Fourier Analysis

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Abstract

The general use of the digital oscilloscope as well as its use to carry out Fourier analysis and fast Fourier transforms in particular was demonstrated. A variety of measurements such as peak-to-peak voltage, period and phase difference were made of waveforms and the oscilloscope was used to analyse an amplitude modulated waveform. The effect of changing the triggering on the oscilloscope was also investigated. The digital oscilloscope was also used in its fast Fourier transform mode to find the fast Fourier transform of a sinewave, square wave, single pulse and repetitive pulse. The effect of aliasing and the different window functions were also investigated, and it was found that there was an almost linear relationship between peak full width half maximum and sampling frequency when analysing a signal with only one frequency component to it.

1 Introduction

Any time-dependent signal, whether or not it is periodic can be represented by a superposition of waves of different frequencies, a Fourier series. A periodic function of period T, $f(t)$, can be expressed as a Fourier series by

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) \tag{1.1}$$

where

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} dt f(t) \ , \tag{1.2}$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} dt f(t) \cos n\omega t \ , \tag{1.3}$$

and

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} dt f(t) \sin n\omega t \ . \tag{1.4}$$

This can also be done by calculating the Fourier transform of the function which can be done continuously or discretely. The Fourier transform of a function is the spectrum of the different frequencies which make up the function. [1], [2]

The continuous Fourier transform, $\tilde{f}(\omega)$ of a function $f(t)$ is given by

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \ . \tag{1.5}$$

In reality, when teaking measurements the precise value of the function is known at only finitely many discrete points. In this case a discrete Fourier transform (DFT) must be carried out. For a periodic function $f(t) = f(t + 2\pi)$ let there be N regularly spaced samples taken of $f(t)$ over the interval $0 \leq x \leq 2\pi$, then

$$x_k = \frac{2\pi k}{N} \tag{1.6}$$

where $k = 0, 1, ..., N-1$.

It is desired to interpolate f between these points with a function,

$$\hat{f}(x) = \sum_{n=0}^{N-1} c_n e^{inx} \tag{1.7}$$

where the condition, $f(x_k) = \hat{f}(x_k)$ is imposed also. The coefficients c_n are given by

$$c_n = \frac{1}{N}\sum_{k=0}^{N-1} f_k exp\left(-\frac{2\pi i n k}{N}\right) \tag{1.8}$$

where $f_k = f(x_k)$ and $n = 0, 1, ..., N-1$. [1]

The DFT of a signal $\boldsymbol{f} = [f_0, f_1, ..., f_{N-1}]^T$ can also be represented by the vector $\hat{\boldsymbol{f}} = [\hat{f}_0, \hat{f}_1, ..., \hat{f_{N-1}}]$ where

$$\hat{f}_n = N c_n = \sum_{k=0}^{N-1} f_k exp\left(-\frac{2\pi i n k}{N}\right) \ . \tag{1.9}$$

This allows the definition of an $N \times N$ Fourier matrix, $\mathbf{F}_N$, where $\hat{\boldsymbol{f}} = \mathbf{F}_N \boldsymbol{f}$ and whose entries, e_{nk} are given by

$$e_{nk} = e^{-2\pi ink/N} = w^{nk} \tag{1.10}$$

where

$$w = w_N = e^{-2\pi i/N} \ . \tag{1.11}$$

This is can be calculated by a computer, however it is computationally intensive for large N, with $O(N^2)$ operations required. The fast Fourier transform (FFT) allows the DFT to be computed practically for large values of N and in the limit towards infinity requires only $O(N)\log_2 N$ operations. [1]

For an FFT the value of N must equal 2^p where p is a positive integer. It allows the transformation to be broken down from a problem of size N to two problems of size $M = N/2$. The vector $\boldsymbol{f}$ is split into tow vectors with M components each, $\boldsymbol{f_e v} = [f_0, f_2, ..., f_{N-2}]^T$ and $\boldsymbol{f_o d} = [f_1, f_3, ..., f_{N-1}]^T$ which contain the even and odd components respectively. The DFT of each is then calculated as

$$\boldsymbol{\hat{f}_{ev}} = \left[\hat{f}_{ev,0}, \hat{f}_{ev,2}, ..., \hat{f}_{ev,N-2}\right]^T = \mathbf{F}_M \boldsymbol{f_{ev}} \tag{1.12}$$

and

$$\boldsymbol{\hat{f}_{od}} = \left[\hat{f}_{od,1}, \hat{f}_{od,3}, ..., \hat{f}_{od,N-1}\right]^T = \mathbf{F}_M \boldsymbol{f_{od}} \ . \tag{1.13}$$

The DFT components, i,e. the elements of $\boldsymbol{\hat{f}}$ are then given by

$$\begin{aligned} \hat{f_n} &= \hat{f_{ev,n}} + w_N^n \hat{f_{od,n}} \qquad n = 0, ..., M-1 \\ f_n \hat{+} M &= \hat{f_{ev,n}} - w_N^n \hat{f_{od,n}} \qquad n = 0, ..., M-1 \ . \end{aligned} \tag{1.14}$$

This can be repeated $p-1$ times to give $N/2$ problems each of size 2, vastly reducing the number of calculations required and speeding up computation time. The digital oscilloscope analyses 2048 points of a time-dependent waveform and produces an FFT of that with 1024 points from 0 Hz to the Nyquist frequency. [1], [3]

The effective sample rate, f_{eff}, for the oscilloscope is the reciprocal of the time between samples. There are 2500 samples taken by the oscilloscope, and 10 horizontal divisions on the screen, therefore the value of f_{eff} is

$$f_{eff} = \frac{2500}{10\text{time/div}} = \frac{250}{\text{time/div}} \ . \tag{1.15}$$

The Nyquist frequency is two times the highest frequency present in a periodic time-varying signal. For a signal to be accurately reproduced without distortion it must be sampled at the Nyquist frequency or more often to avoid aliasing. For the oscilloscope the Nyquist frequency f_{Nq} is the same as f_{eff}. [4]

This lab report has many parts to it, divided into two broad sections, Section 2 Part 1: The Digital Oscilloscope which details a number of experiments undertaken to practice using the oscilloscope, and Section 3 Part 2: Fourier Analysis with the Digital Scope which details a number of experiments using the oscilloscope to perform Fourier analysis on a number of waveforms.

In each section I will describe the experimental details of each part of the experiment and then go through the data analysis and results for each section together.

2 Part 1: The Digital Oscilloscope

2.1 Experimental Details

2.1.1 Measurement of the peak-to-peak voltage, period and frequency of a square wave

The probe was attached from channel 1 on the oscilloscope to pin 1 on the Test Waveforms (TW) box. The peak-to-peak voltage, V_{PP}, and period, T, were measured in three ways: direct measurement from the screen, using the measure function on the oscilloscope, and using the cursors on the oscilloscope.

The coupling of the oscilloscope was switched from DC to AC, the effect was observed and then the oscilloscope was switched back to DC coupling.

2.1.2 Measurement of the time delay between two square waves

Another probe was attached from channel 2 on the oscilloscope to pin 2 on the TW box.

The time difference, Δt, between the waves was measured using direct measurement from the screen and using the cursors. This allowed the phase difference, ϕ, to be calculated from this.

2.1.3 Triggering

The probe connected to channel 2 was disconnected and the probe connected to channel 1 was connected to pin 12 on the TW box. V_{PP} was measured using the measure function.

The trigger source was changed and the effects were noted when it was set to channel 1, channel 2, external and AC line.

With the trigger source set to channel 1 the trigger mode was changed from normal to auto and the effects noted.

