Waves 1

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1 Motivation: Uniform transport

The transport equation models the movement of an object by:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

where c is a fixed constant. It turns out that we can reduce this equation by transforming the coordinate system: instead of viewing x as the position of a fixed object, we can use

$$\xi = x - ct$$

as the position of the object relative to an observer moving at speed c. (train leaving station, etc.)

Writing $u(t, x) = v(t, \xi)$, we have by the chain rule:

$$\frac{\partial u}{\partial t} + c\frac{\partial v}{\partial x} = \frac{\partial v}{\partial t} - c\frac{\partial v}{\partial \xi} + c\frac{\partial v}{\partial \xi} = 0$$

leaving us with the equation $\frac{\partial v}{\partial t} = 0$, which is easily solved by $v(\xi, t) = f(\xi)$, where $f \in C^1$.

Now the solution is given by:

$$u(t, x) = f(x - ct)$$

Note that at time zero, $u(0, x) = f(x)$, so we can have a (unique) solution to the initial value problem when f is specified.

Also note that the solution $u(t, x) = c$ is constant when $x = ct + k$, this is called a **characteristic curve**, which is considered in more detail in the next section.

1.1 Transport with decay

Now consider the slightly different equation:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + au = 0$$

where a is a constant.

Using the same techniques and solving a linear ODE, we find that the solution is given by:

$$u(t, x) = f(x - ct)e^{-at}$$

2 Non-uniform transport

When we replace c with a function $c(x)$, we recall that in the uniform case the characteristic curve is identified with $h(t) = u(t, x(t))$ with $x(t) = ct + k$. (parametrised by t)

On this curve, we have

$$\frac{dh}{dt} = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

(h is constant on the curve)

Following the same idea, we can consider the curve $x(t)$ such that $h(t) = u(t, x(t))$ is constant. Then we set:

$$\frac{dh}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = 0$$

This is valid when:

$$\frac{dx}{dt} = c(x(t))$$

Now we can define the **characteristic curve** as the curve $(t, x(t))$ which satisfies the ODE above.

In other words, the slope of the characteristic curve is given by the function $c(x)$, if c is constant, then we are back to the uniform case.

To solve the equation, recall $h(t) = u(t, x(t))$ is constant on the characteristic curve, so it must be a function of the **characteristic variable**

$$\xi(t) = \beta(x) - t$$

where $\beta(x) = \int \frac{dx}{c(x)} = t + k$ is obtained by solving the ODE above. (the curve is thus $x(t) = \beta^{-1}(t + k)$)

Thus,

$$u(t, x) = f(\beta(x) - t)$$

where $f \in C^1$ is arbitrary.

Example:

Consider the equation:

$$u_t + (x^2 - 1)u_x = 0$$

In this case, the characteristic curve is given by the solutions to the ODE:

$$\frac{dx}{dt} = x^2 - 1$$

which gives $\beta(x) = \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| = t + k$, more explicitly the curve is given by:

$$\left(t, \frac{1 + \exp(2(t + k))}{1 - \exp(2(t + k))}\right)$$

which can only touch the $x - axis$ (we are working with x and t frame) when $k \geq 0$ and admits two horizontal asymptotes $x = 1$ and $x = -1$.

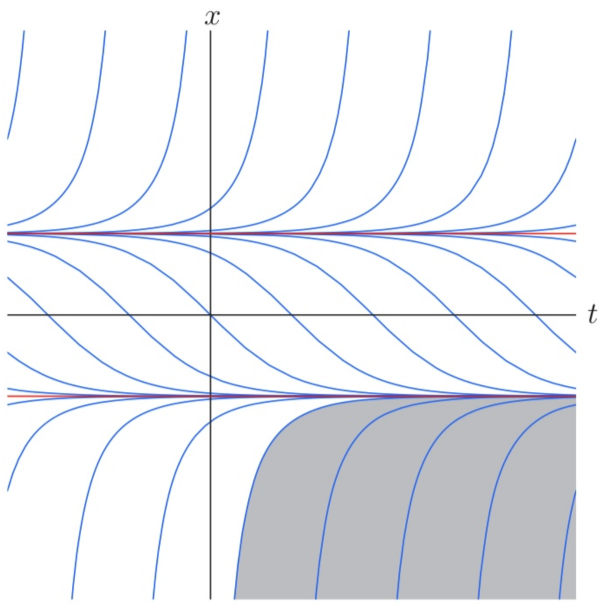

Figure 1: Example taken from Olver

3 Nonlinear Transport and Shocks

Consider the simple nonlinear transport equation, also known as the (inviscid) **Burgers' equation**: (used to model traffic flow, etc.)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

here the speed depends on the magnitude of the wave.

We follow the same method of characteristics, and obtain the ODE:

$$\frac{dx}{dt} = u(t, x(t))$$

Thus, a **characteristic curve** is given by a *straight line*

$$x(t) = ut + k$$

where k is a constant.

