Algorithmic Game Theory and Applications

Lecture 2: Mixed Strategies, Expected Payoffs, and Nash Equilibrium

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Algorithmic Game Theory and Applications

Lecture 1: What is game theory?

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Basic course information

- **Lecturer**: Kousha Etessami;
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- **Lecture times: Monday**&**Thursday**, 11:10-12:00; All lectures are in person in Appleton Tower Lecture Theatre 2 (lectures will be recorded, and the recordings will be accessible via the LEARN page for the course).
  Tutorials: weekly, starting in week 3, based on weekly tutorial sheets. There will be two tutorial groups; current plan is for tutorials on Tuesdays 11:10-12:00 and Wednesdays 11:10 – 12:00 (both to be confirmed).
- **Assessments:** two written courseworks (each counts as 10% of overall mark), and one final exam (counts as 80%).
- **Course's Main Web Page** (with lecture notes/reading list, courseworks, tutorial sheets, etc):
  `http://www.inf.ed.ac.uk/teaching/courses/agta/`

- **No required textbook.** Course based on lecture notes + assigned readings. *<u>Some useful reference textbooks:</u>*

  M. Maschler, E. Solan, & S. Zamir, *Game Theory*, 2013.

  M. Osborne and A. Rubinstein, *A Course in Game Theory*, 1994.

  R. Myerson, *Game Theory: Analysis of conflict*, 1991.

  A. Mas-Colell, M. Whinston, and J. Green, Microeconomic Theory, 1995.

  N. Nisan, T. Roughgarden, E. Tardos, and V. Vazirani (editors), *Algorithmic Game Theory*, 2007. *(Available for free online.)*

  T. Roughgarden *Twenty Lectures on Algorithmic Game Theory*, Cambridge U. Press, 2016. (Available online from Library.)

  Y. Shoham & K. Leyton-Brown, *Multiagent systems: algorithmic, game-theoretic, and logical foundations*, 2009. (Available from Library online.)

  V. Chvátal, *Linear Programming*, 1980.

What is Game Theory?

A general and vague definition:

> *"Game Theory is the formal study of interaction between 'goal-oriented' 'agents' (or 'players'), and the strategic scenarios that arise in such settings."*

What is *Algorithmic Game Theory?*

> *"Concerned with the computational questions that arise in game theory, and that enlighten game theory. In particular, questions about finding efficient algorithms to 'solve' games."*

These vague sentences are best illustrated by looking at examples.

A simple 2-person game: Rock-Paper-Scissors

Player II

Player I		Rock	Paper	Scissors
	Rock	0, 0	−1, 1	1, −1
	Paper	1, −1	0, 0	−1, 1
	Scissors	−1, 1	1, −1	0, 0

- This is a “zero-sum” game: whatever Player I wins, Player II loses, and vice versa.
- What is an “optimal strategy” in this game?
- How do we compute such “optimal strategies” for 2-person zero-sum games?

A non-zero-sum 2-person game: Prisoner's Dilemma

		Player II	
		Cooperate	Defect
Player I	Cooperate	2, 2	3, 0
	Defect	0, 3	1, 1

- For both players Defection is a *"Dominant Strategy"* (regardless what the other player does, you're better off Defecting).
- But if they both Cooperate, they would both be better off.
- Game theorists/Economists worry about this kind of situation as a real problem for society.
- Often, there are no "dominant strategies". What does it mean to "solve" such games?

Nash Equilibria

- A Nash Equilibrium (NE) is a pair (n-tuple) of strategies for the 2 players (n players) such that no player can benefit by unilaterally deviating from its strategy.
- **Nash's Theorem:** Every (finite) game has a *mixed* (i.e., randomized) Nash equilibrium.
- **Example 1:** The pair of dominant strategies (Defect, Defect) is a pure NE in the Prisoner's Dilemma game. (In fact, it is the only NE.)
  In general, there may be many NE, none of which are pure.
- **Example 2:** In Rock-Paper-Scissors, the pair of *mixed* strategies: ((R=1/3, P=1/3,S=1/3), (R=1/3, P=1/3,S=1/3)) is a Nash Equilibrium. (And, we will learn, it is also a *minimax* solution to this zero-sum game. The *"minimax value"* is 0, as it must be because the game is *"symmetric"*.)
- **Question:** How do we compute a Nash Equilibrium for a given game?

