Functional Analysis

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1 Preliminaries

1.1 Norms and Metrics

Definition 1.1. (Metric) Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}^+$ satisfying the following is called a **metric**

- (Positive definitiness) $\forall x, y \in X, d(x, y) \geq 0$ if $x \neq y$ and $d(x, y) = 0 \iff x = y$
- (Symmetry) $\forall x, y \in X, d(x, y) = d(y, x)$
- (Triangle-inequality) $\forall x, y, z \in X, d(x, y) \leq d(x, z) + d(z, y)$

Definition 1.2. (Translation invariant) A metric d is **translation invariant** if $\forall x, y \in X, d(x, y) = d(x + a, y + a)$ for all $a \in X$.

Definition 1.3. (Metric Linear Spaces) A pair (X, d) with X being a linear space over $\mathbb{K}$ and d being a metric is called a **metric linear space** if and only if addition and multiplication by scalar are continuous.

In other words, the following are true:

- $x_n \to x, \quad y_n \to y \implies x_n + y_n \to x + y$
- $\lambda_n \to \lambda, \lambda_n, \lambda \in \mathbb{K}, x_n \to x \implies \lambda_n x_n \to \lambda x$

Definition 1.4. (Norm) Let X be a nonempty set. A function $||\cdot|| : X \to \mathbb{R}^+$ satisfying the following is called a **norm**:

- (Positive definitiness) $\forall x \in X, ||x|| \geq 0$ and $||x|| = 0 \iff x = 0$
- (Triangle-inequality) $\forall x, y \in X, ||x + y|| \leq ||x|| + ||y||$
- (Homogeneity) $\forall x \in X, \forall \lambda \in \mathbb{K}, ||\lambda x|| = |\lambda| ||x||$

Definition 1.5. (Normed Linear Spaces) A pair $(X, ||\cdot||)$ with X being a linear space over $\mathbb{K}$ and $||\cdot||$ being a norm is called a **normed linear space**

1.2 Common Spaces

1.3 Inequalities

Proposition 1.6. (Young)

Corollary 1.7. *(i)(Hölder) (ii)(Minkowski)*

Proposition 1.8. (Jensen)

Proposition 1.9. (Equivalent forms of Jensen)

2 Completeness and Separability

3 Hilbert Spaces

4 Finite Dimensional Spaces

5 Linear Operators

6 Dual Spaces

7 The Hahn Banach Theorems

8 The Uniform Boundedness Theorem

8.1 Baire's Category Theorem

9 The Open Mapping Theorem

10 The Closed Graph Theorem

11 Compact Operators