Calculus Formulae

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1 Integrals

1.1 Integrating Hyperbolic Functions

The *sinh* related:

$$\int sinh(ax)dx = \frac{1}{a}cosh\ ax + C$$
$$\int sinh^2(ax)dx = \frac{1}{4a}sinh\ 2ax - \frac{x}{2} + C$$
$$\int x\ sinh(ax)dx = \frac{1}{a}x\ cosh\ ax - \frac{1}{a^2}sinh\ ax + C$$
$$\int sinh^n(ax)dx = \frac{1}{na}(sinh^{n-1}\ 2ax)(coshax) - \frac{n-1}{n}\int sinh^{n-2}(ax)dx$$

The *cosh* related:

$$\int cosh(ax)dx = \frac{1}{a}sinh\ ax + C$$
$$\int cosh^2(ax)dx = \frac{1}{4a}sinh\ 2ax - \frac{x}{2} + C$$
$$\int x\ cosh(ax)dx = \frac{1}{a}x\ sinh\ ax - \frac{1}{a^2}cosh\ ax + C$$
$$\int cosh^n(ax)dx = \frac{1}{na}(cosh^{n-1}\ 2ax)(sinhax) + \frac{n-1}{n}\int cosh^{n-2}(ax)dx$$

Others:

$$\int tanh(ax)dx = \frac{1}{a}ln(cosh\ ax) + C$$
$$\int coth(ax)dx = \frac{1}{a}ln(sinh\ ax) + C$$

1.2 Integrating to Hyperbolic

$$\int \frac{1}{\sqrt{a^2+x^2}}dx = arcsinh(\frac{x}{a}) + C$$
$$\int \frac{1}{\sqrt{x^2-a^2}}dx = arccosh(\frac{x}{a}) + C$$
$$\int \frac{1}{a^2-x^2}dx = \frac{1}{a}\ arctanh(\frac{x}{a}) + C \qquad (x^2 < a^2)$$
$$\int \frac{1}{a^2-x^2}dx = \frac{1}{a}\ arccoth(\frac{x}{a}) + C \qquad (x^2 > a^2)$$
$$\int \frac{1}{x\sqrt{a^2-x^2}}dx = -\frac{1}{a}\ arcsech(\frac{x}{a}) + C$$
$$\int \frac{1}{x\sqrt{a^2+x^2}}dx = -\frac{1}{a}\ arcsech|\frac{x}{a}| + C$$

1.3 Integrating Trigs

The *sin* related:

$$\int sin^2(ax)dx = \frac{x}{2} - \frac{1}{4a}sin\ 2ax + C$$
$$\int x\ sin(ax)dx = \frac{sin\ ax}{a^2} - \frac{x\ cos\ ax}{a} + C$$

The *cos* related:

$$\int cos^2(ax)dx = \frac{x}{2} + \frac{1}{4a}sin\ 2ax + C$$
$$\int x\ cos(ax)dx = \frac{cos\ ax}{a^2} - \frac{x\ sin\ ax}{a} + C$$

Others:

$$\int tan(x) = ln\ |secx| + C$$

1.4 Integrating to Trigs

$$\int \frac{du}{\sqrt{a^2-u^2}} = arcsin\left(\frac{u}{a}\right) + C$$
$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\ arctan\left(\frac{u}{a}\right) + C$$
$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\ arcsec\left(\frac{|u|}{a}\right) + C$$

Weierstrass Sub

Set $t = tan(\frac{x}{2})$, then

$$
\begin{aligned}
dx &= \frac{2}{1+t^2}\ dt \\
sin(x) &= \frac{2}{1+t^2} \\
cos(x) &= \frac{1-t^2}{1+t^2}
\end{aligned}
$$

2 Vector Calculus

2.1 Tensors

$$
\begin{aligned}
\varepsilon_{ijk}\varepsilon_{klm} &= \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \\
\varepsilon_{ijk}\varepsilon_{ilm} &= \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}
\end{aligned}
$$

2.2 Grad,Div,Curl

2.2.1 Directional Derivative

$$
\begin{aligned}
\frac{\partial \phi}{\partial s} &= \frac{\partial \phi}{\partial n}(\hat{n} \cdot \hat{s}) \\
&= \hat{\mathbf{s}} \cdot \nabla \phi
\end{aligned}
$$

2.2.2 Laplacian

$$
\begin{aligned}
\nabla^2 \phi &= div(\nabla \phi) \\
&= \frac{\partial^2 \phi}{\partial x_i^2}
\end{aligned}
$$

2.3 Green, Divergence, Gauss, Stokes

2.3.1 Green

Closed curve C, L and M are continuously differentiable over R.

$$
\begin{aligned}
\oint_C (L\ dx + M\ dy) &= \int_R (\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y})\ dxdy \\
\oint_C \mathbf{F} \cdot \hat{\mathbf{n}}\ ds &= \int_R \text{div}\mathbf{F}\ dxdy
\end{aligned}
$$

2.3.2 Divergence

Volume τ, closed surface S, outward normal $\hat{\mathbf{n}}$ continuous derivatives.

$$
\int_S \mathbf{A} \cdot \hat{\mathbf{n}}\ dS = \int_\tau \text{div}\mathbf{A}\ d\tau
$$

2.3.3 Gauss

S closed surface with outward unit normal $\hat{\mathbf{n}}$, O is the origin of the coordinate system.

$$\int_S \frac{\mathbf{r} \cdot \hat{\mathbf{n}}}{r^3}\, dS = \begin{cases} 0, \text{ if } O \text{ is outside} \\ 4\pi, \ otherwise \end{cases} \tag{1}$$

