QLunch # 1: April 12, 2017

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Topics: First, Rolando talked about the universal recovery map for approximate Markov chains [1], and then John told us about the equivalence of the quantum approximate Markov chains and Gibbs states of one-dimensional local quantum Hamiltonians [3].

1.1 Universal recovery map for approximate Markov chains [1]

tl;dr: given a state ρ_{ABC}, if the conditional mutual information is small, A is only correlated to C through B up to a small error, in the sense that C can be approximately recovered given the information contained in B only. These states are called *approximate quantum Markov chain.*

1.1.1 Classical case

In classical case, we have a clear relation between *conditional independence* of two random variables and *conditional mutual information.* In particular, we have

Theorem 1. *Two random variables A and B are conditionally independent given B, meaning $\forall a, b, c$,*

$$p_{ABC}(a,b,c) = p_A(a)P_{B|A}(b|a)p_{C|B}(c|b), \tag{1.1}$$

if and only if

$$I(A:C|B) = 0. \tag{1.2}$$

We call such chain of variables, $A - B - C$, an exact Markov chain.

According to (1.1), for each Markov chain with a fixed conditional distribution $p_{C|B}(c|b) = \mathcal{N}(c|b)$, we have

$$p_{ABC}(a,b,c) = N(c|b)p_{AB}(a,b),$$

so determining p_{AB} and a, b, c is enough to get $p_{ABC}(a,b,c)$. In other words, there exists a *recovery map* $\mathcal{R}^{\mathcal{N}}(c|b)$ that maps $p_{AB}(a,b)$ to $p_{ABC}(a,b,c)$.

1.1.2 Quantum case

In quantum case, however, things are not as obvious as before, and the notion of conditional independence is not clear at first glance. Starting with a state ρ_{ABC} on a tripartite quantum system $A \otimes B \otimes C$, we define the conditional mutual information

$$I(A:C|B) = H(AB) + H(BC) - H(B) - H(ABC). \tag{1.3}$$

Then, we have the non-trivial *strong subadditivity* property which says $I(A,C|B) \geq 0$. It has been shown [1] that the states with $I(A,C|B) = 0$ characterize states ρ_{ABC} whose system C can be reconstructed just by acting on B, i.e., as in the classical case, there exists a recovery map $\mathcal{R}_{A\to BC}$ from B to $B \otimes C$ such that

$$\rho_{ABC} = \mathcal{R}_{A\to BC}(\rho_{AB}). \tag{1.4}$$

We call these states (exact) quantum Markov chains. [1] To get some intuition, consider the case when B is classical, i.e.

$$\rho_{ABC} = \sum_b p_B(b)\, |b\rangle\langle b|_B \otimes \rho_{AC,b} \tag{1.5}$$

When $\rho_{BC} = \rho_B \otimes \rho_C$, we expect this state to be Markovian since conditioned on the value of B, the marginal state $\rho_{AC,b}$ is a product state, meaning the two systems A and B are independent – as required by our definition of a Markov chain. One can easily see that in this case indeed we have $I(A;B|C) = 0$. Also, there exists a recovery map $R_{B\to BC}$ such that

$$\mathcal{R}_{B\to BC}(|b\rangle\langle b|) = |b\rangle\langle b| \otimes \rho_{C,b} \quad (\forall b), \tag{1.6}$$

where $\rho_{C,b} = \mathrm{tr}_C(\rho_{AC,b})$. This observation can be generalized as shown in [1] to the following fact

Theorem 2. *For a state ρ_{ABC} satisfying strong subadditivity with equality, i.e. $I(A:C|B) = 0$, the marginal state ρ_{AC} is separable. Conversely, for each separable state ρ_{AC}, there exists an extension ρ_{ABC} such that $I(A:C|B) = 0$.*

Thus, in the exact case of $I(A:C|B) = 0$, we have a meaningful generalization of the conditional independence in terms of recovery maps, but what about the approximate case where $I(A:C|B) \leq \varepsilon$? It turns out [2] that we can still have the same interpretation:

Theorem 3. *For any density operator ρ_{BC} on $B \otimes C$ there exists a trace-preserving completely positive map $\mathcal{R}_{B\to BC}$ such that for any extension ρ_{ABC} on $A \otimes B \otimes C$*

$$I(A:C|B) \geq -2\log_2 F(\rho_{ABC}, \mathcal{R}_{B\to BC}(\rho_{AB}), \tag{1.7}$$

or equivalently

$$F(\rho_{ABC}, \mathcal{R}_{B\to BC}(\rho_{AB})) \geq 2^{-\frac{1}{2}I(A:C|B)}. \tag{1.8}$$

where $F(.,.)$ is the fidelity of the states defined as $F(\rho,\sigma) = \left\|\sqrt{\rho}\sqrt{\sigma}\right\|_1$.