Multiple equilibria

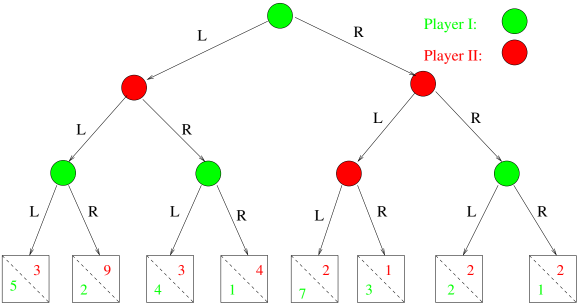
- Many games have > 1 NE. **Example:** A "Coordination Game":

		Player II	
		A	B
Player I	A	2, 2	0, 0
	B	0, 0	1, 1

- There are two **pure** Nash Equilibria: (A, A) and (B, B).
  Are there any other NEs?
  Yes, there's one other *mixed* (randomized) NE.

Games in "Extensive Form"

So far, we have only seen games in "strategic form" (also called "normal form"), where all players choose their strategy simultaneously (independently).
What if, as is often the case, the game is played by a sequence of moves over time? (Think, e.g., Chess.)
Consider the following 2-person game tree:

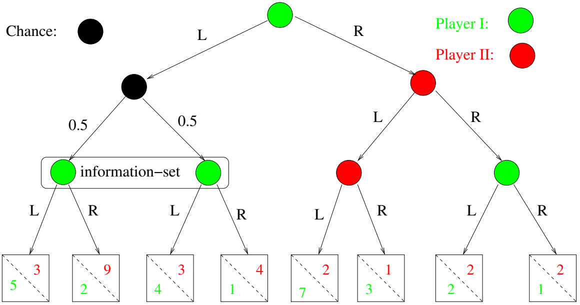


- How do we analyze and compute “solutions” to such *extensive form* games?
- What is their relationship to strategic form games?

chance, and information

Some tree nodes may be *chance* (probabilistic) nodes, controlled by neither player. (Poker, Backgammon.)

Also, a player may not be able to distinguish between several of its “positions” or “nodes”, because not all *information* is available to it. (Think Poker, with opponent’s cards hidden.) Whatever move a player employs at a node must be employed at all nodes in the same “information set”.

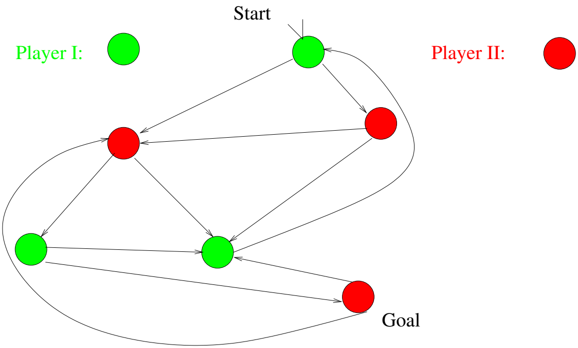


A game where every information set has only 1 node is called a game of *perfect information*.

Theorem Any finite n-person extensive game of perfect information has an "equilibrium in pure strategies".

Again, how do we compute equilibrium solutions for such games?

Extensive games on Graphs



- Does Player I have a strategy to "force" the play to reach the "Goal"?
- Such games have lots of applications.
- Again, how do we compute winning strategies in such games?
- What if some nodes are chance nodes?

Mechanism Design

Suppose you are the game designer. How would you design the game so that the "solutions" will satisfy some "objectives"?

- **Example:** Auctions: (think EBay, or Google Ads) Think of an auction as a multiplayer game between several bidders. If you are the auctioneer, how could you design the auction rules so that, for every bidder, bidding the maximum that an item is worth to them will be a "dominant strategy"?
  A answer: second price, sealed bid *Vickrey auctions*.
- How would you design protocols (such as network protocols), to encourage "cooperation" (e.g., diminish congestion)?
- Many computational questions arise in the study of "good" mechanisms for various goals.
- This is an extremely active area of research (we will only get to scratch its surface).

But why study this stuff?

