Continuous Time Stochastic Processes

22 December, 2022

1 Preliminaries

1.1 Types of Processes

A **right continuous** stochastic process.

There are three types of right continuous processes

- **Normal**
- **Absorption**
- **Explosion**

The **jump times** are random variables

The **holding times** are random variables defined as

A **jump process**

Compute probabilities using **countable union**

A **counting process** is a stochastic process $\{N_t\}_{t\geq 0}$ satisfying

- $N_0 = 0$
- $\forall t \geq 0, N_t \in \mathbb{N}_0$
- (Non-decreasing) If $0 \leq s \leq t$, $N_s \leq N_t$
- (Counting) When $s < t$, $N_t - N_s$ equals the no. of events in $(s, t]$
- (Right continuous) The process is piecewise constant and has upward jumps (single step) of size 1, therefore

$$N_{t^-} = \lim_{s \uparrow t} N_s$$

A **counting process associated the sequence** $(J_n)_{n\in\mathbb{N}_0}$

1.2 Properties of random variables

The **exponential random variable** has

$$f_X(x) = \lambda e^{-\lambda x}$$

and c.d.f.

$$F_X(x) = 1 - e^{-\lambda x}$$

with a nonnegative support.

It has expectation

$$\mathbb{E}[X] = \frac{1}{\lambda}$$

and variance

$$\mathrm{Var}[X] = \frac{1}{\lambda^2}$$

The **memoryless property** of a random variable refers to the fact:

$$\Pr(X > x + y \mid X > x) = \Pr(X > y)$$

- A continuous random variable is memoryless iff it is $\mathrm{Exp}(\lambda)$
- A discrete random variable is memoryless iff it is $\mathrm{Geom}(p)$

The **sum of exponential** $\mathrm{Exp}(\lambda)$ is a $\mathrm{Gamma}(n, \lambda)$ distribution

$$f_{J_n}(t) = \frac{\lambda^n}{\Gamma(n)} t^{n-1} e^{-\lambda t}, \qquad t > 0$$

The **convergence for infinite sum of exponential** has the following criteria

- If $\sum \frac{1}{\lambda_i} < \infty$, then $\Pr(J_\infty < \infty) < 1$
- If $\sum \frac{1}{\lambda_i} = \infty$, then $\Pr(J_\infty = \infty) = 1$

The **minimum of exponential** is

$$H \sim \mathrm{Exp}(\sum_{i=1}^{n} \lambda_i)$$

and the probability of any of the k variables being the minimum is

$$\Pr(H = H_k) = \frac{\lambda_k}{\sum_{i=1}^{n} \lambda_i}$$

The **Laplace Transform** of a random variable X is given by

$$\mathcal{L}_X(u) = \mathbb{E}[e^{-uX}]$$

A list of transformations for common random variables:

- (Poisson) $\exp(\lambda t[e^{-u} - 1])$
- (Exponential) $\dfrac{\lambda}{\lambda + u}$

The **characteristic function** of a random variable X is given by

$$\phi_X(t) = \mathbb{E}[e^{itX}]$$

2 Poisson Processes

2.1 Definitions

A **Poisson process**, denoted $\{N_t\}_{t\geq 0}$, is a non-decreasing stochastic process with nonnegative values satisfying

- $N_0 = 0$
- The increments are independent, $0 \leq t_0 \leq t_1 \leq \ldots \leq t_n$, the random variables $N_{t_0}, N_{t_1} - N_{t_0}, \ldots, N_{t_n} - N_{t_{n-1}}$ are independent
- The increments are stationary

$$\Pr(N_t - N_s = k) = \Pr(N_{t-s} = k)$$

- There is a single arrival (only one arrives in a small interval), for all $t \geq 0$ and $\delta > 0$, $\delta \to 0$

$$\begin{aligned}
&\Pr(N_{t+\delta} - N_t = 1) = \lambda\delta + o(\delta) \\
&\Pr(N_{t+\delta} - N_t \geq 2) = o(\delta) \\
&\Pr(N_{t+\delta} - N_t = 0) = 1 - \lambda\delta + o(\delta)
\end{aligned}$$

This also ensures that a Poisson process is continuous in probability.

