N1

Пусть $z_0 = \pi$

$$\frac{1}{\sin z} = \frac{-1}{\left((z-\pi) - \frac{(z-\pi)^3}{3!} + \dots\right)} = \frac{-1}{(z-\pi)\left(1 - \frac{(z-\pi)^2}{3!}\right)} =$$

$$= \frac{-1}{(z-\pi)}\left(1 + \frac{(z-\pi)^2}{3!} + \dots\right)$$

$$\frac{2z}{z^2-\pi^2} = \frac{2z}{(z-\pi)(z+\pi)} = \frac{2z}{(z-\pi)(z-\pi+2\pi)} =$$

$$= \frac{2z}{2\pi(z-\pi)}\left(1 - \frac{z-\pi}{2\pi} + \frac{(z-\pi)^2}{(2\pi)^2} - \dots\right) =$$

$$= \frac{1}{z-\pi}\left(1 - \frac{z-\pi}{2\pi} + \frac{(z-\pi)^2}{(2\pi)^2} - \dots\right)$$

$$f(z) = \frac{1}{\sin z} + \frac{2z}{z^2-\pi^2} = \frac{-1}{(z-\pi)} + \frac{1}{z-\pi} + \frac{1}{2\pi} + O(z-\pi)$$

$$= \frac{1}{2\pi} + O(z-\pi)$$

Аналогично для $z_0 = -\pi$:

$$f(z) = -\frac{1}{2\pi} + O(z+\pi)$$

устранимая сингул.

N2

1) $f(z) = \dfrac{\sin z}{1-\tan z}$ ~~$z = \frac{\pi}{4} + \pi n$~~

~~$f(z) = \dfrac{\sin z}{1 - 1 - 2(z-\frac{\pi}{4}) - 2(z-\frac{\pi}{4})^2 + \ldots}$~~

$\tan z = 1$

$z = \frac{\pi}{4} + \pi n$

$\sin\left(\frac{\pi}{4} + \pi n\right) = \pm \frac{\sqrt{2}}{2}$

$$f(z) = \frac{\sin z}{1-1-2\left(z-\frac{\pi}{4}+\pi n\right)-2\left(z-\frac{\pi}{4}+\pi n\right)^2}$$

$$f(z) = \frac{\sin\left(\frac{\pi}{4}+\pi n\right)}{-2\left(z-\frac{\pi}{4}+\pi n\right)\left(1+\left(z-\frac{\pi}{4}+\pi n\right)\right)} =$$

$$= \frac{\sin\left(\frac{\pi}{4}+\pi n\right)\left(1-\left(z-\frac{\pi}{4}+\pi n\right)+\left(z-\pi/4+\pi n\right)^2\right)}{-2\left(z-\frac{\pi}{4}+\pi n\right)} =$$

$$= (-1)^{n+1}\frac{\sqrt{2}}{4}\cdot\frac{\left(1-\left(z-\frac{\pi}{4}+\pi n\right)+\left(z-\pi/4+\pi n\right)^2+\ldots\right)}{\left(z-\pi/4+\pi n\right)}$$

– ~~особ.~~ полюса перв. пор.

N2 $z=0$:

$$2)\ f(z) = \frac{e^{c/(z-a)}}{e^{z/a}-1} = \frac{\cancel{1+\frac{c}{z-a}+\frac{1}{2}\left(\frac{c}{z-a}\right)^2+\dots}\ e^{c/z-a}}{1-1+\frac{z}{a}+\left(\frac{z}{a}\right)^2/2+\left(\frac{z}{a}\right)^3/3+\dots}$$

$$= \frac{e^{c/(z-a)}}{\frac{z}{a}\left(1+\frac{1}{2}\frac{z}{a}+\dots\right)} =$$

$$= \frac{a}{z}e^{(c/(z-a))}\left(1-\frac{1}{2}\frac{z}{a}+\frac{1}{4}\frac{z^2}{a^2}+\dots\right)$$ — $z=0$ полюс первого

порядка, аналогично $z=2\pi i n$

$z=a$: $$\frac{1+\frac{c}{z-a}+\dots}{e-1} = \frac{1}{e-1}\sum_{0}^{\infty}\left(\frac{c}{z-a}\right)^n\frac{1}{n}$$

N3

$$ze^{\frac{1}{z}}e^{-\frac{1}{z^2}} = z\left(1+\frac{1}{z}+\dots\right)\left(1-\frac{1}{z^2}+\dots\right)$$

изол. сингул.