With the trigger mode set to normal the oscilloscope was set to trigger on the rising slope, with the level set to 0 V. The screen display was saved. The trigger level was set to approximately half the amplitude of the waveform. The screen display was saved. the oscilloscope was set to trigger on the falling slope of the waveform and the screen display was saved.

The trigger level was set to be higher than the maximum voltage and the screen display was saved. The trigger level was set to be lower than the minimum voltage and the screen display was saved.

The probe was then connected to pin 3 on the TW box. The oscilloscope was set to trigger on the rising slope. The trigger holdoff time was adjusted to find the minimum time needed to produce a stable waveform and the screen display was saved. The holdoff time was then increased again to find the maximum holdoff time which would produce a stable waveform and the screen display was saved.

2.1.4 To capture a single event

The probe connected to channel 1 was connected to pin 9 on the TW box.

This pin produces a single pulse when the botton on the TW box is pressed.

The trigger level was set low enough to capture the waveform, and the voltage per division and time per division were set to 600 mV and 10.0 μs respectively. The pulse was captured by pressing the single sequence button on the oscilloscope and then triggering the pulse form the TW box. The data and screen were saved. The pulse height and period were calculated directly from the screen and from the plot of the pulse.

It was attempted to capture the pulse with the trigger level set to 4 V and then at -1 V.

2.1.5 Amplitude modulated waveform

The probe connected to channel 1 was connected to pin 14 on the TW box.

This pin produces an amplitude modulated signal which can be expressed as

$$V(t) = 2A\cos\omega_{mod}t\cos\omega_{av}t \tag{2.1}$$

or

$$V(t) = A\cos\omega_1 t + A\cos\omega_2 t \tag{2.2}$$

where

$$\omega_{av} = \frac{\omega_1 + \omega_2}{2} \tag{2.3}$$

and

$$\omega_{mod} = \frac{\omega_1 - \omega_2}{2} \tag{2.4}$$

For the derivation of this see Appendix A.1.

The trigger level and time per division were adjusted to produce a stable waveform. The screen and data were saved. From the screen the carrier frequency, ω_{av} and modulation frequency ω_{mod} were calculated using the cursors. These values were also calculated by fitting the data to a function in python.

2.2 Results and analysis

2.2.1 Measurement of the peak-to-peak voltage, period and frequency of a square wave

The waveform recorded for this part of the experiment is shown in figure 2.1.

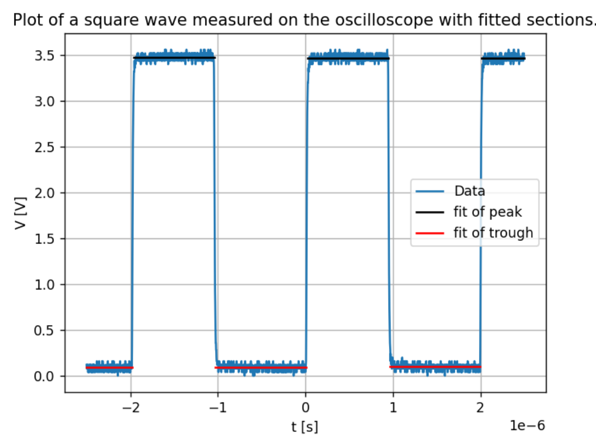


Figure 2.1: The square wave measured on the oscilloscope for part one of the digital oscilloscope experiment.

The peak to peak voltage, V_{PP}, and period, T, of this signal was measured visually from the oscilloscope screen, through the measure function, through the use of the cursors on the oscilloscope and by fitting in python. The frequency, f, was then calculated from these values. These values are tabulated in table 2.1.

The direct measurement is the least precise method. in this method the vertical and horizontal divisions were adjusted so that the waveform took up an integer number of divisions, allowing V_{PP} and T to be found visually through multiplication. This is the least accurate method, and it is impossible to quantify the uncertainty in values generated by this method.

The cursor method involved using the cursors on the oscilloscope and positioning them on the screen so that they intersected the waveform in the desired locations and then finding the difference in their positions either vertically (voltage-wise) or horizontally (time-wise).

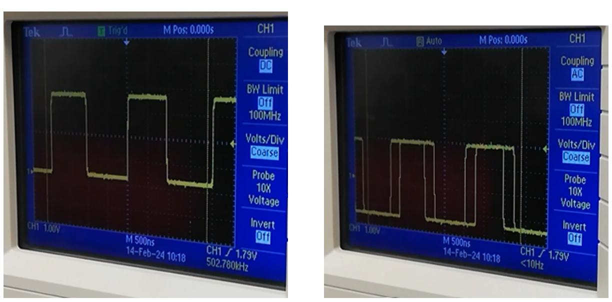
The noise in the data was particularly problematic when measuring V_{PP} directly or using the cursors and likely led to some error in the values calculated.

The python fitting is detailed in Appendix E.

Table 2.1: The different values of V_{PP}, T and f measured by the different methods

Method	V_{PP} [V]	T [μs]	f [kHz]
Direct	3.32	2.0	500
Measure	3.54(4)	1.989(1)	502.8(3)
Cursors	3.34(6)	2.00(6)	500(15)
Python	3.375(3)	1.989(14)	502.8(3)

When the coupling was switched from DC to AC the waveform was seen to shift downwards and oscillate backwards and forwards, this can be seen in figure 2.2.



(a) DC coupling. (b) AC coupling.

Figure 2.2: Photographs of the oscilloscope screen for DC and AC coupling.

2.2.2 Measurement of the time delay between two square waves

The waveform recorded for this part of the experiment is shown in figure 2.3.

It can be seen that the signal from pin 2 (channel 2) lags behind the signal from pin 1 (channel 1) in

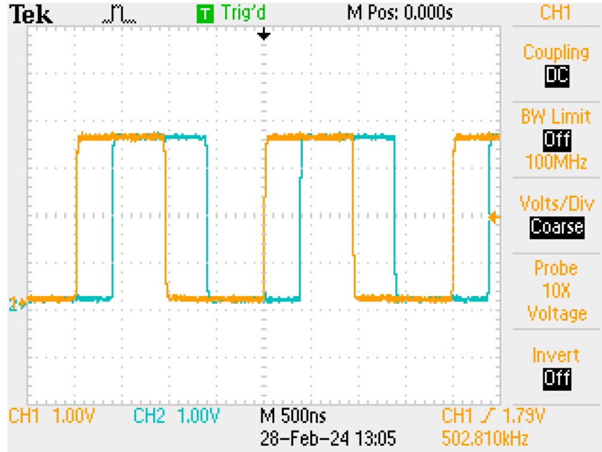


Figure 2.3: The two square waves measured on the oscilloscope for part two of the digital oscilloscope experiment.

this graph.

The time difference, Δt, and hence the phase difference, ϕ, was calculated directly from the screen and through the use of the cursors on the screen. The values are tabulated in table 2.2.

The phase difference is given by

$$\phi = 2\pi f \Delta t \tag{2.5}$$

and is shown in table 2.2 also.

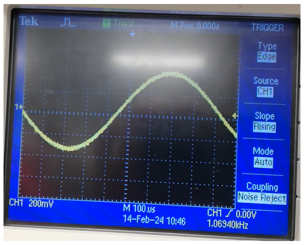
Table 2.2: The different values of Δt and ϕ measured by the different methods.

Method	ΔT [μs]	ϕ
Direct	0.375	1.18
Cursors	0.378 ± 0.002	1.2 ± 0.2

2.2.3 Triggering

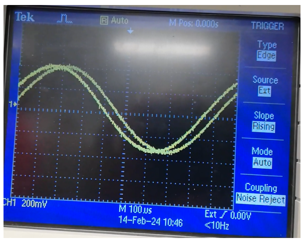
From the measure function V_{PP} was found to be 760 mV.