Now the solution is *implicitly* given by:

$$u(t, x) = f(x - ut)$$

where f is arbitrary in C^1.

While we can approach the initial value problem by setting $u(0, x) = f(x)$, it is more intuitive to consider through each point $(0, y)$,

$$x = tf(y) + y$$

with slope $u(0, y) = f(y)$ (since the value along the characteristic curve is specified by the initial function) and we check

$$u(t, tf(y) + y) = f(y)$$

for all t, which ensures the solution will have the same value as the initial function along the characteristic curve.

Example:

If initially $f(y) = y$, then $u(t, x) = u(t, ty + y) = y$ for all t, so $u(t, x) = \frac{x}{t+1}$ is the solution.

3.1 Types of Characteristic Lines

Recall that characteristic lines determine the solution to the PDE. In the ideal case, we have a unique solution along each characteristic line. However, this is not always true.

Parallel characteristic lines: in this case, the solution is constant along each characteristic line and we have uniqueness.

Diverging characteristic lines: This situation induces a wave called the **rarefaction wave**. We have the derivative of initial $f'(x) \geq 0$ for all x, so the characteristic lines are diverging. (do not intersect for $t > 0$)

Intersecting characteristic lines: This situation induces a wave called **compression wave** at first and then deviates away from the classical solution (shock is produced). We have the derivative of initial $f'(x) < 0$ for some x, so the characteristic lines are converging. (intersect for $t > 0$)

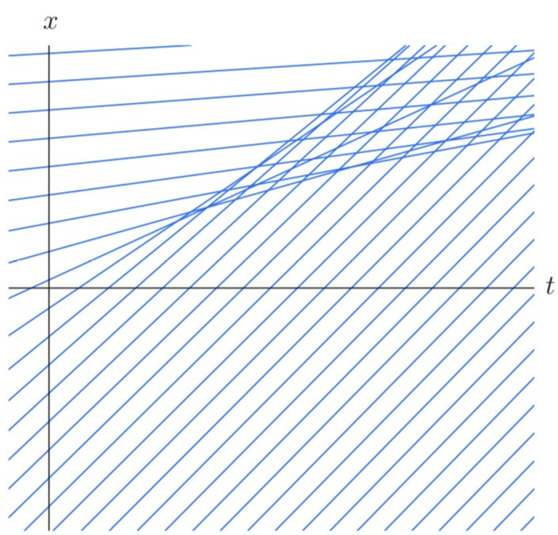

Figure 2: Also from Olver

An important observation is that the characteristic line intersects when the solution wave steepens to be vertical from a compression, which occurs when the derivative

$$\frac{\partial u}{\partial x}(t, x_\star) \to \infty \qquad \text{as} \qquad t \to t_\star$$

where $x_\star$ is the point where the characteristic lines intersect and $t_\star$ is the time when the characteristic lines intersect.

To derive this, we consider the intersection of characteristic lines, take an arbitrary y:

$$tf(y) + y = tf(y + \Delta y) + y + \Delta y$$

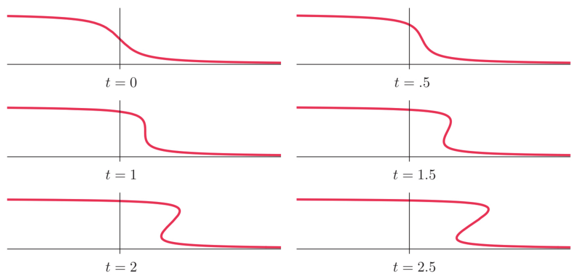

Figure 3: Still from Olver

which gives $t = -\frac{\Delta y}{f(y+\Delta y)-f(y)}$, taking the limit as $\Delta y \to 0$, we obtain the result $t_\star = -\frac{1}{f'(y)}$.

So as soon as we can have a negative derivative for some y, we will be in trouble, thus the critical time is

$$t_\star = \min\{-\frac{1}{f'(y)} \mid f'(y) < 0\}$$

Alternatively, we can derive the critical time by differentiating the solution

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x}(x - ut) = f'(x - ut)(1 - t\frac{\partial u}{\partial x})$$

which gives $\frac{\partial u}{\partial x} = \frac{f'(x-ut)}{1+f'(x-ut)t}$, so the critical time is given by $1 + f'(x - ut)t = 0$, which gives $t_\star = -\frac{1}{f'(x-ut)}$.

3.2 Shock Dynamics

When the solution produces a discontinuity, the solution graph on the u, x becomes vertical and evovles into something that is not a classical function.

Thus, we need to transform this into a solution according to *some law* by specifying its behaviour after this transformation. We will modify the multi-valued solution by introducing a **shock**, which is a discontinuity in the solution.

Since we are modelling physical phenomena, it is natural to consider various conservation laws.

