Computer Simulation Assignment 4

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1 Poisson Distribution

The three Poisson distributions as described by

$$P(n) = \frac{\langle n \rangle^n}{n!} e^{-\langle n \rangle} \tag{1}$$

for $\langle n \rangle = 1, 5, 10$ are shown in figure 1.

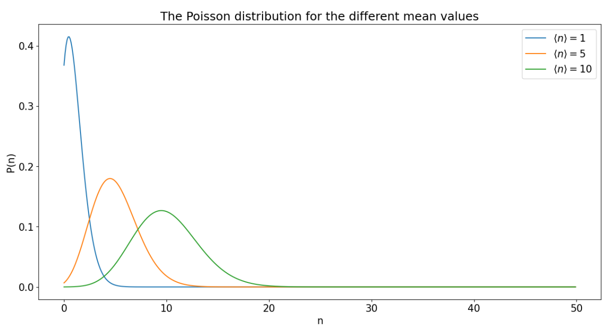


Figure 1: The Poisson distribution for $\langle n \rangle = 1, 5, 10$ according to equation 1

The peaks are seen to occur at the mean values, as expected.

2 Calculation of the sum of probabilities, the first and second moments, the variances and the standard deviations.

The sum of the probabilities for N points (n) in the Poisson distribution is given by

$$\sum_{n=0}^{N} P(n) \tag{2}$$

the first moment ($\langle n \rangle$), which for the Poisson distribution is equal to the mean, is given by

$$\langle n \rangle = \sum_{n=0}^{N} nP(n) \tag{3}$$

and the second moment ($\langle n^2 \rangle$) is given by

$$\langle n^2 \rangle = \sum_{n=0}^{N} n^2 P(n) \tag{4}$$

The variance of the distribution σ^2 is given by

$$\sigma^2 = \langle n^2 \rangle - \langle n \rangle \tag{5}$$

and the standard deviation is given by

$$\sigma = \sqrt{\langle n^2 \rangle - \langle n \rangle} \tag{6}$$

The results of these for the different values of $\langle n \rangle$ is shown below in table 1.

Table 1: The sum of the elements of the Poisson distribution and the first and second moments of the distributions for $\langle n \rangle = 1, 5, 10$.

$\langle n \rangle$	$\sum_{n=0}^{N} P(n)$	$\langle n \rangle$	$\langle n^2 \rangle$	σ^2	σ
1	1	1	2	1	1
5	1	5	30	5	$\sqrt{5}$
10	1	10	110	10	$\sqrt{10}$

As can be seen in table 1 the distributions are all normalised. The variance and standard deviation are equal to the mean and square root of the mean values respectively, as is expected with Poisson distributions.

3 Dart throwing

The python program to simulate throwing N=50 darts into L=100 bins T=10 times in order to calculate $H(n)$, the distribution of regions with n darts and $\langle n \rangle$ the mean number of darts in each region, which was found to equal 500 in this instance.

$H(n)$ was normalised to give $P_{sim}(n)$, and was plotted along with the Poisson distribution for $\langle n \rangle$ equal to the mean of $P_{sim}(n)$. It was found that $\langle n \rangle$ = 0.5. This graph is shown in figure 2.

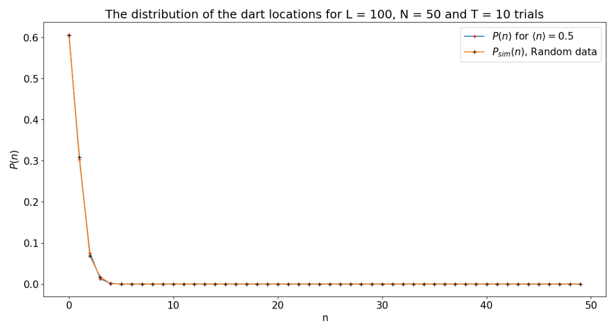
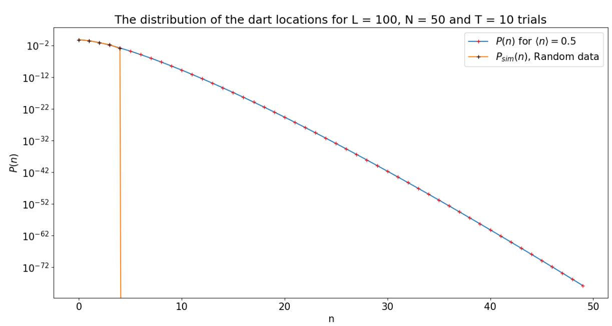


Figure 2: The graph of $P_{sim}(n)$ for 10 trials and 100 bins along with the Poisson distribution for $\langle n \rangle = 0.5$.