The display when the triggering source was changed can be seen in figure 2.4. The display was found to be stable when the trigger source was set to channel 1. When set to channel 2 it was seen to move rapidly and jump at random intervals. It often appeared as two slightly out of phase waveforms, however whenever the screen was paused only one was seen. This held for the external and AC line sources as well. External triggering was observed to be almost identical to channel 2 triggering. AC line triggering was almost a more extreme version of channel 2/external triggering, what appeared to be many waveforms were seen at once, they also moved and jumped rapidly and at random.

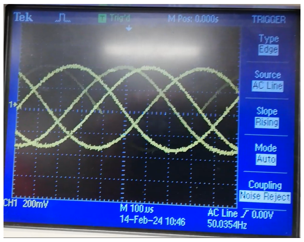


(a) Trigger source: ch 1.

(b) Trigger source: ch 2.



(c) Trigger source: external.



(d) Trigger source: AC line.

Figure 2.4: The oscilloscope display for different trigger sources with the oscilloscope running and not paused, with the trigger mode set to auto.

When the mode was changed from auto to normal there was no effect on channel 1 or AC line, while channel 2 and external stopped triggering altogether.

The effect of changing the trigger level and the slope from rising to falling is shown in figure 2.5.

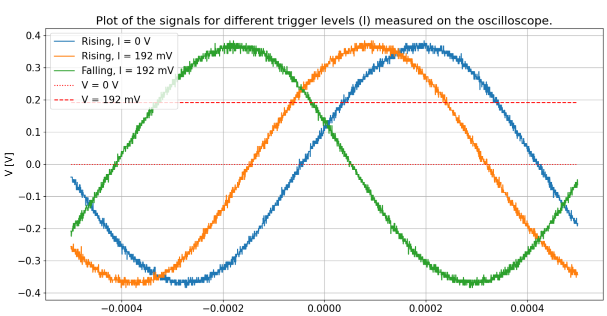
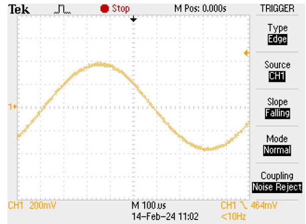
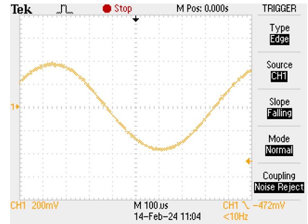


Figure 2.5: The signals measured on the oscilloscope with triggering on the rising and falling slopes and the trigger level set to different voltages.

When the trigger level was set to higher than the maximum or lower than the minimum voltage of the signal there was no signal recorded on the oscilloscope. The oscilloscope still displayed a waveform on the screen, however it was faded, and was simply the last signal which had been recorded, the screens can be seen in figure 2.6 below.



(a) Trigger level greater than maximum voltage.



(b) Trigger level lower than minimum voltage.

Figure 2.6: The oscilloscope display for the trigger levels outside the range of the signal.

When connected to pin 3 on the TW box it was found that the minimum holdoff time required to produce a stable waveform was 19.95(3) μ s and the maximum allowed holdoff time was 29.77(3) μ s. The plot produced can be seen in figure 2.7. The minimum holdoff time is the point at which the final voltage maximum begins, while the maximum holdoff time is the point just before the first voltage maximum in the next cycle begins.

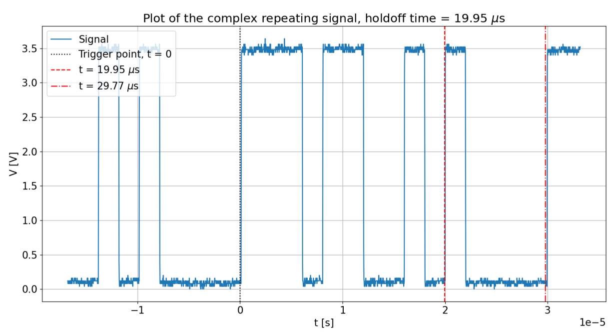


Figure 2.7: The complex repeating signal with maximum and minimum holdoff times marked on the plot.

If it was to trigger before the minimum it would be triggered by other rising edges of the signal occurring before this, including that of the final voltage maximum, and lead to a chaotic appearing waveform. If it was triggered after the first voltage maximum in the next cycle that would cause it to be triggered by rising edges in the middle of the waveform, again leading to a chaotic seeming waveform.

2.2.4 To capture a single event

The pulse was captured and can be seen in figure 2.8.

The pulse width and height were calculated from the screen directly, and by analysing the data in python.

These values are equal to or approximately equal

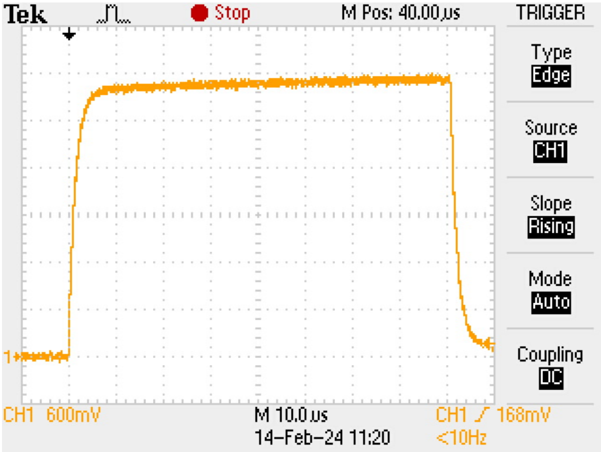


Figure 2.8: The single pulse.

Table 2.3: The different values of pulse width (ΔT) and pulse height (ΔV) calculated for the single pulse.

Method	ΔT [μs]	ΔV [V]
Direct	80	3.6
Python	81.24	3.6

to the values given in the lab handout, so are likely accurate.

2.2.5 Amplitude modulated waveform

The amplitude modulated waveform is described by

$$\begin{aligned} V(t) &= 2A\cos\omega_{mod}t\cos\omega_{av}t \\ &= A\cos\omega_1 t + A\cos\omega_2 t \end{aligned} \tag{2.6}$$

where

$$\omega_{mod} = \frac{\omega_1 + \omega_2}{2} \tag{2.7}$$

and

$$\omega_{av} = \frac{\omega_1 - \omega_2}{2}\ . \tag{2.8}$$

For the derivation of this see Appendix A.1.

The amplitude modulated waveform shown in figure 2.9 was analysed using python, a two functions, one of which was the sum of two cosine functions, the other a product of two cosine functions were fit to the data and the associated frequencies were extracted. The graph with the fitted functions is shown in figure 2.10

The values of ω_{mod}, ω_{av}, ω_1 and ω_2 are shown in table 2.4

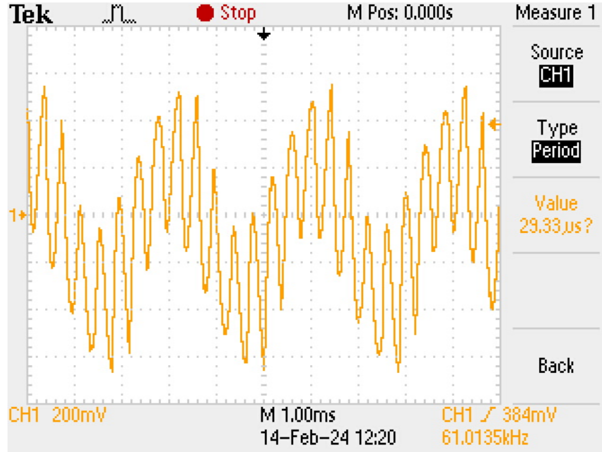


Figure 2.9: The amplitude modulated waveform.

