

# Continuous Time Stochastic Processes

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## 1 Preliminaries

### 1.1 Types of Processes

A **right continuous** stochastic process  $\{X_t\}_{t \geq 0}$  is such that for any  $\omega \in \Omega$  and  $t \geq 0$ , there is  $\varepsilon > 0$ , such that

$$X_t(\omega) = X_s(\omega) \quad \forall s \in [t, t + \varepsilon]$$

Can be thought as closed point on the left and open point on the right.

There are three types of right continuous processes

- **Normal:** infinitely many jumps but only finitely many in a finite time interval
- **Absorption:** Only has finitely many jumps, gets absorbed at some point (stay at one state)
- **Explosion:** Infinitely many jumps in a finite time interval.

The **jump times** are random variables  $J_{n+1} = \inf\{t \geq J_n : X_t \neq X_{J_n}\}$ .

The **holding times** are random variables defined as:

$$H_n = \begin{cases} J_n - J_{n-1} & \text{if } J_{n-1} < \infty \\ \infty & \text{otherwise} \end{cases}$$

from which it follows that  $J_n = \sum_{i=1}^n H_i$ .

The **explosion time** is

$$J_\infty = \sup_{n \in \mathbb{N}_0} J_n = \sum_{n=1}^{\infty} H_n$$

A **jump process** or jump chain is a discrete time stochastic process  $Z_n = X_{J_n}$ , where  $J_n$  is the  $n$ th jump time.

#### 1.1.1 Relating continuous process to its jump process

A **counting process** is a stochastic process  $\{N_t\}_{t \geq 0}$  satisfying

- $N_0 = 0$
- $\forall t \geq 0, N_t \in \mathbb{N}_0$
- (Non-decreasing) If  $0 \leq s \leq t$ ,  $N_s \leq N_t$
- (Counting) When  $s < t$ ,  $N_t - N_s$  equals the no. of events in  $(s, t]$

- (Right continuous) The process is piecewise constant and has upward jumps (single step) of size 1, therefore

$$N_{t-} = \lim_{s \uparrow t} N_s$$

A **counting process** associated the sequence  $(J_n)_{n \in \mathbb{N}_0}$

## 1.2 Properties of random variables

The **Poisson random variable** has (pmf)

$$f_X(x) = \frac{\lambda^x}{x!} e^{-\lambda}, \quad x \in \mathbb{N}_0$$

It has expectation

$$\mathbb{E}[X] = \lambda$$

and variance

$$\text{Var}[X] = \lambda$$

The **exponential random variable** has

$$f_X(x) = \lambda e^{-\lambda x}$$

and c.d.f.

$$F_X(x) = 1 - e^{-\lambda x}$$

with a nonnegative support.

It has expectation

$$\mathbb{E}[X] = \frac{1}{\lambda}$$

and variance

$$\text{Var}[X] = \frac{1}{\lambda^2}$$

The **memoryless property** of a random variable refers to the fact:

$$\Pr(X > x + y \mid X > x) = \Pr(X > y)$$

- A continuous random variable is memoryless iff it is  $\text{Exp}(\lambda)$
- A discrete random variable is memoryless iff it is  $\text{Geom}(p)$

The **sum of exponential**  $\text{Exp}(\lambda)$  is a  $\text{Gamma}(n, \lambda)$  distribution

$$f_{J_n}(t) = \frac{\lambda^n}{\Gamma(n)} t^{n-1} e^{-\lambda t}, \quad t > 0$$

The **convergence for infinite sum of exponential** has the following criteria

- If  $\sum \frac{1}{\lambda_i} < \infty$ , then  $\Pr(J_\infty < \infty) < 1$
- If  $\sum \frac{1}{\lambda_i} = \infty$ , then  $\Pr(J_\infty = \infty) = 1$

The **minimum of exponential** is

$$H \sim \text{Exp}\left(\sum_{i=1}^n \lambda_i\right)$$

and the probability of any of the  $k$  variables being the minimum is

$$\Pr(H = H_k) = \frac{\lambda_k}{\sum_{i=1}^n \lambda_i}$$

The **Laplace Transform** of a random variable  $X$  is given by

$$\mathcal{L}_X(u) = \mathbb{E}[e^{-uX}]$$

A list of transformations for common random variables:

