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MATH 379, Spring 2019
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A Computational Model of Firefly Synchronization

Introduction:

The synchronization of the flashing of fireflies is a phenomenon observed in many parts of the world. In North America, the Synchronous Firefly (*Photinus carolinus*) is the unique firefly species with this ability. In *Nonlinear Dynamics and Chaos* (Strogatz, 2015), the author presents a simple model that simulates the influence of a constant-frequency stimulus on a single firefly. Inspired by this idea, our goal was to simulate the synchronization phenomenon for an arbitrary number of fireflies, and produce a model that relies only on influence from the other fireflies' flashes, and not continuously on their frequencies or phases.

Strogatz model modified for two fireflies:

Before creating our own model, we examined the model presented by Strogatz in more detail and created an extended version for two fireflies (instead of one firefly and a stimulus with constant frequency). We modeled the second firefly with a differential equation analogous to that of the first firefly. We then discretized the differential equations in order to create a simulation and plotted frequency and sine of the phase against time to visualize the behaviour of the system. The model presents a number of different behaviours for varying values of the resetting strength parameter A, as shown in Figure 1.

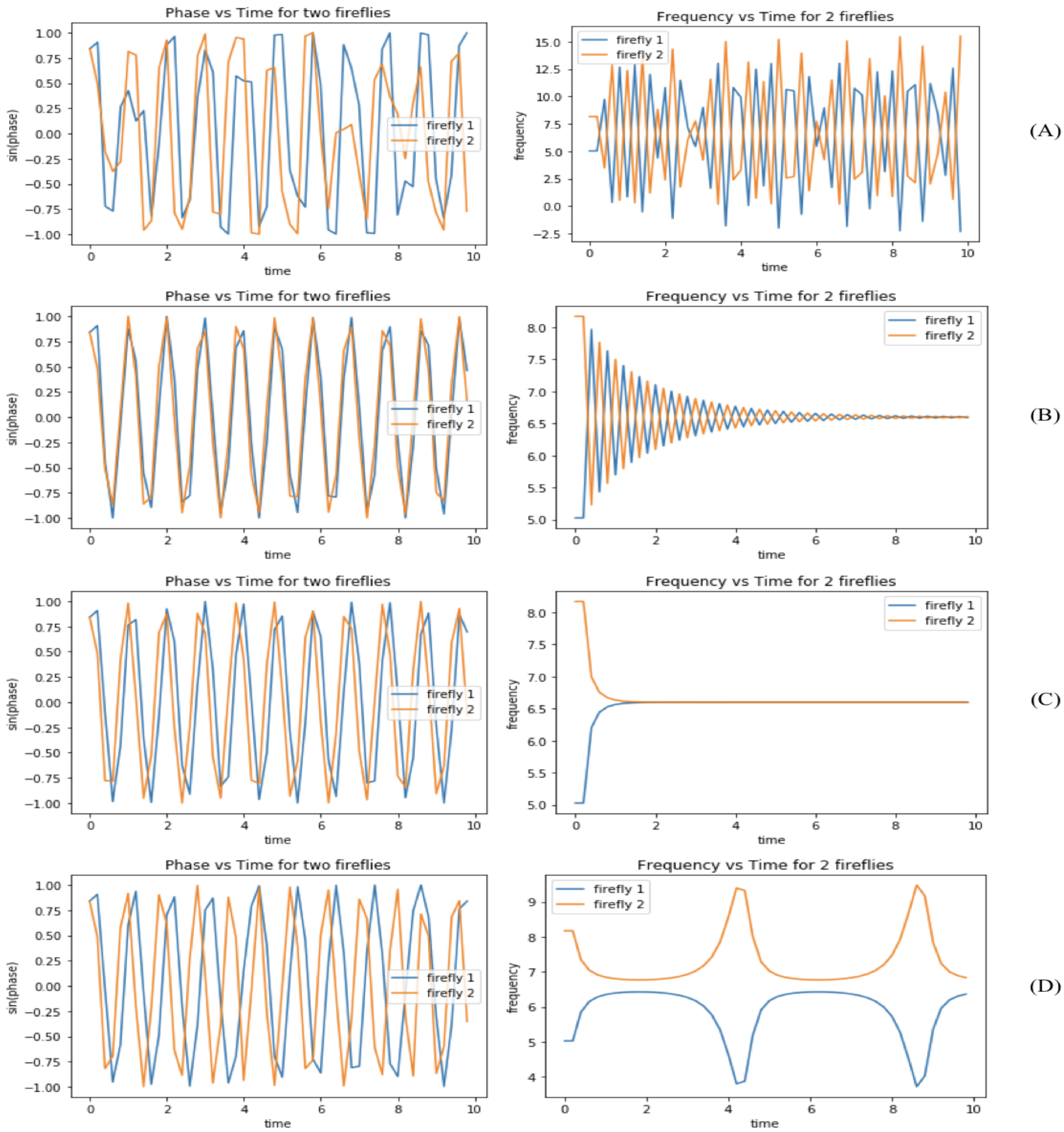


Figure 1: Phase and frequency progression with varying values of parameter A for Strogatz's model adapted for two fireflies (A) A = 8, (B) A = 5, (C) A = 2, (D) A = 1.4