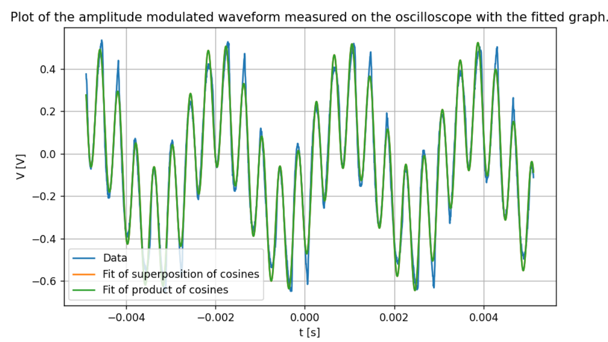


Figure 2.10: The amplitude modulated waveform with the functions fit to it, only one of the fits is visible as they are equivalent and produce identical graphs and so overlap perfectly.

Table 2.4: The values of ω_{mod}, ω_{av}, ω_1 and ω_2.

	[Hz]
ω_{av}	8907.2(1.1)
ω_{mod}	6720.8(1.1)
ω_1	15627.9(1.6)
ω_2	2186.4(1.6)

3 Part 2: Fourier Analysis with the Digital Scope

3.1 Experimental Details

3.1.1 FFT of a Sinewave

The oscillator was connected to channel one of the oscilloscope, with DC coupling selected. The FFT function was chosen from the math menu on the oscilloscope and the window was set to Hanning. The oscillator was set to output a sinewave with

$V_{PP} = 4.0$ V and frequency 10.01 kHz.

The values of time/division, frequency span, Nyquist frequency, peak frequency and peak magnitude were recorded for a number of sampling rates.

The peak magnitude dBV is related to the root mean square value of the voltage by

$$dBV = 20 \log V_{rms} \ . \tag{3.1}$$

3.1.2 Aliasing

The oscillator was set to produce a sinewave with frequency 10.00 kHz and peak to peak amplitude of 4.0 V. The sample rate of the FFT was set to 50 kS/s and the window was set to flattop. The peak frequency f_{pk} was measured.

The oscillator frequency was increased until the peak reached the right hand side of the screen and f_{pk} was measured again, as was the frequency being produced by the oscillator f_o. As the oscillator frequency was increased the behaviour of the display was observed. Once the peak had reached 2 divisions from the right hand side of the screen f_{pk} was measured again, as was f_o.

The frequency was increased again and when the peak was two divisions from the left hand side f_{pk} and f_o were both measured.

The sampling rate was increased to 100 kS/s and f_{pk} and f_o were both measured.

3.1.3 The dependence of peak width on sampling rate and window function

The oscillator was set to produce a sinewave with frequency 10.00 kHz and peak to peak amplitude of 4.0 V. The sample rate of the FFT was set to 50 kS/s. The frequency spectrum was recorded for each of the window modes, flattop, Hanning and rectangular, from this the peak frequency and amplitude were recorded.

The FWHM of the peak Δf was determined for the flattop window for sample rates between 50 kS/s and 500 kS/s. This was plotted against f_{eff}.

3.1.4 Frequency spectrum of a square wave

The oscillator was set to produce a square wave with frequency 10.05 kHz and peak to peak amplitude of 4.0 V. The frequency spectrum was recorded at sample rates of 1 MS/s and at 10 MS/s.

The frequency and magnitude of the first five peaks visible for a sample rate of 1 MS/s were recorded using the cursors on the oscilloscope.

3.1.5 Frequency spectrum of an isolated pulse

The same single pulse was recorded as in section 2.1.4. Terminal 9 on the TW box was connected to channel 1 on the oscilloscope, and the frequency spectrum of the single pulse was recorded and saved, in the same manner that the single pulse was recorded previously, with the window set to rectangular.

The frequencies of the first three minima and the amplitudes of the first three maxima were measured and recorded.

3.1.6 Frequency spectrum of a repetitive pulse

A 10 V DC supply and an oscillator set to produce a square wave of frequency 20 kHz and peak to peak amplitude of 4.0 V were connected to the repetitive pulse unit, which was then connected to channel 1 on the oscilloscope.

The effect of changing the oscillator frequency on the waveform was observed.

The oscillator frequency was reset to 20.16 kHz and the period T and pulse width τ were recorded.

The frequency spectrum for a sample rate of 10 MS/s was recorded. The frequency and amplitude of the first three peaks was measured, as was the frequency of the first three minima of the spectrum.

The frequency spectrum for a sample rate of 2.5 MS/s was recorded. The frequency of the first four peaks was measured. It was attempted to measure the frequency of the first three minima of the envelope curve of the peaks, however no envelope curve was found so this was not possible. However the magnitude of the first three maxima of this envelope curve were approximated by the magnitude of the peaks of the FFT spectrum. The envelope curve would be a smooth curve across the top of the spectrum signal, however I was unable to calculate it.

The sampling rate was set to 1 MS/s and the frequency of the oscillator was decreased to 14.99 kHz and the spectrum was saved.

3.2 Results and analysis

3.2.1 FFT of a Sinewave

Table 3.1 shows the values of time/division, frequency span, Nyquist frequency, peak frequency, peak magnitude and the root mean square of the signal which were recorded for a number of sampling rates.

From direct measurement of the waveform the value of V_{rms} is 1.47(4) V. The majority of peaks which are not at 0 Hz agree with this value.

One of the oscilloscope displays is shown in figure 3.1. The peak at 10 kHz is easily and clearly visible.

Table 3.1: The values of time/division t_{div}, frequency span f_{span}, Nyquist frequency f_{Nq}, peak frequency f_{pk}, peak magnitude DBV and the RMS value of the peak V_{rms} for a number of sampling rates R_S measured in samples per second S/s.

R_S [MS/s]	t_{div} [μs]	f_{span} [MHz]	f_{Nq} [MHz]	f_{pk} [kHz]	dBV	V_{rms} [V]
1000	0.250	2-500	10^9	0	-11.0(2)	0.282(6)
500	0.500	1-250	500	0	-11.0(2)	0.282(6)
250	1	0.5-125	250	0	-11.0(2)	0.282(6)
100	2.5	0.2-50.00	100	0	-5.97(4)	0.503(2)
50	5.0	0.1-25.00	50	0	-1.37(4)	0.854(4)
25	10.0	0.05-12.50	25	0	3.03(4)	1.42(7)
10	25.0	0.02-5	10	0	3.43(4)	1.48(7)
5	50.0	0.01-2.5	5	10	3.43(4)	1.48(7)
2.5	100	0.005-1.250	2.5	10	3.03(8)	1.417(13)
1.0	250	0.002-0.5	1	10	3.03(8)	1.417(13)
0.5	500	0.001-0.25	0.5	10	2.23(8)	1.293(12)
0.25	1000	0.0005-0.125	0.25	10	3.43(8)	1.484(14)

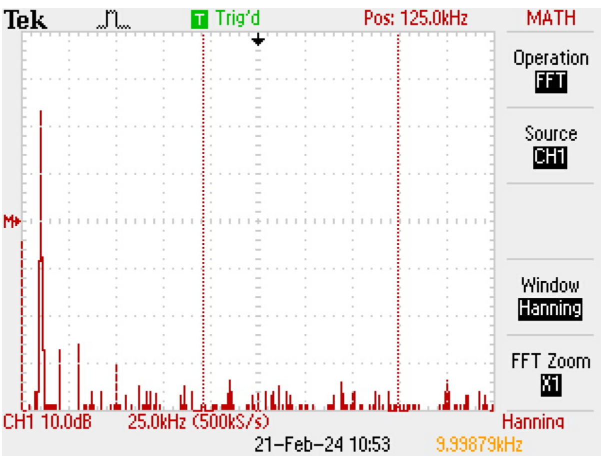


Figure 3.1: The oscilloscope display for $R_S = 500$ kS/s.