Notice that (1.7) immediately gives both the strong subadditivity inequality, and the existence of the recovery map in the exact Markov chain discussed before. Also, the recovery map is *universal* in the sense that the result holds if the recovery map is chosen depending on ρ_{BC} only, rather than on ρ_{ABC}.

1.2 Equivalence of the quantum approximate Markov chains and Gibbs states

We can extend the definition of Markov chain to an n-partite system and say $\rho_{A_1,\dots A_n}$ is a quantum ε-approximate Markov chain if

$$I(A_i,\dots,A_{i-1} : A_{i+1},\dots,A_n|A_i) \leq \varepsilon \quad i \in [n] \tag{1.9}$$

1.2.1 Classical case

Classically, Markov chains are equivalent to the set of Gibbs states of 2-local Hamiltonians:

$$p_{A_1,\dots A_n}(a_1,\dots a_n) = \frac{\exp\{-\sum_i h_i(a_i,a_{i+1})\}}{Z} \tag{1.10}$$

This result is called Hammersley-Clifford Theorem.

[1]The specific form of the recovery map is also known, and is of the form $X_B \mapsto \rho_{BC}^{\frac{1}{2}}(\rho_B^{-\frac{1}{2}} X_B \rho_B^{-\frac{1}{2}} \otimes \mathrm{id}_C)\rho_{BC}^{\frac{1}{2}}$.

1.2.2 Quantum case

It has been shown that the same theorem remains valid in the case of an exact quantum Markov chain. Namely, a full rank quantum state $\rho_{A_1,\ldots,A_n}$ is a quantum Markov chain if, and only if, it can be written as

$$\rho = \frac{\exp(-\sum_i h_i(a_i, a_{i+1}))}{Z} \tag{1.11}$$

where each $h_{i,i+1}$ only acts on subsystems A_i, A_{i+1}, such that $[h_{i,i+1}, h_{j,j+1}] = 0$ for all i, j. Therefore we have a characterization of full rank quantum Markov chains as Gibbs states of 1D commuting local quantum Hamiltonians. The main result of [3] is that if we relax the condition of the previous result from exact Markov chains to approximate case, we get the following:

- Let $H = \sum_i h_i$ be a short-range one-dimensional Hamiltonian with $\|h_i\| \leq 1$. Then for every tripartite split of the system ABC, if we choose the region B sufficiently large, the Gibbs state can be approximately recovered from the partial trace over C by performing a recovery map on B. In turn, this implies the conditional mutual information between two regions A and C given a middle region B decays exponentially with the square root of the length of B.

- Conversely, let $\rho_{A_1,\ldots,A_n}$ be an quantum ε-approximate Markov chain. Then there exists a local Hamiltonian $H = \sum_i h_{A,A_{i+1}}$, such that

$$S(\rho || \frac{e^{-H}}{\mathrm{tr}(e^{-H})}) \leq \varepsilon n \tag{1.12}$$

the combination of the two results gives as a variant of the Hammersley-Clifford theorem for quantum approximate Markov chains.

References

[1] O. Fawzi and R. Renner. *Quantum conditional mutual information and approximate markov chains.* Communications in Mathematical Physics, 340(2):575611, 2015. https://arxiv.org/pdf/1504.07251.pdf

[2] P. Hayden, R. Jozsa, D. Petz, and A. Winter. *Structure of states which satisfy strong subadditivity of quantum entropy with equality.* Communications in Mathematical Physics, 246(2):359374, 2004. https://arxiv.org/pdf/quant-ph/0304007.pdf

[3] Kohtaro Kato and F.G.S.L. Brandao. *Quantum approximate markov chains are thermal.* arXiv preprint: https://arxiv.org/pdf/1609.06636.pdf