An **equivalent definition** replaces the last condition with the variable being Poisson with rate N_t

$$\Pr(N_t = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

Another **equivalent definition** characterizes Poisson process $\{N_t\}_{t\geq 0}$ explicitly

- Let $H_1, H_2, \ldots$ denote i.i.d. $\mathrm{Exp}(\lambda)$ random variables
- Let $J_0 = 0$ and $J_n = \sum_{i=1}^n H_i$
- We define

$$N_t = \sup\{n \in \mathbb{N}_0 : J_n \leq t\}$$

2.2 Properties of Poisson Process

2.2.1 Inter-arrival times

The inter-arrival times are **i.i.d.** $\mathrm{Exp}(\lambda)$ random variables

2.2.2 Time to n^{th} event

The time to n^{th} event is defined as

$$J_n = \sum_{i=1}^n H_i$$

which follows a $\mathrm{Gamma}(n, \lambda)$ distribution

$$f_{J_n}(t) = \frac{\lambda^n}{\Gamma(n)} t^{n-1} e^{-\lambda t}, \qquad t > 0$$

2.2.3 Conditional distribution of arrival times

The conditional joint density of $(J_1, \ldots, J_n)$ is given by the order statistic

$$f_{(J_1,\ldots,J_n)}(t_1, \ldots, t_n \mid N_t = n) = \begin{cases} \dfrac{n!}{t^n} & 0 < t_1 < \ldots < t_n \\ 0 & \text{otherwise} \end{cases}$$

The expectation of the k^{th} value of n uniformly distributed order statistics on $[0, t]$ is

$$\mathbb{E}[X_{(k)}] = \frac{tk}{n+1} = \mathbb{E}[J_k \mid N_t = n]$$

2.3 Extensions to Poisson Processes

2.3.1 Superposition

Given n *independent* Poisson processes $\{N_t^{(1)}\}_{t\geq 0}, \ldots, \{N_t^{(n)}\}_{t\geq 0}$, with respective rates $\lambda_1, \ldots, \lambda_n > 0$,

$$N_t = \sum_{i=1}^{n} N_t^{(i)}$$

is also a Poisson process with rate $\lambda = \sum_{i=1}^{n} \lambda_i$.

This is called a **superposition of Poisson processes**.

2.3.2 Thinning

Each arrival of a Poisson Process $\{N_t\}_{t\geq 0}$ is marked as a type k event with probability p_k, for $k = 1, \ldots, n$, where $\sum_{k=1}^{n} p_k = 1$. Then let $N_t^{(k)}$ denote the number of type k events up to time t (in $[0, t]$). Then each $N_t^{(k)}$ is a Poisson process with rate λp_k.

Each process is called a **thinned Poisson Process**.

2.4 Non-homogenous Poisson processes

Let $\lambda : [0, \infty) \to (0, \infty)$ denote a non-ngative and locally integrable function. Then the process $N = \{N_t\}_{t\geq 0}$ is a **non-homogenous Poisson process** with intensity function $\lambda(t)$ if

- $N_0 = 0$
- N has independent increments
- Single arrival; for all $t \geq 0$ and $\delta > 0$,

$$\Pr(N_{t+\delta} - N_t = 1) = \lambda(t)\delta + o(\delta)$$
$$\Pr(N_{t+\delta} - N_t \geq 2) = o(\delta)$$

Each N_t follows a **Poisson distribution with rate** $m(t)$, where

$$m(t) = \int_0^t \lambda(s) ds$$

The stationarity also changes. We have

$$N_t - N_s \sim \text{Poisson}(\int_s^t \lambda(u) du) = \text{Poisson}(m(t) - m(s))$$

2.5 Compound Poisson processes

Let $\{N_t\}_{t\geq 0}$ be a Poisson process with rate $\lambda > 0$ and $\{Y_n\}_n$ be a sequence of identically, indepdently distributed random variables that are also *independent* of $\{N_t\}_{t\geq 0}$.

$$S_t = \sum_{n=1}^{N_t} Y_n$$

is a **compound Poisson process**.

The mean and variance of S_t are

$$\begin{aligned}\mathbb{E}[S_t] &= \lambda t\ \mathbb{E}[Y_1] \\ \mathrm{Var}[S_t] &= \lambda t\ \mathbb{E}[Y_1^2]\end{aligned}$$

This is proveb by conditioning on N_t and using the fact that Y_n are independent.