3.2.2 Aliasing

The values of f_{pk} and f_o for the different recording times are shown in table 3.2.

As can be seen, once the frequency increases beyond $f_{eff}/2 = 25$ kHz the peak displayed by the FFT is not the same as the actual peak frequency produced by the oscillator.

As the frequency of the oscillator was increased beyond $f_{eff}/2$ the peak was seen to move towards the left of the screen from the right, essentially appearing to decrease in frequency.

When R_S was increased to 100 kS/s, meaning $f_{eff}/2 = 50$ kHz, the value of f_{pk} once again agreed with f_o as the waveform was now being adequately sampled.

3.2.3 The dependence of peak width on sampling rate and window function

The spectra for the three different window functions are shown in figure 3.2.

From these spectra the peak frequency and amplitude were measured and their values are shown in table 3.3.

The full width at half maximum values, Δf for the peaks at the different sample rates were calculated, by assuming the peaks were Gaussian in nature and then fitting the data to a Gaussian graph, whose values of Δf are given by

$$\Delta f = 2\sqrt{2\ln 2}\sigma \tag{3.2}$$

where σ is the standard deviation of the Gaussian. These values are shown in table 3.4 with the associated values of f_{eff}. Figure 3.3 shows the relationship between Δf and f_{eff} with a linear graph fit to it.

There are only a limited number of data points, however it appears from this that the full width at half maximum for the peak in the FFT of a signal with one frequency increases more or less linearly with the sampling rate.

3.2.4 Frequency spectrum of a square wave

The spectra recorded for sample rates of 1 MS/s and 10 MS/s are shown in figure 3.4

There are no harmonics clearly visible in the frequency spectrum for $R_S = 1$ MS/s. The two spectra are clearly different, with distinct peaks visible for $R_S = 1$ MS/s but none clearly visible for $R_S = 10$ MS/s. This is due simply to the frequency resolution

Table 3.2: The values of f_{pk} and f_o for the different recording scenarios.

Scenario	f_{pk} [kHz] ($\Delta = 0.1$ kHz)	f_o [kHz] ($\Delta = 0.01$ kHz)
Initial recording	10.0	10.00
Peak at right of screen	25.2	25.19
2 divisions from right	20.2	30.36
2 divisions from left	5.17	45.34
$R_S = 100$ kS/s	45.3	45.35

Table 3.3: The peak frequencies f_{pk} and rms values of the peak for the different frequency spectra.

Spectrum	f_{pk} [kHz]	V_{rms}
Flattop	9.99(2)	1.48(7)
Hanning	10.01(2)	1.29(6)
Rectangular	10.01(2)	1.07(5)

Table 3.4: The values of Δf and f_{eff} for different sampling rates, R_S.

R_S [kS/s]	Δf	f_{eff} [kHz]
50	174(11)	50
100	320(12)	100
250	710(60)	250
500	1710(70)	500

of the oscilloscope in the FFT mode. The frequency f and magnitude A of the first five peaks are shown in table 3.2.

Table 3.5: The values of f and A in both decibels and volts for the first five peaks in the square wave FFT.

Peak	f [kHz]	A [dB]	A [V]
1	10.0(2)	5.43(4)	1.869(9)
2	30.0(2)	-4.17(4)	0.619(3)
3	50.2(2)	-8.57(4)	0.373(17)
4	70.4(2)	-11.8(1)	0.257(3)
5	90.6(2)	-13.7(1)	0.207(2)

When the Fourier series for this square wave was calculated manually (see Appendix B for the calculation) the function was found to equal

$$f(t) = \frac{4A}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n} \sin \omega n x \tag{3.3}$$

and the following values for the amplitude A and frequency f were found, after taking $\omega = 10.05$ kHz and $\frac{4A}{\pi} = 1.869$ V, as shown in table 3.6.

These values are clearly very close to the values which were measured from the frequency spectrum in figure 3.4a.

Table 3.6: The calculated values of f and A for the square wave signal.

n	f [kHz]	A [V]
1	10.05	1.869
3	30.15	0.623
5	50.25	0.3738
7	70.35	0.267
9	90.45	0.2077

When the sampling frequency was decreased to 500 kS/s the graph in figure 3.5. Aliasing can be seen in the lower amplitude peaks which can be seen between the main peaks peaks in the spectrum, it is expected for harmonics with frequencies ≥ 250 kHz, i.e. $n \geq 25$, or the 13$^{\text{th}}$ or greater harmonic.

3.2.5 Frequency spectrum of an isolated pulse

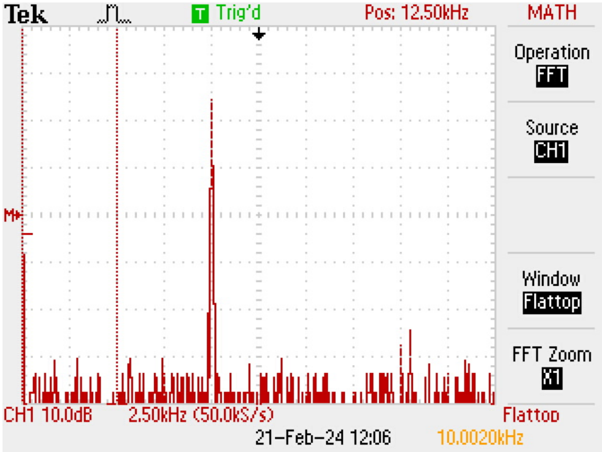
The frequency spectrum of this pulse is shown in figure 3.6. From this the frequencies of the first three minima as well as the amplitudes of the first three maxima were found, their values are shown in tables 3.7 and 3.8.

When the Fourier transform of the single pulse is found the expected amplitudes of the maxima and expected frequencies of the minima can be found, these values are shown in the respective tables also (see Appendix C for the calculation).

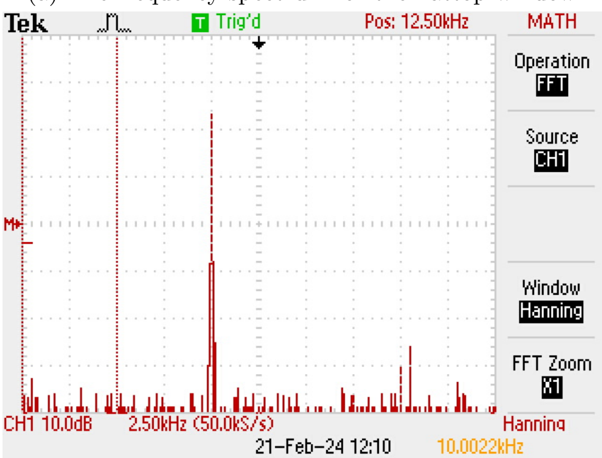
Table 3.7: The frequencies of the first three minima, f, for the single pulse, and their expected values.

Minimum	f [kHz]	Expected f [kHz]
1	12.8(3)	112
2	25.2(7)	273
3	37.7(9)	431

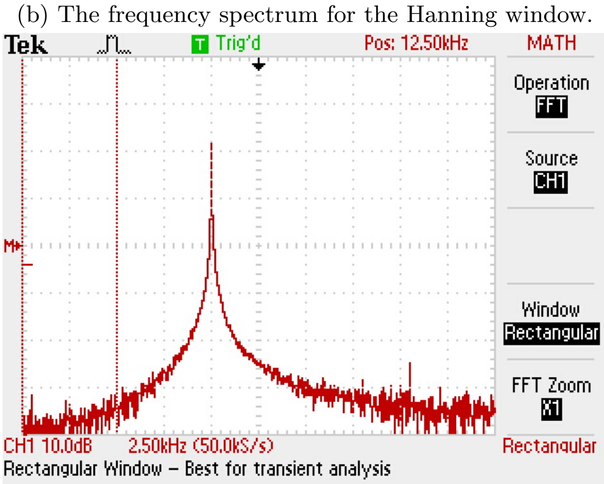
None of these values are close to the expected values. Either there is a mistake which I cannot find in my calculation of $\tilde{f}(\omega)$ or there was a mistake when measuring the values from the oscilloscope, however



(a) The frequency spectrum for the flattop window.



