Stellar Structure Continuous Assessment 2

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November 8, 2023

1 Gaia

Throughout Gaia's initial mission it astrometrically, photometrically and spectroscopically measured one billion stars, on average seventy times each, allowing for multiple measurements of the magnitudes of different variable stars such as RR Lyrae variables.[1] [2]

RR Lyraes are pulsating A-F giants, with masses of $M \sim 0.7M_{\odot}$, pulsation periods of 0.2-1.2 days, and are older than about 10 Gyr.[3] [4]

Figure 1 shows the light curve of an RR Lyrae star - the periodic changes in magnitude are clear.

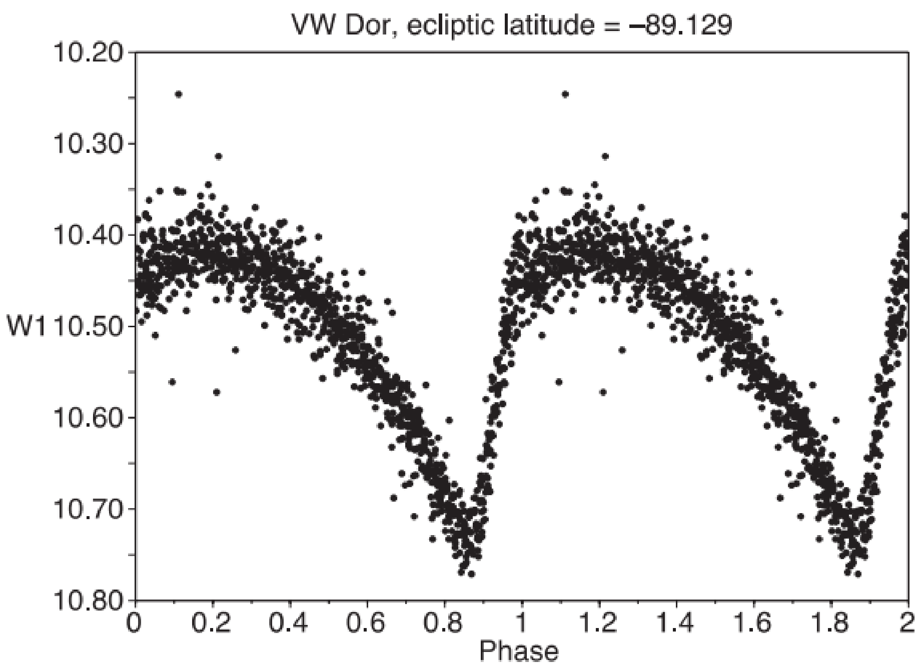

Figure 1: The light curve of the star VW Dor as measured by WISE in the W1 band. (Figure 5 in [4])

Their average magnitudes are related to the fraction of iron present in the stars. This relationship is assumed to be linear in nature, of the form $\langle M(RR)\rangle = a[\mathrm{Fe/H}] + b$. However, the values of a and b are not agreed upon[2] [4] The relation appears to be different for stars with $[\mathrm{Fe/H}] > -1.5$ than for stars with $[\mathrm{Fe/H}] < -1.5$.[5] RR Lyraes often pulsate in the first overtone, so the fundamental frequency must sometimes be found when observing in the K_S band, as at this range the average magnitude depends on the fundamental period (P_F) also.[3] [4]

The plot in figure 2 shows the intrinsic colour index $((\langle V\rangle - \langle K_S\rangle)_0)$ $-2.33 \log P_F$ against [Fe/H] for a sample of RR Lyraes where [4]

$$
\begin{aligned}
(\langle V\rangle - \langle K_S\rangle)_0 &= \langle M_V\rangle - \langle M_{K_S}\rangle \\
&= (a_v - a_k) + (b_v - b_k)[Fe/H] + 2.33 \log P_F \\
&= (1.863 \pm 0.024) + (0.159 \pm 0.013)[Fe/H] + 2.33 log P_F
\end{aligned} \tag{1}
$$

