

Calculus Formulae

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1 Integrals

1.1 Integrating Hyperbolic Functions

The sinh related:

$$\begin{aligned}\int \sinh(ax)dx &= \frac{1}{a} \cosh ax + C \\ \int \sinh^2(ax)dx &= \frac{1}{4a} \sinh 2ax - \frac{x}{2} + C \\ \int x \sinh(ax)dx &= \frac{1}{a} x \cosh ax - \frac{1}{a^2} \sinh ax + C \\ \int \sinh^n(ax)dx &= \frac{1}{na} (\sinh^{n-1} 2ax)(\cosh ax) - \frac{n-1}{n} \int \sinh^{n-2}(ax)dx\end{aligned}$$

The cosh related:

$$\begin{aligned}\int \cosh(ax)dx &= \frac{1}{a} \sinh ax + C \\ \int \cosh^2(ax)dx &= \frac{1}{4a} \sinh 2ax - \frac{x}{2} + C \\ \int x \cosh(ax)dx &= \frac{1}{a} x \sinh ax - \frac{1}{a^2} \cosh ax + C \\ \int \cosh^n(ax)dx &= \frac{1}{na} (\cosh^{n-1} 2ax)(\sinh ax) + \frac{n-1}{n} \int \cosh^{n-2}(ax)dx\end{aligned}$$

Others:

$$\begin{aligned}\int \tanh(ax)dx &= \frac{1}{a} \ln(\cosh ax) + C \\ \int \coth(ax)dx &= \frac{1}{a} \ln(\sinh ax) + C\end{aligned}$$

1.2 Integrating to Hyperbolic

$$\begin{aligned}\int \frac{1}{\sqrt{a^2 + x^2}} dx &= \sinh^{-1}\left(\frac{x}{a}\right) + C \\ \int \frac{1}{\sqrt{x^2 - a^2}} dx &= \cosh^{-1}\left(\frac{x}{a}\right) + C \\ \int \frac{1}{a^2 - x^2} dx &= \frac{1}{a} \tanh^{-1}\left(\frac{x}{a}\right) + C \quad (x^2 < a^2) \\ \int \frac{1}{a^2 - x^2} dx &= \frac{1}{a} \coth^{-1}\left(\frac{x}{a}\right) + C \quad (x^2 > a^2) \\ \int \frac{1}{x\sqrt{a^2 - x^2}} dx &= -\frac{1}{a} \operatorname{arcsech}\left(\frac{x}{a}\right) + C \\ \int \frac{1}{x\sqrt{a^2 + x^2}} dx &= -\frac{1}{a} \operatorname{arcsech}\left|\frac{x}{a}\right| + C\end{aligned}$$

1.3 Integrating Trigs

The sin related:

$$\begin{aligned}\int \sin^2(ax) dx &= \frac{x}{2} - \frac{1}{4a} \sin 2ax + C \\ \int x \sin(ax) dx &= \frac{\sin ax}{a^2} - \frac{x \cos ax}{a} + C\end{aligned}$$

The cos related:

$$\begin{aligned}\int \cos^2(ax) dx &= \frac{x}{2} + \frac{1}{4a} \sin 2ax + C \\ \int x \cos(ax) dx &= \frac{\cos ax}{a^2} - \frac{x \sin ax}{a} + C\end{aligned}$$

Others:

$$\int \tan(x) = \ln |\sec x| + C$$

1.4 Integrating to Trigs

$$\begin{aligned}\int \frac{du}{\sqrt{a^2 - u^2}} &= \arcsin\left(\frac{u}{a}\right) + C \\ \int \frac{du}{a^2 + u^2} &= \frac{1}{a} \arctan\left(\frac{u}{a}\right) + C \\ \int \frac{du}{u\sqrt{u^2 - a^2}} &= \frac{1}{a} \sec^{-1}\left(\frac{|u|}{a}\right) + C\end{aligned}$$

1.5 Weierstrass Sub

Set $t = \tan\left(\frac{x}{2}\right)$, then

$$\begin{aligned}
dx &= \frac{2}{1+t^2} dt \\
\sin(x) &= \frac{2}{1+t^2} \\
\cos(x) &= \frac{1-t^2}{1+t^2}
\end{aligned}$$

2 Vector Calculus

2.1 Tensors

2.1.1 Levi-Civita

$$\begin{aligned}
\varepsilon_{ijk}\varepsilon_{klm} &= \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \\
\varepsilon_{ijk}\varepsilon_{ilm} &= \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}
\end{aligned}$$

2.1.2 Common expressions

- $[A \times B]_i = \varepsilon_{ijk}A_jB_k$
- $A \cdot B = A_iB_i$
- (Divergence) $\text{div}A = \frac{\partial A_i}{\partial x_i}$, *solenoidal* if zero
- (Gradient) $[\nabla\phi]_i = \frac{\partial\phi}{\partial x_i}$
- (Curl) $[\text{curl}A]_i = \varepsilon_{ijk}\frac{\partial\phi}{\partial x_j}A_k$, *irrotational* if zero, and has a *potential* (the path integral is independent of the path chosen and integral over closed curve is 0).