- (Poisson)  $\exp(\lambda t[e^{-u} - 1])$
- (Exponential)  $\frac{\lambda}{\lambda + u}$

The **characteristic function** of a random variable  $X$  is given by

$$\phi_X(t) = \mathbb{E}[e^{itX}]$$

## 2 Poisson Processes

### 2.1 Definitions

A **Poisson process**, denoted  $\{N_t\}_{t \geq 0}$ , is a non-decreasing stochastic process with nonnegative values satisfying

- $N_0 = 0$
- The increments are independent,  $0 \leq t_0 \leq t_1 \leq \dots \leq t_n$ , the random variables  $N_{t_0}, N_{t_1} - N_{t_0}, \dots, N_{t_n} - N_{t_{n-1}}$  are independent
- The increments are stationary

$$\Pr(N_t - N_s = k) = \Pr(N_{t-s} = k)$$

- There is a single arrival (only one arrives in a small interval), for all  $t \geq 0$  and  $\delta > 0$ ,  $\delta \rightarrow 0$

$$\begin{aligned}
\Pr(N_{t+\delta} - N_t = 1) &= \lambda\delta + o(\delta) \\
\Pr(N_{t+\delta} - N_t \geq 2) &= o(\delta) \\
\Pr(N_{t+\delta} - N_t = 0) &= 1 - \lambda\delta + o(\delta)
\end{aligned}$$

This also ensures that a Poisson process is continuous in probability.

An **equivalent definition** replaces the last condition with the variable being Poisson with rate  $N_t$

$$\Pr(N_t = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

Another **equivalent definition** characterizes Poisson process  $\{N_t\}_{t \geq 0}$  explicitly

- Let  $H_1, H_2, \dots$  denote i.i.d.  $\text{Exp}(\lambda)$  random variables
- Let  $J_0 = 0$  and  $J_n = \sum_{i=1}^n H_i$
- We define

$$N_t = \sup\{n \in \mathbb{N}_0 : J_n \leq t\}$$

## 2.2 Properties of Poisson Process

### 2.2.1 Inter-arrival times

The inter-arrival times are **i.i.d.**  $\text{Exp}(\lambda)$  random variables

### 2.2.2 Time to $n^{\text{th}}$ event

The time to  $n^{\text{th}}$  event is defined as

$$J_n = \sum_{i=1}^n H_i$$

which follows a  $\text{Gamma}(n, \lambda)$  distribution

$$f_{J_n}(t) = \frac{\lambda^n}{\Gamma(n)} t^{n-1} e^{-\lambda t}, \quad t > 0$$

### 2.2.3 Conditional distribution of arrival times

The conditional joint density of  $(J_1, \dots, J_n)$  is given by the order statistic

$$f_{(J_1, \dots, J_n)}(t_1, \dots, t_n \mid N_t = n) = \begin{cases} \frac{n!}{t^n} & 0 < t_1 < \dots < t_n \\ 0 & \text{otherwise} \end{cases}$$

The expectation of the  $k^{\text{th}}$  value of  $n$  uniformly distributed order statistics on  $[0, t]$  is

$$\mathbb{E}[X_{(k)}] = \frac{tk}{n+1} = \mathbb{E}[J_k \mid N_t = n]$$

## 2.3 Extensions to Poisson Processes

### 2.3.1 Superposition

Given  $n$  independent Poisson processes  $\{N_t^{(1)}\}_{t \geq 0}, \dots, \{N_t^{(n)}\}_{t \geq 0}$ , with respective rates  $\lambda_1, \dots, \lambda_n > 0$ ,

$$N_t = \sum_{i=1}^n N_t^{(i)}$$

is also a Poisson process with rate  $\lambda = \sum_{i=1}^n \lambda_i$ .

This is called a **superposition of Poisson processes**.

### 2.3.2 Thinning

- Each arrival of a Poisson Process  $\{N_t\}_{t \geq 0}$  is marked as a type  $k$  event with probability  $p_k$ , for  $k = 1, \dots, n$ , where  $\sum_{k=1}^n p_k = 1$ .
- Then let  $N_t^{(k)}$  denote the number of type  $k$  events up to time  $t$  (in  $[0, t]$ ).
- Every  $N_t^{(k)}$  is a Poisson process with rate  $\lambda p_k$ .

