Computer Simulation Assignment 3

ismisebrendan

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1 Numerical methods

1.1 The Simple Euler Method

This is a method of solving ordinary differential equations (ODEs). A single order ODE might have the form

$$\frac{d}{dt}x(t) = f(x,t) \tag{1}$$

and the aim is generally to evaluate x(t) for given starting values of x and t, t_0 and $x(t = t_0) = x_0$.
Euler's method works by taking the Taylor series expansion of x about the point t

$$x(t + \Delta t) = x(t) + \left.\frac{dx}{dt}\right|_t \Delta t + \frac{1}{2}\left.\frac{d^2x}{dt^2}\right|_t (\Delta t)^2 + \frac{1}{3!}\left.\frac{d^3x}{dt^3}\right|_t (\Delta t)^3 + ... \tag{2}$$

and dropping any terms of higher order than Δt. This gives

$$x(t + \Delta t) = x(t) + \left.\frac{dx}{dt}\right|_t \Delta t \tag{3}$$

If this is rewritten using $t_i = i\Delta t$, $x_i = x(t_i)$ and $\frac{d}{dt}x_i = f(x_i, t_i)$ this gives

$$x_{i+1} = x_i + f(x_i, t_i)\Delta t \tag{4}$$

for the Simple Euler Method.

The Simple Euler Method is a zeroth order approximation globally, meaning that the error has a leading term with zeroth order.

When this is implemented in a programming language, Δt is the step size between evaluations and the various values of x_i can be evaluated by looping over equation 4. Starting from $t = 0$, if $x(t = b)$ is to be evaluated, $N = \frac{b}{\Delta t}$ steps are needed. Smaller values of Δt give smaller errors but longer runtimes, and eventually lead to rounding errors when machine precision limits are reached, as such Δt must be chosen carefully to balance all of these considerations.

1.2 The Improved Euler Method

If equation 1 is integrated with respect to t, from the point t_i to t_{i+1} we get

$$x_{i+1} = x_i + \int_{t_i}^{t_{i+1}} f(x_i, t_i)dt \tag{5}$$

According to the mean value theorem there is some value of t, t_m, in the interval $[t_i, t_{i+1}]$ such that equation 5 can be rewritten as

$$x_{i+1} = x_i + f(x_m, t_m)\Delta t \tag{6}$$

where t_m (and therefore x_m) is unknown. In the Simple Euler Method t_m is replaced by t_i (and so $x_m = x_i$).

The Improved Euler Method uses the trapezoidal rule to evaluate the integral in equation 5. It evaluates $f(x, t)$ at the initial point, (x_i, t_i), and at the endpoint approximated by the Simple Euler Method, $f(x_i + f(x_i, t_i)\Delta t, t_{i+1})$ and takes the average of the sum these values multiplied by the step size

$$x_{i+1} = x_i + (f(x_i, t_i) + f(x_i + f(x_i, t_i)\Delta t, t_{i+1}))\frac{\Delta t}{2} \tag{7}$$

This is the Improved Euler Method and is locally a second order approximation, while globally is a first order approximation.

1.3 Fourth-Order Runge-Kutta Method

Runge-Kutta Methods are a group of methods used to evaluated ODEs. The Fourth-Order Runge-Kutta Method is expressed as

$$x_{i+1} = x_i \frac{\Delta t}{6}(f(x_1', t_1') + 2f(x_2', t_2') + 2f(x_3', t_3') + f(x_4', t_4')) \tag{8}$$

where

$$\begin{aligned} x_1' &= x_i & t_1' &= t_i \\ x_2' &= x_i + \frac{1}{2} f(x_1', t_1')\Delta t & t_2' &= t_i + \frac{\Delta t}{2} \\ x_3' &= x_i + \frac{1}{2} f(x_2', t_2')\Delta t & t_3' &= t_i + \frac{\Delta t}{2} \\ x_4' &= x_i + f(x_3', t_3')\Delta t & t_4' &= t_i + \Delta t \end{aligned} \tag{9}$$

In this expression of the Fourth-Order Runge-Kutta Method, $f(x_1', t_1')$ is the slope of the equation at the beginning of the interval $[t_i, t_{i+1}]$ as evaluated by Euler's Method, $f(x_2', t_2')$ is the slope of the equation in the middle of the interval, at the point $t_i + \frac{\Delta t}{2}$ evaluated at x_1', while $f(x_3', t_3')$ is the slope of the equation again in the middle of the interval but this time evaluated at x_2'. Finally, $f(x_4', t_4')$ is the slope at the end of the interval evaluated using x_3' and t_{i+1}.

The value calculated for x_{i+1} is the the weighted value of the four slopes added to x_i, the previous value.

