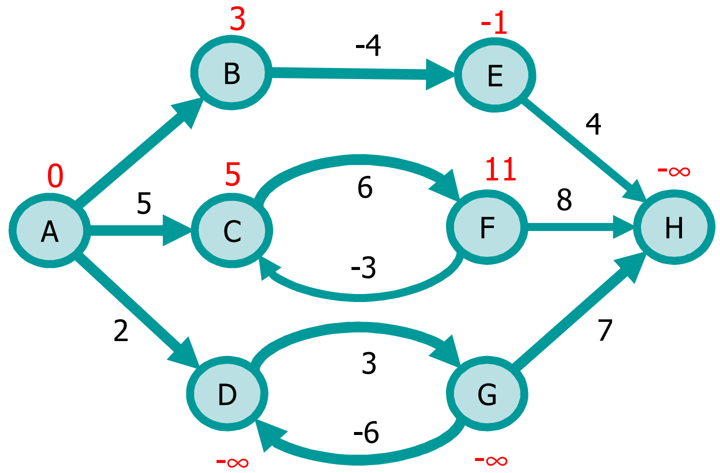
Negative Weight Edges

- If there are edges with negative weight but there are no cycles with negative weight
  - Djikstra's algorithm
    - Optimum solution not guaranted
  - Bellman-Ford's algorithm
    - Optimum solution guaranted
- It there are cycle with negative weight
  - The problem is not defined (there is no solution)
  - Dijkstra's algorithm
    - Meaningless result
  - Bellman-Ford's algorithm
    - Find cycles with negative weights

Example



Original graph



Shortest-paths

Representing Shortest Paths

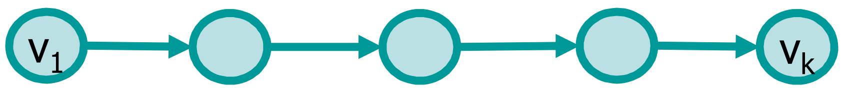
- Often we wish to compute vertices on shorterst path, not only weights
  - A few representations are possible
- Array of predecessors v.pred
  - $\forall v \in V \quad v.pred = \begin{cases} \text{parent(v) if } \exists \\ \text{NULL otherwise} \end{cases}$
- Predecessor's sub-graph
  - $G_{pred}=(V_{pred}, E_{pred})$, where
    - $V_{pred} = \{v \in V\text{: } v.pred \neq NULL\} \cup \{s\}$
    - $E_{pred} = \{(v.pred, v) \in E : v \in V_{pred} - \{s\}\}$

Representing Shortest Paths

- Shortest-Paths Tree
  - G′ = (V′, E′)
    - Where $V' \subseteq V$ && $E' \subseteq E$
    - V′ is the set of vertices reachable from s
    - S is the tree root
    - $\forall v \in V'$ the unique simple path from s to v in G′ is a minimum weight from s to v in G

Theoretical Background

- Lemma
  - Sub-paths of shortest paths are shortest paths
  - G = (V, E)
    - Directed, weighted $w: E \rightarrow R$
  - $P = <v_1, v_2, \ldots, v_k>$
    - Is a shortest path from v_1 to v_k
  - $\forall i, j\ 1 \leq i \leq j \leq k,\ p_{ij} = <v_i, v_{i+1}, \ldots, v_j>$
    - Sub-path of p from v_i to v_j
  - The p_{ij} is a shortest path from v_i to v_j



Theoretical Background

- Corollary
  - G = (V, E)
    - Directed, weighted w: $E \rightarrow R$
  - A shortest path p from s to v may be decomposed into
    - A shortest sub-path from s to u
    - An edge (u,v)
  - Then
    - $\delta(s,v) = \delta(s,u) + w(u,v)$

Theoretical Background

- Lemma
  - $G = (V, E)$
    - Directed, weighted $w: E \rightarrow R$
  - $\forall (u,v) \in E$
    - $\delta(s,v) \leq \delta(s,u) + w(u,v)$
  - A shortest path from s to v cannot have a weight larger than the path formed by a shortest path from s to u and an edge (u, v)

Relaxation

- The algorithms we are going to anayse use the technique of **relaxation**
- For each vertex we mantain an estimate **v.d** (superior limit) of the weight of the path from s to v