For a range in A, the frequencies of the two fireflies converge. Within this range, higher A means the frequency adjustments "overshoot" which leads to oscillating frequency convergence (Figure 1C) and lower values of A cause direct, non-oscillatory convergence (Figure 1B). Certain values of A in this range also result in phase locking, such as in Figure 1C, where the frequencies synchronize and the phases differ slightly but are relatively constant. If A is too small however, the frequencies fail to converge; they approach convergence and then diverge periodically. The periodic divergences result from the phase of one firefly periodically "lapping" the phase of the other one, similar to the phase drift phenomenon that Strogatz presents. For A slightly larger than the convergence range (Figure 1A), the frequencies constantly oscillate, but the behaviour is similar to the behaviour for small A: the frequencies periodically approach convergence and diverge. As A increases beyond this point both the frequency and phase appear increasingly erratic with time and no pattern, periodicity, or synchrony is apparent.

Model development:

Variables and Parameters for Simulation Model	
N	*Number of fireflies*
a	*Scale factor for effect of frequency difference*
ε	*Scale factor for effect of stimulus on phase response*
Δt	*Time step*
ω_i	*Frequency of firefly i*
θ_i	*Phase of firefly i*

Our primary challenge, and where we were attempting to differ from Strogatz's model, was to create a model where the phase and frequency of a firefly does not update continuously, but instead updates instantaneously only when it senses a flash from another firefly. This is more realistic since a firefly is only able to synchronize based on the flashes it sees and does not have access to the phase and frequency of every other firefly.

Initially, we tried to develop a model that could be expressed entirely with differential equations. After some reading it became clear that this was well beyond both the scope of this course and our abilities and would likely require the use of Dirac's delta function in order to discretize the changes in phase and frequency with flashes (Ermentrout, 1991). Instead, we created a temporally discrete simulation where a flash occurs whenever the phase of a firefly passes through zero. Whenever a firefly flashes, all other fireflies update their frequencies and phases in an attempt to synchronize.

In any time step where no fireflies flash, all phases update according to equation (1), which is the expected progression of phase with time given the frequency of each firefly. If any firefly flashes, the phases update according to equation (2). This equation models a small instantaneous phase jump towards the phase of the firefly that flashed. Because the flash point is always at $\theta = 0$, the phase jump should be negative if the firefly's phase is in the first half of the cycle and positive if it is in the last half of the cycle. The magnitude of the phase jump should also be related to the difference in phase: if a firefly's phase differs significantly from the flashing firefly the phase jump should be greater than if the phase differs by a small amount. We chose to model these phase jumps with a negative sine function, scaled by the parameter ε. The factor of 2π in the argument is because of our choice to measure phase in our model in [0,1] instead of in [0,2π]. This avoids some of the rounding error resulting from frequently repeated operations and is otherwise equivalent.

The functioning of this model can be seen in Figure 2 which plots the phase synchronization of three fireflies. Each time a firefly passes through the 0-phase point, the phase jumps of the other two fireflies are clearly towards this point and related in magnitude to the distance of the phase from zero.