GT is a core foundation of mathematical economics.
But what does it have to do with Computer Science? More than you might think: GT ideas have played an important role in CS:

- Games in AI: modeling "rational agents" and their interactions. (Similar to Econ. view.)
- Games in Modeling and analysis of reactive systems: computer-aided verification: formulations of model checking via games, program inputs viewed "adversarially", etc.
- Games in Algorithms: several GT problems have a very interesting algorithmic status (e.g., in NP, but not known to be NP-complete, etc).
- Games in Logic in CS: GT characterizations of logics, including modal and temporal logics (Ehrenfeucht-Fraisse games and bisimulation).

- Games in Computational Complexity: Many computational complexity classes are definable in terms of games: Alternation, Arthur-Merlin games, the Polynomial Hierarchy, etc.. Boolean circuits, a core model of computation, can be viewed as games (between AND and OR).

More recently:

- Games, the Internet, and E-commerce: An extremely active research area at the intersection of CS and Economics. Basic idea: "The internet is a HUGE experiment in interaction between agents (both human and automated)". How do we set up the rules of this game to harness "socially optimal" results?

I hope you are convinced: knowledge of the principles and algorithms of game theory will be useful to you for carrying on future work in many CS diciplines.

Ok, let's get started

Definition A <u>*strategic form game*</u> Γ, with n players, consists of:

1. A set $N = \{1, \ldots, n\}$ of players.
2. For each $i \in N$, a set S_i of *(pure) strategies*.
   Let $S = S_1 \times S_2 \times \ldots \times S_n$ be the set of possible combinations of (pure) strategies.
3. For each $i \in N$, a *payoff (utility) function* $u_i : S \mapsto \mathbb{R}$, describes the payoff $u_i(s_1, \ldots, s_n)$ to player i under each combination of strategies.

(Each player prefers to maximize its own payoff.)

Definition A <u>zero-sum</u> game Γ, is one in which for all $s = (s_1, \ldots, s_n) \in S$,

$$u_1(s) + u_2(s) + \ldots + u_n(s) = 0.$$

Some “food for thought”

Food for Thought (**the “guess half the average game”**): Consider a strategic-form game Γ with n-players. Each player has to guess a whole number from 1 to 1000. The player who guesses a number that is closest to half of the average guess of all players wins a payoff of 1. All other players get a payoff of 0. (If there are ties for who is closest, those who are closest share the payoff of 1 equally amongst themselves; alternatively, all who are closest get payoff 1.)

Question: What would your strategy be in such a game?

Question: What is a “Nash Equilibrium” of such a game?

Finite Strategic Form Games

Recall the “strategic game” definition, now “finite”:

Definition A finite *strategic form game* Γ, with n-players, consists of:

1. A set $N = \{1, \ldots, n\}$ of Players.
2. For each $i \in N$, a finite set $S_i = \{1, \ldots, m_i\}$ of *(pure) strategies*.
   Let $S = S_1 \times S_2 \times \ldots \times S_n$ be the set of possible combinations of (pure) strategies.
3. For each $i \in N$, a *payoff (utility) function*:
   $u_i : S \mapsto \mathbb{R}$, describes the payoff $u_i(s_1, \ldots, s_n)$ to player i under each combination of strategies.

(Each player wants to maximize its own payoff.)

Mixed (Randomized) Strategies

We define “mixed” strategies for general finite games.

Definition A **mixed** (i.e., **randomized**) **strategy** x_i for Player i, with $S_i = \{1, \ldots, m_i\}$, is a probability distribution over S_i. In other words, it is a vector $x_i = (x_i(1), \ldots, x_i(m_i))$, such that $x_i(j) \geq 0$ for $1 \leq j \leq m_i$, and

$$x_i(1) + x_i(2) + \ldots + x_i(m_i) = 1$$

Intuition: Player i uses randomness to decide which strategy to play, based on the probabilities in x_i.
Let X_i be the set of mixed strategies for Player i.
For an n-player game, let

$$X = X_1 \times \ldots \times X_n$$

denote the set of all possible combinations, or “**profiles**”, of mixed strategies.