(Single) source

```
initialize_single_source (G, s)
  for each v ∈ V
    v.d = ∞
    v.pred = NULL
  s.d = 0
```

v.pred = predecessor

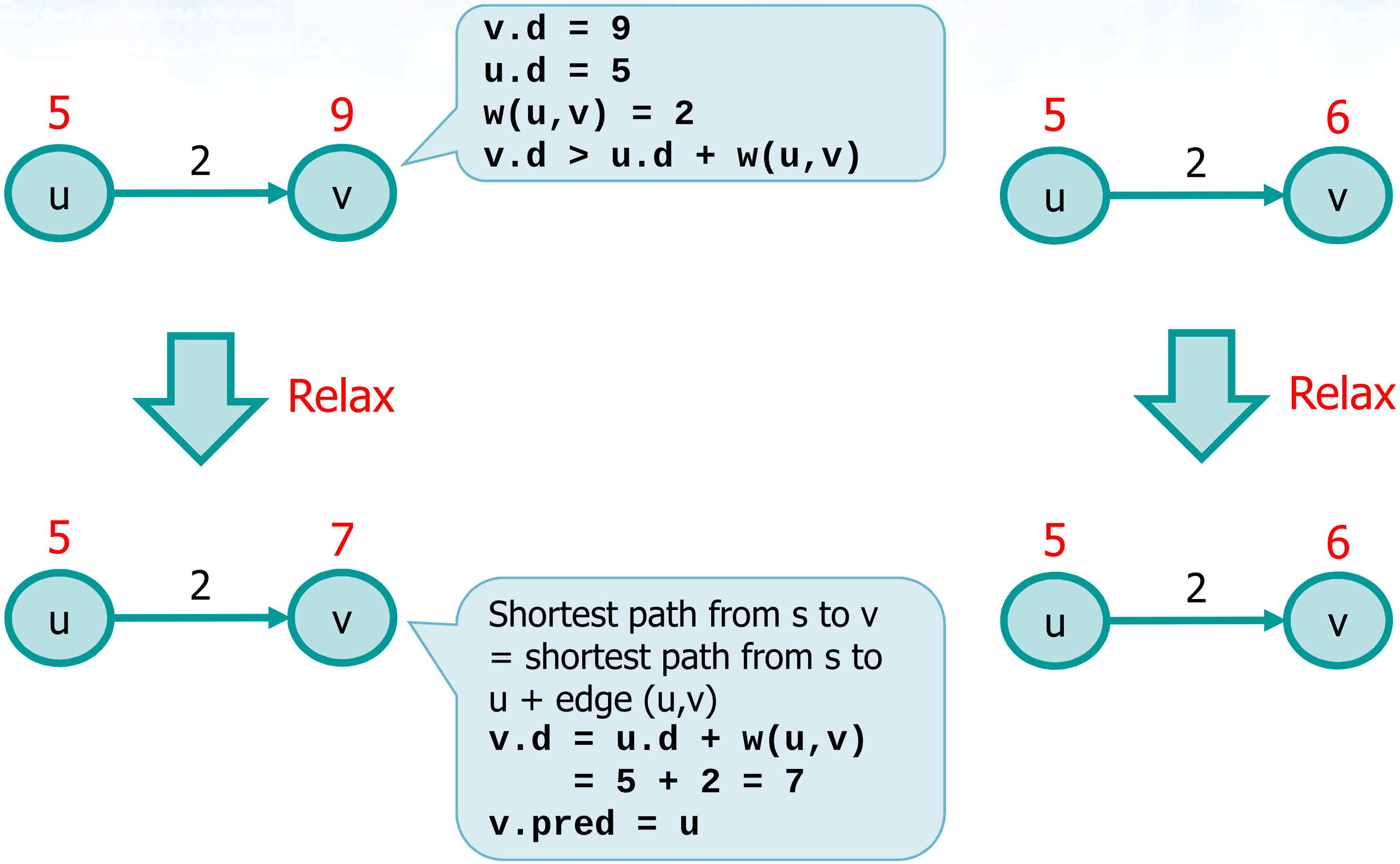
v.d
= shortest path estimate =
upper bound on the weight of
a shortest path from s to v