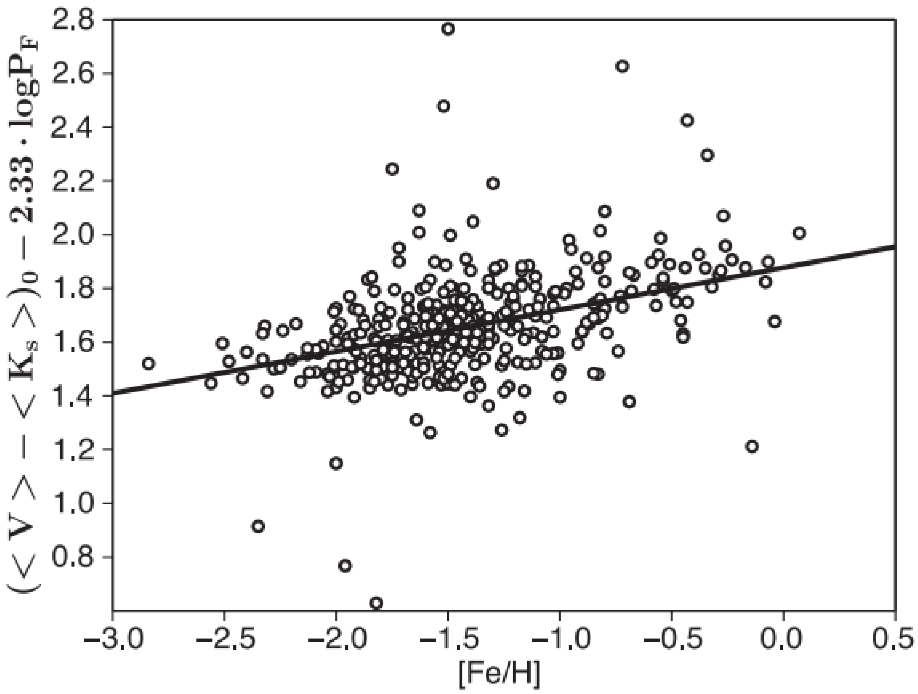

Figure 2: A plot of the intrinsic colour index against [Fe/H] for a sample of 403 RR Lyrae stars in the Galaxy. (Figure 9 in [4])

RR Lyrae stars are found in older populations of stars such as globular clusters, elliptical galaxies and the halos and thick discs of spiral galaxies. This, along with their high luminosities of $L = 40 - 50 L_{\odot}$ allows them to be used as standard candles for inter- and extra-galactic distance measurement.[4]

The uncertainty of the $\langle M_V(RR)\rangle$-[Fe/H] relationship causes difficulties in measuring distances using these stars as standard candles, so more accurate data of the distances to them, which can be found by Gaia, and their spectra, would allow fine-tuning of the $\langle M_V(RR)\rangle$-[Fe/H] relationship.

Better modelling of the $\langle M_V(RR)\rangle$-[Fe/H] relationship is also necessary to fine-tune the distances measured using RR Lyraes, and therefore improve the accuracy of the cosmic distance ladder.

2 Ionisation fraction in stars

2.1 Partition function

The partition function, Z, is given by

$$Z = \sum_{n=1}^{\infty} g_n e^{-(E_n - E_1)/kT} \tag{2}$$

Where g_n is the degeneracy of the nth energy level ($g_n = 2n^2$ in H I), E_n is the energy of the nth energy level, k is the Boltzmann constant and T is the temperature. The first three terms of the partition function for H I are

$$
\begin{aligned}
Z_{H_I} =& g_1 e^{-(E_1-E_1)/kT} + g_2 e^{-(E_2-E_1)/kT} + g_3 e^{-(E_3-E_1)/kT} \\
=& 2 + 8e^{-51eV/5kT} + 18e^{-544eV/45kT}
\end{aligned} \tag{3}
$$

