Markov Chains

30 October, 2022

1 Basics

1.1 Chapman-Kolmogorov (CK) equations

The **n-step** transition probability is

$$p_{ij}(n) = \Pr(X_{m+n} = j | X_m = i)$$

1.2 First passage and hitting times

The **first passage time** is

$$T_j = \min\{n \in \mathbb{N} : X_n = j\}$$

In other words, $\{T_j = n\} = \{X_n = j, X_{n-1} \neq j, \cdots, X_1 \neq j\}$, if $X_n \neq j, \forall n \in \mathbb{N}$, then $T_j = \infty$.

The **first passage probability** is

$$f_{ij}(n) = \Pr(T_j = n | X_0 = i), n \in \mathbb{N}_0$$

from which the hitting probability follows

$$f_{ij} = \Pr(T_j < \infty | X_0 = i) = \sum_{n=0}^{\infty} f_{ij}(n)$$

With the special case being $f_{ij}(0) = 0$.

1.3 Generating Functions of Markov Chain

Recall the **probability generating function**

$$G_X(s) = \sum_{x=0}^{\infty} s^x \Pr(X = x)$$

where this holds on the support

$$\mathcal{S}_{\mathcal{X}} = \left\{ s \in \mathbb{R} : \sum_{x=0}^{\infty} |s|^x \Pr(X = x) < \infty \right\}$$

The generating functions here are

$$G_{p_{ij}(n)} = \sum_{n=0}^{\infty} p_{ij}(n) s^n$$
$$G_{f_{ij}(n)} = \sum_{n=0}^{\infty} f_{ij}(n) s^n$$

By arguing using equating coefficients and an identity, we have a **theorem**

$$G_{p_{ij}(n)} = \delta_{ij} + G_{f_{ij}(n)}(s) G_{p_{ij}(n)}$$

The identity used is

$$p_{ij}(n) = \sum_{l=1}^{n} f_{ij}(l) p_{jj}(n-l)$$

2 Recurrence and Transience

A state j is **recurrent** if and only if

$$\sum_{n=1}^{\infty} p_{jj}(n) = \infty$$

A state j is **transient** if and only if

$$\sum_{n=1}^{\infty} p_{jj}(n) < \infty$$

Examples: Examples of transient, irreducible chains

The **number of periods** that the chain is in state j (or **number of visits** to j) is

$$N_j = \sum_{n=1}^{\infty} I_n(j)$$

where $I_n(j)$ is the indicator function taking value 1 if $X_n = j$ and 0 otherwise.

The **expected number of visits** to state j given $X_0 = j$ is

$$\mathbb{E}[N_j | X_0 = j] = \sum_{n=0}^{\infty} p_{jj}(n)$$

proof using generating functions:

Taking $s \to 1$ and using Abel's theorem, we can deduce…

2.1 Properties of recurrent/transient states

Theorem (Number of visits is geometric for transient states)

If j is transient, then

$$\Pr(N_j = n | X_0 = j) = f_{jj}^{n-1}(1 - f_{jj}), n \in \mathbb{N}$$

Let $i \neq j$, then

$$\Pr(N_j = n | X_0 = i) = \begin{cases} 1 - f_{ij} & n = 0 \\ f_{ij} f_{jj}^{n-1}(1 - f_{jj}) & n \geq 1 \end{cases}$$

Intuition is that the chain visits j for the first time and returns to it for $n-1$ times, then leaves it.

Therefore, it follows that for $i \neq j$,

$$\mathbb{E}[N_j | X_0 = i] = \frac{f_{ij}}{1 - f_{jj}}$$

and

$$\mathbb{E}[N_j | X_0 = j] = \frac{1}{1 - f_{jj}}$$

Theorem(Unlikely to visit a transient state)

If j is transient, then

$$\lim_{n \to \infty} p_{ij}(n) = 0, \forall j \in E$$

2.2 Mean recurrence time, null and positive recurrence

The **mean recurrence time** μ_j is

$$\mu_j = \mathbb{E}[T_j | X_0 = j] = \sum_{n=1}^{\infty} n f_{jj}(n)$$

where we recall that $\{T_j = n\} = \{X_n = j, X_{n-1} \neq j, \cdots, X_1 \neq j\}$.

Similarly, we can define the **mean first passage time**

$$\mu_{ij} = \mathbb{E}[T_j | X_0 = i] = \sum_{n=1}^{\infty} n f_{ij}(n)$$

those expectations can be finite or infinite; for transient states, they must be infinite.

Theorem(mean first passage time)

For a recurrent state j, it is called **null recurrent** if $\mu_j = \infty$ and **positive recurrent** if $\mu_j < \infty$.

Theorem (unlikely to visit null recurrent state) If j is null recurrent, then

$$\lim_{n \to \infty} p_{jj}(n) = 0, \forall j \in E$$

In addition,

$$\lim_{n\to\infty} p_{ij}(n) = 0, \forall i \neq j \in E$$

2.3 Examples

3 Aperiodicity and Ergodicity

The **period** of a state j is

$$d(j) = gcd\{n : p_{jj}(n) > 0\}$$

It is not necessarily true that $p_{jj}(d(j)) > 0$ (cf. Notes Pg. 36).

A state is **ergodic** if it is positive recurrent and aperiodic.