This method is globally a fourth order approximation.[1]

2 Findings

The direction field for the ODE

$$f(x, t) = \frac{dx}{dt} = (1 + t)x + 1 - 3t + t^2 \tag{10}$$

was generated first and is shown in figure 1.

The starting values are $t_0 = 0$ and $x(t_0) = x_0 = 0.0655$. The value of x_0 is very close to the critical point of $x_c = 0.065923....$ If $x(0) > x_c$ the solutions tend towards $+\infty$, while if $x(0) < x_c$ the solutions tend towards $-\infty$. Clearly therefore, all of the graphs of solutions to this ODE should tend towards $-\infty$ if evaluated accurately.

The Simple Euler, Improved Euler, and Fourth-Order Runge-Kutta Methods were used to solve the ODE for these initial conditions. The step sizes used were $\Delta t = 0.04$ and $\Delta t = 0.02$.

A graph of the solutions produced by evaluating each of the methods with both step sizes were produced, as were two graphs of the solutions produced by all three methods for the same step size.

The graph of the solutions produced by each method for a step size of $\Delta t = 0.04$ is shown in figure 2a.

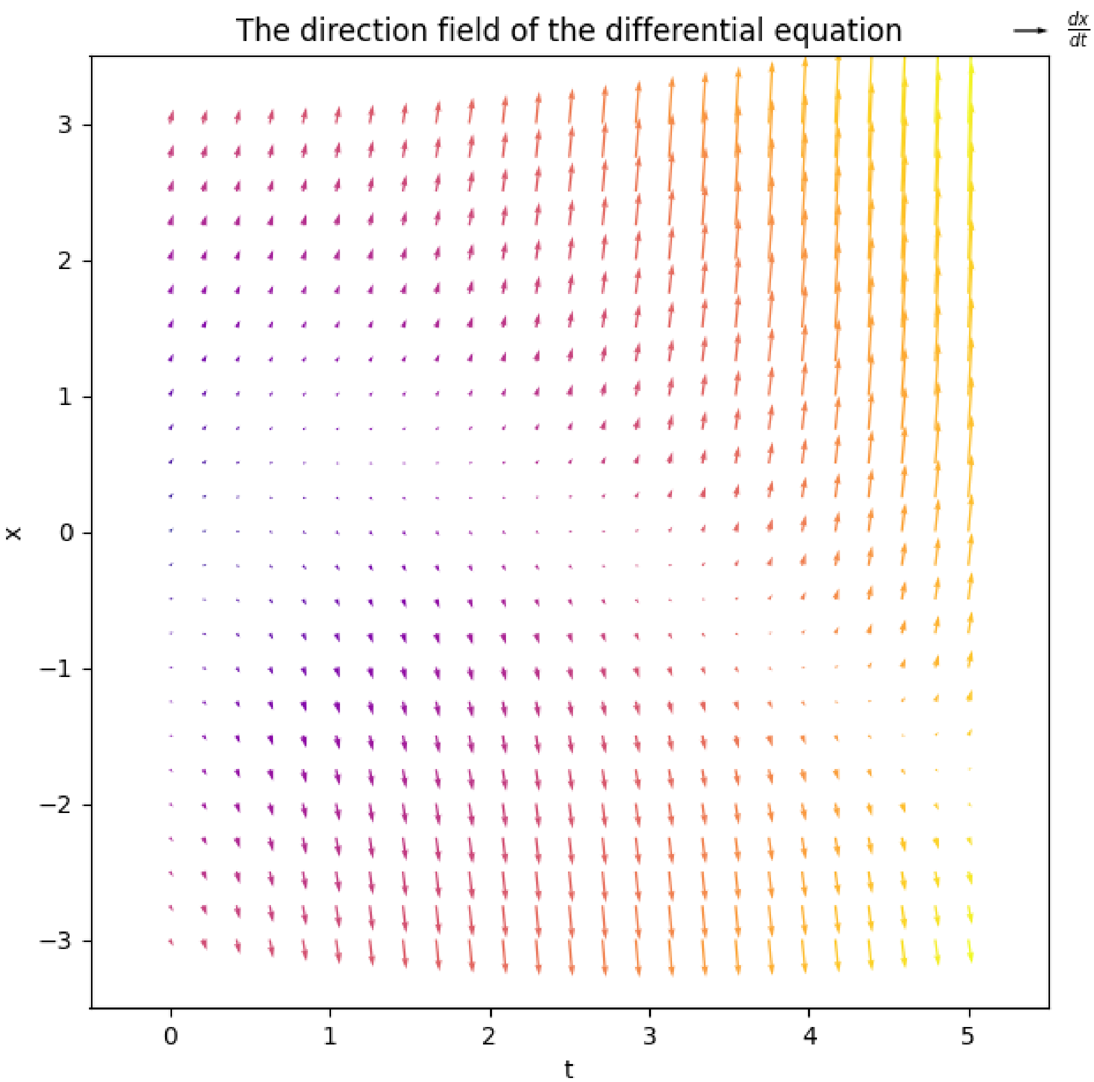
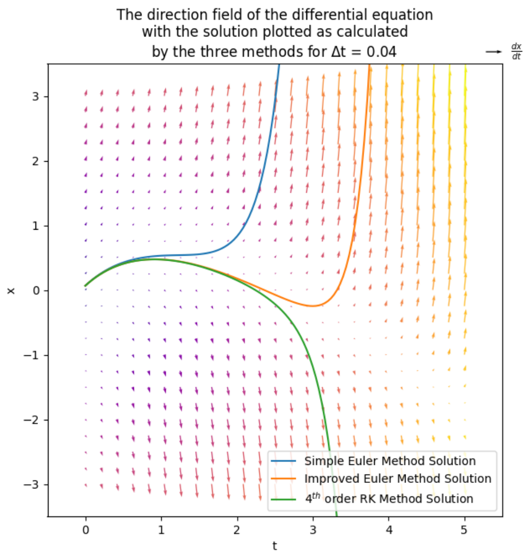
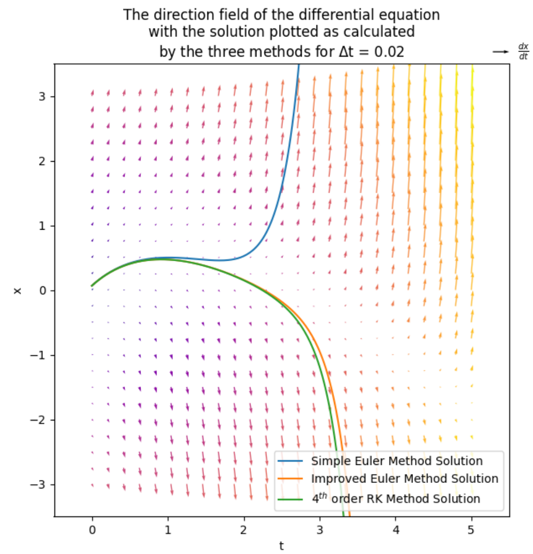