We also recall the laws of total expectation and total variance.

$$\begin{aligned}\mathbb{E}[S_t] &= \mathbb{E}[\mathbb{E}[S_t \mid N_t]] \\ \mathrm{Var}[S_t] &= \mathbb{E}[\mathrm{Var}[S_t \mid N_t]] + \mathrm{Var}[\mathbb{E}[S_t \mid N_t]]\end{aligned}$$

2.6 Cramer-Lundberg

An application of the compound Poisson process is the Cramer-Lundberg model.

For an insurance company, there are **claims** S_t (expense to pay when there are accidents) modelled by a **compound Poisson process**, **initial capital** u, and **premiums** ct (money collected from customers with rate c).

We define the **risk process** to be

$$U_t = u + ct - S_t, \qquad t \geq 0$$

The company goes bankrupt if $U_t < 0$.

Thus, the **ruin probability** is defined as

$$\psi(u, T) = \Pr(U_T < 0 \text{ for } \text{ some } \text{ t} \leq T), \qquad T > 0, u \geq 0$$

The **total claim amount** $\{S_t\}_{t\geq 0}$ is

$$S_t = \begin{cases} \sum_{n=1}^{N_t} Y_n & \text{if } N_t \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

where N_t is a Poisson process with rate λ and Y_n are independent and identically distributed random variables with finite mean μ and variance σ^2.

We can compute the expected value of the risk process.

$$\mathbb{E}[U_t] = u + ct - \lambda t \mu$$

Therefore, a minimal requirement for this company to choose premium rate could be

$$c > \lambda\mu$$

this is called the **net profit condition**.

2.7 Coalescent Process

The coalescent process describes the merging of n offspring into a single ancestor occuring at random times.

- We have n individuals at time $t = 0$
- Each pair of individuals merge according to a Poisson process with rate $\lambda = 1$ and there are $\binom{n}{2}$ pairs
- The time of first coalescence follows $\text{Exp}(\binom{n}{2})$ distribution
- There are $n - 1$ coalescences
- The process is in fact a death process

We can compute the time to the most recent common ancestor (i.e. the time of the last coalescence).

$$\mathbb{E}\left(\sum_{k=1}^{n-1} H_k\right) \qquad n \in \mathbb{N}, n \geq 2$$

with

$$H_k \sim \text{Exp}(\binom{n-(k-1)}{2})$$

So it follows that

$$\mathbb{E}\left(\sum_{k=1}^{n-1} H_k\right) = \sum_{k=1}^{n-1} \frac{2}{k(k+1)} = 2(1 - \frac{1}{n})$$

Comparing with the last coalescence time, we have

$$\mathbb{E}(H_{n-1}) = 1 > 2(1 - \frac{1}{n})$$

showing that the last coalescence time is larger than half of the expected total coalescence time.

3 Continuous-time Markov chains

A continuous-time process $\{X_t\}_{t\in[0,\infty)}$ satisfies the **Markov property** if

$$\Pr(X_{t_n} = j \mid X_{t_1} = i_1, \ldots, X_{t_{n-1}} = i_{n-1}) = \Pr(X_{t_n} = j \mid X_{t_{n-1}} = i_{n-1})$$

for all $j, i_1, \ldots, i_{n-1} \in E$ and for any sequence $0 \leq t_1 < \cdots < t_n < \infty$.

The **transition probability**

The chain is **homogeneous**

Theorem The family is a **stochastic semigroup** if:

- $\mathbf{P}_0 = I_{K\times K}$

- $\mathbf{P}_t$ is stochastic
- Chapman-Kolmogorov

The semigroup $\{P_t\}$ is called **standard** if

$$\lim_{t \downarrow 0} \mathbf{P}_t = \mathbf{I}$$

The Poisson process is a continuous time Markov chain.