Each process is called a **thinned Poisson Process**.

## 2.4 Non-homogenous Poisson processes

Let  $\lambda : [0, \infty) \rightarrow (0, \infty)$  denote a non-negative and locally integrable function. Then the process  $N = \{N_t\}_{t \geq 0}$  is a **non-homogenous Poisson process** with intensity function  $\lambda(t)$  if

- $N_0 = 0$
- $N$  has independent increments
- Single arrival; for all  $t \geq 0$  and  $\delta > 0$ ,

$$\begin{aligned}\Pr(N_{t+\delta} - N_t = 1) &= \lambda(t)\delta + o(\delta) \\ \Pr(N_{t+\delta} - N_t \geq 2) &= o(\delta)\end{aligned}$$

Each  $N_t$  follows a **Poisson distribution with rate  $m(t)$** , where

$$m(t) = \int_0^t \lambda(s) ds$$

The stationarity also changes. We have

$$N_t - N_s \sim \text{Poisson}\left(\int_s^t \lambda(u) du\right) = \text{Poisson}(m(t) - m(s))$$

### 2.4.1 Deriving the forward equations

An important technique for deriving concrete probability mass functions using the single arrival property.

$$\begin{aligned}
p_n(t + \delta) &= \Pr(N_{t+\delta} = n) = \sum_{k=0}^n \Pr(N_{t+\delta} = n \mid N_t = k) \Pr(N_t = k) \\
&= \sum_{k=0}^n \Pr(N_{t+\delta} - N_t = n - k \mid N_t = k) \Pr(N_t = k) \\
&= \sum_{k=0}^n \Pr(N_{t+\delta} - N_t = n - k) \Pr(N_t = k) \\
&= (1 - \lambda(t)\delta)p_n(t) + \lambda(t)\delta p_{n-1}(t) + o(\delta)
\end{aligned}$$

Note the use of independence of increments and the single arrival property.

This gives the differential equation

$$\frac{dp_n(t)}{dt} = \lambda(t)p_{n-1}(t) - \lambda(t)p_n(t)$$

When  $n = 0$ ,

$$\frac{dp_0(t)}{dt} = -\lambda(t)p_0(t)$$

## 2.5 Compound Poisson processes

Let  $\{N_t\}_{t \geq 0}$  be a Poisson process with rate  $\lambda > 0$  and  $\{Y_n\}_n$  be a sequence of identically, indepdently distributed random variables that are also *independent* of  $\{N_t\}_{t \geq 0}$ .

$$S_t = \sum_{n=1}^{N_t} Y_n$$

$\{S_t\}_{t \geq 0}$  is a **compound Poisson process**.

The mean and variance of  $S_t$  are

$$\begin{aligned}
\mathbb{E}[S_t] &= \lambda t \mathbb{E}[Y_1] \\
\text{Var}[S_t] &= \lambda t \mathbb{E}[Y_1^2]
\end{aligned}$$

This is proven by conditioning on  $N_t$  and using the fact that  $Y_n$  are independent.

We also recall the laws of total expectation and total variance.

$$\begin{aligned}
\mathbb{E}[S_t] &= \mathbb{E}[\mathbb{E}[S_t \mid N_t]] \\
\text{Var}[S_t] &= \mathbb{E}[\text{Var}[S_t \mid N_t]] + \text{Var}[\mathbb{E}[S_t \mid N_t]]
\end{aligned}$$

## 2.6 Cramer-Lundberg

An application of the compound Poisson process is the Cramer-Lundberg model.

For an insurance company, there are - **Claims**  $S_t$  (expense to pay when there are accidents) modelled by a **compound Poisson process**

- **Initial capital**  $u$
- **Premiums**  $ct$  (money collected from customers with rate  $c$ )

We define the **risk process** to be

$$U_t = u + ct - S_t, \quad t \geq 0$$

The company goes bankrupt if  $U_t < 0$ .

Thus, the **ruin probability** is defined as

$$\psi(u, T) = \Pr(U_t < 0 \text{ for some } t \leq T), \quad T > 0, u \geq 0$$

The **total claim amount**  $\{S_t\}_{t \geq 0}$  is

$$S_t = \begin{cases} \sum_{n=1}^{N_t} Y_n & \text{if } N_t \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

where  $N_t$  is a Poisson process with rate  $\lambda$  and  $Y_n$  are independent and identically distributed random variables with finite mean  $\mu$  and variance  $\sigma^2$ .