(b) The frequency spectrum for the Hanning window.



(c) The frequency spectrum for the rectangular window.

Figure 3.2: The frequency spectra for the three different window functions.

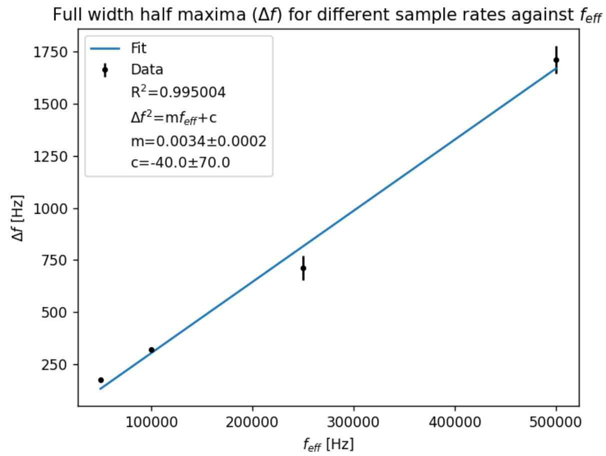


Figure 3.3: The graph of Δf against f_{eff} with a linear graph fit to the data.

this does not appear likely given the spectrum which was produced, and is shown in figure 3.6.

Table 3.8: The amplitudes of the first three maxima, A, for the single pulse.

Maximum	A [dBV]	A [V]	Expected A [V]
1	-33.9(1)	0.0202(2)	1.24×10^{-4}
2	-46.3(1)	0.00484(6)	1.60×10^{-5}
3	-49.1(1)	0.00351(4)	8.83×10^{-6}

3.2.6 Frequency spectrum of a repetitive pulse

The waveform which is being investigated is shown in figure 3.7, and its FFT frequency spectrum is shown in figure 3.8.

It was found that increasing the oscillator frequency caused the pulse width and frequency to decrease.

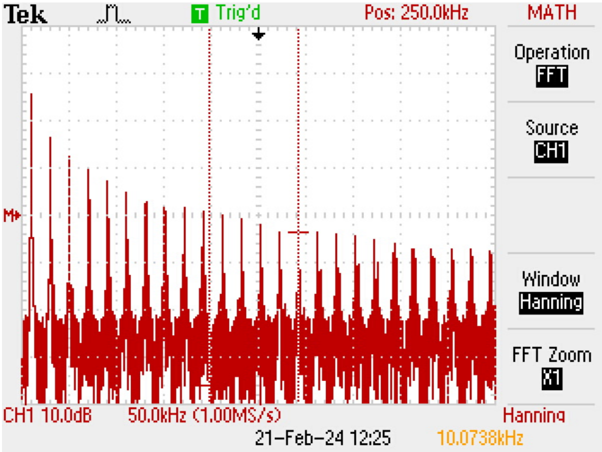
With the oscillator set to 20.16 kHz T was found to be 49.66(6) μs, which is in agreement with the expected period of $T = 49.60$ μs, and the pulse width τ was found to be 24.8(8).

From the frequency spectrum in figure 3.8 the position of the minima and amplitudes of the maxima were found and are shown in tables 3.9 and 3.10.

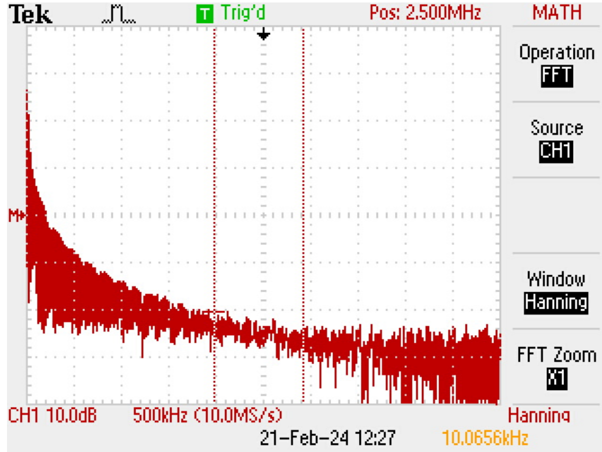
Table 3.9: The frequencies f of the first three minima, for the repetitive pulse for $R_S = 10$ MS/s.

Maximum	f [kHz]	
0	0 1	38(2)
2	78(2)	
3	116(2)	

The ratio of the amplitudes of maxima 1 and 2 is 3.16 and the ratio of the amplitudes of maxima 1 and 3 is 5.25. This suggests that again, in this case



(a) Frequency spectrum for 1 MS/s.



(b) Frequency spectrum for 10 MS/s.

Figure 3.4: The frequency spectra of the square wave for the different sample rates.

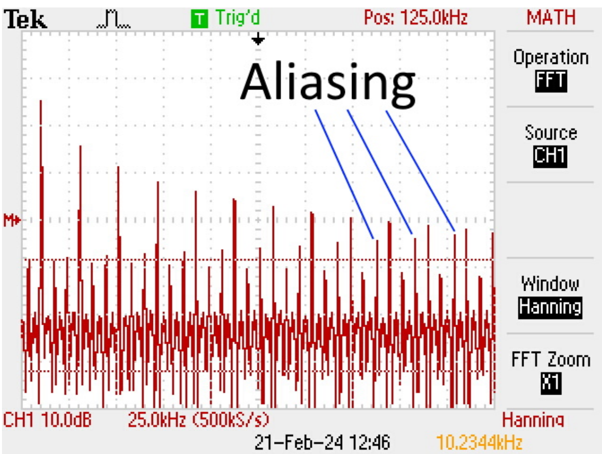


Figure 3.5: The frequency spectrum of the square wave for a sampling rate of 500 kS/s with some of the cases of aliasing indicated.

Table 3.10: The amplitudes A of the first three maxima, for the repetitive pulse for $R_S = 10$ MS/s.

Maximum	A [dBV]	A [V]
1	0.21(8)	1.024(9)
2	-9.79(8)	0.324(3)
3	-14.2(8)	0.195(18)

it is the odd harmonics which are involved in the construction of this signal.

The frequency spectrum when the sampling rate is set to 2.5 MS/s is shown in figure 3.9. The sharp peaks are visible here when they were not previously due to an increased sampling resolution in this spectrum. The frequencies of the first four of these max-

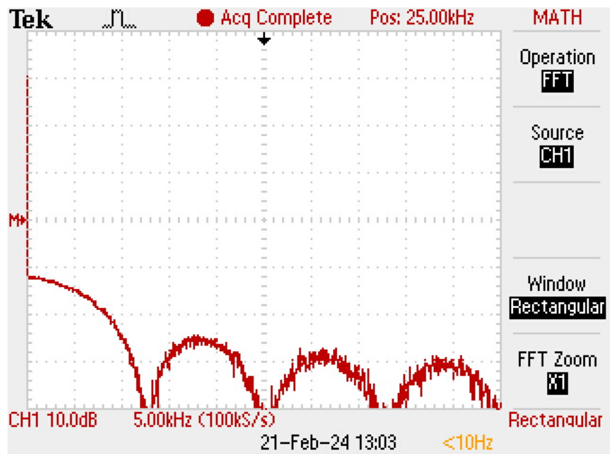


Figure 3.6: The frequency spectrum of the single pulse.

ima are shown in table 3.11

Table 3.11: The frequencies f of the first four peaks, for the repetitive pulse for $R_S = 2.5$ MS/s.