N4

$$1)\ \underbrace{\oint_C \frac{ze^z}{\tan z^2}dz}_{I} \qquad I = \frac{ze^z}{z^2+\frac{z^6}{3}+\dots} =$$

$$\frac{e^z}{z(1+\frac{z^5}{3}+\dots)} = \frac{e^z}{z}\left(1+\frac{z^5}{3}\right)$$

N6

~~$\oint\limits_C \frac{z^5 dz}{1+z^6}$~~ Ряд Лорана Вычет : $\left.\frac{e^z\left(1+\frac{z^5}{3}\right)}{z}\right|_{z\to 0} =$

$1 \Rightarrow \oint\limits_C \frac{z e^z}{\tan z^2} = 1\cdot 2\pi i$

2) $\oint\limits_C e^{-1/z}\sin\left(\frac{1}{z}\right)dz = \oint\limits_C \left(1 - \frac{1}{z} + \frac{1}{2z^2} + \dots\right)\cdot\left(\frac{1}{z} + \dots\right) dz$

$= \oint\limits_C \frac{1}{z}dz = 1\cdot 2\pi i$

3) $\oint\limits_C \frac{e^z}{z^n}dz = \oint\limits_C \frac{1+z+\frac{z^2}{2}+\dots+\frac{z^{n-1}}{(n-1)!}+\dots}{z^n}dz =$

$= \frac{2\pi i}{(n-1)!}$

N5

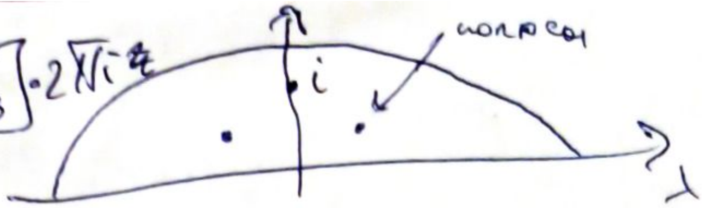
1) $I = \int\limits_{-\infty}^{\infty} \frac{x^4}{1+x^6}dx =$

$I = \lim\limits_{L\to\infty} \int\limits_{-L}^{L} \frac{x^4}{1+x^6}dx$

$I = \lim\limits_{L\to\infty}\left[\int\limits_{-L}^{L} f(x)\,dx + \int\limits_{C_1} f(z)\,dz\right] =$

0

$= \int_C \frac{x^4}{1+x^6} = [R_1 + R_2 + R_3]\cdot 2\pi i$



полюсы: $x^6 = -1$ $\qquad \sqrt[6]{-1} = \left(\cos\frac{\pi + 2\pi k}{6} + i\sin\frac{\pi + 2\pi k}{6}\right)$

$$\begin{cases} x_1 = \frac{\sqrt{3}}{2} + i\frac{1}{2} \\ x_2 = i \\ x_3 = -\frac{\sqrt{3}}{2} + \frac{1}{2}i \end{cases}$$

$$\frac{x^4}{1+x^6} \simeq \frac{f(z)}{g(z)} \approx \frac{f(z_0) + \cancel{f'(z_0)(z-z_0)}^{\to 0}}{\underbrace{\cancel{g(z_0)}}_{0} + g'(z_0)(z-z_0)} =$$

