MATH50006 Week 3 and 4

18/02/2022

1 The Lebesgue Measure

Defined as the Hahn-Caratheodory extension of pre-measure $\tilde{\lambda}$ on $\mathcal{A}$ the **algebra** of collections of **elementray figures**

$(a, b) = \prod(a_k, b_k)$, with $a_k < b_k$ is the **interval**, whose finite disjoint union forms **elementary figures**

$$\tilde{\lambda}([a, b]) = b - a$$

$\tilde{\lambda}(\cup I) = \sum \tilde{\lambda}(I_i)$ for disjoint I_j, is a pre-measure

1.0.1 $\tilde{\lambda}$ is a Pre-measure

- Only need to verify sigma additivity*
- $\tilde{\lambda}(I) \leq \sum^{\infty} I_i$ already by taking limits of finite additivity
- The other direction is by compactness argument
- Case where I is interval first
- Take $\bar{I}_L = \bar{I} \cap [-L, L]^N$ to ensure boundedness, hence compactness, and for taking L to ∞
- Aim to find an open cover for $\bar{I}_L$
- Operate on the I_i's, cover by I_i^ϵ, then cover I_i^ϵ by open set $\tilde{I}_i^\epsilon$, where
- $\tilde{\lambda}(\tilde{I}_i^\epsilon) \leq (1 + \epsilon)^n \lambda(I_i) + \epsilon\ 2^{-i}$
- So can take finite subcover formed by $\tilde{I}_i^\epsilon$
- $\tilde{\lambda}(\bar{I}_L) \leq (1 + \epsilon)^n \sum \lambda(I_i) + \epsilon$, taking first ϵ to 0 then $L \to \infty$

1.0.2 Borel sigma algebra contained in extension algebra

$$\mathcal{B}(\mathbb{R}^n) \subset \Sigma$$

where Σ is defined from extension thm. - Note from extension thm., $\mathcal{A} \subset \Sigma$ - So suffice to show for open set $\mathcal{O} \in \mathcal{B}(\mathbb{R}^n)$, $\mathcal{O} \in \sigma(\mathcal{A})$ - Choose half open cubes $[\epsilon, \epsilon + 2^{-m})$ at each step in $\mathcal{O}$ and disjoint with previous (finitely many each step)

- Above result finishes Lebesgue measure construction as original domain is Σ and borel algebra
-

1.1 Restriction of a Measure

Restriction to $A \subset \mathcal{F}$ in a measure space $(X, \mathcal{F}, \mu)$ with the $\sigma - algebra$, $\mathcal{F}|_A = \{A \cup B : B \subset \mathcal{F}\}$

Example: $A = [0, 1]$, with Lebesgue measure is a probability measure, and for $[a, b]$, use $\frac{1}{b-a}\lambda(A)$.

-

1.2 Properties of Lebesgue Measure

1.2.1 Approximating measure

For all $A \in \mathcal{B}(\mathbb{R}^n)$,

$$\lambda(A) = \inf_{A \subset G} \lambda(G), \qquad G \quad \text{open}$$

Proof by
- $\lambda(A) \leq \lambda(G)$ from monotonicity

- Show $\lambda(A) \geq \lambda(G)$ using property $\lambda^* = \lambda$, and take open cover $\lambda(G) \leq \sum \lambda(\tilde{I}_j) \leq \sum \lambda(I_j) + \epsilon$, where the open sets are chosen similarly to above, finally by definition of $\sum \lambda(I_j) \leq \lambda^*(A) + \epsilon$; take to zero

It also follows there is $\lambda(G \setminus A) < \epsilon$, for some G

More commonly it's closed set with in $F \subset A \subset G$, s.t. $\lambda(G \setminus F) < \epsilon$

1.2.2 Approximating with general extended measure

For all $A \in \sigma(A)$, we have mutually disjoint sets A_i with

$$\mu(\cup_{i=1}^{\infty} A_i \setminus A) < \epsilon$$

Translational Invariance
Define the translation of sets as $\Phi_{x_0}(A) = A + x_0$ in borel sigma algebra,

$$\lambda(\Phi_{x_0}(A)) = \lambda(A)$$

Proof by considering intervals first, then open sets written as union of intervals, finally use open set approximation for arbitrary sets.

1.2.3 Vitali Sets