$$\theta_i = \theta_i + \omega_i \Delta t \tag{1}$$

$$\theta_i = \theta_i - \varepsilon sin(2\pi\, \theta_i) \tag{2}$$

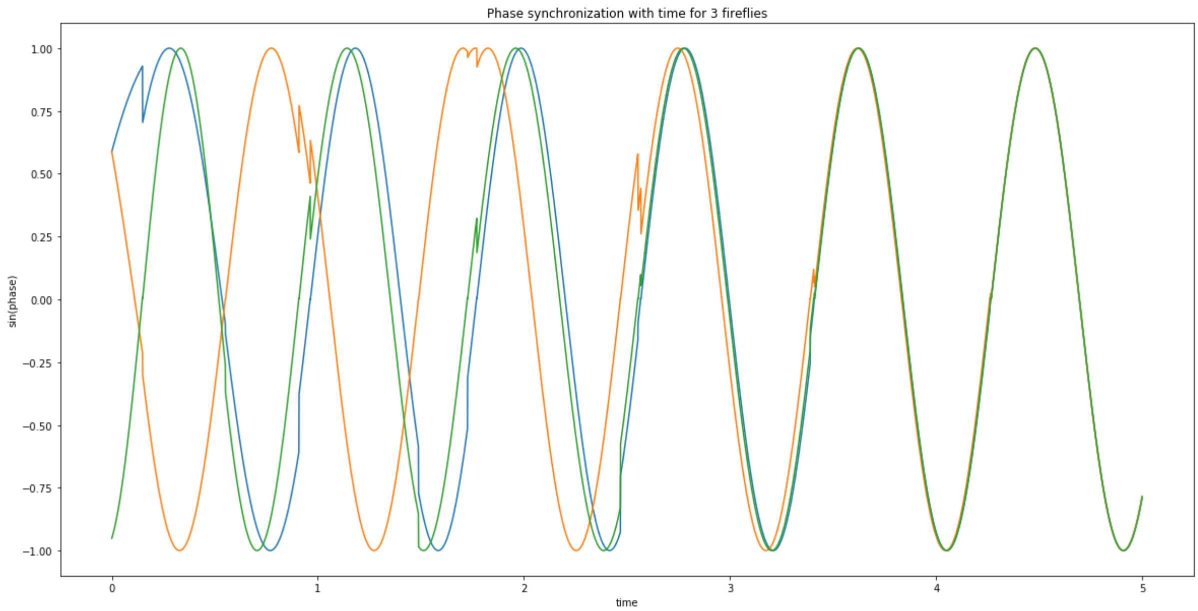


Figure 2: Demonstration of phase synchronization for a 3-firefly system

The frequency of a firefly updates according to equation (3) and only updates when another firefly has flashed and it has not. The direction and magnitude of the change in frequency for a firefly is determined by the sum of the frequency difference between that firefly and every firefly that flashes in the given time step. This way, a larger difference in frequency between fireflies will lead to a larger contribution to change in frequency when one of them flashes. The frequency adjustments are scaled by the parameter a.

$$\omega_i = \omega_i + \sum_j a(\omega_j - \omega_i)\text{, } where\ firefly\ j\ \ has\ flashed \tag{3}$$

Figure 3 visualizes this frequency synchronization for three fireflies. The discrete jumps in frequency which correspond to the flashes of other fireflies are visible, and it is apparent that the more the frequencies of two fireflies differ, the greater the frequency adjustment will be.

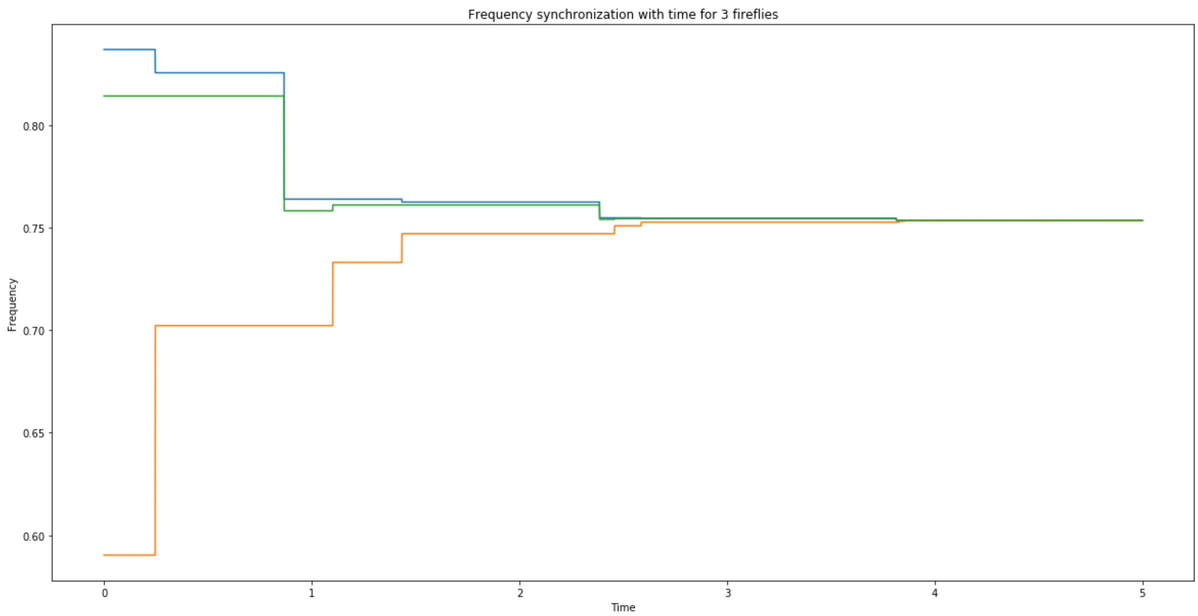


Figure 3: frequency synchronization for a 3-firefly system with $a < 1$

Figure 4 shows both the phase and frequency synchronization process for 100 fireflies. The large number of fireflies leads to an interesting effect in the phase plot where the phases of the fireflies appear to "flow in" towards the 0-phase point corresponding to a flash. Both the frequency and phase updates appear almost continuous at first when all the fireflies are flashing at different times and become more discrete as the fireflies synchronize.