Relaxation

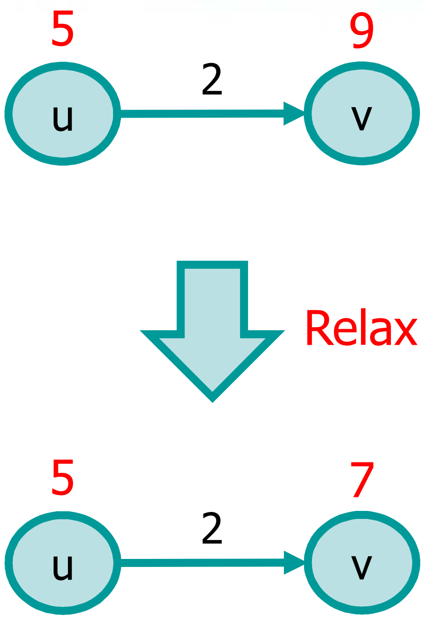
- Relaxation
  - Update v.d and v.pred by testing whether it is possibile to improve the shortest path to v found so far by going through the edge e = (u,v), where w(u,v) is the weigth of the edge

```
relax (u, v, w)
  if ( v.d > (u.d + w(u, v)) )
    v.d = u.d + w (u, v)
    v.pred = u
```

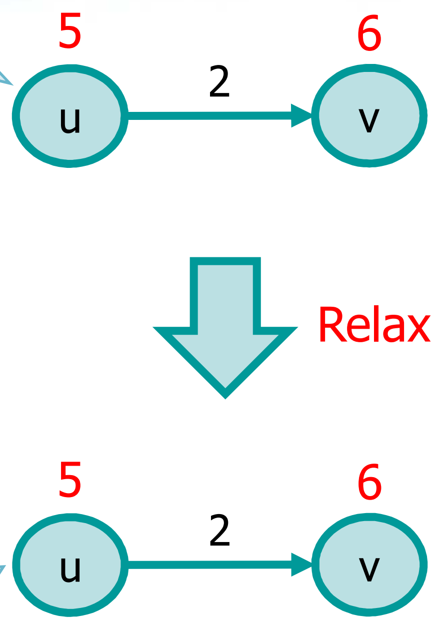
Example


5
9
2
u
v
v.d = 9
u.d = 5
w(u,v) = 2
v.d > u.d + w(u,v)
Relax
5
7
2
u
v
Shortest path from s to v = shortest path from s to u + edge (u,v)
v.d = u.d + w(u,v)
= 5 + 2 = 7
v.pred = u
5
6
2
u
v
Relax
5
6
2
u
v

Example




```
v.d = 6
u.d = 5
w (u,v) = 2
v.d < u.d + w(u,v)
```



Relaxation has no effect

```
v.d = unchanged
    = 6
v.pred = unchanged
```

Properties

- Lemma
  - Given $G=(V,E)$
  - Directed, weighted $w: E \rightarrow R$, with $e = (u,v) \in E$
- After relaxing $e = (u,v)$ we have
  - $v.d \leq u.d + w(u, v)$
- That is, after relaxing e, v.d cannot increase
  - Either v.d is unchanged (relaxation with no effect)
  - Or v.d is decreased (effective relaxation)

Properties

- Lemma
  - Given $G=(V,E)$, directed, weighted $w: E \rightarrow R$, with source $s \in V$
  - After a proper initialization of v.d and v.pred
- $\forall\ v \in V \quad v.d \geq \delta(s,v)$
  - For all relaxation steps on the edges
  - When $v.d = \delta(s,v)$, then v.d does not change any more

Properties

- Lemma
  - Given G=(V,E) directed, weighted w: E$\rightarrow$R, with source s $\in$ V
  - After a proper initialization of v.d and v.pred
- The shortest path from s to v is made-up of
  - Path from s to u
  - Edge e = (u, v)
- Application of relaxation on e=(u,v)
  - If before relaxation u.d = δ(s,u)
  - After relaxation v.d = δ(s,v)

Dijkstra's Algorithm

- It works on graphs with no negative weigth
- It is a greedy strategy
  - It applies relaxation once for all edges
- Algorithm
  - S: set of vertices whose shortest path from s has already been computed
  - V-S: priority queue Q of vertices till to estimate
  - Stop when Q is empty
    - Extract u from V-S (u.d is minimum)
    - Insert u in S
    - Relax all outgoing edges from u