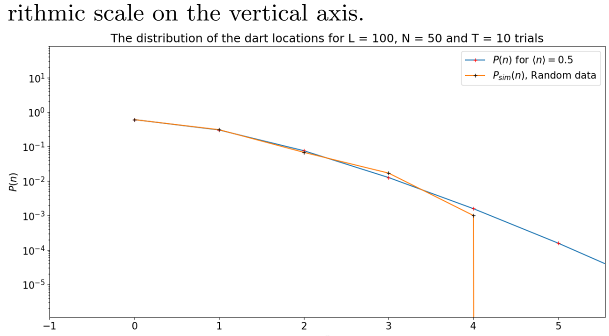
4 Log plot of distributions

The two distributions were plotted with a log scale on the vertical axis, and this is shown in figure 3a, with a zoomed in version visible in figure 3b.

As can be seen in figure 3b, in this case the randomly simulated data approximates the Poisson distribution down to values of $P(n) \approx 1 \times 10^{-3}$, or values of n up to 4.



(a) The graph of $P_{sim}(n)$ for 10 trials and 100 bins along with the Poisson distribution for $\langle n \rangle = 0.5$ with a logarithmic scale on the vertical axis.



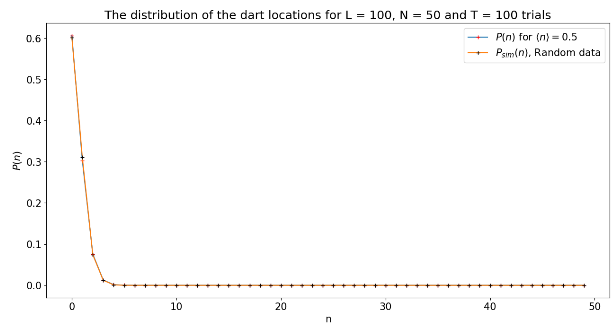
(b) The same graph as in figure 3a, zoomed into the point at which the distributions begin to diverge.

Figure 3: The graph of $P_{sim}(n)$ for 10 trials and 100 bins along with the Poisson distribution for $\langle n \rangle = 0.5$ with a logarithmic scale on the vertical axis.

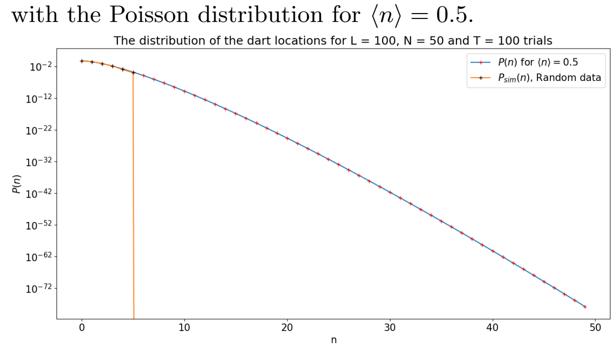
5 Repeated simulations for T = 100, 1000, 10000

The above procedure was repeated for a number of trials (T) equal to 100, 1000 and 10000. The value of $\langle n \rangle$ was found to equal 0.5 for all of the normalised distributions.

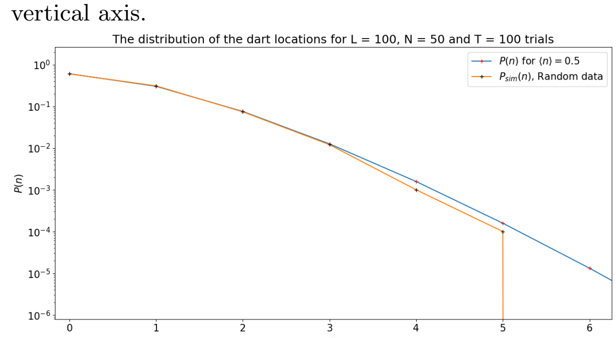
5.1 For T = 100



(a) The graph of $P_{sim}(n)$ for 100 trials and 100 bins along with the Poisson distribution for $\langle n \rangle = 0.5$.