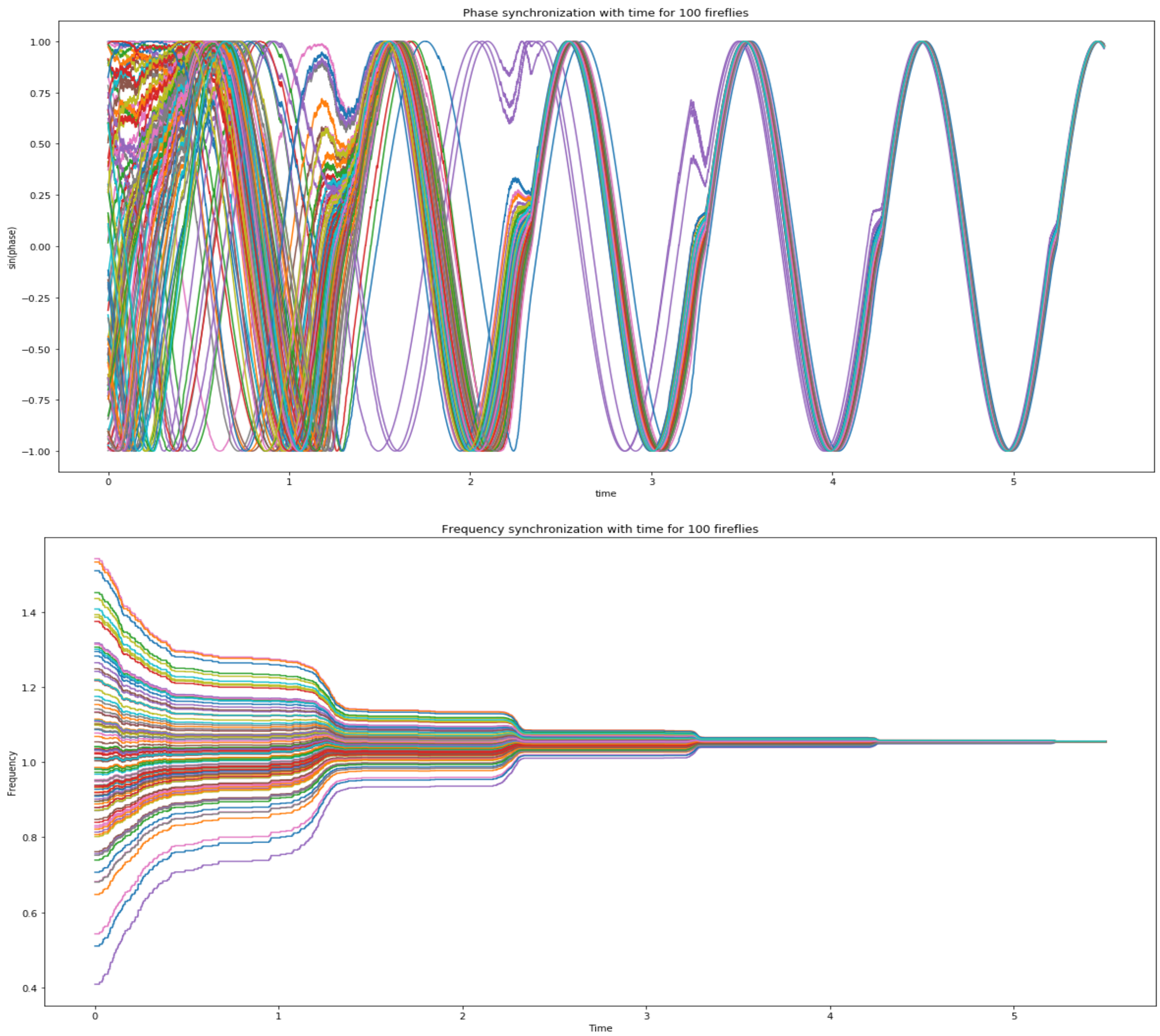
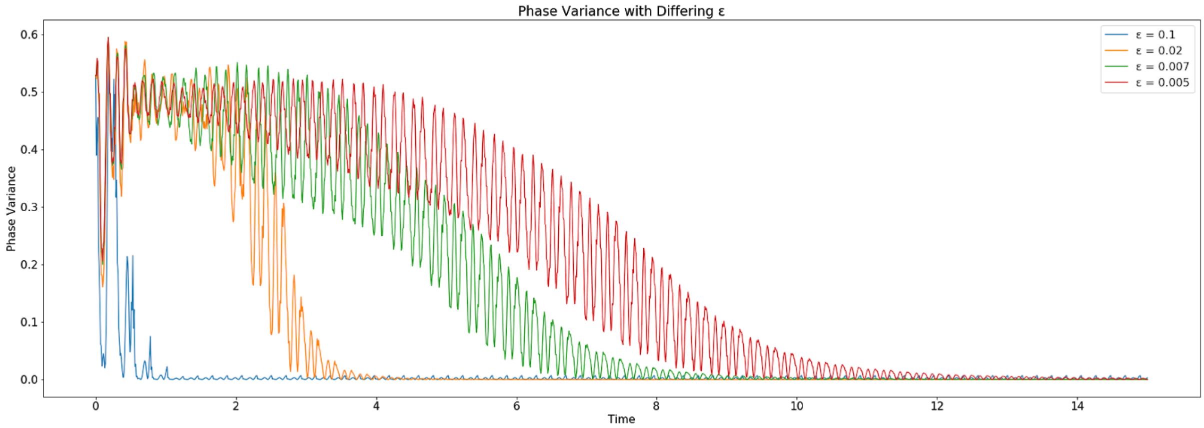