We can compute the expected value of the risk process.

$$\mathbb{E}[U_t] = u + ct - \lambda t \mu$$

Therefore, a minimal requirement for this company to choose premium rate could be

$$c > \lambda \mu$$

this is called the **net profit condition**.

## 2.7 Coalescent Process

The coalescent process describes the merging of  $n$  offspring into a single ancestor occurring at random times.

- We have  $n$  individuals at time  $t = 0$
- Each pair of individuals merge according to a Poisson process with rate  $\lambda = 1$  and there are  $\binom{n}{2}$  pairs
- The time of first coalescence follows  $\text{Exp}(\binom{n}{2})$  distribution
- There are  $n - 1$  coalescences
- The process is in fact a death process

We can compute the time to the most recent common ancestor (i.e. the time of the last coalescence).

$$\mathbb{E} \left( \sum_{k=1}^{n-1} H_k \right) \quad n \in \mathbb{N}, n \geq 2$$

with

$$H_k \sim \text{Exp} \left( \binom{n - (k - 1)}{2} \right)$$

So it follows that

$$\mathbb{E} \left( \sum_{k=1}^{n-1} H_k \right) = \sum_{k=1}^{n-1} \frac{2}{k(k+1)} = 2 \left( 1 - \frac{1}{n} \right)$$

Comparing with the last coalescence time, we have

$$\mathbb{E}(H_{n-1}) = 1 > 2 \left( 1 - \frac{1}{n} \right)$$

showing that the last coalescence time is larger than half of the expected total coalescence time.

### 3 Continuous-time Markov chains

A continuous-time stochastic process  $\{X_t\}_{t \in [0, \infty)}$  satisfies the **Markov property** if

$$\Pr(X_{t_n} = j \mid X_{t_1} = i_1, \dots, X_{t_{n-1}} = i_{n-1}) = \Pr(X_{t_n} = j \mid X_{t_{n-1}} = i_{n-1})$$

for all  $j, i_1, \dots, i_{n-1} \in E$  and for **any** sequence  $0 \leq t_1 < \dots < t_n < \infty$ .

The **transition probability** is  $p_{ij}(s, t)$ , for  $s \leq t$ ,  $i, j \in E$

$$p_{ij}(s, t) = \Pr(X_t = j \mid X_s = i)$$

The chain is **homogeneous** if

$$p_{ij}(s, t) = p_{ij}(0, t - s)$$

In this course, it is always assumed that the chain is homogeneous, thus we always denote  $p_{ij}(t) = p_{ij}(0, t)$ .

**Theorem** The family is a **stochastic semigroup** if:

- $\mathbf{P}_0 = I_{K \times K}$
- $\mathbf{P}_t$  is stochastic
- Chapman-Kolmogorov equations are satisfied

$$p_{ij}(s + t) = \sum_{k \in E} p_{ik}(s) p_{kj}(t)$$

The semigroup  $\{P_t\}$  is called **standard** if

$$\lim_{t \downarrow 0} \mathbf{P}_t = \mathbf{I}$$

*The Poisson process is a continuous time Markov chain.*

#### 3.1 Holding times

We define the **holding time at state i** as

$$H_i = \inf\{s \geq 0 : X_{t+s} \neq i\}$$

**Theorem** The holding time follows an **exponential distribution** (due to its memoryless property)



### 3.1.1 Exponential Alarm Clocks

- For each state  $i \in E$ , it can reach  $n_i$  states
- Set  $n_i$  independent exponential alarm clocks with rates  $q_{ij}$
- The state transfers to the index of the first alarm clock that rings
- Transfer to state  $j$  with probability  $\frac{q_{ij}}{\sum_k q_{ik}}$  (ordering of exponential random variables)

## 3.2 The generator

The **generator**  $\mathbf{G} = (g_{ij})_{i,j \in E}$  of the Markov chain with stochastic semigroup  $\mathbf{P}_t$  is defined as the  $\text{card}(E) \times \text{card}(E)$  matrix

$$\mathbf{G} = \lim_{\delta \downarrow 0} \frac{1}{\delta} [\mathbf{P}_\delta - \mathbf{I}]$$

where  $\mathbf{P}_t$  is differentiable at  $t = 0$ .