Expected Payoffs

Let $x = (x_1, \ldots, x_n) \in X$ be a profile of mixed strategies.
For $s = (s_1, \ldots, s_n) \in S$ a combination of pure strategies, let

$$x(s) := \prod_{j=1}^{n} x_j(s_j)$$

be the probability of combination s under mixed profile x. (We're assuming players make their random choices independently.)
Definition: The **expected payoff** of Player i under a mixed strategy profile $x = (x_1, \ldots, x_n) \in X$, is:

$$U_i(x) \; := \sum_{s \in S} x(s) * u_i(s)$$

I.e., the "weighted average" Player i's payoff under each pure combination s, weighted by the probability of that combination.
Key Assumption: Every player's goal is to maximize its own expected payoff. (This can somtimes be a dubious assumption.)

some notation

We call a mixed strategy $x_i \in X_i$ <u>*pure*</u> if $x_i(j) = 1$ for some $j \in S_i$, and $x_i(j') = 0$ for $j' \neq j$. We denote such a pure strategy by $\pi_{i,j}$.
I.e., the “mixed” strategy $\pi_{i,j}$ does not randomize at all: it picks (with probability 1) exactly one strategy, j, from the set of pure strategies for player i.
Given a profile of mixed strategies $x = (x_1, \ldots, x_n) \in X$, let

$$x_{-i} = (x_1, x_2, \ldots, x_{i-1}, \texttt{empty}, x_{i+1}, \ldots, x_n)$$

I.e., x_{-i} denotes everybody's strategy except that of player i.
For a mixed strategy $y_i \in X_i$, let $(x_{-i}; y_i)$ denote the new profile:

$$(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) \quad (x_{-i}; y_i)$$

i的变成 y_i

In other words, $(x_{-i}; y_i)$ is the new profile where everybody's stategy remains the same as in x, except for player i, who switches from mixed strategy x_i, to mixed strategy y_i.

Best Responses

Definition: A (mixed) strategy $z_i \in X_i$ is a **best response** for Player i to x_{-i} if for all $y_i \in X_i$,

$$U_i(x_{-i}; z_i) \geq U_i(x_{-i}; y_i)$$

zi 是 i 的 highest payoff

Clearly, if any player were given the opportunity to "cheat" and look at what other players have done, it would want to switch its strategy to a best response.

Of course, players in a strategic form game can't do that: players pick their strategies simultaneously/independently.

But suppose, somehow, the players "arrive" at a profile where everybody's strategy is a best response to everybody else's.

Then no one has any incentive to change the situation.

We will be in a "stable" situation: an "*Equilibrium*".

That's what a "Nash Equilibrium" is.

每个人都是 best response

Nash Equilibrium

Definition: For a strategic game Γ, a strategy profile $x = (x_1, \ldots, x_n) \in X$ is a mixed **Nash Equilibrium** if for every player, i , x_i is a best response to x_{-i}.
In other words, for every Player $i = 1, \ldots, n$, and for every mixed strategy $y_i \in X_i$,

$$U_i(x_{-i}; x_i) \geq U_i(x_{-i}; y_i)$$

In other words, *no player can improve its own payoff by unilaterally deviating from the mixed strategy profile* $x = (x_1, \ldots, x_n)$.

x is called a **pure Nash Equilibrium** if in addition every x_i is a pure strategy $\pi_{i,j}$, for some $j \in S_i$.

[illegible] pure strategy

Nash's Theorem

This can, agruably, be called

"The Fundamental Theorem of Game Theory"

Theorem(Nash 1950) Every finite *n*-person strategic game has a mixed Nash Equilibrium.

[illegible] finite [illegible] mNE

We will prove this theorem next time.
To prove it, we will "cheat" and use a fundamental result from topology: the Brouwer Fixed Point Theorem.

The crumpled sheet experiment

Let's all please conduct the following experiment:

1. Take two identical rectangular sheets of paper.
2. Make sure neither sheet has any holes in it, and that the sides are straight (not dimpled).
3. "Name" each point on both sheets by its "(x, y)-coordinates".
4. Crumple one of the two sheets any way you like, *but make sure you don't rip it in the process*.
5. Place the crumpled sheet completely on top of the other flat sheet.

Fact! There must be a point named (a, b) on the crumpled sheet that is directly above the same point (a, b) on the flat sheet. (Yes, really!)

As crazy as it sounds, this fact, in its more formal and general form, will be the key to why every game has a mixed Nash Equilibrium.

Algorithmic Game Theory and Applications

Lecture 3: Nash's Theorem

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The Brouwer Fixed Point Theorem

We will use the following to prove Nash's Theorem.