Figure 4: Phase and frequency synchronization for 100 fireflies

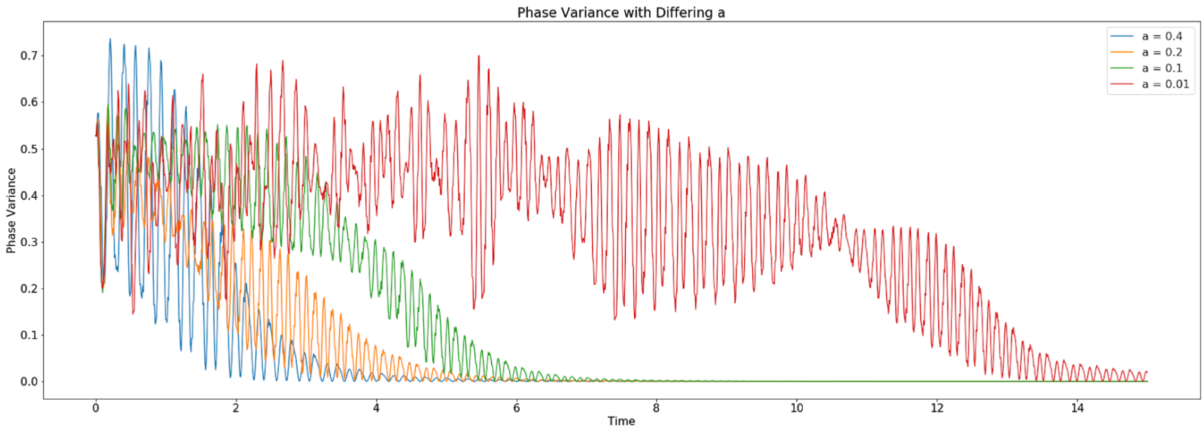
Dependence on parameters:

To visualize the speed of firefly synchronization, we plotted the variance in phase and frequency over time. We used this to examine the effect of varying parameters on the behaviour of the model. Smaller time steps gave a simulation that better approximated continuous time, the aim of our model, but also took much longer to run. We found $\Delta t = 0.001$ to be an appropriate value since time steps smaller than this did not tend to change the behaviour of the model significantly, and we could still run simulations in a reasonable amount of time using this value.

As expected, both computation time and time to synchronize increased dramatically with the number of fireflies. The scale parameters *a* and ε affect the synchronization of the fireflies in different ways. The parameter *a* mainly affects the convergence of frequencies and ε mainly affects the convergence of phases. Figure 5 shows the convergence of the phases for varying values of *a* and of ε. The phase variance has an interesting behaviour: the convergence is almost always oscillatory and in the shape of an S-curve, however the time to converge and the length of the initial noise before the convergence begins is dependent on parameter values.



N = 15, *a* = 0.1, dt = 0.001



N = 15, ε = 0.01, dt = 0.001

Figure 5: Phase variance with time for different scale values, holding other parameters constant

Larger values of both *a* and ε lead to faster convergence, however if they are too large the convergence fails and the behaviour can become erratic and the frequencies may diverge to extreme values, either positive or negative. With large numbers of fireflies, the system becomes much more sensitive and even relatively small parameter values can lead to this frequency “blow-up”.