Where E_n is given by

$$
E_n = -\frac{13.60eV}{n^2} \tag{4}
$$

2.2 Ionisation fraction of hydrogen for different temperatures

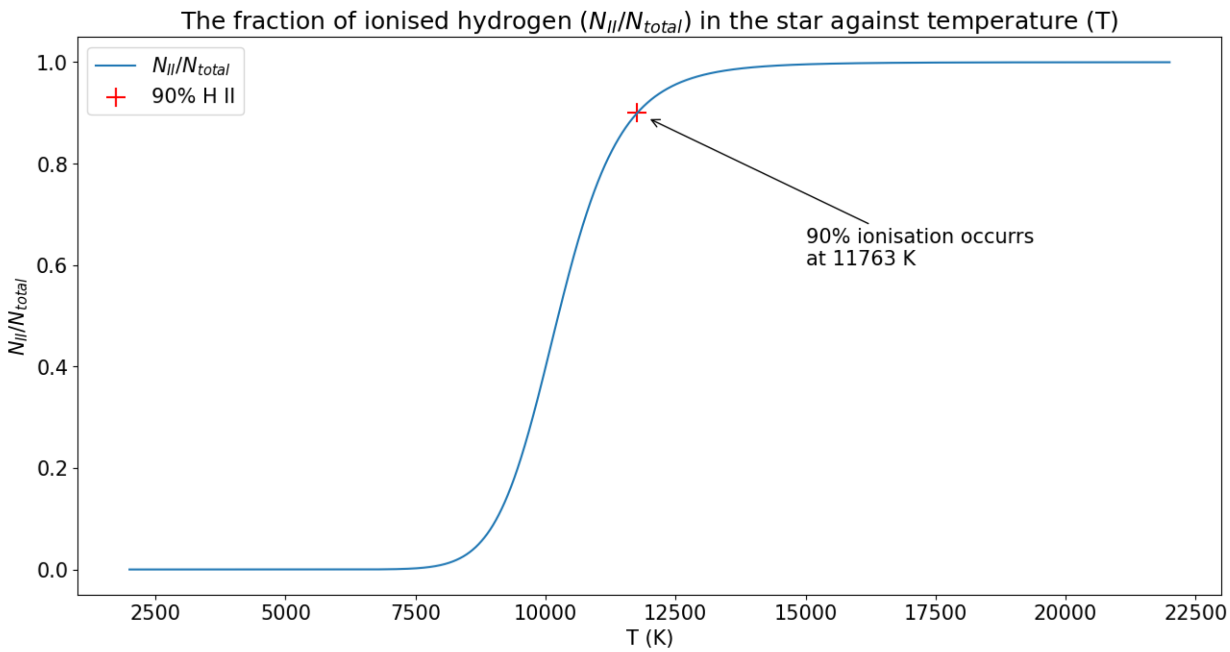

Figure 3: The ionisation fraction of hydrogen for a star with $n_e = 51 \times 10^{19}$ m^{-3} for the temperature range 2000 to 22000 K. The point at which 90% of the hydrogen is ionised is also shown and occurs when the temperature is approximately 11763 K according to python.

The hydrogen in a star is either H I, neutral hydrogen, or H II, singly ionised hydrogen as hydrogen has only one electron. The fraction of H II in a star is given by

$$
\frac{N_{II}}{N_{total}} = \frac{N_{II}}{N_I + N_{II}} = \frac{N_{II}/N_I}{1 + N_{II}/N_I} \tag{5}
$$

where N_I is the number of H I atoms in the star, N_{II} is the number of H II atoms in the star and N_{total} is the total number of hydrogen atoms in the star, which is equal to $N_{total} = N_I + N_{II}$ as there are no further ionisation states available to hydrogen.

The third version of this equation is more useful, as the Saha equation can be used to find the ratio N_{II}/N_I. The Saha equation is given by

$$
\frac{N_{i+1}}{N_i} = \frac{2Z_{i+1}}{n_e Z_i} \left(\frac{2\pi m_e k_B T}{h^2} \right)^{3/2} e^{-\chi_i/k_B T} \tag{6}
$$

Where N_i is the number of atoms in the i^{th} state (with $i = I$ being the ground state, $i = II$ being the first ionised state, etc.), Z_i is the partition function for the atoms in this state and χ_i is the energy needed to remove an electron from an atom in the i^{th} state and bring it to the $(i+1)^{th}$ state.