Figure 1: The direction field for the ODE in equation 10 represented for $t = [0, 5]$ and $x = [-3, -3]$.



(a) The solution to the ODE evaluated by the Simple Euler, Improved Euler and Fourth-Order Runge-Kutta methods for initial conditions $t_0 = 0$ and $x_0 = 0.0655$ and step size $\Delta t = 0.04$.



(b) The solution to the ODE evaluated by the Simple Euler, Improved Euler and Fourth-Order Runge-Kutta methods for initial conditions $t_0 = 0$ and $x_0 = 0.0655$ and step size $\Delta t = 0.02$.

Figure 2: The solutions to the ODE as evaluated by the Simple and Improved Euler Methods and the Fourth-Order Runge-Kutta Method for two different step sizes.

The solutions are expected to all tend towards $-\infty$, however only the solution evaluated by the Fourth-Order Runge-Kutta methods tends towards this value as expected while the others tend towards $+\infty$. This is due to the relative inaccuracy of the two Euler methods.

The Fourth-Order Runge-Kutta is a fourth-order approximation, while the Simple and Improved Euler Methods are zeroth and first order respectively. This leads to smaller errors in the evaluation of the solutions by the Runge-Kutta Method than by either of the Euler Methods. The three solutions begin close to one another before diverging. The Simple Euler Method's solution diverges first and tends towards $+\infty$, while the Improved Euler Method's solution diverges later but still tends towards $+\infty$. This error is due to the accumulation of smaller errors, eventually leading to complete divergence.

The graph of the solutions produced by each method for a step size of $\Delta t = 0.02$ is shown in figure 2b.

The solutions are again expected to all tend towards $-\infty$, however the solution evaluated by the Simple Euler Method tends towards $+\infty$, due to its inaccuracy.

Clearly the use of smaller step sizes improves the accuracy of the various methods of solving the ODE. This can be seen clearly in the fact that upon halving the step size the solution as evaluated by the Improved Euler Method changes from tending towards $+\infty$ to tending towards $-\infty$ as is expected, or the fact that the solution evaluated by the Simple Euler Method diverges at a higher value of t for $\Delta t = 0.02$ than it does for $\Delta t = 0.04$, this can be seen more clearly in figure 3.

However, for a given step size it is better to use higher order approximations as they are more accurate than the lower order approximations. The Fourth-Order Runge-Kutta Method produces a solution which behaves as expected for steps of both evaluated sizes, and at least for the range visible the solution for $\Delta t = 0.04$ and $\Delta t = 0.02$ appears to be the same.

