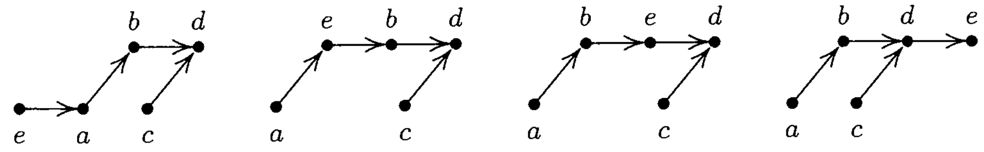
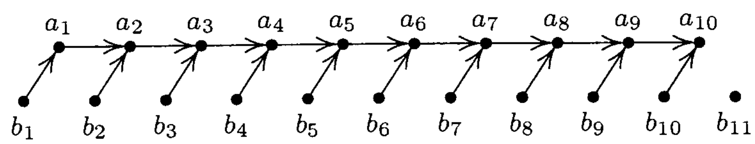
indicating that $a < b < d$ and $c < d$. (It is convenient to represent known ordering relations between elements by drawing directed graphs such as this, where x is known to be less than y if and only if there is a path from x to y in the graph.) At this point we insert the fifth element $K_5 = e$ into its proper place among $\{a, b, d\}$; only two comparisons are needed, since we may compare it first with b and then with a or d. This leaves one of four possibilities,



(6)

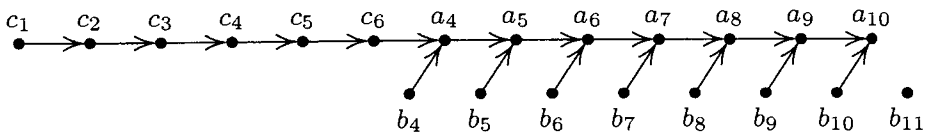
and in each case we can insert c among the remaining elements less than d in two more comparisons. This method for sorting five elements was first found by H. B. Demuth [Ph.D. thesis, Stanford University (1956), 41–43].

Merge insertion. A pleasant generalization of the method above has been discovered by Lester Ford, Jr. and Selmer Johnson. Since it involves some aspects of merging and some aspects of insertion, we shall call it *merge insertion.* For example, consider the problem of sorting 21 elements. We start by comparing the ten pairs $K_1 : K_2, K_3 : K_4, \ldots, K_{19} : K_{20}$; then we sort the ten larger elements of the pairs, using merge insertion. As a result we obtain the configuration



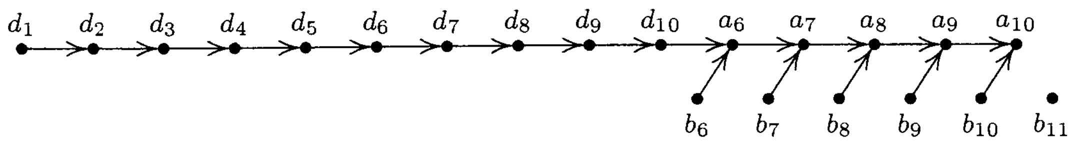
(7)

analogous to (5). The next step is to insert b_3 among $\{b_1, a_1, a_2\}$, then b_2 among the other elements less than a_2; we arrive at the configuration



(8)

Let us call the upper-line elements the *main chain.* We can insert b_5 into its proper place in the main chain, using three comparisons (first comparing it to c_4, then c_2 or c_6, etc.); then b_4 can be moved into the main chain in three more steps, leading to



(9)

The next step is crucial; is it clear what to do? We insert b_{11} (*not* b_7) into the main chain, using only four comparisons. Then b_{10}, b_9, b_8, b_7, b_6 (in this order) can also be inserted into their proper places in the main chain, using at most four comparisons each.

A careful count of the comparisons involved here shows that the 21 elements have been sorted in at most $10 + S(10) + 2 + 2 + 3 + 3 + 4 + 4 + 4 + 4 + 4 + 4 = 66$

steps. Since

$$2^{65} < 21! < 2^{66},$$

we also know that no fewer than 66 would be possible in any event; hence

$$S(21) = 66. \tag{10}$$

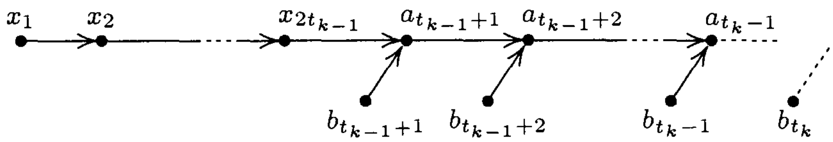
(Binary insertion would have required 74 comparisons.)