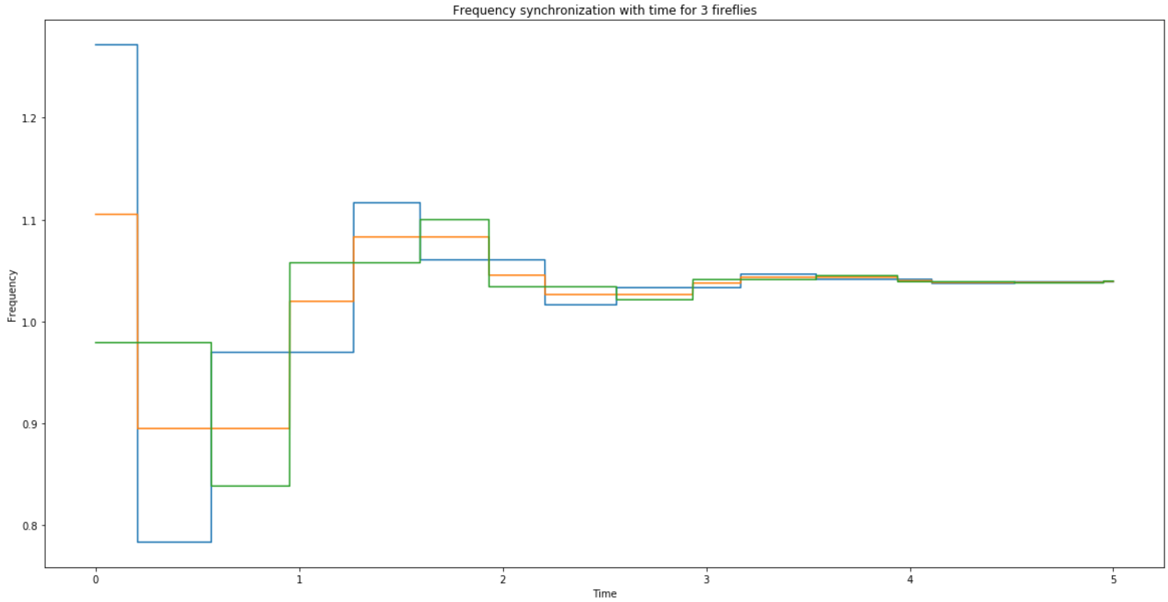
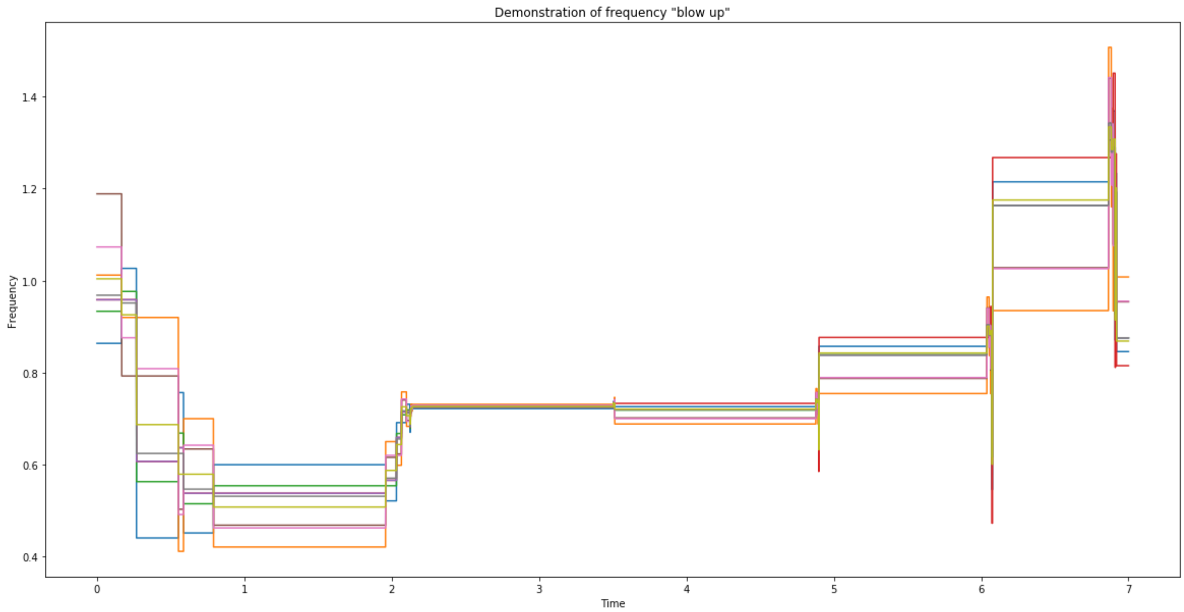