Peak	f [kHz]
1	19(1)
2	59(1)
3	98(1)
4	139(8)

These again appear to be odd-numbered harmonics, i.e. the first, third, fifth and seventh harmonics. T was found to be 49.66(6) μs, and the frequencies of the harmonics appear to be given by $f = n/T$ where n is an odd natural number.

The amplitudes of the approximated maxima of

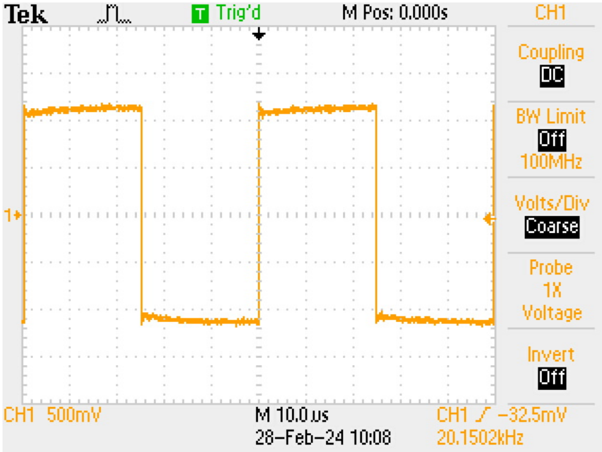


Figure 3.7: The repetitive pulse.

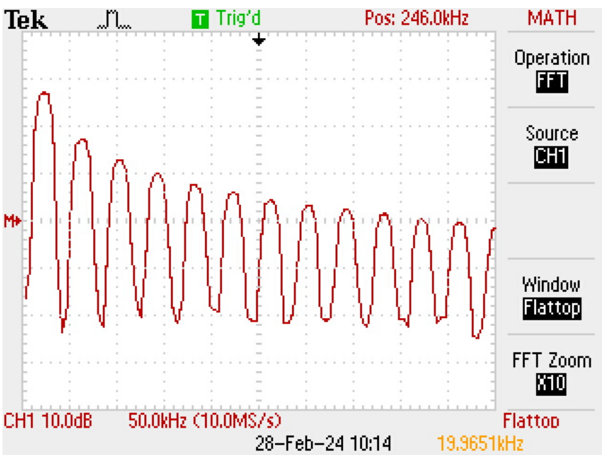


Figure 3.8: The frequency spectrum of the repetitive pulse for $R_S = 10$ MS/s.

the envelope curve are shown in table 3.12. They are seen to be the same as obtained for a sampling rate of 10 MS/s, and to be significantly larger than the values obtained for the single pulse.

Table 3.12: The amplitudes A of the first three maxima, for the repetitive pulse for $R_S = 2.5$ MS/s.

Maximum	A [dBV]	A [V]
1	0.21(4)	1.024(5)
2	-9.79(4)	0.3240(15)
3	-14.2(1)	0.195(2)

The FFT frequency spectra for $R_S = 1$ M S/s are shown in figure 3.10 for oscillator frequencies of 21.3 kHz and 14.99 kHz respectively.

I could not find the envelope function yet again, and as such I cannot say for certain, however it does

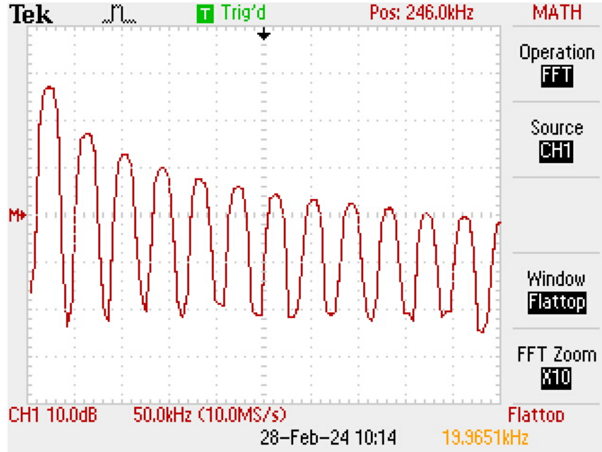


Figure 3.9: The frequency spectrum of the repetitive pulse for $R_S = 2.5$ MS/s.

appear that the positions of the minima do change as the frequency is changed. It makes sense that the envelope curve would be frequency dependent.

The frequency spectrum of a repetitive pulse appears (visually at least) to be a combination of the spectrum of an isolated pulse and the spectrum of a square wave.

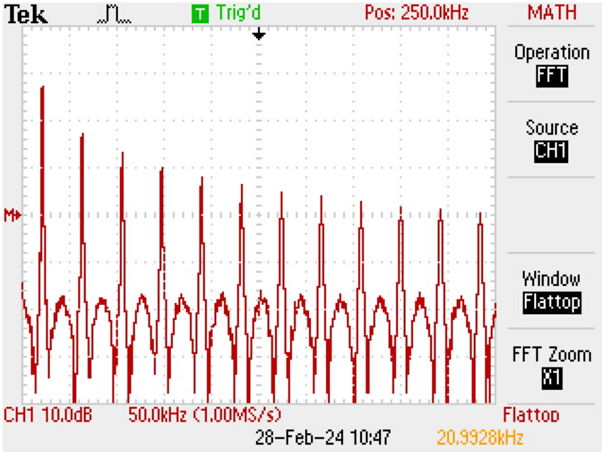
4 Conclusions

The peak-to-peak voltage and frequency of a square wave can be found in a number of different ways using an oscilloscope, as can the measurement of the phase shift between two signals. The triggering mode on the oscilloscope can have a very important impact on the signal which is produced and shown on the oscilloscope, changing the apparent phase shift, and/or dictating if the signal is visible at all. Oscilloscopes can also be used to analyse complex repeating signals, as well as singular pulses.

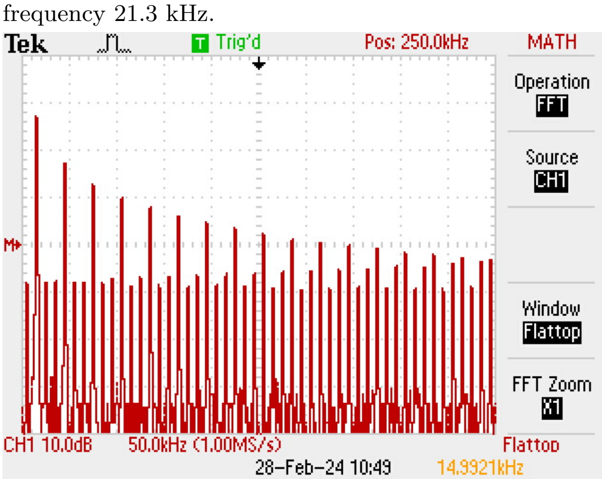
The FFT of a sinewave was measured and found to be simply a singular peak.

Aliasing can seriously affect the signal being analysed and cause a signal with a frequency which appears to significantly lower than it is to be produced, and it must therefore either be corrected to find its actual frequency, or ignored altogether.

It was found that there is an almost linear relationship between the full width half maximum of a frequency peak and the sampling frequency, when analysing a signal with only a single frequency component, further investigation of other signals will determine if this is the case for signals of multiple frequencies.



(a) Frequency spectrum for R_S =1 MS/s and oscillator frequency 21.3 kHz.



(b) Frequency spectrum for R_S =1 MS/s and oscillator frequency 14.99 kHz.

Figure 3.10: The frequency spectra of the repetitive pulse for the different frequencies.

