Functional Analysis

Typed by: tw1320@ic.ac.uk

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1 Preliminaries

1.1 Norms and Metrics

Definition 1.1. (Metric) Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}^+$ satisfying the following is called a **metric**

- (Positive definitiness) $\forall x, y \in X, d(x, y) \geq 0$ if $x \neq y$ and $d(x, y) = 0 \iff x = y$
- (Symmetry) $\forall x, y \in X, d(x, y) = d(y, x)$
- (Triangle-inequality) $\forall x, y, z \in X, d(x, y) \leq d(x, z) + d(z, y)$

Definition 1.2. (Translation invariant) A metric d is **translation invariant** if $\forall x, y \in X, d(x, y) = d(x + a, y + a)$ for all $a \in X$.

To introduce the idea of a metric linear space, we need to define metrics on product spaces.

Definition 1.3. (Metric on Product Spaces) Given metric ρ on a vector space V over $\mathbb{K}$, a metric on $V \times V$ is defined by:

$$d((a, b), (c, d)) = (\rho(a, c)^p + \rho(b, d)^p)^{1/p}, p \in [1, \infty)$$

and on $\mathbb{K} \times V$ by:

$$d((\lambda, a), (\lambda', a')) = \max\{|\lambda - \lambda'|, \rho(a, a')\}$$

Definition 1.4. (Metric Linear Spaces) A pair (X, d) with X being a linear space over $\mathbb{K}$ and d being a metric is called a **metric linear space** if and only if addition and multiplication by scalar are continuous.

In other words, the following are true:

- $x_n \to x, \quad y_n \to y \implies x_n + y_n \to x + y$
- $\lambda_n \to \lambda, \lambda_n, \lambda \in \mathbb{K}, x_n \to x \implies \lambda_n x_n \to \lambda x$

It is easily verified and if d is translation invariant, then addition of vectors is continuous: namely, $d(x_n + y_n, x + y) = d(x_n - x, y - y_n) \leq d(x_n - x, 0) + d(y_n - y, 0)$. However, a translation invariant metric does not guarantee that multiplication by scalar is continuous.

Example 1.5. Let X be the space of all sequences in $\mathbb{R}$ and $d(x, y) = \sup_{i \in \mathbb{N}} |x^i - y^i|^{1/i}$, where the x^i denotes the i^{th} element of the sequence x. Then d is a metric on X and it is translation invariant.

Take $(x_n^i)_{i \in \mathbb{N}} = (a)_{i \in \mathbb{N}}$, a constant sequence with $a > 1$, and a scalar $\lambda_n = \xi^n, \xi \in (0, 1)$, so that $\lambda_n \to 0$ and $\lambda_n x_n \to 0$.

$$d(\lambda_n x_n, 0) = \sup_{i \in \mathbb{N}} |\xi|^{n/i} |a|^{1/i} \geq 1$$

So multiplication by scalar is not continuous.

Definition 1.6. (Norm) Let X be a nonempty set. A function $|| \cdot || : X \to \mathbb{R}^+$ satisfying the following is called a **norm**:

- (Positive definitiness) $\forall x \in X, ||x|| \geq 0$ and $||x|| = 0 \iff x = 0$
- (Triangle-inequality) $\forall x, y \in X, ||x + y|| \leq ||x|| + ||y||$
- (Homogeneity) $\forall x \in X, \forall \lambda \in \mathbb{K}, ||\lambda x|| = |\lambda| ||x||$

Remark 1.7. Norm is a continuous function.

Definition 1.8. (Normed Linear Spaces) A pair $(X, || \cdot ||)$ with X being a linear space over $\mathbb{K}$ and $|| \cdot ||$ being a norm is called a **normed linear space**

Note that every normed linear space is a metric space, since every norm can induce a metric by $d(x,y) = ||x-y||$. However, not every metric is a norm.

Example 1.9. Let X be the space of all sequences in $\mathbb{R}$ and $z > 1$. A translation invariant metric d is defined by

$$d(x,y) = \sum_{i=1}^{\infty} z^{-n} \frac{|x_i - y_i|}{1 + |x_i - y_i|}$$

But d is not a norm, as it is not homogenous.

Example 1.10. Let $X = \mathbb{R}$ be the real numbers and $|\cdot|$ the Euclidean norm. Another example of a metric that is not a norm is given by:

$$d(x,y) = \min\{|x-y|, 1\}$$

this is not a norm because it is not homogenous. (Note also that it is not translation invariant.)

1.2 Common Spaces

l_p **Spaces** For $p \in [1, \infty)$, the space l_p is defined as the set of all sequences $(x_n)_{n\in\mathbb{N}}$ such that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty$$

the function

$$||x||_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

defines a norm on l_p.

l_∞ **Spaces** The space l_∞ is defined as the set of all sequences $(x_n)_{n\in\mathbb{N}}$ such that

$$\sup_{n\in\mathbb{N}} |x_n| < \infty$$

the function

$$||x||_\infty = \sup_{n\in\mathbb{N}} |x_n|$$

defines a norm on l_∞.

1.3 Inequalities

Proposition 1.11. (Young)

Corollary 1.12. *(i)(Hölder) (ii)(Minkowski)*

Proposition 1.13. (Jensen)

Proposition 1.14. (Equivalent forms of Jensen)

2 Completeness and Separability

3 Hilbert Spaces

4 Finite Dimensional Spaces

5 Linear Operators

6 Dual Spaces

7 The Hahn Banach Theorems

8 The Uniform Boundedness Theorem

8.1 Baire's Category Theorem

9 The Open Mapping Theorem

10 The Closed Graph Theorem

11 Compact Operators