(b) The graph of $P_{sim}(n)$ along with the Poisson distribution for $\langle n \rangle = 0.5$ with a logarithmic scale on the vertical axis.



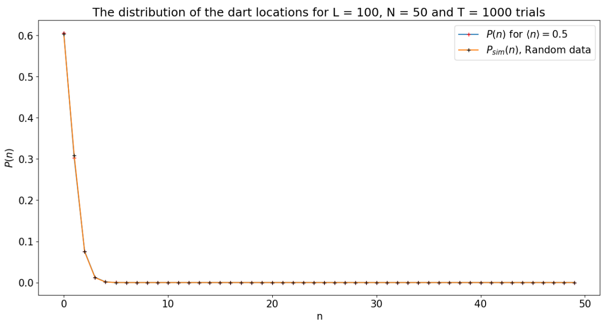
(c) The same graph as in figure 4b, zoomed into the point at which the distributions begin to diverge.

Figure 4: The graph of $P_{sim}(n)$ for 100 trials and 100 bins along with the Poisson distribution for $\langle n \rangle = 0.5$ with a logarithmic scale on the vertical axis.

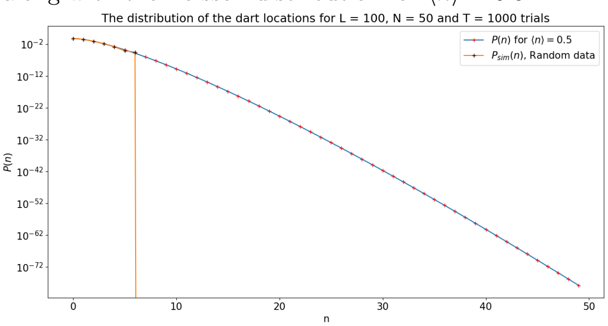
The graph of this data on a linear vertical axis is shown in figure 4a. The version of the data with a logarithmic scale is shown in figure 4b.

As seen in figure 4c, with 100 trials the Poisson distribution can be approximated down to values of $P(n) \approx 1.0 \times 10^{-4}$, or values of n up to 5.

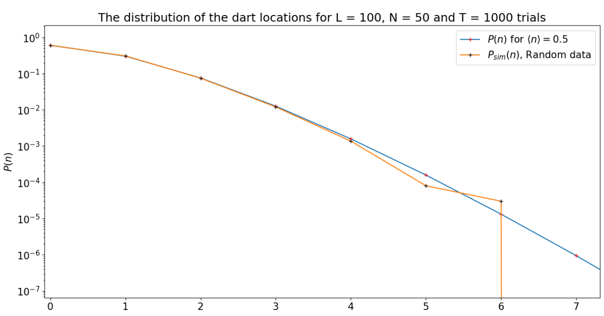
5.2 For T = 1000



(a) The graph of $P_{sim}(n)$ for 1000 trials and 100 bins along with the Poisson distribution for $\langle n \rangle = 0.5$.



(b) The graph of $P_{sim}(n)$ along with the Poisson distribution for $\langle n \rangle = 0.5$ with a logarithmic scale on the vertical axis.



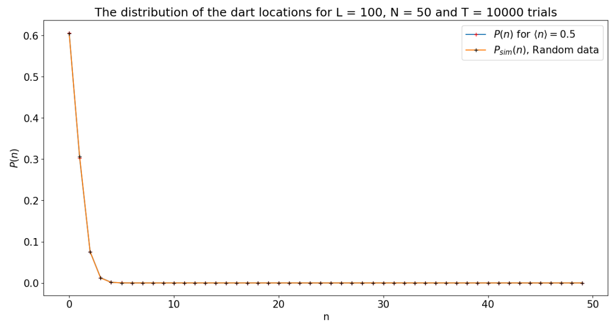
(c) The same graph as in figure 5b, zoomed into the point at which the distributions begin to diverge.

Figure 5: The graph of $P_{sim}(n)$ for 1000 trials and 100 bins along with the Poisson distribution for $\langle n \rangle = 0.5$ with a logarithmic scale on the vertical axis.