It was found that the frequency spectrum of a square wave is made up of odd harmonics, the frequencies and amplitudes of which agree well with the theoretical values calculated from the Fourier series of the square wave.

The same cannot be said for the FFT spectrum of a singular pulse, the measured and theoretical values do not agree in the slightest, for some undetermined reason.

The FFT spectrum of a repeated pulse has a shape which appears to be a mix between that of a singular pulse and that of a square wave, it has the distinctive strong odd harmonic peaks of the square wave as well as the distinctive shape of the singular pulse. It was not possible to analyse these spectra fully as the envelope curve was not producible.

Further investigation should be made into the FFT of the singular and repeated pulses to determine why the singular pulse FFT does not behave as expected, and to determine the envelope curve of the repeated pulse FFT.

A Derivations of requested equations

A.1 Amplitude modulated waveform

The voltage is represented by

$$V(t) = A_{mod} \cos \omega_{av} t \tag{A.1}$$

where

$$A_{mod} = 2A \cos \omega_{mod} t \tag{A.2}$$

therefore

$$V(t) = 2A \cos \omega_{mod} t \cos \omega_{av} t \ . \tag{A.3}$$

By trigonometry,

$$V(t) = A \left(\cos \left(\left(\omega_{mod} + \omega_{av} \right) t \right) + \cos \left(\left(\omega_{mod} - \omega_{av} \right) t \right) \right) \ . \tag{A.4}$$

Let

$$\omega_1 = \omega_{mod} + \omega_{av} \tag{A.5}$$

and

$$\omega_2 = \omega_{mod} - \omega_{av} \ . \tag{A.6}$$

Therefore,

$$V(t) = A \cos \omega_1 t + A \cos \omega_2 t \tag{A.7}$$

and

$$\omega_{mod} = \frac{\omega_1 + \omega_2}{2} \tag{A.8}$$

$$\omega_{av} = \frac{\omega_1 - \omega_2}{2} \ . \tag{A.9}$$

B Fourier series calculation

Taking the square wave as the function

$$f(t) = \begin{cases} -A & \text{if } -\frac{1}{T} < x \leq 0 \\ A & \text{if } 0 < x \leq \frac{1}{T} \end{cases} \tag{B.1}$$

and $f(t+T) = f(t)$ where A is the amplitude of the wave.

As this is an odd function for all n, $a_n = 0$. Thus, equation 1.4 will be used to find all the coefficients of the Fourier series in equation 1.1. Also, the relation $\omega = 2\pi/T$ will be used to all for cancellation of some terms

$$\begin{aligned}
b_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\frac{2\pi nt}{T}dt \\
&= -\frac{2}{T}\int_{-T/2}^{0} A\sin\frac{2\pi nt}{T}dt + \frac{2}{T}\int_{0}^{T/2} A\sin\frac{2\pi nt}{T}dt \\
&= -\frac{2}{T}\frac{AT}{2\pi n}\left[-\cos\frac{2\pi nt}{T}\right]_{-T/2}^{0} + \frac{2}{T}\frac{AT}{2\pi n}\left[-\cos\frac{2\pi nt}{T}\right]_{0}^{T/2} \\
&= -\frac{A}{\pi n}\left[-\cos\frac{2\pi nt}{T}\right]_{-T/2}^{0} + \frac{A}{\pi n}\left[-\cos\frac{2\pi nt}{T}\right]_{0}^{T/2} \\
&= -\frac{A}{\pi n}\left[-\cos 0 + \cos \pi n\right] + \frac{A}{\pi n}\left[-\cos \pi n + \cos 0\right] \\
&= \frac{A}{\pi n}\left[+\cos 0 - \cos \pi n - \cos \pi n + \cos 0\right] \\
&= \frac{A}{\pi n}\left[1 - (-1)^n - (-1)^n + 1\right] \\
&= \frac{2A}{\pi n}\left[1 - (-1)^n\right]
\end{aligned} \tag{B.2}$$

so

$$b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4A}{\pi n} & \text{if } n \text{ is odd} \end{cases} \tag{B.3}$$

and the function is given by

$$f(t) = \frac{4A}{\pi}\sum_{n=1,3,5,...}^{\infty}\frac{1}{n}\sin\omega nx\ . \tag{B.4}$$

C Fourier transformation calculation

Taking the single pulse as the function

$$f(t) = \begin{cases} 3.6 & \text{if } -40\times 10^{-6} < x \leq 40\times 10^{-6} \\ 0 & \text{otherwise} \end{cases} \tag{C.1}$$

using equation 1.5 the Fourier transformation of this is found as below

$$\begin{aligned}
\tilde{f}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \\
&= \frac{3.6}{\sqrt{2\pi}}\int_{-40\times10^{-6}}^{40\times10^{-6}} e^{-i\omega t}dt \\
&= \frac{3.6}{\sqrt{2\pi}}\frac{i}{\omega}\left[e^{-i\omega t}\right]_{-40\times10^{-6}}^{40\times10^{-6}} \\
&= \frac{3.6}{\sqrt{2\pi}}\frac{i}{\omega}\left[\cos\omega t - i\sin\omega t - \cos\omega t - i\sin\omega t\right]_{-40\times10^{-6}}^{40\times10^{-6}} \\
&= \frac{7.8}{\sqrt{2\pi}}\frac{1}{\omega}\sin\left(40\times10^{-6}\omega\right)
\end{aligned} \tag{C.2}$$

D Propagation of Uncertainty

Here, I will be using δ to denote uncertainty in the interests of clarity as there are a number of terms elsewhere that use Δ to denote other values.

This shows the equations used to calculate the uncertainty in some of the more complicated cases, i.e. not for simple addition/subtraction, exponentiation, multiplication and division of two values, etc.

These all make use of Gauss's law of error propagation. If there is a function $f(x_1, x_2, ...)$ then the uncertainty in f, δf, is,

$$\delta f = \sqrt{\left(\frac{\partial f}{\partial x_1}\delta x_1\right)^2 + \left(\frac{\partial f}{\partial x_2}\delta x_2\right)^2 + ...} \tag{D.1}$$

where δx_i is the uncertainty in x_i.

E Code

This section contains some examples of the code used in this lab report, however all of the code as well as the original data can be found here:
https://github.com/ismisebrendan/Fourier_Analysis

E.1 Fitting

The data was fit to the functions through the use of the function below.

```
def fitting(p, x, y, func, s=1):
    """
    Fit data to a function.

    Parameters
    ----------
    p : array_like
        Initial guess at the values of the
    coefficients to be fit
```

```
    x : array_like
        The x data to be fit.
    y : array_like
        The y data to be fit.
    func : callable
        The function to fit the data to.
    s : float, default 1
        The standard deviation.

    Returns
    -------
    p_fit : numpy.ndarray
        The array of the coefficients after fitting the function to the data.
    chi : float
        The chi squared value of this fit.
    unc_fit : numpy.ndarray
        The array of the uncertainties in the values of p_fit.
    """
    # Fit the data and find the uncertainties
    r = opt.least_squares(residuals, p, args=(func, x, y, s))
    p_fit = r.x
    hessian = np.dot(r.jac.T, r.jac) # estimate the hessian matrix
    K_fit = np.linalg.inv(hessian) # covariance matrix
    unc_fit = np.sqrt(np.diag(K_fit)) # stdevs

    # rescale
    beta = np.sqrt(np.sum(residuals(p_fit, func, x, y)**2) / (x.size - len(p_fit)))
    unc_fit = unc_fit * beta
    K_fit = K_fit * beta

    # find the chi2 score
    chi = np.sum(residuals(p_fit, func, x, y, beta)**2) / (x.size - len(p_fit))

    return p_fit, chi, unc_fit
```

References

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