2.2 Grad, Div, Curl

2.2.1 Directional Derivative

Surface ϕ in direction of $\hat{s} := \vec{PQ}$ with length $s = |PQ|$.

$$\begin{aligned}
\frac{\partial\phi}{\partial s} &= \frac{\partial\phi}{\partial n}(\hat{n} \cdot \hat{s}) \\
&= \hat{s} \cdot \nabla\phi
\end{aligned}$$

For Cartesian,

$$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

2.2.2 Laplacian

$$\begin{aligned}
\nabla^2\phi &= \text{div}(\nabla\phi) \\
&= \frac{\partial^2\phi}{\partial x_i^2}
\end{aligned}$$

2.2.3 Some Results

- **(Linearity)** $\nabla(\phi_1 + \phi_2) = \nabla\phi_1 + \nabla\phi_2$, the same holds for div and curl
- **(Chain-like ∇)** $\nabla(\phi\psi) = \psi \nabla(\phi) + \phi \nabla(\psi)$
- **(Chain-like div)** $\text{div}(\phi\mathbf{A}) = \phi \text{div}(\mathbf{A}) + \nabla\phi \cdot \mathbf{A}$, with \mathbf{A} being a vector field
- **(Chain-like curl)** $\text{curl}(\phi\mathbf{A}) = \phi \text{curl}(\mathbf{A}) + \nabla\phi \times \mathbf{A}$, with \mathbf{A} being a vector field
- $\text{div}(\text{curl } \mathbf{A}) = 0$
- $\text{curl}(\text{curl}(\mathbf{A})) = \nabla(\text{div}(\mathbf{A})) - \nabla^2 \mathbf{A}$

2.3 Green, Divergence, Gauss, Stokes

2.3.1 Projection

Choose plane of projection Σ , e.g. $z = 0$ with normal \mathbf{k} , only valid when the plane S with normal \hat{n} is not orthogonal to plane of projection

$$\int_S f(P) dS = \int_{\Sigma} f(P) \frac{dx dy}{|\hat{n} \cdot \mathbf{k}|}$$

2.3.2 Green

Closed curve C , L and M are continuously differentiable over \mathbb{R} .

$$\oint_C (L dx + M dy) = \int_R \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) dx dy$$

$$\oint_C \mathbf{F} \cdot \hat{\mathbf{n}} ds = \int_R \text{div} \mathbf{F} dx dy$$

The second identity is *Divergence Thm.* in 2 - D .

2.3.3 Divergence

Volume τ , closed surface S , outward normal $\hat{\mathbf{n}}$ continuous derivatives.

$$\int_S \mathbf{A} \cdot \hat{\mathbf{n}} dS = \int_{\tau} \text{div} \mathbf{A} d\tau$$

2.3.4 Gauss

S closed surface with outward unit normal $\hat{\mathbf{n}}$, O is the origin of the coordinate system.

$$\int_S \frac{\mathbf{r} \cdot \hat{\mathbf{n}}}{r^3} dS = \begin{cases} 0, & \text{if } O \text{ is outside} \\ 4\pi, & \text{otherwise} \end{cases}$$

2.3.5 Stokes

S an open surface, simple closed curve γ , \mathbf{A} with continuous partial derivatives, outward normal $\hat{\mathbf{n}}$ determined by right-hand rule

$$\oint_{\gamma} \mathbf{A} \cdot d\mathbf{r} = \int_S \text{curl} \mathbf{A} \cdot \hat{\mathbf{n}} dS$$

3 Curvilinear Coordinates

3.1 Basics

For transformation $u_i = u_i(x_1, x_2, x_3)$

$$h_i \mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial u_i}$$

Scale factor is the norm of $\frac{\partial \mathbf{r}}{\partial u_i}$.