Informally, we have  $g_{ij} = q_{ij} = p'_{ij}(0)$ , so when the time interval  $\delta$  is small enough, we have the estimates for transition probabilities:

$$p_{ij}(\delta) \approx g_{ij}\delta = q_{ij}\delta p_{ii}(\delta) \approx 1 + g_{ii}\delta = 1 - \sum_{j \in E} q_{ij}\delta$$

## 3.3 Forward and backward equations

### Theorem

Subject to regularity conditions, a continuous-time Markov chain with stochastic semigroup  $\{P_t\}$  and generator  $\mathbf{G}$  satisfies the **Kolmogorov forward equation** and the **Kolmogorov backward equation**

$$\begin{aligned} \mathbf{P}'_t &= \mathbf{P}_t \mathbf{G} \\ \mathbf{P}'_t &= \mathbf{G} \mathbf{P}_t \end{aligned}$$

This allows us to write

$$\mathbf{P}_t = \exp(t\mathbf{G})$$

using matrix exponential.

## 3.4 Irreducibility and stationarity

The chain is **irreducible** if for all  $i, j \in E$ , there exists  $t > 0$  such that  $p_{ij}(t) > 0$ .

### Theorem (No periodicity in continuous)

If  $p_{ij}(t) > 0$  for some  $t > 0$ , then  $p_{ij}(t) > 0$  for all  $t > 0$ .

A distribution is the **stationary distribution** if it satisfies

$$\pi \mathbf{P}_t = \pi$$

for all  $t \geq 0$ .

A distribution  $\pi$  is the **limiting distribution** if for all  $i, j \in E$

$$\lim_{t \rightarrow \infty} p_{ij}(t) = \pi_j$$

**Theorem (find stationary distr)**

Subject to regularity conditions,  $\pi = \pi \mathbf{P}_t$  for all  $t \geq 0$  if and only if  $\pi \mathbf{G} = 0$ .

**Theorem (Ergodicity in continuous time)**

1. If there exists a stationary distribution, then it is *unique* and  $\forall i, j \in E$

$$\lim_{t \rightarrow +\infty} p_{ij}(t) = \pi_j$$

2. If there is no stationary distribution then

$$\lim_{t \rightarrow +\infty} p_{ij}(t) = 0$$

### 3.5 Jump chain and explosion

#### 3.5.1 From continuous to discrete

Assume the generator is known.

- $J_n$  being the  $n^{th}$  change in value of the chain  $X$ ,  $J_0 = 0$
- Values right after the jump  $Z_n = X_{J_n+}$  form a discrete time Markov chain
- Construct transition matrix  $p_{ij}^Z = \frac{g_{ij}}{-g_{ii}}$  and 0 if absorption (all the diagonal entries are 0)
- $\{Z_n\}_{n \geq 0}$  is the **jump chain**

#### 3.5.2 From discrete to continuous

Assume the transition matrix is known.

- Let  $p_{ii}^Z = 0$  to avoid jumps to itself in the discrete chain
- Construct generator matrix with arbitrary nonnegative  $g_i$  for each  $i$

$$g_{ij} = \begin{cases} g_i p_{ij}^Z & i \neq j \\ -g_i & i = j \end{cases}$$

- Condition on  $Z_i$ , let  $H_i \sim \text{Exp}(g_{Z_{i-1}})$  be the ‘holding times’
- Then at time  $t$ , check if between two jump times

$$X_t = \begin{cases} Z_n & J_n \leq t < J_{n+1} \\ \infty & \text{otherwise} \end{cases}$$

The chain explodes if  $\Pr(J_\infty < \infty) > 0$ .