3.1 Holding times

We define the **holding time at state i** as

$$H_{|i} = \inf\{s \geq 0 : X_{t+s} \neq i\}$$

Theorem The holding time follows an **exponential distribution** (due to its memoryless property)

3.1.1 Exponential Alarm Clocks

- For each state $i \in E$, it can reach n_i states
- Set n_i independent exponential alarm clocks with rates q_{ij}
- The state transfers to the index of the first alarm clock that rings
- Transfer to state j with probability $\dfrac{q_{ij}}{\sum_k q_{ik}}$ (ordering of exponential random variables)

3.2 The generator

The **generator** $\mathbf{G} = (g_{ij})_{i,j \in E}$ of the Markov chain with stochastic semigroup P_t is defined as the card(E) $\times$ card(E) matrix

$$\mathbf{G} = \lim_{\delta \downarrow 0} \frac{1}{\delta}[\mathbf{P}_\delta - \mathbf{I}]$$

Hence we have the estimates for transition probabilities

$$p_{ij}(\delta) \approx g_{ij}\delta = q_{ij}\delta p_{ii}(\delta) \approx 1 + g_{ii}\delta = -\sum_{j \in E} q_{ij}\delta$$

3.3 Forward and backward equations

Theorem Subject to regularity conditions, a continuous-time Markov chain with stochastic semigroup $\{P_t\}$ and generator $\mathbf{G}$ satisfies the **Kolmogorov forward equation** and the **Kolmogorov backward equation**

$$\mathbf{P}'_t = \mathbf{P}_t\mathbf{G}$$
$$\mathbf{P}'_t = \mathbf{G}\mathbf{P}_t$$

This allows us to write

$$\mathbf{P}_t = \exp(t\mathbf{G})$$

using matrix exponential.

3.4 Irreducibility and stationarity

The chain is **irreducible** if for all $i, j \in E$ there exists $t > 0$ such that $p_{ij}(t) > 0$.

A distribution is the **stationary distribution** if it satisfies

$$\pi \mathbf{P}_t = \pi$$

for all $t \geq 0$.

A distribution π is the **limiting distribution** if for all $i, j \in E$

$$\lim_{t\to\infty} p_{ij}(t) = \pi_j$$

Theorem (find stationary distr)
Subject to regularity conditions, $\pi = \pi\mathbf{P}_t$ for all $t \geq 0$ if and only if $\pi\mathbf{G} = 0$,.

Theorem (Ergodicity in continuous time)

1. If there exists a stationary distribution, then it is *unique* and $\forall i, j \in E$

$$\lim_{t\to+\infty} p_{ij}(t) = \pi_j$$

2. If there is no stationary distribution then

$$\lim_{t\to+\infty} p_{ij}(t) = 0$$

3.5 Jump chain and explosion

3.5.1 From continuous to discrete

- J_n being the n^{th} change in value of the chain X, $J_0 = 0$
- Values right after the jump $Z_n = X_{J_n+}$ form a discrete time Markov chain
- Construct transition matrix $p^Z_{ij} = \dfrac{g_{ij}}{-g_{ii}}$ and 0 if absorption (all the diagonal entries are 0)
- $\{Z_n\}_{n\geq 0}$ is the **jump chain**

3.5.2 From discrete to continuous

- Let $p^Z_{ii} = 0$ to avoid jumps to itself in the discrete chain
- Construct generator matrix with arbitrary nonnegative g_i for each i

$$g_{ij} = \begin{cases} g_i p^Z_{ij} & i \neq j \\ -g_i & i = j \end{cases}$$

- Condition on Z_i, let $H_i \sim \mathrm{Exp}(\mathrm{g_{Z_{i-1}}})$ be the 'holding times'
- Then at time t, check if between two jump times

$$X_t = \begin{cases} Z_n & J_n \leq t < J_{n+1} \\ \infty & \text{otherwise} \end{cases}$$

The chain explodes if $\Pr(J_\infty < \infty) > 0$.

Notation	Element	Meaning and Conditions
q_{ij}	$q_i = \sum_{j \in E} q_{ij}$	The **exponential rates** $q_{ij} > 0$ when $i \neq j$ and $i \leftrightarrow j$, zero otherwise
$\mathbf{G}$	$g_{ij} = q_{ij}$ and $g_{ii} = -q_{ii}$	**generator**, $\mathbf{P}_t = \exp(t\mathbf{G})$, not stochastic, row sum is 0
$\mathbf{P}_t$	$p_{ij}(t) = \exp(tG)_{ij}$	the **stochastic semigroup**, transition matrix at time t, a stochastic matrix
$\mathbf{P}^Z$	$p_{ij}^Z = -g_{ij}/g_{ii} = q_{ij}/q_i$	transition matrix of **jump chain**, a stochastic matrix

