Topological Dynamical Systems

12 October, 2022

0.1 Continuous Maps

In this part, only discrete dynamical systems will be considered.

A **dynamical system** is an iteration of a continuous map f that operates on state space X, namely $f : X \to X$.

We use the notation

$$f^n = f \circ \cdots \circ f$$

for iterations.

Definition The **forward orbit** is

$$O_f^+(x) = \{x, f(x), f^2(x), \cdots\}$$

Definition $O_f^+(x)$ has **period** p if

- $f^p(x) = x$
- $O_f^+(x) = \{x, f(x), f^2(x), \cdots f^{p-1}(x)\}$

where p is the least positive integer that satisfies the above two properties.

Theorem (3-period implies all periods) *Let $f : I \to I$ be a continuous map of the interval with a periodic orbit of period 3. Then f has periodic orbits of any period.*

(Sketch of proof)

In fact, the theorem above is a corollary of the following more powerful theorem.

$$\begin{array}{cccccccc}
3 & 5 & 7 & 9 & 11 & \ldots & (2n+1)\cdot 2^0 & \ldots \\
3\cdot 2 & 5\cdot 2 & 7\cdot 2 & 9\cdot 2 & 11\cdot 2 & \ldots & (2n+1)\cdot 2^1 & \ldots \\
3\cdot 2^2 & 5\cdot 2^2 & 7\cdot 2^2 & 9\cdot 2^2 & 11\cdot 2^2 & \ldots & (2n+1)\cdot 2^2 & \ldots \\
3\cdot 2^3 & 5\cdot 2^3 & 7\cdot 2^3 & 9\cdot 2^3 & 11\cdot 2^3 & \ldots & (2n+1)\cdot 2^3 & \ldots \\
 & \vdots & & & & & & \\
\ldots & 2^n & \ldots & 2^4 & 2^3 & 2^2 & 2 & 1
\end{array}$$

Figure 1: Sharkovskii ordering of integers

The ordering consists of all positive integers, placing the odd ones first and order others according the power of 2 they contain. This is a *total-order* but not a *well-order*.

We write $n \succ m$ if n comes before m in this ordering.

Theorem (Sharkovskii) $f : I \to \mathbb{R}$ is a continuous function on an interval I. If f has a periodic orbit of period n, then f has m-periodic points for all m, such that $n \succ m$.

0.2 Attractors

0.3 Chaos