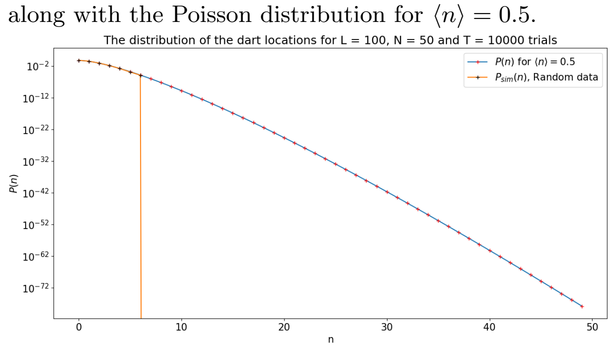
The graph of this data on a linear vertical axis is shown in figure 5a. The version of the data with a logarithmic scale is shown in figure 5b.

As seen in figure 5c, with 1000 trials the Poisson distribution can be approximated down to values of $P(n) \approx 3.0 \times 10^{-5}$, or for values of n up to 6.

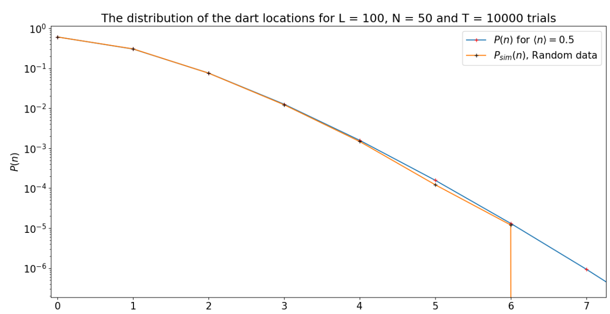
5.3 For T = 10000



(a) The graph of $P_{sim}(n)$ for 10000 trials and 100 bins along with the Poisson distribution for $\langle n \rangle = 0.5$.



(b) The graph of $P_{sim}(n)$ along with the Poisson distribution for $\langle n \rangle = 0.5$ with a logarithmic scale on the vertical axis.



(c) The same graph as in figure 6b, zoomed into the point at which the distributions begin to diverge.

Figure 6: The graph of $P_{sim}(n)$ for 10000 trials and 100 bins along with the Poisson distribution for $\langle n \rangle = 0.5$ with a logarithmic scale on the vertical axis.

The graph of this data on a linear vertical axis is shown in figure 6a. The version of the data with a logarithmic scale is shown in figure 6b. As seen in figure 6c, with 10000 trials the Poisson distribution can be approximated down to values of $P(n) \approx 1.2 \times 10^{-5}$, or for values of n up to 6.

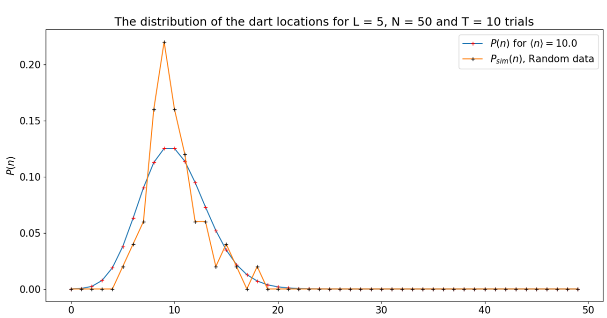
This is a very small increase in accuracy as compared to the previous increases after increasing the number of trials. Previously, when the number of trials increased by a factor of 10, $P_{sim}(n)$ approximated $P(n)$ down to a value approximately one power of 10 lower. The randomness of the data means that it wasn't exact powers of 10 each time, however it was close to this. It should also be noted that the randomness of this data means that the values of $P(n)$ which can be approximated for each value of T will vary slightly each time the program is run.

This might suggest a limit to the accuracy of approximation to the Poisson distribution which can be achieved with random numbers. For the values used here, with 100 locations chosen randomly 50 times, that limit may be $\sim 1.2 \times 10^{-5}$.

6 Repeated simulations for 5 bins

The above procedure was repeated for a number bins (L) equal to 5, and numbers of trials (T) equal to 10, 1000 and 10000. The value of $\langle n \rangle$ was found to equal 10 for all of the normalised distributions.