$= \frac{f(z_0)}{g'(z_0)} \cdot \frac{1}{z - z_0}$ – вычет $\qquad$ вычет: $\frac{z_0^4}{6z_0^5} = \frac{1}{6z_0}$

$$z_0 = \begin{cases} \frac{\sqrt{3}}{2} + \frac{1}{2}i \\ i \\ -\frac{\sqrt{3}}{2} + \frac{1}{2}i \end{cases}$$

т.е. $\int_C \frac{x^4}{x^6+1} = \frac{2\pi i}{6}\left[\frac{1}{\frac{\sqrt{3}}{2} + \frac{1}{2}i} + \frac{1}{i} + \frac{1}{-\frac{\sqrt{3}}{2} + \frac{1}{2}i}\right]$

$= \frac{2\pi i}{3}\left[\frac{\sqrt{3}}{2} - i/2 - i - \frac{\sqrt{3}}{2} - \frac{i}{2}\right] =$

$= \frac{-\pi i^2}{3} 2 = \frac{2\pi}{3}$

2)

$$\int_0^{2\pi} \frac{\cos 2\theta}{2+\cos\theta}\,d\theta = \left[\begin{array}{l} z=e^{i\theta} \\ \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)\end{array}\right.$$

$$= \oint_C \frac{\frac{1}{2}\left(z^2+\frac{1}{z^2}\right)dz}{zi\left(2+\frac{1}{2}\left(z+\frac{1}{z}\right)\right)} = -\frac{i2}{2}\oint_C \underbrace{\frac{(z^4+1)\,dz}{\cancel{2}z^{2}(\cancel{4}z^2+4z+1)}}_{I} =$$

$$= \left[R_1+R_2+R_3\right]2\pi i\cdot(-i) =$$

$$z = \left[\begin{array}{l} 0 \\ -2\pm\sqrt{3}\end{array}\right. \qquad z=0:\ I \approx \frac{1+z^4}{z^2}\left(1-z^2-4z\right)$$

$$R_1 = -4$$

$$R_2 = \frac{f(z_0)}{g'(z_0)} = +\frac{7}{\sqrt{3}}$$

$-2-\sqrt{3}$ – вне контурА

$$\Rightarrow 2\pi\left[-4+\frac{7}{\sqrt{3}}\right] = -8\pi + \frac{14}{\sqrt{3}}\pi.$$

N6

$$\int\limits_C \frac{z^5}{1+z^6}dz$$

Аналогично n. 1. только

$$R = \frac{f(z_0)}{g'(z_0)} = \frac{1\, z_0^5}{6\, z_0^5} = \frac{1}{6}$$

всего 6 ~~3~~ вычетов в верхн. полуплоск. $\Rightarrow$

$$J = 6R \cdot 2\pi i = \frac{12\pi i}{6} = 2\pi i$$

N7

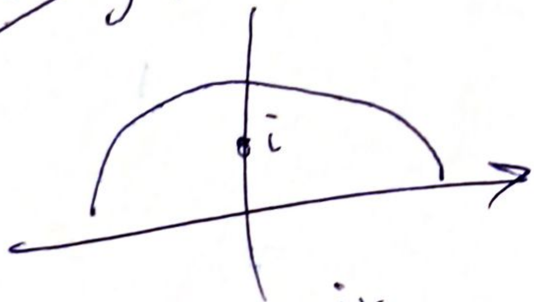
$$\int\limits_{-\infty}^{\infty} \frac{\sin^2 x\, dx}{x^2(x^2+1)} = ?$$ ~~накенедела~~ Написана в 4 n.