Implementation

Pseudo-code

```
sssp_Dijkstra (G, w, s)
  initialize_single_source (G, s)
  S = ϕ
  Q = V
  while Q ≠ ϕ
    u = extract_min (Q)
    S = S ∪ {u}
    for each vertex v ∈ adjacency list of u
      relax (u, v, w)
```

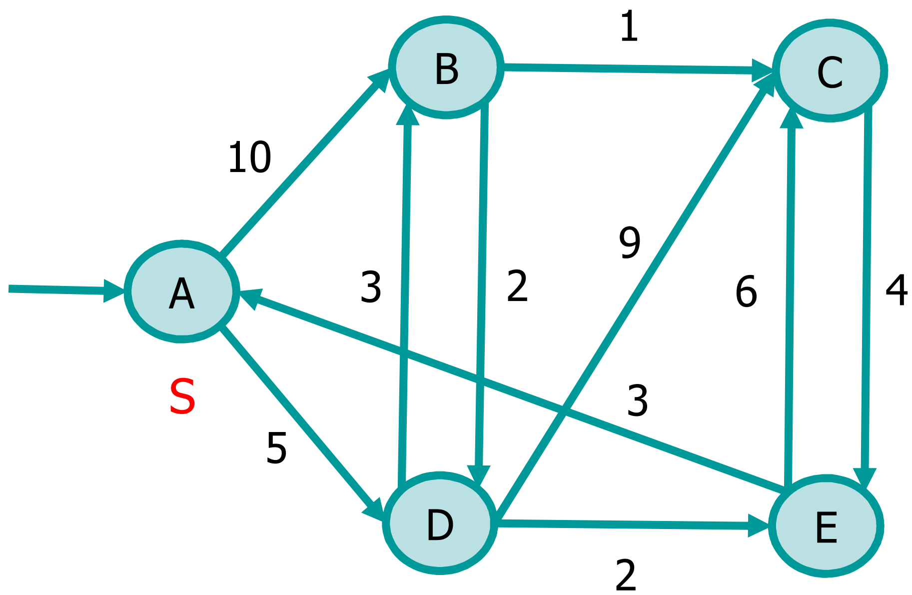
For all vertices starting from s

Extract vertex with minimum distance

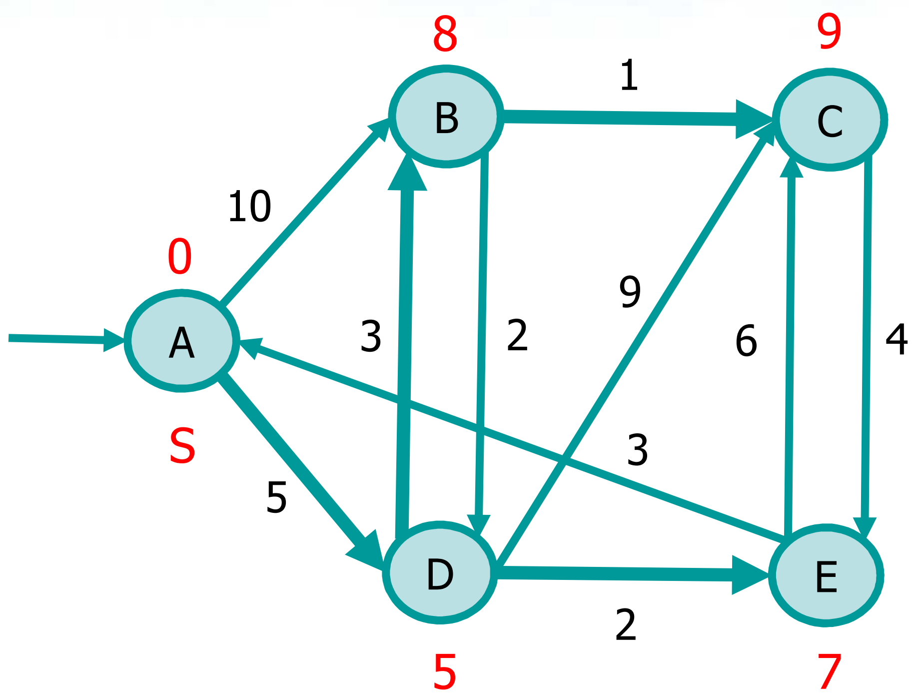
Insert if in S

Relax all adjacency vertices

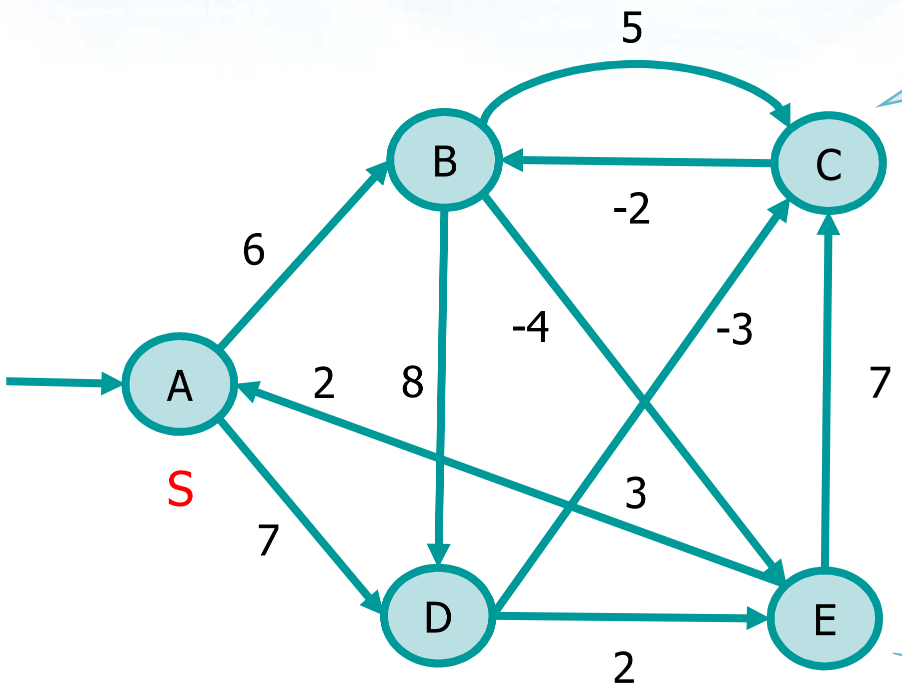
Example 1



Example 1



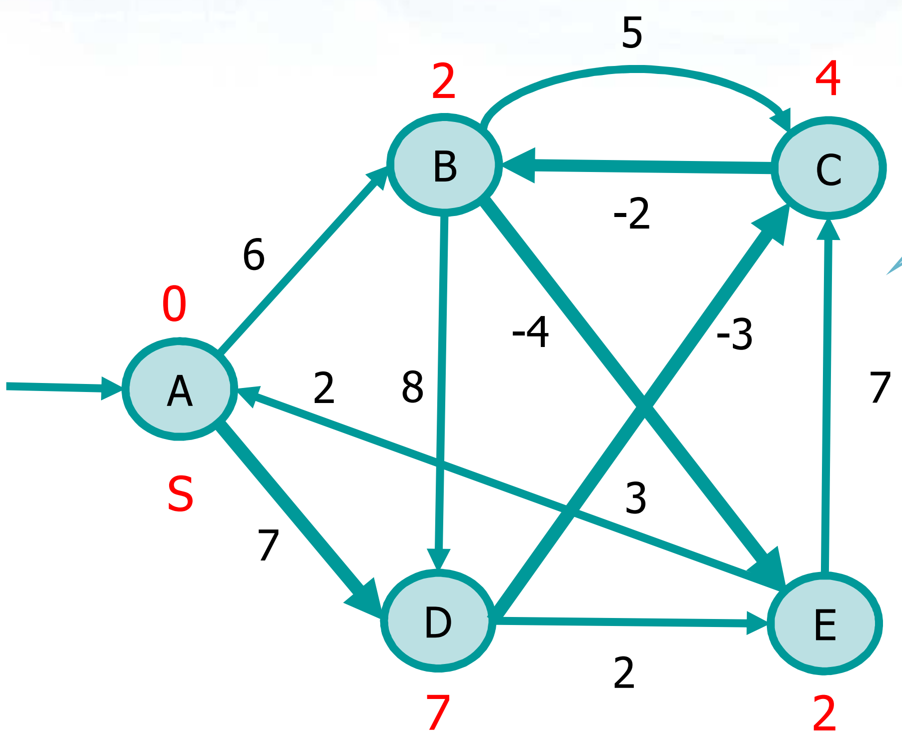
Example 2: Negative edges



There are edges with negative weight

There are no cycles with negative weight

Example 2: Negative edges