Theorem(Brouwer, 1909) Every continuous function $f : D \to D$ mapping a compact and convex, nonempty subset $D \subseteq \mathbb{R}^m$ to itself has a "fixed point", i.e., there is $x^* \in D$ such that $f(x^*) = x^*$.

不动点

Explanation:

- A "continuous" function is intuitively one whose graph has no "jumps". MLT 隐含定理
- For our current purposes, we don't need to know exactly what "compact and convex" means.

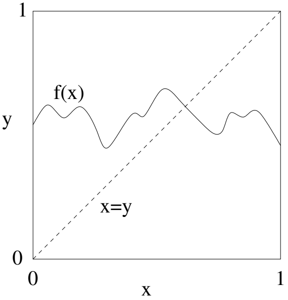
(See the appendix of this lecture for definitions.)

We only state the following fact:

Fact The set of profiles $X = X_1 \times \ldots \times X_n$ is a compact and convex subset of R^m, where $m = \Sigma_{i=1}^{n} m_i$, with $m_i = |S_i|$.

Simple cases of Brouwer's Theorem

To see a simple example of what Brouwer's theorem says, consider the interval $[0, 1] = \{x \mid 0 \leq x \leq 1\}$.
$[0, 1]$ is compact and convex. ($[0, 1]^n$ is also compact & convex.)
For a continuous $f : [0, 1] \rightarrow [0, 1]$, you can "visualize" why the theorem is true. Here's the "visual proof" in the 1-dimensional case:



For $f : [0, 1]^2 \rightarrow [0, 1]^2$, the theorem is already far less obvious: "the crumpled sheet experiment".

brief remarks

- Brouwer's Theorem is a deep and important result in topology.
- It is not very easy to prove, and we won't prove it.
- If you are desperate to see a proof, there are many. See, e.g., any of these:
  - [Milnor'66] (Differential Topology). (uses, e.g., Sard's 数学. Theorem).
  - [Scarf'67 & '73, Kuhn'68, Border'89], uses **Sperner's Lemma**.
  - [Rotman'88] (Algebraic Topology). (uses homology, etc.)
  - [D. Gale'79], ~~possibly my favorite proof:~~ uses the fact that the game of (n-dimensional) HEX is a finite "win-lose" game.

proof of Nash's theorem

Proof: (Nash's 1951 proof)
We will define a continuous function $f : X \to X$, where $X = X_1 \times \ldots \times X_n$, and we will show that if $f(x^*) = x^*$ then $x^* = (x_1^*, \ldots, x_n^*)$ must be a Nash Equilibrium.

By Brouwer's Theorem, we will be done.

(In fact, it will turn out that x^* is a Nash Equilibrium if and only if $f(x^*) = x^*$.)

We start with a claim.

Claim: A profile $x^* = (x_1^*, \ldots, x_n^*) \in X$ is a Nash Equilibrium if and only if, for every player i, and every pure strategy $\pi_{i,j}$, $j \in S_i$:

$$U_i(x^*) \geq U_i(x^*_{-i}; \pi_{i,j}).$$

Proof of claim: If x^* is a NE then, it is obvious by definition that $U_i(x^*) \geq U_i(x^*_{-i}, \pi_{i,j})$.
For the other direction: by calculation it is easy to see that for any mixed strategy $x_i \in X_i$,

$$U_i(x^*_{-i}; x_i) = \sum_{j=1}^{m_i} x_i(j) * U_i(x^*_{-i}; \pi_{i,j})$$

By assumption, $U_i(x^*) \geq U_i(x^*_{-i}; \pi_{i,j})$, for all j.
So, clearly $U_i(x^*) \geq U_i(x^*_{-i}; x_i)$, for any $x_i \in X_i$, because $U_i(x^*_{-i}; x_i) = \sum_{j=1}^{m_i} x_i(j) * U_i(x^*_{-i}; \pi_{i,j}) \leq \sum_{j=1}^{m_i} x_i(j) * U_i(x^*) = U_i(x^*)$.