Filling this in for the I and II states,

$$\frac{N_{II}}{N_I} = \frac{2Z_{II}}{n_e Z_I}\left(\frac{2\pi m_e k_B T}{h^2}\right)^{3/2} e^{-\chi_I/k_B T} \tag{7}$$

Clearly then, equations 5 and 7 can be used to find N_{II}/N_I and therefore the ionisation fraction.

The fraction of ionised hydrogen in a star with an electron number density $n_e = 51 \times 10^{19}$ m^{-3} for the temperature range 2000 to 22000 K is shown in figure 3. In programming this it was assumed that the partition function for H II, Z_{II}, was equal to 1, as H II is only a proton, so it only has one energy state, and that only the first three energy levels of H I contribute. Z_I was also taken from equation 3.

The code used to produce this graph is shown in Appendix A.

2.3 Temperature at which 90% of the hydrogen is ionised

The temperature at which 90% of the hydrogen is ionised is found by finding the point at which the y-value of the graph in figure 3 is equal to 0.9.

This was found by the python program in Appendix A to be equal to 11763 K. And is marked as a red cross on the graph in figure 3.

2.4 Ionisation fraction of calcium as a function of temperature

The ionisation fraction of calcium as function of temperature between 2000 K and 22000 K was found again using equations 5 and 7, with the appropriate values inserted for Ca. $Z_I = 1.32$, $Z_{II} = 2.30$ and $\chi_I = 6.11$ eV, and $n_e = 51 \times 10^{19}$ m^{-3} as above.

Again, the code in Appendix A was used to produce a graph of this, as shown in figure 4

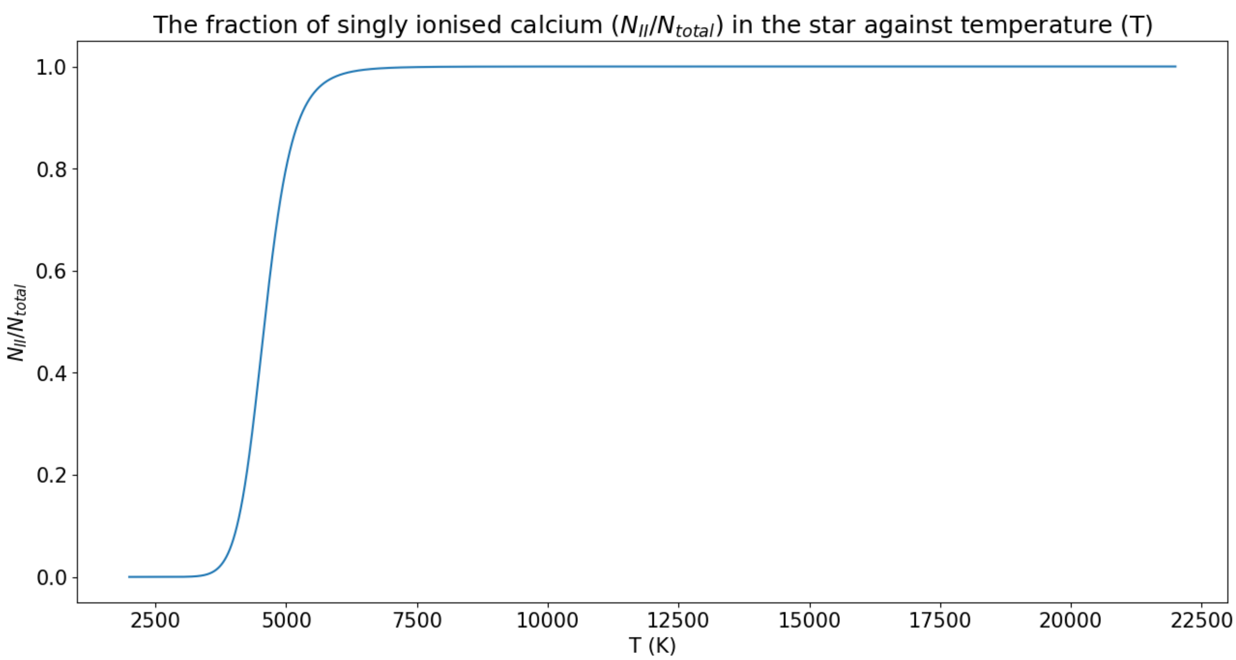

Figure 4: The ionisation fraction of calcium for a star with $n_e = 51 \times 10^{19}$ m^{-3} for the temperature range 2000 to 22000 K.