3.2 Cylindrical

The Jacobian determinant is r .

$$x = r \cos \phi \quad y = r \sin \phi \quad z = z$$

The **scale factors**:

$$h_1 = 1$$

$$h_2 = r$$

$$h_3 = 1$$

The **path element**

$$d\mathbf{r} = dr + r d\phi + dz$$

The **volume element**

$$d\tau = r dr d\phi dz$$

The **surface element**

Keep one of them constant e.g. u_1

$$dS = r d\phi dz$$

The **gradient**

$$\nabla = \hat{r} \frac{\partial}{\partial r} + \frac{\hat{\phi}}{r} \frac{\partial}{\partial \phi} + \hat{k} \frac{\partial}{\partial z}$$

The **divergence**:

$$\operatorname{div} \mathbf{A} = \frac{\partial A_1}{\partial r} + \frac{A_1}{r} + \frac{1}{r} \frac{\partial A_2}{\partial \phi} + \frac{\partial A_3}{\partial z}$$

The **curl**:

$$\operatorname{curl} \mathbf{A} = \frac{1}{r} \begin{vmatrix} \hat{\mathbf{r}} & r\hat{\phi} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_1 & rA_2 & A_3 \end{vmatrix}$$

The **Laplacian**:

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{1}{r} \frac{\partial^2 \Phi}{\partial z^2}$$

3.3 Spherical

The Jacobian determinant is $r^2 \sin \theta$.

$$x = r \sin \theta \cos \phi \quad y = r \sin \theta \sin \phi \quad z = r \cos \theta$$

The **scale factors**:

$$\begin{aligned} h_1 &= 1 \\ h_2 &= r \\ h_3 &= r \sin \theta \end{aligned}$$

The **path element**

$$d\mathbf{r} = dr + r d\theta + r \sin \theta d\phi$$

The **volume element**

$$d\tau = (r^2 \sin \theta) d\theta dr d\phi$$

The **surface element**

$$dS = (r^2 \sin \theta) d\theta d\phi$$

The **gradient**

$$\nabla = \hat{r} \frac{\partial}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r \sin \theta} \frac{\partial}{\partial \phi}$$

The **divergence**:

$$\operatorname{div} \mathbf{A} = \frac{1}{r^2 \sin \theta} \left\{ \frac{\partial}{\partial r} (r^2 \sin \theta A_1) + \frac{\partial}{\partial \theta} (r \sin \theta A_2) + \frac{\partial}{\partial \phi} (r A_3) \right\}$$

The **curl**:

$$\text{curl} \mathbf{A} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r \sin \theta \hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_1 & rA_2 & r \sin \theta A_3 \end{vmatrix}$$

The **Laplacian**:

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r} \frac{\partial \Phi}{\partial r} + \frac{\cot \theta}{r^2} \frac{\partial \Phi}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2}$$

3.4 General

The **path element**

$$d\mathbf{r} = h_1 du_1 \mathbf{e}_1 + h_2 du_2 \mathbf{e}_2 + h_3 du_3 \mathbf{e}_3$$

The **volume element**

$$d\tau = (h_1 h_2 h_3) du_1 du_2 du_3$$

The **surface element**

Keep one of them constant e.g. u_1

$$dS = (h_2 h_3) du_2 du_3$$

The **gradient**:

$$\nabla = \frac{\hat{e}_1}{h_1} \frac{\partial}{\partial u_1} + \frac{\hat{e}_2}{h_2} \frac{\partial}{\partial u_2} + \frac{\hat{e}_3}{h_3} \frac{\partial}{\partial u_3}$$

The **divergence**:

$$\text{div} \mathbf{A} = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u_1} (A_1 h_2 h_3) + \frac{\partial}{\partial u_2} (A_2 h_3 h_1) + \frac{\partial}{\partial u_3} (A_3 h_1 h_2) \right\}$$

The **curl**:

$$\text{curl} \mathbf{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\mathbf{e}}_1 & h_2 \hat{\mathbf{e}}_2 & h_3 \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

The **Laplacian**:

$$\nabla^2 \Phi = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \Phi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial \Phi}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_2 h_1}{h_3} \frac{\partial \Phi}{\partial u_3} \right) \right\}$$

3.5 Vector Jacobian

S parametrised by u_1, u_2 .

$$x = x(u_1, u_2) \quad y = y(u_1, u_2) \quad z = z(u_1, u_2)$$

$$dS = |\mathbf{J}| du_1 du_2$$

where \mathbf{J} is $\frac{\partial \mathbf{r}}{\partial u_1} \times \frac{\partial \mathbf{r}}{\partial u_2}$, $\mathbf{r} = x \hat{i} + y \hat{j} + z \hat{k}$

e.g. Parametrising a sphere's surface with θ, ϕ , and $|\det J| = r^2 \sin \theta$

4 Calculus of variations

Family of curves $y(x, \epsilon) = Y(x) + \epsilon \eta(x)$

With *functional* $L := L(x, y, y')$

4.1 One Dimensional

$$\frac{\partial L}{\partial y} - \frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) = 0$$

4.2 Independent of y

$$\frac{\partial L}{\partial y'} = K$$

4.3 Independent of y'

$$\frac{\partial L}{\partial y} = 0$$

4.4 Independent of x

$$L - y' \frac{\partial L}{\partial y'} = K$$

4.5 Multivariate

A system of **E-L** equations

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt} \frac{\partial L}{\partial x'_i} = 0$$