When $\Delta t = 0.04$ the solution produced by the Fourth-Order Runge-Kutta Method is the only one which behaves as expected, and when $\Delta t = 0.02$ the solution produced by the Improved Euler Method still diverges from that of the Fourth-Order Runge-Kutta Method.

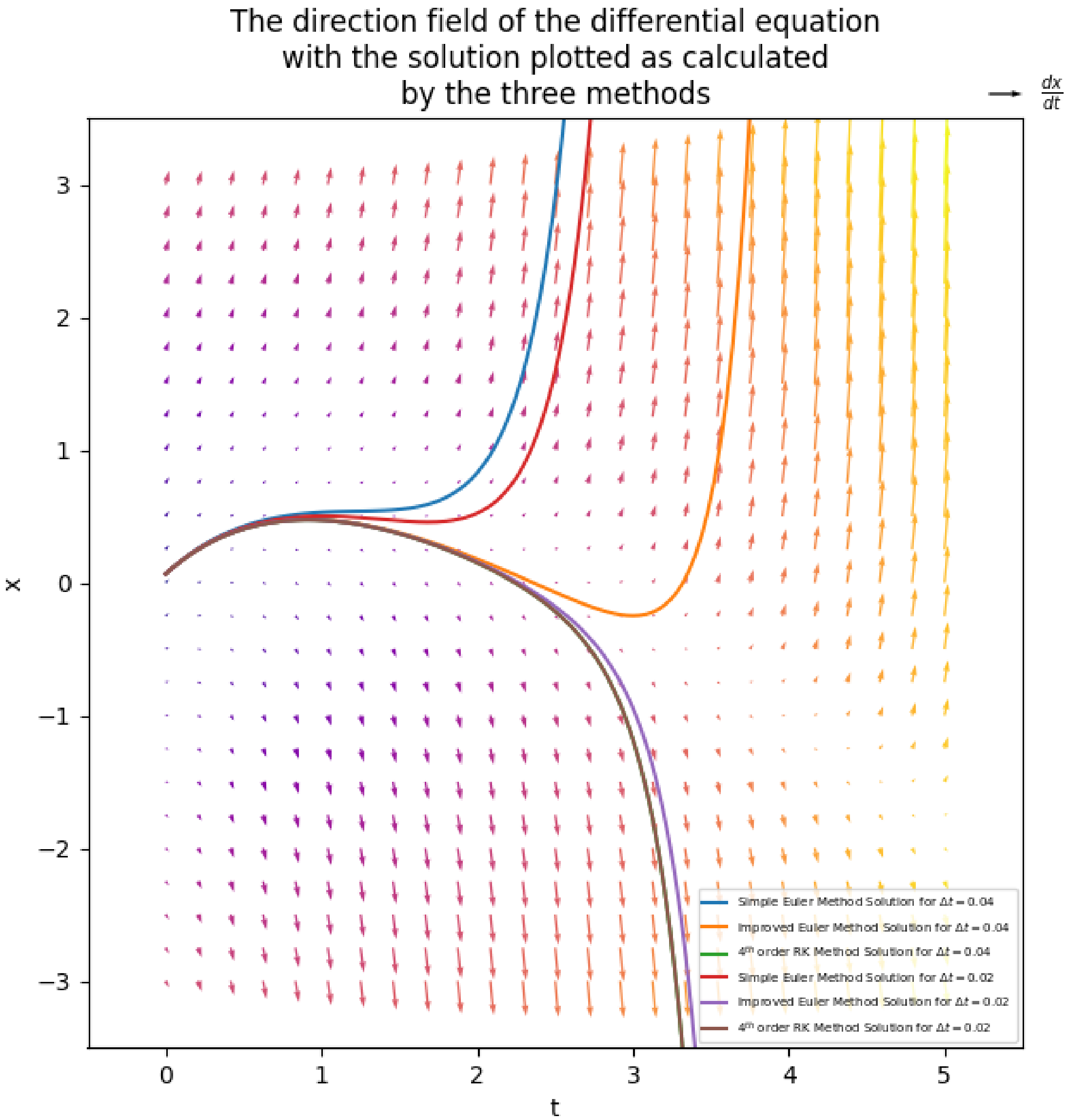


Figure 3: The solutions to the ODE as evaluated by the Simple and Improved Euler Methods and the Fourth-Order Runge-Kutta Method for both step sizes.

3 Conclusions

The use of more accurate integration schemes allows for more accurate solutions with larger step sizes, therefore saving on computation time and power.

Smaller step sizes can be used to improve the accuracy of a given integration scheme also. However, if possible it is better to simply use a more accurate integration scheme.

References

[1] G. Dahlquist, Å. Björck, *Numerical Methods.* USA: Dover Publications, Inc., 2003. Translated by N. Anderson