4 Communicating classes

We say that a state j is **accessible** from state i if the chain can reach j at some time, written as $i \to j$.
Two states i and j are **communicating** if there exists a state k such that $i \to k$ and $k \to j$, we write $i \leftrightarrow j$; this is an **equivalence relation**.
If $i \neq j$, then $i \to j$ if and only if $f_{ij} > 0$.

4.1 Properties preserved by Communicating Classes

- Same period
- Same transience/recurrence
- Null recurrence

For a **set of states** C:

- C is **closed** if $\forall i \in C$, $j \notin C$, $p_{ij} = 0$
- C is **irreducible** if all states in the set communicate with each other

Therefore, an irreducible set of states share the same properties described above.

Theorem (Recurrence and closed) If C is a communicating class of recurrent states, then C is closed.

Theorem (Stochastic matrix on closed states) The stochastic matrix P restricted to a closed set of closed states C is still a stochastic matrix.

4.2 Decomposition of Chains

The state space can be partitioned into communicating classes.

$$E = T \cup \left(\bigcup_i C_i \right)$$

where T is the set of transient states and C_i's are irreducible closed sets of recurrent states.

4.3 Class Properties

The **classes** refer to communicating classes.

Theorem (Finite Chains have recurrent) When state space is **finite**, at least one state is *recurrent* and all *recurrent* states are **positive**

Remark This combined with later results on stationarity makes a chain with finite state space particularly nice.

Remark It follows that there are no null recurrent states in a finite state space.

Theorem (Finite and closed) If C is a finite, closed communicating class, then all states are positive recurrent.

Communicating class properties

Type of Class	Finite	Infinite
Closed	positive recurrent	positive/null recurrent, transient
Not closed	transient	transient

5 Gambler's Ruin

6 Stationarity

We are interested in the equilibrium states of a chain

6.1 Distribution

- Distribution is a row vector λ with $\Sigma_j \lambda_j = 1$
- If $\lambda P = \lambda$ then it is called *invariant*

6.2 Stationary distributions of irreducible chains

Theorem Every irreducible chain has a **stationary distribution** π if and only if all states are **positive recurrent** - π is unique - $\pi = \mu_i^{-1}$ the inverse of mean recurrence time

We first have some lemmas:

$$l_{ji}(n) = \Pr(X_n = i, T_j \geq n | X_0 = j)$$

being the probability that the chain reaches i in n steps without returning to j
Lemma

$$f_{jj}(m+n) = \sum_{i \in E, i \neq j} l_{ji}(m) f_{ij}(n)$$

from which $f_{jj}(m+n) \geq l_{ji}(m) f_{ij}(n)$ follows

Lemma We also have the following recurrence relation

Lemma: A positive recurrent chain has a stationary distribution.

Proof: (constructive)

- **(Step1 Construction)** Let $N_i(j)$ be the number of visits to state i before state j ; the sum of such numbers over i is equal to the hitting time T_j

- Define $\rho_i(j)$ to be the expected number of visits to the state i between two successive visits to state j (in this step the **recurrence** of the chain is used, as the T_j is finite with probability 1)

$$\begin{aligned}\rho_i(j) &= \mathbb{E}[N_i(j)|X_0 = j] \\ &= \sum_n \Pr(X_n = i, T_j \geq n|X_0 = j) \\ &= \sum_n l_{ij}(n)\end{aligned}$$

- Now the mean hitting time can be computed as

$$\begin{aligned}\mu_j &= \mathbb{E}\big[\sum_i N_i(j)|X_0 = j\big] \\ &= \sum_i \rho_i(j)\end{aligned}$$

- which can be written as sum of $\rho_i(j)$ by Tonelli and linearity of conditional expectation
- **(Step2 Finiteness)** Use a lemma to bound $\rho_i(j)$ so it's finite
- Namely write $\rho_i(j) = \sum_n l_{ji}(n)$ and bound using the fact that the chain is irreducible, so there exists $f_{ij}(n^*) > 0$, so $f_{jj}(m + n^*) \geq l_{ji}(m)f_{ij}(n^*)$
- **(Step3 Stationarity)** Use a recurrence to show

$$\begin{aligned}\rho_i(j) &= \sum_n l_{ji}(n) \\ &= p_{ji} + \sum_{n=2} \sum_{r,r\neq j} p_{ri}l_{jr}(n-1) \\ &= p_{ji}\rho_i(j) + \sum_{n=1} \sum_{r,r\neq j} p_{ri}l_{jr}(n) \\ &= p_{ji}\rho_i(j) + \sum_{r,r\neq j} p_{ri} \sum_{n=1} l_{jr}(n) \\ &= \sum_r \rho_r(j)p_{ri}\end{aligned}$$

- This $\rho_i(j)$ does not necessarily give a probability vector when the chain is not positive recurrent.
- Now if the chain is positive recurrent, we have μ_j finite for every j, we have

$$\pi_i = \frac{\rho_i(j)}{\mu_j}$$

Lemma If a stationary distribution exists, then the chain is positive recurrent and the distribution must be given by $\pi_i = \mu_i^{-1}$

proof: ...

6.3 Limiting Distribution

6.4 Ergodic Theorem

6.5 Summary of properties of irreducible chains

7 Time reversibility