N7

$$\int_{-\infty}^{\infty}\frac{\sin^2 x\,dx}{x^2(x^2+1)} = \underbrace{\int\frac{\sin^2 x\,dx}{x^2}}_{I_1} - \underbrace{\int\frac{\sin^2 x\,dx}{x^2+1}}_{I_2} =$$

$$\cancel{I_1 = \frac{1}{2}\int\frac{dx}{x^2} - \frac{1}{2}\int\frac{\cos 2x}{x^2}\,dx}\Big]_2 = -\frac{1}{2}\int\frac{dx}{x^2+1} + \frac{1}{2}\int\frac{\cos 2x}{x^2+1}\,dx =$$

$$= \cancel{-\frac{1}{2\cdot 2}\mathrm{Re}\int\frac{e^{i2x}}{x^2}\,dx} - \frac{1}{2}\pi + \frac{1}{2\cdot 2}\mathrm{Re}\int\frac{e^{i2x}}{x^2+1}\,dx$$



$\varphi[0;\pi]$

$$= \cancel{\mathrm{Re} - \frac{1}{2}\pi i\,\mathrm{Res}\,\frac{e^{ix}}{x^2}} - \frac{1}{2}\pi + \mathrm{Re}\,\frac{1}{2\cdot 2}\mathrm{Res}\,\frac{e^{i2x}}{x^2+1} =$$

$$= \frac{\pi}{2}\left(-1 + e^{-2}\right)$$

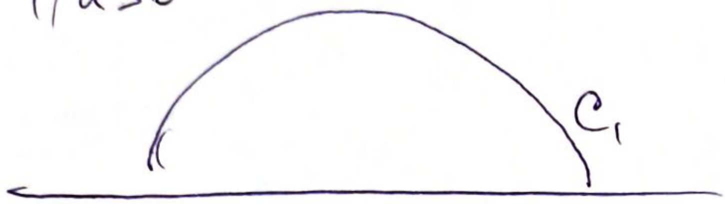
$I_1 = \pi$ (инт. Дирихле)

$$I = \frac{\pi}{2}\left(1 + e^{-2}\right)$$

N8 $\int\limits_0^\infty \frac{x\sin ax}{x^2+k^2}dx = \frac{1}{2}\int\limits_{-\infty}^{\infty}\frac{x\sin ax\,dx}{x^2+k^2} =$

$= \frac{1}{2}\int\limits_{-\infty}^{\infty}\frac{x e^{iax}dx}{x^2+k^2} - \int\limits_{-\infty}^{\infty}\frac{x e^{-iax}}{x^2+k^2}dx = \frac{1}{2}\operatorname{Im}\int\frac{x e^{iax}dx}{x^2+k^2} =$

1) $a>0$



$\left|\int\limits_{C_1}\frac{z e^{iaz}dz}{z^2+k^2}\right| < \int\limits_{C_1}\left|\frac{z e^{iaz}}{z^2+k^2}\right|dz < \int\limits_{C_1}\frac{dz}{|z|}$

грубо

$e^{iaz}f(z)\,dz \to 0$ по лемме Жордана

$|f|<M$

$a>0$

$\int\limits_C \frac{x e^{iax}dx}{x^2+k^2} = 2\pi i\,\frac{ik\,e^{iaik}}{2ik} = \pi i e^{-ak}$

$\Rightarrow J = \frac{1}{2}\operatorname{Im}\pi i e^{-ak} = \frac{1}{2}\pi e^{-ak}$

$\forall a, \forall k: \; J = \frac{\pi}{2}e^{-|a||k|}\operatorname{sgn}(a)$

№10

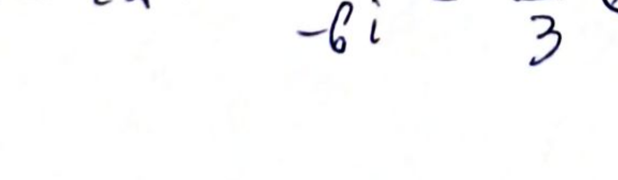
1) $\int_0^{\infty}\frac{x-\sin x}{x^3}dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{x-\sin x}{x^3}dx = -\frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin x}{x^3} =$

$= \operatorname{Im} -\frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{ix}}{x^3}dx = \left[\operatorname{Im} -\frac{1}{2\cdot 2}\left\{\operatorname{Res}_{0}\frac{e^{ix}}{x^3}\right\}\cdot 2\pi i\right]$