Hence, each x_i^* is a best response strategy to x^*_{-i}. In other words, x^* is a Nash Equilibrium. $\square$

So, rephrasing our goal, we want to find $x^* = (x_1^*, \dots, x_n^*)$ such that

$$U_i(x_{-i}^*; \pi_{i,j}) \leq U_i(x^*)$$

i.e., such that

$$U_i(x_{-i}^*; \pi_{i,j}) - U_i(x^*) \leq 0$$

for all players $i \in N$, and all $j = 1, \dots, m_i$.
For a mixed profile $x = (x_1, x_2, \dots, x_n) \in X$: let

$$\varphi_{i,j}(x) = \max\{0, U_i(x_{-i}; \pi_{i,j}) - U_i(x)\}$$

Intuitively, $\varphi_{i,j}(x)$ measures "how much better off" player i would be if he/she picked $\pi_{i,j}$ instead of x_i (and everyone else remained unchanged).

Define $f : X \to X$ as follows: For $x = (x_1, x_2, \ldots, x_n) \in X$, let

$$f(x) = (x_1', x_2', \ldots, x_n')$$

where for all i, and $j = 1, \ldots, m_i$,

$$x_i'(j) = \frac{x_i(j) + \varphi_{i,j}(x)}{1 + \sum_{k=1}^{m_i} \varphi_{i,k}(x)}$$

Facts:

1. If $x \in X$, then $f(x) = (x_1', \ldots, x_n') \in X$.
2. $f : X \to X$ is continuous.

(These facts are not hard to check.)

Thus, by Brouwer, there exists $x^* = (x_1^*, x_2^*, \ldots, x_n^*) \in X$ such that $f(x^*) = x^*$.
Now we have to show x^* is a NE.

For each i, and for $j = 1, \ldots, m_i$,

$$x_i^*(j) = \frac{x_i^*(j) + \varphi_{i,j}(x^*)}{1 + \sum_{k=1}^{m_i} \varphi_{i,k}(x^*)}$$

thus,

$$x_i^*(j)(1 + \sum_{k=1}^{m_i} \varphi_{i,k}(x^*)) = x_i^*(j) + \varphi_{i,j}(x^*)$$

hence,

$$x_i^*(j) \sum_{k=1}^{m_i} \varphi_{i,k}(x^*) = \varphi_{i,j}(x^*)$$

We will show that in fact this implies $\varphi_{i,j}(x^*)$ must be equal to 0 for all j.

Claim: For any mixed profile x, for each player i, there is some j such that $x_i(j) > 0$ and $\varphi_{i,j}(x) = 0$.
Proof of claim: For any $x \in X$,

$$\varphi_{i,j}(x) = \max\{0, U_i(x_{-i}; \pi_{i,j}) - U_i(x)\}$$

Since $U_i(x)$ is the "weighted average" of $U_i(x_{-i}; \pi_{i,j})$'s, based on the "weights" in x_i, there must be some j used in x_i, i.e., with $x_i(j) > 0$, such that $U_i(x_{-i}; \pi_{i,j})$ is no more than the weighted average. I.e.,

$$U_i(x_{-i}; \pi_{i,j}) \leq U_i(x)$$

I.e.,

$$U_i(x_{-i}; \pi_{i,j}) - U_i(x) \leq 0$$

Therefore,

$$\varphi_{i,j}(x) = \max\{0, U_i(x_{-i}; \pi_{i,j}) - U_i(x)\} = 0$$

$\square$

Thus, for such a j, $x_i^*(j) > 0$ and

$$x_i^*(j) \sum_{k=1}^{m_i} \varphi_{i,k}(x^*) = 0 = \varphi_{i,j}(x^*)$$

TTDS.

But, since $\varphi_{i,k}(x^*)$'s are all ≥ 0, this means $\varphi_{i,k}(x^*) = 0$ for all $k = 1, \ldots, m_i$. Thus, for all players i, and for $j = 1, \ldots, m_i$,

$$U_i(x^*) \geq U_i(x^*_{-i}; \pi_{i,j})$$

Q.E.D. (Nash's Theorem)

In fact, since $U_i(x^*) = \sum_{j=1}^{m_i} x_i^*(j) \cdot U_i(x^*_{-i}; \pi_{i,j})$ is the "weighted average" of $U_i(x^*_{-i}, \pi_{i,j})$'s, we see that:

Useful Corollary for Nash Equilibria:

$U_i(x^*) = U_i(x^*_{-i}, \pi_{i,j})$, whenever $x_i^*(j) > 0$.

