Functional Analysis

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1 Preliminaries

1.1 Norms and Metrics

Definition 1.1. (Metric) Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}^+$ satisfying the following is called a **metric**

- (Positive definitiness) $\forall x, y \in X, d(x, y) \geq 0$ if $x \neq y$ and $d(x, y) = 0 \iff x = y$
- (Symmetry) $\forall x, y \in X, d(x, y) = d(y, x)$
- (Triangle-inequality) $\forall x, y, z \in X, d(x, y) \leq d(x, z) + d(z, y)$

Definition 1.2. (Translation invariant) A metric d is **translation invariant** if $\forall x, y \in X, d(x, y) = d(x + a, y + a)$ for all $a \in X$.

Example 1.3. The Euclidean metric on $\mathbb{R}^n$ is translation invariant. But the metric $d(x, y) = |x^3 - y^3|$ on $\mathbb{R}$ is not translation invariant.

To introduce the idea of a metric linear space, we need to define metrics on product spaces.

Definition 1.4. (Metric on Product Spaces) Given metric ρ on a vector space V over $\mathbb{K}$, a metric on $V \times V$ is defined by:

$$d((a, b), (c, d)) = (\rho(a, c)^p + \rho(b, d)^p)^{1/p}, p \in [1, \infty)$$

and on $\mathbb{K} \times V$ by:

$$d((\lambda, a), (\lambda', a')) = \max\{|\lambda - \lambda'|, \rho(a, a')\}$$

Definition 1.5. (Metric Linear Spaces) A pair (X, d) with X being a linear space over $\mathbb{K}$ and d being a metric is called a **metric linear space** if and only if addition and multiplication by scalar are continuous.

In other words, the following are true:

- $x_n \to x, \quad y_n \to y \implies x_n + y_n \to x + y$
- $\lambda_n \to \lambda, \lambda_n, \lambda \in \mathbb{K}, x_n \to x \implies \lambda_n x_n \to \lambda x$

It is easily verified and if d is translation invariant, then addition of vectors is continuous: namely, $d(x_n + y_n, x + y) = d(x_n - x, y - y_n) \leq d(x_n - x, 0) + d(y_n - y, 0)$. However, a translation invariant metric does not guarantee that multiplication by scalar is continuous.

Example 1.6. Let X be the space of all sequences in $\mathbb{R}$ and $d(x, y) = \sup_{i \in \mathbb{N}} |x^i - y^i|^{1/i}$, where the x^i denotes the i^{th} element of the sequence x. Then d is a metric on X and it is translation invariant.

Take $(x_n^i)_{i \in \mathbb{N}} = (a)_{i \in \mathbb{N}}$, a constant sequence with $a > 1$, and a scalar $\lambda_n = \xi^n, \xi \in (0, 1)$, so that $\lambda_n \to 0$ and $\lambda_n x_n \to 0$.

$$d(\lambda_n x_n, 0) = \sup_{i \in \mathbb{N}} |\xi|^{n/i} |a|^{1/i} \geq 1$$

So multiplication by scalar is not continuous.

Definition 1.7. (Norm) Let X be a nonempty set. A function $||\cdot|| : X \to \mathbb{R}^+$ satisfying the following is called a **norm**:

- (Positive definitiness) $\forall x \in X, ||x|| \geq 0$ and $||x|| = 0 \iff x = 0$
- (Triangle-inequality) $\forall x, y \in X, ||x + y|| \leq ||x|| + ||y||$
- (Homogeneity) $\forall x \in X, \forall \lambda \in \mathbb{K}, ||\lambda x|| = |\lambda| ||x||$

Remark 1.8. Norm is a continuous function.

Definition 1.9. (Normed Linear Spaces) A pair $(X, ||\cdot||)$ with X being a linear space over $\mathbb{K}$ and $||\cdot||$ being a norm is called a **normed linear space**

Note that every normed linear space is a metric space, since every norm can induce a metric by $d(x,y) = ||x-y||$. However, not every metric is a norm.

Example 1.10. Let X be the space of all sequences in $\mathbb{R}$ and $z > 1$. A translation invariant metric d is defined by

$$d(x,y) = \sum_{i=1}^{\infty} z^{-n} \frac{|x_i - y_i|}{1 + |x_i - y_i|}$$

But d is not a norm, as it is not homogenous.

Example 1.11. Let $X = \mathbb{R}$ be the real numbers and $|\cdot|$ the Euclidean norm. Another example of a metric that is not a norm is given by:

$$d(x,y) = \min\{|x-y|, 1\}$$

this is not a norm because it is not homogenous. (Note also that it is not translation invariant.)

1.2 Common Spaces

l_p Spaces For $p \in [1, \infty)$, the space l_p is defined as the set of all sequences $(x_n)_{n\in\mathbb{N}}$ such that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty$$

the function

$$||x||_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

defines a norm on l_p.

Remark 1.12. $l_p \subset l_q$ when $p < q$. And $\lim_{p\to\infty} ||x||_p = ||x||_\infty$

l_∞ Spaces The space l_∞ is defined as the set of all sequences $(x_n)_{n\in\mathbb{N}}$ such that

$$\sup_{n\in\mathbb{N}} |x_n| < \infty$$

the function

$$||x||_\infty = \sup_{n\in\mathbb{N}} |x_n|$$

defines a norm on l_∞.