2.3.4 Stokes

S an open surface, simple closed curve γ, $\mathbf{A}$ with continuous partial derivatives, outward normal $\hat{\mathbf{n}}$ determined by right-hand rule

$$\oint_\gamma \mathbf{A} \cdot \mathbf{dr} = \int_S curl\mathbf{A} \cdot \hat{\mathbf{n}}\, dS$$

3 Curvilinear Coordinates

3.1 Cylindrical

The Jacobian determinant is r.

$$x = rcos\phi \qquad y = rsin\phi \qquad z = z$$

The scale factors:

$$h_1 = 1$$
$$h_2 = r$$
$$h_3 = 1$$

The graident

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\phi}}{r}\frac{\partial}{\partial \phi} + \hat{k}\frac{\partial}{\partial z}$$

The divergence:

$$\text{div}\mathbf{A} = \frac{\partial A_1}{\partial r} + \frac{A_1}{r} + \frac{1}{r}\frac{\partial A_2}{\partial \phi} + \frac{\partial A_3}{\partial z}$$

The curl:

$$curl\mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\phi} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_1 & rA_2 & A_3 \end{vmatrix}$$

The Laplacian:

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial \phi^2} + \frac{1}{r}\frac{\partial^2\Phi}{\partial z^2}$$

3.2 Spherical

The Jacobian determinant is $r^2 sin\theta$.

$$x = rsin\theta cos\phi \qquad y = rsin\theta sin\phi \qquad z = rcos\theta$$

The scale factors:

$$\begin{aligned} h_1 &= 1 \\ h_2 &= r \\ h_3 &= rsin\theta \end{aligned}$$

The graident

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{rsin\ \theta}\frac{\partial}{\partial \phi}$$

The divergence:

$$\text{div}\mathbf{A} = \frac{1}{r^2\ sin\ \theta}\left\{\frac{\partial}{\partial r}(r^2\ sin\ \theta\ A_1) + \frac{\partial}{\partial \theta}(r\ sin\theta\ A_2) + \frac{\partial}{\partial \phi}(r\ A_3)\right\}$$

The curl:

$$curl\mathbf{A} = \frac{1}{r^2\ sin\ \theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\theta} & r\ sin\theta\ \hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_1 & rA_2 & r\ sin\ \theta\ A_3 \end{vmatrix}$$

The Laplacian:

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{2}{r}\frac{\partial\Phi}{\partial r} + \frac{cot\ \theta}{r^2}\frac{\partial\Phi}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{1}{r^2 sin^2\ \theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

3.3 General

The graident

$$\nabla = \frac{\hat{e_1}}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{e_2}}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{e_3}}{h_3}\frac{\partial}{\partial u_3}$$

The divergence:

$$\text{div}\mathbf{A} = \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial u_1}(A_1\ h_2\ h_3) + \frac{\partial}{\partial u_2}(A_2\ h_3\ h_1) + \frac{\partial}{\partial u_3}(A_3\ h_1\ h_2)\right\}$$

The curl:

$$curl\mathbf{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{\mathbf{e_1}} & h_2\hat{\mathbf{e_2}} & h_3\hat{\mathbf{e_3}} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1\ A_1 & h_2\ A_2 & h_3\ A_3 \end{vmatrix}$$

The Laplacian:

$$\nabla^2\Phi = \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial u_1}(\frac{h_2\ h_3}{h_1}\ \frac{\partial\Phi}{\partial u_1}) + \frac{\partial}{\partial u_2}(\frac{h_1\ h_3}{h_2}\ \frac{\partial\Phi}{\partial u_2}) + \frac{\partial}{\partial u_3}(\frac{h_2\ h_1}{h_3}\ \frac{\partial\Phi}{\partial u_3})\right\}$$

3.4 Vector Jacobian

S parametrised by u_1, u_2.

$$x = x(u_1, u_2) \qquad y = y(u_1, u_2) \qquad z = z(u_1, u_2)$$

$$dS \;=\; |\mathbf{J}| \; du_1 du_2$$

where $\mathbf{J}$ is $\frac{\partial \mathbf{r}}{\partial u_1} \times \frac{\partial \mathbf{r}}{\partial u_2}$, $\mathbf{r} = x \; \hat{\imath} + y \; \hat{\jmath} + z \; \hat{k}$

4 Calculus of variations

4.1 Independent of y

$$\frac{\partial L}{\partial y'} = K$$

4.2 Independent of y'

$$\frac{\partial L}{\partial y} = 0$$

4.3 Independent of x

$$L - y' \; \frac{\partial L}{\partial y'} = K$$

4.4 Multivariate

A system of **E-L** equations

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt} \frac{\partial L}{\partial x_i'} = 0$$

4.5 With constraint

$$\frac{\partial}{\partial y}(L + \lambda g) - \frac{d}{dx}\left(\frac{\partial}{\partial y'}(L + \lambda g)\right) = 0$$

4.6 Higher dimensions

$$\frac{\partial L}{\partial f} - \; \mathrm{div}(\nabla_{\nabla \mathbf{f}} \; L) = 0$$

where $\nabla_{\mathbf{p}} A = i \frac{\partial}{\partial p_1} + j \frac{\partial}{\partial p_2}$

5 Differential Equations

5.1 Euler-Cauchy

$$\mathcal{L}[y] = \beta_k x^k \frac{d^k y}{dx^k} + \cdots + \beta_1 x \frac{dy}{dx} = f(x)$$

Try substitution $x = e^z$:

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz}$$
$$\frac{d^2y}{dx^2} = \frac{1}{x^2}[\frac{d^2y}{dz^2} - \frac{dy}{dz}]$$
$$\frac{d^3y}{dx^3} = \frac{1}{x^3}[\frac{d^3y}{dz^3} - 3\ \frac{d^2y}{dz^2} + 2\ \frac{dy}{dz}]$$

Note the $a(a-1)(a-2)$ factorial like pattern here.