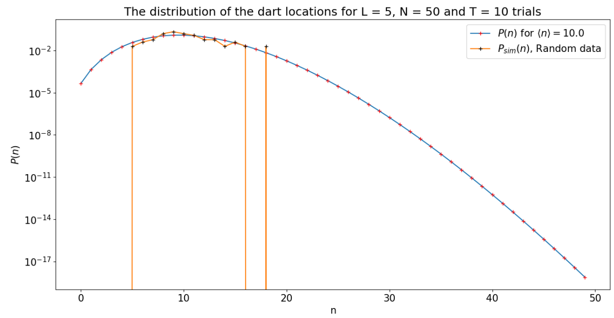
6.1 For T = 10



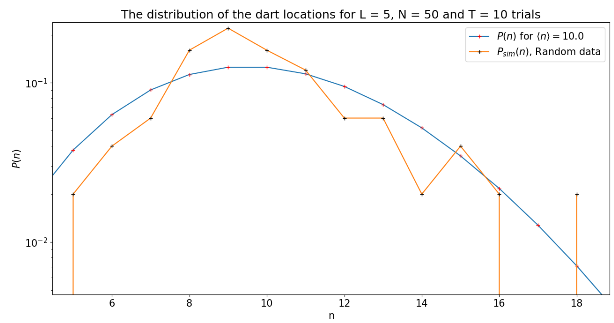
(a) The graph of $P_{sim}(n)$ for 10 trials and 5 bins along with the Poisson distribution for $\langle n \rangle = 10$.

The graph of this data on a linear vertical axis is shown in figure 7a. The version of the data with a logarithmic scale is shown in figure 7b.

As seen in figure 7c, with 10 trials the Poisson distribution can be approximated down to values of $P(n) \approx 2.0 \times 10^{-2}$. Of course, $P_{sim}(n)$ varies wildly from $P(n)$ throughout the distribution (likely due



(b) The graph of $P_{sim}(n)$ along with the Poisson distribution for $\langle n \rangle = 10$ with a logarithmic scale on the vertical axis.



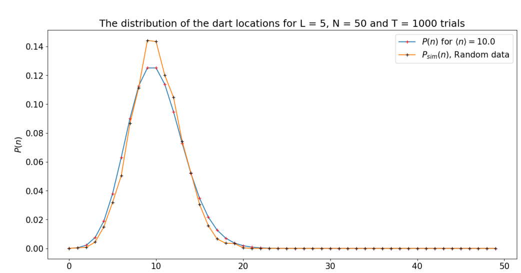
(c) The same graph as in figure 7b, zoomed into the point at which the distributions begin to diverge.

Figure 7: The graph of $P_{sim}(n)$ for 10 trials and 5 along with the Poisson distribution for $\langle n \rangle = 10$ with a logarithmic scale on the vertical axis.

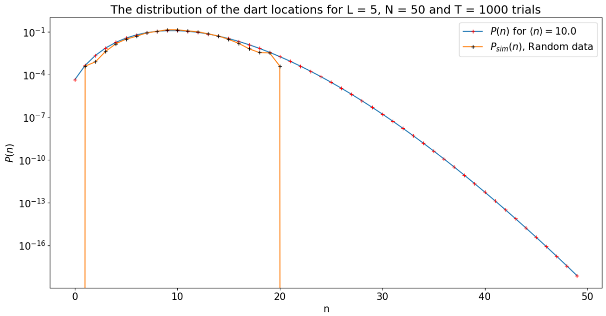
to the small number of trials), however it at least approaches $P(n)$ for values greater than 2.0×10^{-2}.

6.2 For T = 1000

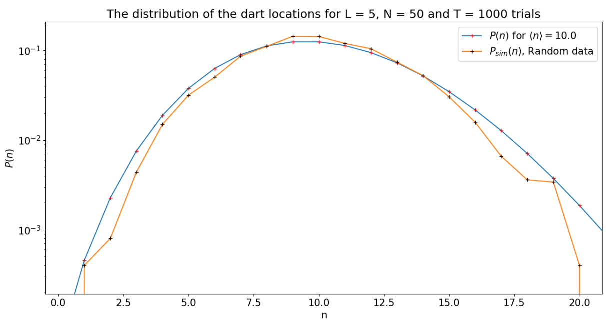
The graph of this data on a linear vertical axis is shown in figure 8a. The version of the data with a logarithmic scale is shown in figure 8b.



(a) The graph of $P_{sim}(n)$ for 1000 trials and 5 bins along with the Poisson distribution for $\langle n \rangle = 10$.



(b) The graph of $P_{sim}(n)$ along with the Poisson distribution for $\langle n \rangle = 10$ with a logarithmic scale on the vertical axis.



(c) The same graph as in figure 8b, zoomed into the point at which the distributions begin to diverge.