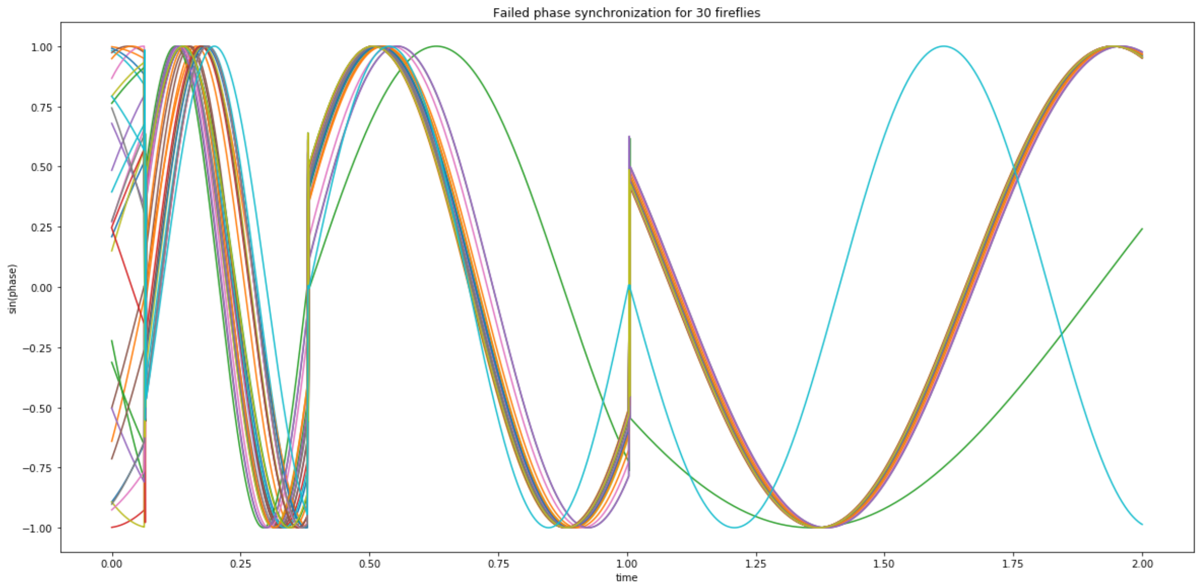
Figure 6: Oscillatory frequency convergence for three fireflies with *a* > 1

In general, a value of *a* greater than 1 means the frequency adjustments will overshoot their goal, leading to oscillatory convergence, as seen in Figure 6. However, for more fireflies these large *a* values almost always lead to divergence: if the frequency of a single firefly begins to diverge, the others will quickly follow, thus amplifying the effect. With more fireflies, the onset of this process becomes increasingly likely. Figure 7 shows the beginning of this phenomenon for a 9-firefly system: convergence of frequencies is nearly reached before the system begins to diverge, and once the divergence begins the effect is amplified with each flash. Figure 8 shows failed convergence for 30 fireflies with large values of both *a* and ε. It is apparent in both the phase and frequency plots how these excessively large adjustments lead to blow-up and erratic behaviour.



N = 9, *a* = 1.72, ε = 0.07, dt = 0.001

Figure 7: Demonstration of frequency instability and divergence for nine fireflies, *a* > 1