Rephrased: In a Nash Equilibrium x^*, if $x_i^*(j) > 0$ then $U_i(x^*_{-i}; \pi_{i,j}) = U_i(x^*)$; i.e., each such $\pi_{i,j}$ is itself a "best response" to x^*_{-i}.

This is a subtle but very important point. It will be useful later when we want to compute NE's.

Remarks

- The proof using Brouwer gives ostensibly no clue how to compute a Nash Equilibrium. It just says it exists!
- We will come back to the question of computing Nash Equilibria in general games later in the course.
- We start next time with a special case: 2-player zero-sum games (e.g., of the Rock-Paper-Scissor's variety). These have an elegant theory (von Neumann 1928), predating Nash.
- To compute solutions for 2p-zero-sum games, Linear Programming will come into play.
  Linear Programming is a very important tool in algorithms and optimization. Its uses go FAR beyond solving zero-sum games. So it will be a good opportunity to learn about LP.

NE need not be "Pareto optimal"

Given a profile $x \in X$ in an n-player game, the **"(purely utilitarian) social welfare"** is: $U_1(x) + U_2(x) + \ldots + U_n(x)$. A profile $x \in X$ is **pareto efficient** (a.k.a., **pareto optimal**) if there is no other profile x' such that $U_i(x) \leq U_i(x')$ for all players i, and $U_k(x) < U_k(x')$ for some player k.

Note: The Prisoner's Dilemma game shows NE need not optimize social welfare, nor be Pareto optimal.

		Player II	
		Cooperate	Defect
Player I	Cooperate	2, 2	0, 3
	Defect	3, 0	1, 1

Indeed, there is a unique NE, (Defect, Defect), and it neither optimizes social welfare nor is Pareto optimal, because (Cooperate, Cooperate) gives a higher payoff to both players.

application in biology: evolution as a game

- One way to view how we might "arrive" at a Nash equilibrium is through a process of evolution.
- John Maynard Smith (1972-3,'82) introduced game theoretic ideas into evolutionary biology with the concept of an Evolutionarily Stable Strategy.
- Your extra reading (for fun) is from Straffin(1993) which gives an amusing introduction to this.
- Intuitively, a mixed strategy can be viewed as percentages in a population that exhibit different behaviors (strategies).
- Their behaviors effect each other's survival, and thus each strategy has a certain survival value dependent on the strategy of others.
- The population is in "evolutionary equilibrium" if no "mutant" strategy could invade it and "take over".

The Hawk-Dove Game

		Player II	
		Hawk	Dove
Player I	Hawk	-15, -15	50, 0
	Dove	0, 50	25, 25

Large population of same “species”, each behaving as either “hawk” or “dove”.
What proportions will behaviors eventually stabilize to (if at all)?

Definition of ESS

Definition: A 2-player game is **symmetric** if $S_1 = S_2$, and for all $s_1, s_2 \in S_1$, $u_1(s_1, s_2) = u_2(s_2, s_1)$.

Definition: In a 2p-sym-game, mixed strategy x_1^* is an **Evolutionarily Stable Strategy** (ESS), if:

1. x_1^* is a best response to itself, i.e., $x^* = (x_1^*, x_1^*)$ is a symmetric Nash Equilibrium, &
2. If $x_1' \neq x_1^*$ is another best response to x_1^*, then $U_1(x_1', x_1') < U_1(x_1^*, x_1')$.

Nash (1951, p. 289) also proves that every symmetric game has a symmetric NE, (x_1^*, x_1^*). (However, not every symmetric game has a ESS.)