$= \frac{\pi}{4}$

2) $\int_{-\infty}^{\infty}\frac{e^{iz}dz}{z^2+9}$, $\sin\varphi<0$, чтобы $\int\to 0$ при $R\to\infty$

$\Rightarrow$ нижнюю полуплоск $\Rightarrow z_0=-3i$



$I = -2\pi i\cdot\frac{e^{-3}}{-6i} = \frac{\pi}{3}e^{-3}$

N11

$$\cos\frac{1}{z-2} = 1 - \frac{1}{2(z-2)^2} + \frac{1}{24(z-2)^4} + \dots =$$

$$= 1 - \frac{2}{z^2}\left(1 + \frac{4}{z} + \frac{(-2)(-3)}{z^2}\cdot\frac{21}{2} + \dots\right) + \frac{1}{24z^4}\left(1 + \frac{8}{z}\dots\right.$$

$$= \dots + \frac{143}{24}z^{-4} + \dots$$

$$f(z) = \dots + \frac{143}{24}z^{-1} + \dots$$

$$\operatorname{Res} f = \frac{143}{24}$$

N12.

$$f(z) = \frac{1}{z^3 - z^5}$$

$z=-1$

$z=0$

$z=1$

$z=\infty$

$$\operatorname{Res}(-1) = \frac{f(-1)}{g'(-1)} = -\frac{1}{2}$$

$$\operatorname{Res}(1) = \frac{f(1)}{g'(1)} = -\frac{1}{2}$$

$$\operatorname{Res}(0) = \operatorname{Res}\left(\frac{1}{z^3(1-z^2)}\right) =$$

$$= \operatorname{Res}\left(\frac{1}{z}\right) = 1$$

$$\operatorname{Res}(\infty) = \operatorname{Res}\left(\frac{-1}{z^5\left(1-\frac{1}{z^2}\right)}\right) = \operatorname{Res}\left(-\frac{1}{z^5}\left(1+\frac{1}{z^2}+\dots\right)\right) = 0$$

12

N13

$$f(z) = \frac{\sin\frac{1}{z}}{1-z}; z=0$$

$$f = \left(\frac{1}{1-z}\right)\left(\frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5}\right) = \left(1+z+z^2 ...\right)\left(\frac{1}{z} - \frac{1}{6z^3} + \frac{1}{120z^5} + ...\right)$$

$$\sum_{n=0}^{\infty} z^n \sum_{p=0}^{\infty} \frac{z^{-(2p+1)}(-1)^p}{(2p+1)!}$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} = \sin 1$$

$$g(z) = \exp\left[-\exp\left[\frac{1}{z}\right]\right]$$

$$g(z) = 1 + \left(-1 - \frac{1}{z} - \frac{1}{2z^2} - ...\right) + \frac{1}{2}\left(-1 - \frac{1}{z} - \frac{1}{2z} - ...\right)^2 + ...$$

$$a_{-1} = e^{-1} = -1 + \frac{1}{2}\cdot 2 - \frac{1}{6}\cdot 3 + ...$$

N14.

1) $$\lim_{R\to\infty} \int_{C_R} e^{iz}\,dz = \lim_{R\to\infty} \frac{e^{iz}}{i}\Big|_{R}^{R} = \lim_{R\to\infty} \frac{e^{iR} - e^{iR}}{i} =$$

$= \lim 2\sin R$ — не существ.