Non optimal solution
With edge (B,E),
E would go to -2

Complexity

Pseudo-code

```
sssp_Dijkstra (G, w, s)
  initialize_single_source (G, s)
  S = ϕ
  Q = V
  while Q ≠ ϕ
    u = extract_min (Q)
    S = S ∪ {u}
    for each vertex v ∈ adjacency list of u
      relax (u, v, w)
```

O(|V|)

Executed |V| times

O (lg |V|) → **O(|V| log|V|)**

Overall O(|E|)

O(lg |V|) → **O(|E|log|V|)** due to PQ change

Overall running time complexity $T(n) = O((|V|+|E|) \cdot \lg |V|)$

Complexity

- In general
  - $T(n) = O((|V|+|E|) \cdot \lg |V|)$
- This can be reduced to
  - $T(n) = O(|E| \cdot \lg |V|)$

  if all vertices are reachable from the source s

Bellman-Ford's Algorithm

- Bellman-Ford may run on graph
  - With negative weight edges
  - If there is a cycle with negative weight it detects it
  - It applies relaxation more than once for all edges
  - |V|-1 step of relaxation on all edges
  - At the i-th relaxation step either
    - It decreases at least one estimate

    or

    - It has already found an optimal solution and it can stop returning an optimum solution

Implementation

Pseudo-code

```
sssp_Bellman_Ford (G, w, s)
  initialize_single_source (G, s)
  for i = 1 to |V| – 1
    for each edge (u, v) ∈ E
      relax (u, v, w)
  for each edge (u, v) ∈ E
    if ( v.d > (u.d + w(u, v)) )
      return FALSE
return TRUE
```

