

MATH50006 Week2 and 3

2/1/2022

1 Week 2

1.1 Constructing Measure

1.1.1 Pre-measure

- Maps from an **algebra** to $[0, \infty]$
- $\mu(\emptyset) = 0$
- Countably additive whenever that countable union is in the algebra

1.1.2 Outer-measure

- Maps from 2^X to $[0, \infty]$
- $\mu(\emptyset) = 0$
- Sigma-sub-additive

1.1.3 Cover

A family in 2^X wrt X - Has empty set - Countably many sets' union equals X - An algebra is easily a cover -
e.g. the union of all intervals cover R^n

1.1.4 Extend Pre to Outer

\mathcal{K} is a cover, $\tilde{\mu} : \mathcal{K} \rightarrow [0, \infty]$ function on the sets in this cover with $\tilde{\mu}(\emptyset) = 0$, outer measure for a set is the infimum of the sum of all possible coverings of that set

$$\mu^*(A) = \inf \left\{ \sum_{j=1}^{\infty} \tilde{\mu}(K_j) : K_j \in \mathcal{K}, A \subset \bigcup_{j=1}^{\infty} K_j \right\} \quad A \in 2^X$$

This is well-defined, as $A \subset X$ is always covered by some sequence of sets, to show sigma sub-additivity, $A \subset \bigcup_{k=1}^{\infty} A_k$, and $A_k \subset \bigcup_{j=1}^{\infty} K_{k,j}$, by infimum $\sum_{j=1}^{\infty} \tilde{\mu}(K_{k,j}) \leq \mu^*(A_k) + 2^{-k}\epsilon$, then sum over k

1.1.5 Generate σ -algebra from outer-measure

From outer measure μ^* define:

$$\Sigma = \{A \subset X : \mu^*(B) = \mu^*(B \cap A) + \mu^*(B \setminus A), \forall B \subset X\}$$

The equality can be replaced by \geq by sigma sub-additivity.

1.1.6 Hahn-Caratheodory Extension

Extending a pre-measure to a real measure given an algebra.

Conditions

- X with \mathcal{A} an algebra over it
- $\tilde{\mu} : \mathcal{A} \mapsto [0, \infty]$ a pre-measure
- Define μ^* with \mathcal{A} being the cover, and Σ by previous construction
- Limit μ^* to Σ , then $\mu = \mu^*|_{\Sigma}$ is a measure

Results

- (X, Σ, μ) is a measure space
- $\mathcal{A} \subset \Sigma$
- $\mu(A) = \mu^*(A) = \tilde{\mu}(A), \quad \forall A \in \mathcal{A}$

Proof

- Verify μ is a measure. Already satisfy $\mu(\emptyset) = 0$ and range $[0, \infty]$.
 - To show **finite additivity**, use definition of Σ .
 - To show **sigma additivity**, use sub-additivity of μ^* and other direction of inequality by finite additivity.

- Show $\mathcal{A} \subset \Sigma$.

>- $\forall A \in \mathcal{A}$, show $A \in \Sigma$, equiv to $\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \setminus A)$ >- Use $\mu^*(K) \leq \tilde{\mu}(K)$ >- Use *inf* definition, and take cover of B , $\sum \tilde{\mu}(K_i) \leq \mu^*(B) + \epsilon$ >- by additivity of pre-measure. >-

$$\mu^*(B \cap A) + \mu^*(B \setminus A) \leq \sum \tilde{\mu}(K_i \cap A) + \tilde{\mu}(K_i \setminus A) = \sum \tilde{\mu}(K_i)$$

- Show $\mu^*(A) = \tilde{\mu}(A)$ in \mathcal{A} . >- Only need $\mu^*(A) \geq \tilde{\mu}(A)$ >- Consider cover of A , K_i 's, made into disjoint \tilde{K}_i 's, intersected with A , $\tilde{\tilde{K}}_i$'s. >-

$$\tilde{\mu}(A) = \sum \tilde{\mu}(\tilde{\tilde{K}}_i) \leq \sum \tilde{\mu}(\tilde{K}_i) \leq \sum \tilde{\mu}(K_i)$$

>- Finish by taking infimum.

Proof of uniqueness

Assume there is another pre-measure with $\nu|_{\mathcal{A}} = \tilde{\mu}$, then we show their extensions are equal, namely on the same sigma algebra, $\nu|_{\Sigma} = \mu$

- $\nu(A) \leq \mu(A)$

Taking cover as usual,

$$\nu(A) \leq \sum \nu(K_i) = \sum \tilde{\mu}(K_i)$$

Taking infimum, obtain $\nu(A) \leq \mu^*(A) = \mu(A)$

- $\nu(A) \geq \mu(A)$

Suppose first S finite

$$\nu(A) + \nu(S \setminus A) \leq \mu(A) + \mu(S \setminus A) = \mu(S) = \tilde{\mu}(S) = \nu(A) \leq \nu(A) + \nu(S \setminus A)$$

So

$$\mu(A) = \nu(A) + [\nu(S \setminus A) - \mu(S \setminus A)] \leq \nu(A)$$

For case S infinite, use disjoint covering sets of A

$$\nu(A) \geq \lim \nu(\cup K_i) = \lim \mu(\cup K_i) = \mu(A)$$