3.6 Relation between common quantities

3.7 Birth Processes

A **birth process** with intensities $\lambda_1, \lambda_2, \ldots$ is a continuous Markov chain $\{N_t\}_{t \geq 0}$ with nonnegative values such that

- It is non-decreasing
- There is 'single arrival'

$$\Pr(N_{t+\delta} = n + m \mid N_t = n) = \begin{cases} 1 - \lambda_n \delta + o(\delta) & m = 0 \\ \lambda_n \delta + o(\delta) & m = 1 \\ o(\delta) & m > 1 \end{cases}$$

- Conditional on N_s, the increment $N_t - N_s$ is independent of all arrivals prior to time s, where $t > s$.

A birth process with constant intensity is a Poisson process. (Poisson process is a special case of birth process.)

It has generator $\mathbf{G}$

$$\mathbf{G} = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & \cdots & 0 & \cdots \\ 0 & -\lambda_1 & \lambda_1 & 0 & 0 & \cdots \\ 0 & 0 & -\lambda_2 & \lambda_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

3.7.1 Simple Birth Process

- We take intensities $\lambda_n = n\lambda$

$$\Pr(N_{t+\delta} = n + m \mid N_t = n) = \binom{n}{m} (\lambda\delta)^m (1 - \lambda\delta)^{n-m} + o(\delta)$$

which gives

$$\Pr(N_{t+\delta} = n + m \mid N_t = n) = \begin{cases} (1 - \lambda\delta)^n + o(\delta) & m = 0 \\ n\lambda\delta(1 - \lambda\delta)^{n-1} + o(\delta) & m = 1 \\ o(\delta) & m > 1 \end{cases}$$

The **Forward & Backward** equations are given by

$$p'_{ij}(t) = -\lambda_j p_{ij}(t) + \lambda_{j-1} p_{i,j-1}(t)$$
$$p'_{ij}(t) = -\lambda_i p_{ij}(t) + \lambda_i p_{i+1,j}(t)$$

Theorem The forward equation has a unique solution, which is also satisfied by the backward equation.

3.8 Birth-Death Processes

The **birth-death process** $\{X_t\}_{t\geq 0}$ is a continuous-time Markov chain taking values in $\mathbb{N}_0$ such that

- The birth rates λ_n and death rates μ_n are nonnegative with $\mu_0 = 0$
- The infinitesimal transition probabilities are

$$\Pr(X_{t+\delta} = n + m \mid X_t = n) = \begin{cases} 1 - (\lambda_n + \mu_n)\delta + o(\delta) & m = 0 \\ \lambda_n \delta + o(\delta) & m = 1 \\ \mu_n \delta + o(\delta) & m = -1 \\ o(\delta) & |m| > 1 \end{cases}$$

The **stationary distribution** of a birth-death process is

$$\pi_n = \frac{\lambda_0 \times \cdots \lambda_{n-1}}{\mu_1 \times \cdots \times \mu_n} \pi_0$$

with normalizing constant when the sum $\sum_{n=0}^{\infty} \pi_n < \infty$

$$\pi_0 = \frac{1}{\sum_{n=0}^{\infty} \pi_n}$$

3.8.1 Immigration

4 Brownian Motions

A real-valued stochastic process $B = \{B_t\}_{t\geq 0}$ is a **Brownian Motion** if

- $B_0 = 0$ almost surely
- B has independent increments
- B has stationary increments
- The increments are Gaussian, for $0 \leq s < t$

$$B_t - B_s \sim N(0, t - s)$$

- The samples paths are a.s. continuous. ($t \mapsto B_t$ is a.s. continuous)

4.1 Construction of Brownian Motion

Consider the random walk $X_n = \sum_{i=1}^{n} Y_n$ with $Y_i \in \{-1, 1\}$, from the central limit theorem, we have

$$\frac{X_n}{\sqrt{n}} \xrightarrow{d} N(0,1)$$

We define the Brownian motion as a limit when $n \to \infty$

$$B_t^{(n)} = \frac{X_{\lfloor nt \rfloor}}{\sqrt{n}} \xrightarrow{d} N(0,t)$$

by Slutsky's Theorem. So $\frac{X_{\lfloor nt \rfloor}}{\sqrt{n}} \xrightarrow{d} B_t$

4.2 Properties