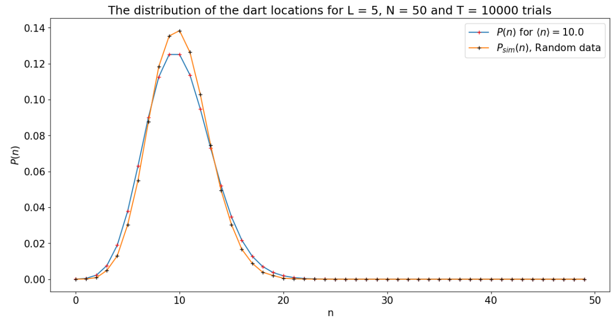
Figure 8: The graph of $P_{sim}(n)$ for 1000 trials and 5 along with the Poisson distribution for $\langle n \rangle = 10$ with a logarithmic scale on the vertical axis.

As seen in figure 8c, with 1000 trials the Poisson distribution can be approximated down to values of $P(n) \approx 4.0 \times 10^{-4}$. Due to the greater number of trials $P_{sim}(n)$ for this distribution is closer to the ideal $P(n)$ than it was when there were only 10 trials carried out.

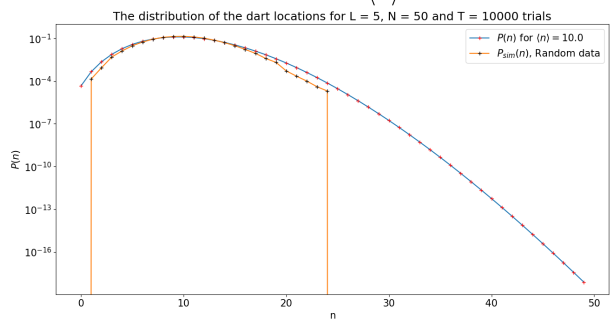
6.3 For T = 10000

The graph of this data on a linear vertical axis is shown in figure 9a. The version of the data with a logarithmic scale is shown in figure 9b.

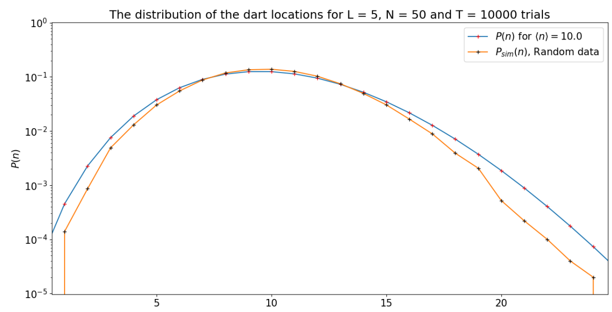
As seen in figure 9c, with 10000 trials the Poisson distribution can be approximated down to values of $P(n) \approx 2.0 \times 10^{-5}$. As for the simulations with 100 bins it was found that the values at which the simulated random data no longer approximated the Poisson distribution decreased as the number of trials increased.



(a) The graph of $P_{sim}(n)$ for 10000 trials and 5 bins along with the Poisson distribution for $\langle n \rangle = 10$.



(b) The graph of $P_{sim}(n)$ along with the Poisson distribution for $\langle n \rangle = 10$ with a logarithmic scale on the vertical axis.



(c) The same graph as in figure 9b, zoomed into the point at which the distributions begin to diverge.

Figure 9: The graph of $P_{sim}(n)$ for 10000 trials and 5 along with the Poisson distribution for $\langle n \rangle = 10$ with a logarithmic scale on the vertical axis.

7 Conclusions

In all cases the more trials which are carried out, the lower the values of $P(n)$ which can be simulated are. This makes sense as it allows chances for the less likely outcomes to occur and populate $H(n)$ and therefore $P_{sim}(n)$.

It does not appear that the number of bins for the random numbers makes a large difference in the value of $P(n)$ which can be simulated with the random numbers. However, simply from these simulations, it appears that for larger numbers of bins (L = 100) the approximations are either very accurate, or do not appear at all, as can be seen in figures 2 - 6.

While for lower numbers of bins (L = 5) the approximations can be made for more or less the same values of $P(n)$ as with a large number of bins, but they tend to be less accurate than those approximations made with a larger number of potential bins.

This can be seen clearly in figures 7 to 9, the approximations extend to the same values of $P(n)$ as they do for L = 100, however they clearly stick less closely to the distribution - in fact, when T = 10 in this report the value of $P_{sim}(17) = 0$.