Iterates |V|-1 times

Relaxes all edges

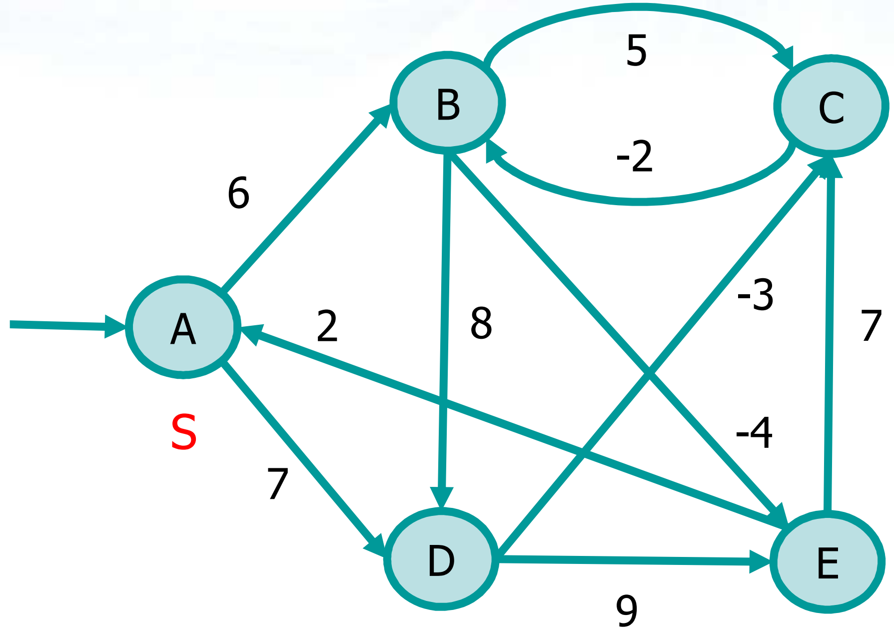
Checks for negative weight cycles

Returns FALSE if a negative weight cycle is detected

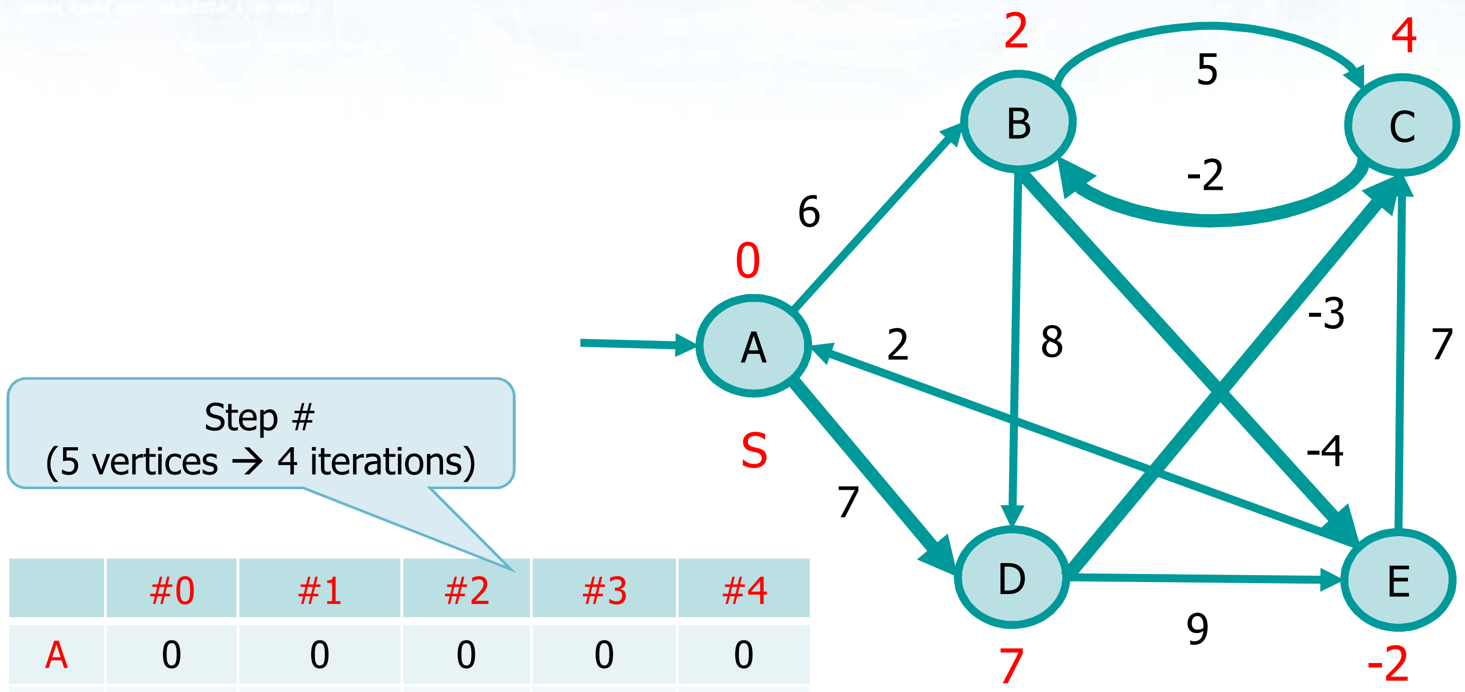
Returns TRUE otherwise

Example 1

Lessicographic order of the edges:
(A,B)
(A,D)
(B,C)
(B,D)
(B,E)
(C,B)
(D,C)
(D,E)
(E,A)
(E,C)