2.5 Spectrum of a star with photospheric temperature 7000 K

I would expect to see only some visible-wavelength absorption lines from H I at this temperature. Only hydrogen atoms with electrons in the n = 2 energy level produce Balmer absorption, the visible-range absorption in hydrogen. And there is only some excited hydrogen in stars of this temperature, meaning there is not a strong hydrogen absorption although there is still likely a small amount.[6]

I would expect some visual-wavelength absorption lines from Ca II in a star such as this with a photospheric temperature of 7000 K, as at this temperature the majority of the calcium present is at least singly ionised, as can be seen clearly in figure 4 respectively. Calcium absorption lines are some of the most common in stellar spectra, and two prominent lines are the so-called K and H lines at 393 nm and 397 nm respectively, both of which are within the visual wavelength range, in the violet.[6] [3] (If - of course calcium is present).

Absorption line production relies on the presence of particular elements in the star, as of course if calcium for example is not present in the star it cannot absorb any light. The strength of absorption lines in a spectrum can act as an indicator for the abundance of the elements in a star.[6]

Absorption line production also relies on the presence of a diffuse (relatively) cool gas between a source of a continuous spectrum and the observer. The elements in the gas absorb the light from the continuous source, producing the dark absorption lines.[3]

3 Stellar interiors and main-sequence evolution

3.1 Mean molecular weight for H II

The mean molecular weight of a star composed of completely ionised hydrogen is given by

$$\mu \simeq \frac{N_H m_H}{N_H(1+z_H)m_H} = \frac{1}{1+1} = \frac{1}{2} \tag{8}$$

where N_H is the number of hydrogen atoms in the star, m_H is the mass of a hydrogen atom, and z_H is the number of free electrons resulting from complete ionisation of hydrogen, which is one electron.

3.2 Mean molecular weight for H II and He III

The mean molecular weight of a star composed of a mixture of completely ionised species is

$$\mu_i \simeq \frac{\sum_j N_j A_j}{\sum_j N_j(1+z_j)} \tag{9}$$

where N_j is the number of atoms of species j, $A_j \equiv m_j/m_H$, m_j is the mass of an atom of type j, m_H is the mass of a hydrogen atom, and z_j is the number of free electrons resulting from complete ionisation of atom j. This can be rewritten in terms of the mass fraction x_j where $x_j = M_j/M_{total}$ and $\sum_j x_j = 1$, for M_j the total mass of the atoms of type j and M_{total} is the total mass of all species. Therefore μ_i is given by

$$\mu_i \simeq \frac{\sum_j x_j A_j}{\sum_j x_j(1+z_j)} \tag{10}$$

Then,

$$\frac{1}{\mu_i} \simeq \frac{\sum_j x_j(1+z_j)}{\sum_j x_j A_j} = \frac{\sum_j x_j/A_j(1+z_j)}{\sum_j x_j} = \sum_j \frac{x_j}{A_j}(1+z_j) \tag{11}$$

Solving for ${}^1_1$H and ${}^4_2$He with mass fractions X and Y respectively we get

$$\frac{1}{\mu_i} \simeq 2X + \frac{3}{4}Y \tag{12}$$

3.3 Main sequence star

For a star with $X = 0.65$ and $Y = 0.35$ and which is completely ionised, the mean molecular weight is approximately

$$\mu_i \simeq \left(2X + \frac{3}{4}Y\right)^{-1} = \left(2(0.65) + \frac{3}{4}(0.35)\right)^{-1} = 0.64 \tag{13}$$

3.4 After some H fusion

The star likely underwent fusion through the proton-proton chain, where overall four $^1_1$H atoms fuse to form one $^4_2$He atom.