4.6 With constraint

Constraint $J_0 = \int_{x_1}^{x_2} g(x, y, y') dx$ being a fixed constant

$$\frac{\partial}{\partial y}(L + \lambda g) - \frac{d}{dx} \left(\frac{\partial}{\partial y'}(L + \lambda g) \right) = 0$$

4.7 Higher dimensions

$$\frac{\partial L}{\partial f} - \text{div}(\nabla_{\nabla f} L) = 0$$

where $\nabla_{\mathbf{p}} A = i \frac{\partial}{\partial p_1} + j \frac{\partial}{\partial p_2}$

5 Differential Equations

5.1 Euler-Cauchy

$$\mathcal{L}[y] = \beta_k x^k \frac{d^k y}{dx^k} + \cdots + \beta_1 x \frac{dy}{dx} = f(x)$$

Try substitution $x = e^z$:

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{x} \frac{dy}{dz} \\ \frac{d^2 y}{dx^2} &= \frac{1}{x^2} \left[\frac{d^2 y}{dz^2} - \frac{dy}{dz} \right] \\ \frac{d^3 y}{dx^3} &= \frac{1}{x^3} \left[\frac{d^3 y}{dz^3} - 3 \frac{d^2 y}{dz^2} + 2 \frac{dy}{dz} \right] \end{aligned}$$

Note the $a(a-1)(a-2)$ factorial like pattern here.

5.2 Linear 1st order ODE

$$\frac{dx}{dt} + a(t)x = f(t)$$

solved by

$$\lambda(t) = e^{-\int a(t) dt} \left[\int e^{\int a(t) dt} f(t) dt + c \right]$$

5.3 Trace and Det. Rule

- $tr < 0, \det > 0 \implies Re\lambda < 0, \text{stable}$
- $tr^2 - 4\det < 0 \implies \text{Non-real}$
- $tr > 0, \det > 0 \implies Re\lambda > 0, \text{unstable}$
- $\det < 0 \implies \text{saddle}$ (opposite signs)

5.4 Solution to Linear Systems

$$\dot{x} = Ax$$

is solved by

$$x(t) = \exp(At)x_0$$

Lyapunov Exponents

Defined for nontrivial solutions starting in E_j (cf. Pg.57&PS5)

$$\sigma_{Lyap}(\varphi(\cdot, x)) = \lim_{t \rightarrow \infty} \frac{\ln \|\varphi(t, x)\|}{t}$$

Variation of Constants

General solution to $\dot{x} = Ax + g(t)$, g continuous.

$$\lambda(t, t_0, x) = e^{A(t-t_0)}x_0 + \int_{t_0}^t e^{A(t-s)}g(s)ds \quad \forall t, t_0 \in I, x_0 \in \mathbb{R}^d$$

5.5 Complex Jordan Forms

Eigenvalues $\lambda = a \pm ib$ and eigenvectors $u \pm iv$ resp.

$$J = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, T = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$$

T has columns as basis vectors.

$$A = TJT^{-1}$$

Easier to see the sign of b by writing (either is fine)

$$Au = au - bv$$

$$Av = bu + av$$

5.6 Matrix Exponentials

Real

d is dimension of this block

$$\exp\left(\begin{pmatrix} \lambda & 1 & \cdots & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}\right) = e^{\lambda t} \begin{pmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{d-1}}{(d-1)!} \\ 0 & 1 & t & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2} \\ 0 & & & 1 & t \\ 0 & 0 & 0 & & 1 \end{pmatrix}$$

In \mathbb{R}^2

$$\exp(t \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}) = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$$

Complex

Eigenvalues $\lambda = a \pm ib$ and $G(t) = \begin{pmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{pmatrix}$

$$\exp\left(\begin{pmatrix} C & I_2 & \cdots & 0 \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ 0 & & & C \end{pmatrix}\right) = e^{at} \begin{pmatrix} G(t) & tG(t) & \frac{t^2}{2}G(t) & \cdots & \frac{t^{d-1}}{(d-1)!}G(t) \\ 0 & G(t) & tG(t) & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2}G(t) \\ 0 & & & G(t) & tG(t) \\ 0 & 0 & 0 & & G(t) \end{pmatrix}$$

In \mathbb{R}^2

$$\exp(t \begin{pmatrix} a & b \\ -b & a \end{pmatrix}) = e^{at} \begin{pmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{pmatrix}$$

5.7 Ansatz für Lyapunov Functions

$$V(x, y) = ax^2 + 2bxy + cy^2$$

with $a > 0$ and $ac - b^2 > 0$.