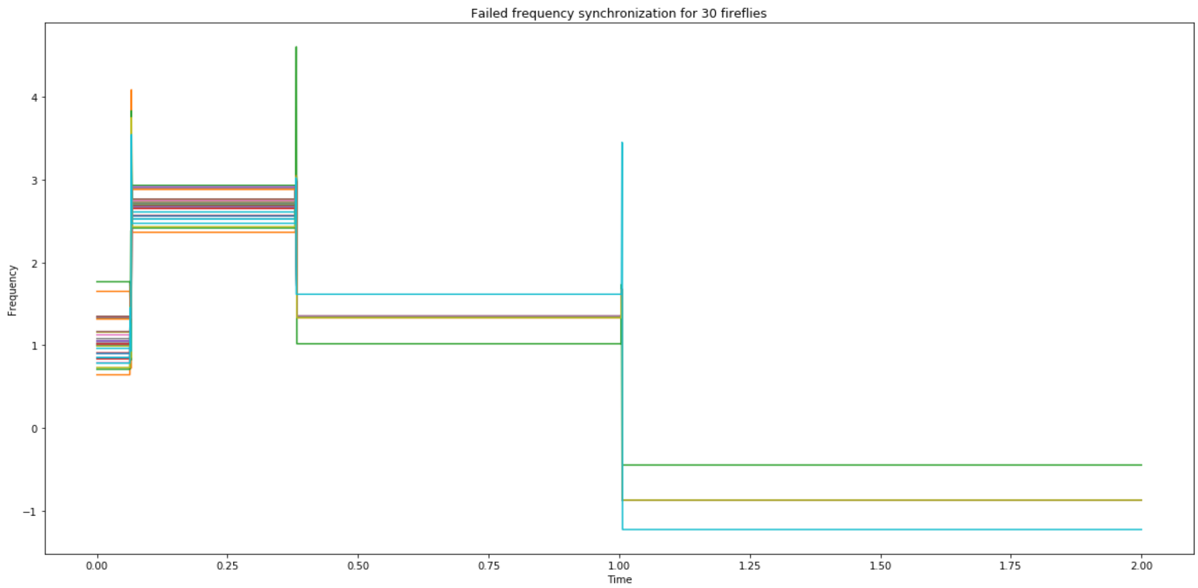


Figure 8: Failed synchronization and demonstration of erratic behaviour for large parameter values

Conclusion:

Developing this model taught us about the practicality and tradeoffs of a discrete time simulation versus a continuous time model. Although it is impossible to model true continuous time on a computer, we were able to create a successful model that produced the expected behaviour. Exploring different time steps also revealed the importance that discretizing can have on results. Our model also produced some behaviours we were not expecting and we were able to interpret these within the scope of the problem.

To further expand on our model, it would be interesting to incorporate a spatial element where the fireflies' impact on one another is scaled by the distance between them. While working on our project we came across a simulation that incorporates this aspect (Fogleman, 2013). When visualized, this results in wave-like behaviour similar to that which can be observed in footage of real fireflies synchronizing.

References:

Ermentrout, B. (1991). An adaptive model for synchrony in the firefly Pteroptyx malaccae. *Journal of Mathematical Biology,29*(6), 571-585. doi:10.1007/bf00164052

Fogleman, M. (2013, January 16). Sync: Virtual, Synchronizing Fireflies. Retrieved from https://www.youtube.com/watch?v=1G6GHQ-EbJI

Strogatz, S. H. (2015). 4.5 Fireflies. In *Nonlinear Dynamics and Chaos With Applications to Physics, Biology, Chemistry, and Engineering* (2nd ed.)

Appendix A: code for simulation model

```
import random
import numpy

def update_omegas(omegas,a,flashed):
    n = len(omegas)
    orig = omegas.copy()
    for i in range(n):
        if not flashed[i]:
            for j in range(n):
                if flashed[j]:
                    omegas[i] += a*(orig[j] - orig[i])
    return omegas

def update_thetas_time(time, omegas, thetas):
    thetas = [t + o*dt for o,t in zip(omegas, thetas)]
    return thetas

def update_thetas_flash(thetas,a):
    theta_cp = thetas.copy()
    thetas = thetas-e*numpy.sin([2*numpy.pi*theta for theta in theta_cp])
    return thetas

def fireflies(n, a, e, dt, MAXTIME):
    time = 0
    flashed = [False for _ in range(n)]
    flash = False
    thetas = [random.uniform(0,1) for _ in range(n)]
    omegas = [random.gauss(1,0.2) for _ in range(n)]
    while (time < MAXTIME):
        if flash:
            thetas = update_thetas_flash(thetas,a)
            omegas = update_omegas(omegas,a,flashed)
            flash = False
        else:
            thetas = update_thetas_time(time, omegas, thetas)
        flashed = [False for _ in range(n)]
        for i in range(n):
            if thetas[i]>=1:
                flashed[i] = True
                flash = True
                thetas[i] = thetas[i]%1
        time = time + dt
```

Appendix B: two firefly model adapted from Strogatz

```
import math
import numpy

W_0 = 0.8*2*pi #entrainment frequency
w_0 = 1.3*2*pi #firefly's natural frequency
A = 1.4 #parameter
Th_0 = 1 #initial phase of entrainment
th_0 = 1 #initial phase of firefly
a = 1 #scale factor

dTh = lambda Th,th: W_0 + A*numpy.sin(th-Th)
dth = lambda Th,th: w_0 + A*numpy.sin(Th-th)

#set initial conditions

Th = [Th_0]
th = [th_0]
t = [0] #stores corresponding time values
w = [w_0]
W = [W_0]

dt = 0.2 #time step
t_end = 10

for i in range(math.floor(t_end//dt)):
    Th.append(Th[i]+dTh(Th[i],th[i])*dt/a)
    th.append(th[i]+dth(Th[i],th[i])*dt/a)
    t.append(t[i]+dt)
    w.append(dth(Th[i],th[i]))
    W.append(dTh(Th[i],th[i]))
```

Appendix C: Visualization with 100 fireflies

Video on Youtube at: https://www.youtube.com/watch?v=2hzznVBbRWY