In general, merge insertion proceeds as follows for n elements:

i) Make pairwise comparisons of $\lfloor n/2 \rfloor$ disjoint pairs of elements. (If n is odd, leave one element out.)
ii) Sort the $\lfloor n/2 \rfloor$ larger numbers, found in step (i), by merge insertion.
iii) Name the elements $a_1, a_2, \ldots, a_{\lfloor n/2 \rfloor}$, $b_1, b_2, \ldots, b_{\lceil n/2 \rceil}$ as in (7), where $a_1 \le a_2 \le \cdots \le a_{\lfloor n/2 \rfloor}$ and $b_i \le a_i$ for $1 \le i \le \lfloor n/2 \rfloor$; call b_1 and the a's the "main chain." Insert the remaining b's into the main chain, using binary insertion, in the following order, leaving out all b_j for $j > \lceil n/2 \rceil$:

$$b_3, b_2;\ b_5, b_4;\ b_{11}, b_{10}, \ldots, b_6;\ \ldots;\ b_{t_k}, b_{t_k - 1}, \ldots, b_{t_{k-1}+1};\ \ldots. \tag{11}$$

We wish to define the sequence $(t_1, t_2, t_3, t_4, \ldots) = (1, 3, 5, 11, \ldots)$, which appears in (11), in such a way that each of $b_{t_k}, b_{t_k - 1}, \ldots, b_{t_{k-1}+1}$ can be inserted into the main chain with at most k comparisons. Generalizing (7), (8), and (9), we obtain the diagram



where the main chain up to and including $a_{t_k - 1}$ contains $2t_{k-1} + (t_k - t_{k-1} - 1)$ elements. This number must be less than 2^k; our best bet is to set it equal to $2^k - 1$, so that

$$t_{k-1} + t_k = 2^k. \tag{12}$$

Since $t_1 = 1$, we may set $t_0 = 1$ for convenience, and we find that

$$\begin{aligned} t_k = 2^k - t_{k-1} = 2^k - 2^{k-1} + t_{k-2} = \cdots &= 2^k - 2^{k-1} + \cdots + (-1)^k 2^0 \\ &= \bigl(2^{k+1} + (-1)^k\bigr)/3 \end{aligned} \tag{13}$$

by summing a geometric series. (Curiously, this same sequence arose in our study of an algorithm for calculating the greatest common divisor of two integers; see exercise 4.5.2–36.)

Let $F(n)$ be the number of comparisons required to sort n elements by merge insertion. Clearly

$$F(n) = \lfloor n/2 \rfloor + F(\lfloor n/2 \rfloor) + G(\lceil n/2 \rceil), \tag{14}$$

where G represents the amount of work involved in step (iii). If $t_{k-1} \le m \le t_k$, we have

$$G(m) = \sum_{j=1}^{k-1} j(t_j - t_{j-1}) + k(m - t_{k-1}) = km - (t_0 + t_1 + \cdots + t_{k-1}), \tag{15}$$

summing by parts. Let us set

$$w_k = t_0 + t_1 + \cdots + t_{k-1} = \lfloor 2^{k+1}/3 \rfloor, \tag{16}$$

so that $(w_0, w_1, w_2, w_3, w_4, \ldots) = (0, 1, 2, 5, 10, 21, \ldots)$. Exercise 13 shows that

$$F(n) - F(n-1) = k \qquad \text{if and only if} \qquad w_k < n \le w_{k+1}, \tag{17}$$

and the latter condition is equivalent to

$$\frac{2^{k+1}}{3} < n \le \frac{2^{k+2}}{3},$$

or $k + 1 < \lg 3n \le k + 2$; hence

$$F(n) - F(n-1) = \lceil \lg \tfrac{3}{4}n \rceil. \tag{18}$$

(This formula is due to A. Hadian [Ph.D. thesis, Univ. of Minnesota (1969), 38–42].) It follows that $F(n)$ has a remarkably simple expression,

$$F(n) = \sum_{k=1}^{n} \lceil \lg \tfrac{3}{4}k \rceil, \tag{19}$$

quite similar to the corresponding formula (3) for binary insertion. A closed form for this sum appears in exercise 14.