Assuming there are N_{H_i} initial hydrogen atoms, and N_{He_i} initial helium atoms in the star. Which means that

$$X_i = \frac{N_{H_i}}{N_{H_i} + N_{He_i}} \tag{14}$$

and

$$Y_i = \frac{N_{He_i}}{N_{H_i} + N_{He_i}} \tag{15}$$

where X_i and Y_i are the initial mass fractions of hydrogen and helium respectively.

Then, if n_H atoms are fused into helium, $n_H/4$ new helium atoms are created. And $n_H = 0.02N_{H_i}$ so the new proportions of hydrogen and helium in the star, P_H and P_{He} are

$$P_H = \frac{0.98N_{H_i}}{N_{H_i} + N_{He_i}} = 0.98X_i \tag{16}$$

and

$$\begin{aligned} P_{He} =& \frac{0.005N_{H_i} + N_{He_i}}{N_{H_i} + N_{He_i}} \\ =& \frac{0.005N_{H_i}}{N_{H_i} + N_{He_i}} + \frac{N_{He_i}}{N_{H_i} + N_{He_i}} \\ =& 0.005X_i + Y_i \end{aligned} \tag{17}$$

The new fractions, X_f and Y_f, are therefore

$$\begin{aligned} X_f =& \frac{P_H}{P_H + P_{He}} \\ =& \frac{0.98X_i}{0.98X_i + 0.005X_i + Y_i} \end{aligned} \tag{18}$$

and

$$\begin{aligned} Y_f =& \frac{P_{He}}{P_H + P_{He}} \\ =& \frac{0.005X_i + Y_i}{0.98X_i + 0.005X_i + Y_i} \end{aligned} \tag{19}$$

Checking that these do in fact add up to 1 as would be expected,

$$X_f + Y_f = \frac{0.98X_i}{0.98X_i + 0.005X_i + Y_i} + \frac{0.005X_i + Y_i}{0.98X_i + 0.005X_i + Y_i} = \frac{0.98X_i + 0.005X_i + Y_i}{0.98X_i + 0.005X_i + Y_i} = 1 \tag{20}$$

Therefore the mean molecular weight of the star is now

$$
\begin{aligned}
\mu_f &\simeq \left(2X + \frac{3}{4}Y\right)^{-1} \\
&\simeq \frac{2(0.98X_i) + \frac{3}{4}(0.005X_i + Y_i)}{0.98X_i + 0.005X_i + Y_i} \\
&\simeq \frac{2(0.98)(0.65) + \frac{3}{4}(0.005(0.65) + 0.35)}{0.98(0.65) + 0.005(0.65) + 0.35} \\
&\simeq 0.643
\end{aligned}
\tag{21}
$$

This makes physical sense, as there are now less individual particles. It is also known that the mean molecular weight of a star increases as it evolves on the main sequence.

3.5 How the luminosity of the star changes

The luminosity of a star has the dependency

$$L \propto \frac{\mu^4 M^3}{\overline{\kappa}} \tag{22}$$

where M is the mass of the star and $\overline{\kappa}$ is the average opacity of the star, both of which we will assume remain approximately constant over this timescale. Therefore if L_i is the initial luminosity of the star, and L_f is the new luminosity of the star, then

$$
\begin{aligned}
\frac{L_f}{L_i} &\simeq \frac{\mu_f^4}{\mu_i^4} \\
\implies L_f &\simeq \frac{\mu_f^4}{\mu_i^4} L_i \\
&\simeq \frac{0.643^4}{0.64^4} L_i \\
&\simeq 1.02 L_i
\end{aligned}
\tag{23}
$$

The star's luminosity has increased by approximately 2%.

This of course is not entirely accurate due to our assumptions, as the opacity is composition and temperature dependent, and the temperature would increase as the luminosity does, however it is true that in general stars' luminosities increase as they evolve throughout their lifetimes.[3] So, while the result may not be exactly what is expected, it does make physical sense.