### 3.6 Relation between common quantities

#### 3.7 Birth Processes

A **birth process** with intensities  $\lambda_1, \lambda_2, \dots$  is a continuous time Markov chain  $\{N_t\}_{t \geq 0}$  with nonnegative values such that

- It is non-decreasing

Notation	Element	Meaning and Conditions
$q_{ij}$	$q_i := q_{ii} = \sum_{j \in E} q_{ij}$	The <b>exponential rates</b> $q_{ij} > 0$ when $i \neq j$ and $i \leftrightarrow j$ , zero otherwise <b>generator</b> , $\mathbf{P}_t = \exp(t\mathbf{G})$ , not stochastic, row sum is 0 the <b>stochastic semigroup</b> , transition matrix at time $t$ , a stochastic matrix transition matrix of <b>jump chain</b> , a stochastic matrix
$\mathbf{G}$	$g_{ij} = q_{ij}$ and $g_{ii} = -q_{ii}$	
$\mathbf{P}_t$	$p_{ij}(t) = \exp(t\mathbf{G})_{ij}$	
$\mathbf{P}^Z$	$p_{ij}^Z = -g_{ij}/g_{ii} = q_{ij}/q_{ii}$	

- There is ‘single arrival’

$$\Pr(N_{t+\delta} = n + m \mid N_t = n) = \begin{cases} 1 - \lambda_n \delta + o(\delta) & m = 0 \\ \lambda_n \delta + o(\delta) & m = 1 \\ o(\delta) & m > 1 \end{cases}$$

- Conditional on  $N_s$ , the increment  $N_t - N_s$  is independent of all arrivals prior to time  $s$ , where  $t > s$ .

A birth process with constant intensity is a Poisson process. (Poisson process is a special case of birth process.)

It has generator  $\mathbf{G}$

$$\mathbf{G} = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & \cdots & 0 & \cdots \\ 0 & -\lambda_1 & \lambda_1 & 0 & 0 & \cdots \\ 0 & 0 & -\lambda_2 & \lambda_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

### 3.7.1 Simple Birth Process

- We take intensities  $\lambda_n = n\lambda$

$$\Pr(N_{t+\delta} = n + m \mid N_t = n) = \binom{n}{m} (\lambda\delta)^m (1 - \lambda\delta)^{n-m} + o(\delta)$$

which gives

$$\Pr(N_{t+\delta} = n + m \mid N_t = n) = \begin{cases} (1 - \lambda\delta)^n + o(\delta) & m = 0 \\ n\lambda\delta(1 - \lambda\delta)^{n-1} + o(\delta) & m = 1 \\ o(\delta) & m > 1 \end{cases}$$

Note that the higher order terms are  $o(\delta)$ , so we have  $1 - n\lambda\delta + o(\delta)$  and  $n\lambda\delta + o(\delta)$ .

The **Forward & Backward** equations are given by

$$\begin{aligned} p'_{ij}(t) &= -\lambda_j p_{ij}(t) + \lambda_{j-1} p_{i,j-1}(t) \\ p'_{ij}(t) &= -\lambda_i p_{ij}(t) + \lambda_i p_{i+1,j}(t) \end{aligned}$$

### 3.7.2 Deriving the Forward & Backward Equations

Note here we are looking at the transition probabilities  $p_{ij}(t)$ , not the value of the process  $N_t$ .

We need to use the Chapman-Kolmogorov equations

$$p_{ij}(t + \delta) = \sum_{l \in E} p_{il}(t) p_{lj}(\delta)$$

which gives the forward direction with  $p_{i,j-1}(t) \lambda_{j-1} \delta + p_{ij}(t)(1 - \lambda_j \delta) + o(\delta)$ .

The backward direction is similar but ‘splitting’ in a different way.

$$p_{ij}(t + \delta) = \sum_{l \in E} p_{il}(\delta) p_{lj}(t)$$

with  $p_{i+1,j}(t) \lambda_i \delta + p_{ij}(t)(1 - \lambda_i \delta) + o(\delta)$ .

**Theorem** The forward equation has a unique solution, which is also satisfied by the backward equation.

### 3.8 Birth-Death Processes

The **birth-death process**  $\{X_t\}_{t \geq 0}$  is a continuous-time Markov chain taking values in  $\mathbb{N}_0$  such that

- The birth rates  $\lambda_n$  and death rates  $\mu_n$  are nonnegative with  $\mu_0 = 0$
- The infinitesimal transition probabilities are

$$\Pr(X_{t+\delta} = n + m \mid X_t = n) = \begin{cases} 1 - (\lambda_n + \mu_n)\delta + o(\delta) & m = 0 \\ \lambda_n \delta + o(\delta) & m = 1 \\ \mu_n \delta + o(\delta) & m = -1 \\ o(\delta) & |m| > 1 \end{cases}$$