Equation (19) makes it easy to construct a table of $F(n)$; we have

$n =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\lceil \lg n! \rceil =$	0	1	3	5	7	10	13	16	19	22	26	29	33	37	41	45	49
$F(n) =$	0	1	3	5	7	10	13	16	19	22	26	30	34	38	42	46	50

$n =$	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
$\lceil \lg n! \rceil =$	53	57	62	66	70	75	80	84	89	94	98	103	108	113	118	123
$F(n) =$	54	58	62	66	71	76	81	86	91	96	101	106	111	116	121	126

Notice that $F(n) = \lceil \lg n! \rceil$ for $1 \le n \le 11$ and for $20 \le n \le 21$, so we know that merge insertion is optimum for those n:

$$S(n) = \lceil \lg n! \rceil = F(n) \qquad \text{for } n = 1, \ldots, 11, 20, \text{ and } 21. \tag{20}$$

Hugo Steinhaus posed the problem of finding $S(n)$ in the second edition of his classic book *Mathematical Snapshots* (Oxford University Press, 1950), 38–39. He described the method of binary insertion, which is the best possible way to sort n objects if we start by sorting $n-1$ of them first before the nth is considered; and he conjectured that binary insertion would be optimum in general. Several years later [*Calcutta Math. Soc. Golden Jubilee Commemoration* **2** (1959), 323–327], he reported that two of his colleagues, S. Trybuła and P. Czen, had "recently" disproved his conjecture, and that they had determined $S(n)$ for $n \le 11$. Trybuła and Czen may have independently discovered the method of merge insertion, which was published soon afterwards by Ford and Johnson [*AMM* **66** (1959), 387–389].

After the discovery of merge insertion, the first unknown value of $S(n)$ was $S(12)$. Table 1 shows that 12! is quite close to 2^{29}, hence the existence of a

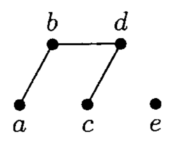
Table 1

VALUES OF FACTORIALS IN BINARY NOTATION

$(1)_2$	$= 1!$
$(10)_2$	$= 2!$
$(110)_2$	$= 3!$
$(11000)_2$	$= 4!$
$(1111000)_2$	$= 5!$
$(1011010000)_2$	$= 6!$
$(1001110110000)_2$	$= 7!$
$(1001110110000000)_2$	$= 8!$
$(1011000100110000000)_2$	$= 9!$
$(1101110101111100000000)_2$	$= 10!$
$(1001100001000101010000000)_2$	$= 11!$
$(1110010001100111111000000000)_2$	$= 12!$
$(1011100110010100011001100000000000)_2$	$= 13!$
$(1010001001100001110110010100000000000)_2$	$= 14!$
$(1001100000111011101110111010110000000000)_2$	$= 15!$
$(1001100000111011101110111010110000000000000)_2$	$= 16!$
$(10100001101111110111011101100110110000000000000)_2$	$= 17!$
$(10110101111101110110011001010011100110000000000000000)_2$	$= 18!$
$(110110000001010111001001100000110100010010000000000000000)_2$	$= 19!$
$(10000111000011011001110111110010000010101101000000000000000000)_2$	$= 20!$

29-step sorting procedure for 12 elements is somewhat unlikely. An exhaustive search (about 60 hours on a Maniac II computer) was therefore carried out by Mark Wells, who discovered that $S(12) = 30$ [*Proc. IFIP Congress 65* **2** (1965), 497–498; *Elements of Combinatorial Computing* (Pergamon, 1971), 213–215]. Thus the merge insertion procedure turns out to be optimum for $n = 12$ as well.

***A slightly deeper analysis.** In order to study $S(n)$ more carefully, let us look more closely at partial ordering diagrams such as (5). After several comparisons have been made, we can represent the knowledge we have gained in terms of a directed graph. This directed graph contains no cycles, in view of the transitivity of the $<$ relation, so we can draw it in such a way that all arcs go from left to right; it is therefore convenient to leave arrows off the diagram. In this way (5) becomes



(21)

If G is such a directed graph, let $T(G)$ be the number of permutations consistent with G, that is, the number of ways to assign the integers $\{1, 2, \ldots, n\}$ to the vertices of G so that the number on vertex x is less than the number on vertex y whenever $x \to y$ in G. For example, one of the permutations consistent with (21) has $a = 1$, $b = 4$, $c = 2$, $d = 5$, $e = 3$. We have studied $T(G)$ for various G in Section 5.1.4, where we observed that $T(G)$ is the number of ways in which G can be sorted topologically.