A Code

The python code used in this assignment is found below.

```python
import numpy as np
import matplotlib.pyplot as plt

plt.rcParams.update({'font.size': 15})

# Constants - all values from Carroll and Ostlie
me = 9.10938215 * 10**(-31) # kg
k = 1.3806504 * 10**(-23) # J K^-1
h = 6.62606896 * 10**(-34) # J s
eV = 1.602176487 * 10**(-19) # J
chi = 13.6 * eV

```

```python
# Taken from my student number
ne = 51 * 10**19 # m^-3

# A list of the temperatures
temp_list = np.arange(2000, 22000, 0.1)

#########
#       #
#   H   #
#       #
#########

# Assume ZII = 1 as H II is just a proton, which has no degeneracy
ZII = 1

# ZI for H as a function of temperature
def ZI_H(T):
    return 2 + 8*np.exp(-51/(5*k*T)) + 18*np.exp(-544/(45*k*T))

# The Saha equation for H
def saha_H(T):
    return 2*ZII/(ne*ZI_H(T)) * ((2*np.pi*me*k*T)/(h**2))**(3/2) * np.exp(-chi/(k*T))

# The ionisation fraction for H
def frac_H(T):
    return saha_H(T)/(1 + saha_H(T))

fraction = frac_H(temp_list)

# Find where 90% of the H atoms are ionised
intersect = np.argwhere(np.diff(np.sign(0.9-fraction))).flatten()
intersect_temp = temp_list[intersect][0]
print("The temperature at which 90% of the H atoms are ionised is "+ str(intersect_temp) + " K")

# Plot the graph
plt.plot(temp_list, fraction, label=r'$N_{II}/N_{total}$')
plt.scatter(intersect_temp, 0.9, s=200, marker='+', label='90% H II', color='red')
plt.xlabel("T (K)")
plt.ylabel(r"$N_{II}/N_{total}$")
plt.title(r"The fraction of ionised hydrogen ($N_{II}/N_{total}$) in the star against temperature (T)")
plt.annotate('90% ionisation occurrs\nat ' + str(np.rint(intersect_temp))[:-2] + ' K', xy=(intersect_temp + 200, 0.89), xytext=(15000, 0.6), arrowprops=dict(arrowstyle="->"))
plt.legend()
plt.show()

##########
#        #
#   Ca   #
#        #
##########

# Values given in the assignment
ZI = 1.32
ZII = 2.3
chi = 6.11 * eV

# The Saha equation for Ca
def saha_Ca(T):
    return 2*ZII/(ne*ZI) * ((2*np.pi*me*k*T)/(h**2))**(3/2) * np.exp(-chi/(k*T))

# The ionisation fraction for Ca
def frac_Ca(T):
    return saha_Ca(T)/(1 + saha_Ca(T))

fraction = frac_Ca(temp_list)
```

```python

# Plot the graph
plt.plot(temp_list, fraction, label=r'$N_{II}/N_{total}$')
plt.xlabel("T (K)")
plt.ylabel(r"$N_{II}/N_{total}$")
plt.title(r"The fraction of singly ionised calcium ($N_{II}/N_{total}$) in the star against temperature (T)")
plt.show()
```

References

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[2] The Gaia Collaboration, "Testing parallaxes with local Cepheids and RR Lyrae stars," *A&A*, vol. 605 A79, Sept. 2017. DOI: https://doi.org/10.1051/0004-6361/201629925

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[4] A. K. Dambis, L. N. Berdnikov, A. Y. Kniazev, V. V. Kravtsov, A. S. Rastorguev, R. Sefako, O. V. Vozyakova, "RR Lyrae variables: visual and infrared luminosities, intrinsic colours and kinematics," *Monthly Notices of the Royal Astronomical Society*, Volume 435, Issue 4, 11 November 2013, Pages 3206–3220, https://doi.org/10.1093/mnras/stt1514

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[6] E. Böhm-Vitense *Introduction to stellar astrophysics. Volume 1: Basic stellar observations and data*, Cambridge, England: Cambridge University Press: 1989.