1.3 Inequalities

Proposition 1.13. (Young) If $p > 1$ and q is defined by $\frac{1}{p} + \frac{1}{q} = 1$ (such p, q are called conjugates), then for $a, b \geq 0$

$$a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q} \tag{1}$$

Proof. (Sketch) Consider the function $f(t) = t^\alpha - \alpha t + \alpha - 1$, where $\alpha \in (0,1), t \geq 0$, $f(1) = 0$ is a maximum and consider $f(\frac{a}{b}) \leq 0$ with $\alpha = \frac{1}{p}$. □

Corollary 1.14. *The following inequalities are results of 1.13:*

- *(Hölder) If p, q are conjugates, then for complex numbers $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$:*

$$\sum_{i=1}^{n} |x_i y_i| \leq \left[\sum_{i=1}^{n} |x_i|^p\right]^{1/p} \left[\sum_{i=1}^{n} |y_i|^q\right]^{1/q} \tag{2}$$

For $x_i \in l_p, y_i \in l_q$, then:

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \left[\sum_{i=1}^{\infty} |x_i|^p\right]^{1/p} \left[\sum_{i=1}^{\infty} |y_i|^q\right]^{1/q} \tag{3}$$

When $p = q = 2$, this is the Cauchy-Schwarz inequality.

For functions $f \in L^p, g \in L^q$, then:

$$f \cdot g \in L^1 \quad \textit{and} \quad ||fg||_{L^1} \leq ||f||_{L^p} ||g||_{L^q} \tag{4}$$

- *(Minkowski) If $p \geq 1$, then for complex numbers $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$:*

$$\left[\sum_{i=1}^{n} |x_i + y_i|^p\right]^{1/p} \leq \left[\sum_{i=1}^{n} |x_i|^p\right]^{1/p} + \left[\sum_{i=1}^{n} |y_i|^p\right]^{1/p} \tag{5}$$

For $x_i, y_i \in l_p$, then:

$$\left[\sum_{i=1}^{\infty} |x_i + y_i|^p\right]^{1/p} \leq \left[\sum_{i=1}^{\infty} |x_i|^p\right]^{1/p} + \left[\sum_{i=1}^{\infty} |y_i|^p\right]^{1/p} \tag{6}$$

For functions $f, g \in L^p$, then:

$$f + g \in L^p \quad \textit{and} \quad ||f + g||_{L^p} \leq ||f||_{L^p} + ||g||_{L^p} \tag{7}$$

Proof. (Sketch) For the Hölder inequality, use Young's inequality with $a = (\dfrac{|x_i|}{||\mathbf{x}||_p})^p$ and $b = (\dfrac{|y_i|}{||\mathbf{y}||_q})^q$ (use L^p norm when proving for functions).
For Minkowski, use $(|x_i + y_i|^{p-1})(|x_i| + |y_i|)$ to break down the LHS, then use Hölder's inequality, $\sum_{i=1}^{n}(|x_i + y_i|^{p-1})|x_i| \leq \left[\sum_{i=1}^{n} |x_i|^p\right]^{1/p} \left[\sum_{i=1}^{n}((|x_i| + |y_i|)^{p-1})^q\right]^{1/q}$ and sum up the inequalities. For the l^p case, first note that $p = 1, \infty$ cases are obvious, then note that $|x_i + y_i|^{p/q}$ is in l^q and use Hölder's inequality as before on $\sum_{i=1}^{\infty}(|x_i + y_i|^{p/q})|x_i| + \sum_{i=1}^{\infty}(|x_i + y_i|^{p/q})|y_i|$. The L^p case is similar. $\square$

Definition 1.15. (Convex functions) A function f is convex if

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$$

$\forall x, y \in V$ and $\forall \alpha \in [0, 1]$.

Concave functions are defined similarly but with the inequality reversed. We also note that all convex functions defined on an open interval is **continuous** on that interval but not every convex function is continuous. An example being $f(x) = -\sqrt{x}, x > 0$ and $f(0) = 1$; it is convex on $[0, 1)$ but clearly not continuous at 0.

Proposition 1.16. (Equivalent forms of convexity) If $f : I \to \mathbb{R}$ is a twice differentiable function,

- If $f''(x) \geq 0, \forall x \in I$.
- If $\forall y \in I$, there exists $\gamma \in \mathbb{R}$, such that $\forall x \in I$, $\gamma(x - y) \leq f(x) - f(y)$.

Proposition 1.17. (Triangle inequality for concave functions) If $f : \mathbb{R}_+ \to \mathbb{R}_+$ is concave and $f(0) = 0$, then for $x, y \in \mathbb{R}_+$:

$$f(x + y) \leq f(x) + f(y)$$

The above proposition is useful when considering different norms on $\mathbb{R}$. For instance, the function $f(x) = x^p$, for $p \in (0, 1)$.

Proposition 1.18. (Jensen) For real continuous convex function f and positive weights satisfying $\sum_{i=1}^{n} \alpha_i = 1$,

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \sum_{i=1}^{n} \alpha_i f(x_i)$$

If the function is concave, then the inequality is reversed. The equality is attained when $x_i's$ are equal or f is linear.

2 Completeness and Separability

3 Hilbert Spaces

4 Finite Dimensional Spaces

5 Linear Operators

6 Dual Spaces

7 The Hahn Banach Theorems

8 The Uniform Boundedness Theorem

8.1 Baire's Category Theorem

9 The Open Mapping Theorem

10 The Closed Graph Theorem

11 Compact Operators