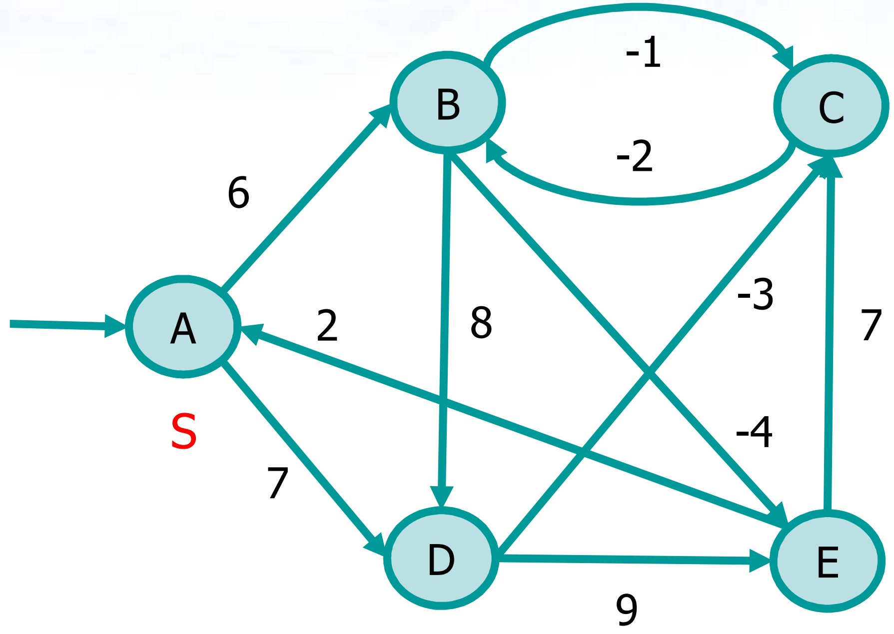
Example 1



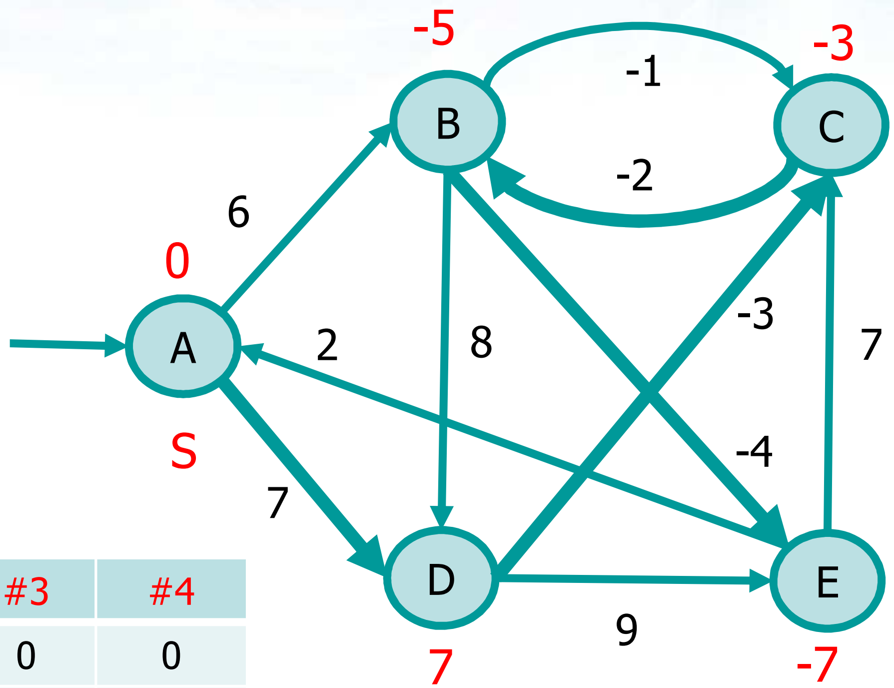
	#0	#1	#2	#3	#4
A	0	0	0	0	0
B	∞	6	2	2	2
C	∞	11→4	4	4	4
D	∞	7	7	7	7
E	∞	2	2	-2	-2

Example 2: Negative cycles

Lessicographic order of the edges:
(A,B)
(A,D)
(B,C)
(B,D)
(B,E)
(C,B)
(D,C)
(D,E)
(E,A)
(E,C)



Example 2: Negative cycles



Step #
(5 vertices → 4 iterations)

	#0	#1	#2	#3	#4
A	0	0	0	0	0
B	∞	6→3	1	-2	-5
C	∞	5→4	3	0	-3
D	∞	7	7	7	7
E	∞	2	-1	-3	-7

At the next iteration, edges BC and CB would make B and C reachable in -8 and -6

Complexity

Pseudo-code

```
sssp_Bellman_Ford (G, w, s)
  initialize_single_source (G, s)
  for i = 1 to |V| – 1
    for each edge (u, v) ∈ E
      relax (u, v, w)
  for each edge (u, v) ∈ E
    if ( v.d > (u.d + w(u, v)) )
      return FALSE
return TRUE
```

O (|V|)

Executed |V|-1 times

Executed |E| times

O(1) → **O(|E|·|V|)**

Executed |E| times → **O(|E|)**

Oerall running time complexity
$T(n) = O(|V| \cdot |E|)$