A little justification of the definition of ESS

Suppose x_1^* is an ESS. Consider the "*fitness function*", $F(x_1)$, for a "mutant" strategy x_1' that "invades" (becoming a small $\epsilon > 0$ fraction of) a current ESS population, x_1^*. Then, **Claim**:

$$
\begin{aligned}
F(x_1') &\doteq (1-\epsilon)U_1(x_1', x_1^*) + \epsilon U_1(x_1', x_1') \qquad (1) \\
&< (1-\epsilon)U_1(x_1^*, x_1^*) + \epsilon U_1(x_1^*, x_1') \doteq F(x^*) \qquad (2)
\end{aligned}
$$

Proof: if x_1' is a best response to the ESS x_1^*, then $U_1(x_1', x_1^*) = U_1(x_1^*, x_1^*)$ and $U_1(x_1', x_1') < U_1(x_1^*, x_1')$, and since we assume $\epsilon > 0$, the strict inequality in (2) follows. If on the other hand x_1' is *not* a best response to x_1^*, then $U_1(x_1', x_1^*) < U_1(x_1^*, x_1^*)$, and for a *small enough* $\epsilon > 0$, we have $(1-\epsilon)(U_1(x_1^*, x_1^*) - U_1(x_1', x_1^*)) > \epsilon(U_1(x_1^*, x_1') - U_1(x_1', x_1'))$. Thus again, the strict inequality in (2) follows. □

So, an ESS x_1^* is "strictly fitter" than any other strategy, when it is already dominant in the society. This is the sense in which it is "evolutionarily stable".

Does an ESS necessarily exist?

- As mentioned, Nash (1951) already proved that every symmetric game has a symmetric NE (x^*, x^*).
- However, not every symmetric game has a ESS.
  Example: Rock-paper-scissors:

$$\begin{pmatrix} (0,0) & (1,-1) & (-1,1) \\ (-1,1) & (0,0) & (1,-1) \\ (1,-1) & (-1,1) & (0,0) \end{pmatrix}$$

  Obviously, $s = (1/3, 1/3, 1/3)$ is the only NE. But it is not an ESS: any strategy is a best reponse to s, including the pure strategy s^1 (rock). We have payoff $U(s^1, s^1) = 0 = U(s, s^1)$, so s is not an ESS.
- But many games do have an ESS. For example, in the Hawk-Dove game, $(5/8, 3/8)$ is an ESS.
- Even when a game does have an ESS, it is not at all obvious how to find one.

How hard is it to detect an ESS?

- It turns out that even deciding whether a 2-player symmetric game has an ESS is hard. It is both NP-hard and coNP-hard, and contained in Σ_2^P:
  K. Etessami & A. Lochbihler, "The computational complexity of Evolutionarily Stable Strategies", *International Journal of Game Theory*, vol. 31(1), pp. 93–113, 2008.
  (And, more recently, it has been shown Σ_2^P-complete, see: V. Conitzer, *"The exact computational complexity of Evolutionary Stable Strategies"*, in *Proceeding of Web and Internet Economics* (WINE), pages 96-108, 2013.)
- For simple 2×2 2-player symmetric games, there is a simple way to detect whether there is an ESS, and if so to compute one (described in the reading from Straffin).
- There is a huge literature on ESS and *"Evolutionary Game Theory"*. See, e.g., the book: J. Weibull, *Evolutionary Game Theory*, 1997.

Appendix: continuity, compactness, convexity

Definition For $x, y \in \mathbb{R}^n$, $\texttt{dist}(x, y) = \sqrt{\sum_{i=1}^{n}(x(i) - y(i))^2}$ denotes the Euclidean distance between points x and y.

A function $f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ is **continuous at a point** $x \in D$ if for all $\epsilon > 0$, there exists $\delta > 0$, such that for all $y \in D$: if $\texttt{dist}(x, y) < \delta$ then $\texttt{dist}(f(x), f(y)) < \epsilon$.

f is called **continuous** if it is continuous at every point $x \in D$.

Definition A set $K \subseteq \mathbb{R}^n$ is **convex** if for all $x, y \in K$ and all $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda)y \in K$.

Fact A set $K \subseteq \mathbb{R}^n$ is **compact** if and only if it is **closed** and **bounded**. (So, we need to define "closed" and "bounded".)

Definition A set $K \subseteq \mathbb{R}^n$ is **bounded** iff there is some non-negative integer M, such that $K \subseteq [-M, M]^n$. (i.e., K "fits inside" a finite n-dimensional box.)

Definition A set $K \subseteq \mathbb{R}^n$ is **closed** iff for all sequences $x_0, x_1, x_2, \ldots$, where $x_i \in K$ for all i, such that $x = \lim_{i \rightarrow \infty} x_i$ for some $x \in \mathbb{R}^n$, then $x \in K$. (In other words, if a sequence of points is in K then its limit (if it exists) must also be in K.)