*The single arrival property rids us of the cancellation of birth and death.*

The **stationary distribution** of a birth-death process is

$$\pi_n = \frac{\lambda_0 \times \cdots \times \lambda_{n-1}}{\mu_1 \times \cdots \times \mu_n} \pi_0$$

with normalizing constant when the sum  $\sum_{n=0}^{\infty} \pi_n < \infty$

$$\pi_0 = \frac{1}{\sum_{n=0}^{\infty} \pi_n}$$

#### 3.8.1 Immigration

- Constant immigration(birth) rate  $\lambda$
- Varying death rate  $\mu_n = n\mu$

equivalent to a birth-death process with  $\lambda_n = \lambda$  and  $\mu_n = n\mu$ . Same formulas above.

## 4 Brownian Motions

A real-valued stochastic process  $B = \{B_t\}_{t \geq 0}$  is a **Brownian Motion** if

- $B_0 = 0$  almost surely
- $B$  has independent increments
- $B$  has stationary increments
- The increments are Gaussian, for  $0 \leq s < t$

$$B_t - B_s \sim N(0, t - s)$$

- The sample paths are a.s. continuous. ( $t \mapsto B_t$  is a.s. continuous)

A Brownian motion with **drift**  $\mu$  and **variance**  $\sigma^2$  is given by

$$Y_t = \mu t + \sigma B_t$$

then we have

$$Y_t - Y_s \sim N(\mu(t - s), \sigma^2(t - s))$$

### 4.1 Construction of Brownian Motion

Consider the random walk  $X_n = \sum_{i=1}^n Y_i$  with  $Y_i \in \{-1, 1\}$ , from the central limit theorem, we have

$$\frac{X_n}{\sqrt{n}} \xrightarrow{d} N(0, 1)$$

We define the Brownian motion as a limit when  $n \rightarrow \infty$

$$B_t^{(n)} = \frac{X_{[nt]}}{\sqrt{n}} = \sqrt{t} \frac{X_k}{\sqrt{k}} \xrightarrow{d} N(0, t)$$

where  $k$  is such that  $k \leq nt < k + 1$  and this follows by Slutsky's Theorem. So  $\frac{X_{[nt]}}{\sqrt{n}} \xrightarrow{d} B_t$

### 4.2 Properties

The covariance of  $B_t$  and  $B_s$  is

$$\text{Cov}(B_t, B_s) = \min(t, s)$$

#### 4.2.1 The symmetries of Brownian motion

Let  $B_t$  be a standard Brownian motion, then each of the following is also a Brownian motion:

- (Reflection)  $\{-B_t\}$
- (Translation)  $\{B_{t+s} - B_s\}$
- (Rescaling) For  $a > 0$ ,  $\{aB_{t/a^2}\}$
- (Inversion)  $\{tB_{1/t}\}$

### 4.2.2 Reflection

The **stopping-time**  $\tau$  is the first time  $B_t$  hits  $x$  for some  $x > 0$ .

$$\tau = \inf\{t \geq 0 \mid B_t \geq x\}$$

The **reflected Brownian motion**  $B_t''$  is given by

$$B_t'' = \begin{cases} B_t & t \leq \tau \\ x - (B_t - x) & t > \tau \end{cases}$$

This is also a Brownian motion.

The **maximum and minimum processes** of a Brownian motion are given by

$$M_t^+ = \max_{0 \leq s \leq t} B_s$$

$$M_t^- = \min_{0 \leq s \leq t} B_s$$

The distribution of  $M_t^+$  is given by

$$\Pr(M_t^+ \geq x) = \Pr(\tau \leq t) = 2 - 2\Phi(x/\sqrt{t})$$

whence the density of  $\tau$  is given by

$$p_\tau(t) = \frac{x}{\sqrt{2\pi t^3}} \exp\left(-\frac{x^2}{2t}\right)$$

### 4.3 A model for asset prices

Let  $S_t$  be the price of an asset at time  $t$ . We can model the price as:

$$S_t = S_0 \exp\left((\mu - \sigma^2/2)t + \sigma B_t\right)$$

where  $S_0$  is the initial price,  $\mu$  is the risk-free interest rate and  $\sigma$  is the volatility (the instantaneous standard deviation of the stock).