Solving System of Linear Equations

Original system	Intermediate System	Solved system
$5a + b = 17$ $4a - 3b = 6$	$a + 0.2b = 3.4$ $b = 2$	$a = 3$ $b = 2$
$\begin{bmatrix} 5 & 1 \\ 4 & -3 \end{bmatrix}$	$\begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Original matrix	upper diagonal matrix row echelon form	diagonal matrix reduced row echelon form

$a + b = 10$
$2a + 2b = 20$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$a + b = 10$
$0a + 0b = 0$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

row echelon form

$5a + b = 11$
$10a + 2b = 22$

$$\begin{bmatrix} 5 & 1 \\ 10 & 2 \end{bmatrix}$$

$a + 0.2b = 2.2$
$0a + 0b = 0$

$$\begin{bmatrix} 1 & 0.2 \\ 0 & 0 \end{bmatrix}$$

row echelon form

Row Echelon Form

$$\begin{bmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Main diagonal can be all 1, all 0 or bunch of 1 and zero

below diagonal should be 0

right of the diagonal : can be anything if diagonal 1
or be zero if diagonal 0

Row operations

Switching Rows

$$\begin{vmatrix} 5 & 1 \\ 4 & 3 \end{vmatrix} \rightarrow \begin{vmatrix} 4 & 3 \\ 5 & 1 \end{vmatrix}$$

$\det = 5.3 - 1.4 = 11$ $\det = 4.1 - 3.5 = -11$

* new det. always negative of the original det.

Multiplying a row by (non-zero) scalar

$$\begin{vmatrix} 5 & 1 \\ 4 & 3 \end{vmatrix}^{\times 10} \rightarrow \begin{vmatrix} 50 & 10 \\ 4 & 3 \end{vmatrix}$$

multiplying make new det., multiplying number x old det.

$\det = 5.3 - 1.4 = 11$ $\det = 5.(10.3) - 1.(10.4) = 10.11$

Adding a row to another row

$$\begin{vmatrix} 5 & 1 \\ 4 & 3 \end{vmatrix} \longrightarrow \begin{vmatrix} 9 & 4 \\ 4 & 3 \end{vmatrix}$$

* adding one row to another row dont change the det

$\det = 5.3 - 4.1 = 11$ $\det = 9.3 - 4.4 = 11$

*** these operations preserve singularity and non-singularity

Rank of a matrix

- The maximum number of its linearly independent colums (or rows) of a matrix

System 1	System 2	System 3
$a + b = 0$	$a + b = 0$	$0a + 0b = 0$
$a + 2b = 0$	$2a + 2b = 0$	$0a + 0b = 0$
Two equations	Two equations	Two equations
Two peaces of information	One piece of information	Zero piece of information
Rank = 2	Rank = 1	Rank = 0

*** a matrix is non-singular if and only if it has full rank

Row echelon form for finding rank

Original matrix → **Row echelon form**

$$\begin{bmatrix} 5 & 1 \\ 4 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0.2 \\ 1 & -0.75 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0.2 \\ 0 & -0.95 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}$$

Divide each row by the leftmost coefficient

$$\begin{array}{rr} & \begin{bmatrix} 1 & -0.75 \end{bmatrix} \\ - & \begin{bmatrix} 1 & 0.2 \end{bmatrix} \\ \hline & \begin{bmatrix} 0 & -0.95 \end{bmatrix} \end{array}$$

Divide the second row by the leftmost non-zero coefficient

rank = 2

*– Rank of a row echelon form matrix is number of 1 in main diagonal

*– A matrix is non-singular if main diagonal only contain 1

Rank of a matrix examples:

Matrix 1

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Rank = 3

Matrix 2

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

Rank = 2

Matrix 3

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$$

Rank = 1

Matrix 4

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank = 0

Row echelon forms

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Number of pivots = 3

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Number of pivots = 2

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Number of pivots = 1

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Number of pivots = 0

Reduced row echelon form

Row echelon form → **Reduced row echelon form**

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Subtract 2 times the second row from the first one

Add 5 times the third row to the first one

Subtract 4 times the third row from the second one