2) $$\lim_{R\to\infty} \int_{C_R} e^{iz^2}\,dz = \lim_{R\to\infty} \int_0^{\pi/4} e^{iR^2(\cos 2\varphi + i\sin 2\varphi)}\, iRe^{i\varphi}\,d\varphi$$

$$I = \left| \int e^{iz^2} dz \right| \le R \int_0^{\pi/4} \left| ie^{i\varphi} e^{R^2(i\cos^2\varphi - \sin^2\varphi)} \right| d\varphi$$

$$\le \int_0^{\pi/4} R e^{-\frac{R^2 4\varphi}{\pi}} d\varphi = \frac{\pi}{4R}(1 - e^{-R^2}) \to 0 \Rightarrow \lim_{R\to\infty} f = 0$$

N 5~~23~~

3) $$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)^2} = \int_C \frac{dz}{(z^2+a^2)(z^2+b^2)^2}$$

$$\operatorname{Res}(ia) = \frac{1}{2ia(b^2-a^2)^2(2a^2+b^2-3a^2)} = \frac{1}{2ia(b^2-a^2)^2}$$

$$\operatorname{Res}(ib): \cancel{\operatorname{Res}} \frac{1}{z^2+a^2} \frac{1}{(z-ib)^2} \frac{1}{(z+ib)^2} = \frac{1}{z^2+a^2} \frac{1}{(z-ib)^2} \frac{1}{(2ib)^2}$$

$$\left(1 - \frac{z-ib}{2ib} + \left(\frac{z-ib}{2ib}\right)^2 + \dots\right)^2 = \left(\frac{1}{a^2-b^2} - \frac{2ib(z-ib)}{(a^2-b^2)^2} + \dots\right) \cdot$$

$$\cdot \frac{1}{4b^2} \frac{1}{(z-ib)^2} \left(-1 + \frac{z-ib}{ib} + \dots\right) = \frac{1}{4b^2} \frac{1}{ib} \frac{1}{(a^2-b^2)^2} \cdot$$

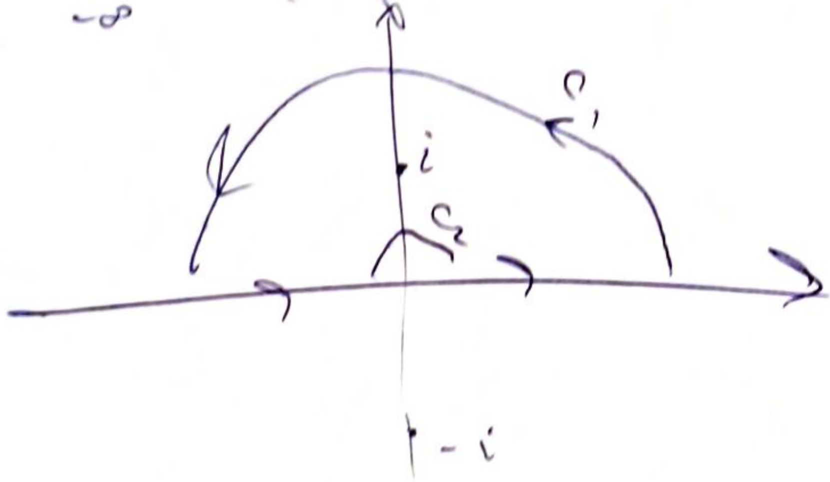
$$\cdot \frac{1}{z-ib}(a^2-b^2-2b^2) = \frac{a^2-3b^2}{4ib^3(a^2-b^2)^2} \cdot \frac{1}{z-ib}$$

$$\Rightarrow I = \frac{2\pi b^3 - 3\pi ab^2 + \pi a^2}{2ab^3(b^2-a^2)^2} = \frac{(a+2b)\pi}{2ab^3(a+b)^2}$$

№9

$$I=\int_{-\infty}^{\infty} \frac{\cos\left(x-\frac{1}{x}\right)}{1+x^2}\,dx = \mathrm{Re}\left\{\int_{-\infty}^{\infty}\frac{\exp\left[iz-\frac{i}{z}\right]}{1+z^2}\,dz\right\}$$

$$I_1=\int_{-\infty}^{\infty}\frac{\exp[zi-i/z]}{1+z^2}\,dz$$



$$I = I_1 + \cancelto{0}{\oint_{C_1}} + \cancelto{0}{\oint_{C_2}}$$

$$I_1 - 2\pi i\,\mathrm{Res}(i) = \pi e^{-2}$$