A **conservation law** in one space dimension is an equation:

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0$$

where T is the *conserved density* and X is the *flux*. Note those are functions $T(t, x, u)$ and $X(t, x, u)$.

An integrated form of the conservation law is given by:

$$\frac{d}{dt}\int_a^b T(t, x, u)dx = X(t, b, u) - X(t, a, u)$$

where we take $x \in [a, b]$. This says that the rate of change of the total mass is equal to the flux at the boundary.

To apply this, we can consider first a heuristic argument: the total area under curve remains unchanged when we modify the solution by introducing a shock.

Example: Consider the inviscid Burgers' equation with initial condition:

$$u(0,x) = \begin{cases} a & x < 0 \\ b & x > 0 \end{cases}$$

where $a > b$, then we can take the mid-point between a and b as the shock value $s = \frac{a+b}{2}$, and we have the solution:

$$u(t,x) = \begin{cases} a & x < \frac{a+b}{2}t \\ b & x > \frac{a+b}{2}t \end{cases}$$

which is a shock wave moving at speed $\frac{a+b}{2}$. In the case where $a < b$, we can connect the two values at the endpoints to get

$$u(t,x) = \begin{cases} a & x < at \\ b & x > bt \\ x/t & at < x < bt \end{cases}$$

Moving to a more mathematical argument, we have the **Rankine-Hugoniot condition** which governs the shock dynamics uniquely.

Rankine-Hugoniot condition: Let $u(t,x)$ be a solution to the nonlinear transport equation (Burger's equation) with discontinuity at $x = \sigma(t)$ with finite, unequal left and right limits

$$u^-(t) = \lim_{x \to \sigma(t)^-} u(t,x) \qquad u^+(t) = \lim_{x \to \sigma(t)^+} u(t,x)$$

Then the conversation of mass deterimes the speed of the shock:

$$\frac{d\sigma}{dt} = \frac{u^-(t) + u^+(t)}{2}$$

The proof simply follows from the conservation law. We approximate the mass $M(t) = \int_a^b u(t,x)dx$ by $u^+(t)(b-a)$ with $\sigma(t) = a, \sigma(t+\Delta t) = b$. Then we have:

$$\frac{dM}{dt} = \lim_{\Delta t \to 0} [u^-(t+\Delta t) - u^+(t)] \frac{\sigma(t+\Delta t) - \sigma(t)}{\Delta t} = \frac{d\sigma}{dt}(u^-(t) - u^+(t))$$

Now the flux is given by $\frac{1}{2}[u(\tau,a)^2 - u(\tau,b)^2] \to \frac{1}{2}[u^-(t)^2 - u^+(t)^2]$ as $\tau \to t$, so setting $\frac{dM}{dt} = \frac{1}{2}[u^-(t)^2 - u^+(t)^2]$, we obtain the result.

Another governing condition is the **Entropy condition**:

$$u^-(t) > \frac{d\sigma}{dt} = \frac{u^-(t) + u^+(t)}{2} > u^+(t)$$

which gives the bound for shock speed (note this can be applied to our example above).

Example: Now consider the triangular wave initial condition

$$u(0,x) = \begin{cases} x & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Then we have $u(t, tf(y) + y) = y$ for $y \in [0,1]$ and $u(t, tf(y) + y) = u(t,y) = 0$ otherwise. This translates to $u(t,x) = \frac{x}{t+1}$ for $x \in [0, t+1]$ and $u(t,x) = 0$ for $\{x \leq 0\} \cup \{x \geq 1\}$, hence a multivalued function appears.

Using the Rankine-Hugoniot condition, we have $u^-(t) = u(t, \sigma(t)^-) = \frac{\sigma(t)}{t+1}$ and $u^+(t) = u(t, \sigma(t)^+) = 0$, so the speed of the shock is given by $\frac{d\sigma}{dt} = \frac{1}{2}\frac{\sigma(t)}{t+1}$, which gives $\sigma(t) = \sqrt{t+1}$, so the solution is given by:

$$u(t,x) = \begin{cases} \frac{x}{t+1} & x \in [0, \sqrt{t+1}] \\ 0 & \text{otherwise} \end{cases}$$

In fact, the existence and uniqueness of a solution is guaranteed by the Rankine-Hugoniot condition and the entropy condition.

Theorem: The existence and uniqueness of a (weak) solution to the nonlinear transport equation is guaranteed by the Rankine-Hugoniot condition and the entropy condition if the initial data is: - $u(0,x) = f(x), \qquad f \in C^1$

- f has finitely many jump discontinuities

3.3 Extensions to general wave speeds

We can extend the discussions above to the more general version of nonlinear transport equation:

$$u_t + c(u)u_x = 0$$

where $c(u)$ is a function of u.

Now to obtain the characteristic curves, we solve:

$$\frac{dx}{dt} = c(u(t,x))$$

So the characteristic curves are given by the solutions to the ODE above as: $tc(u) + k = x$, which gives $u(t,x) = f(x - tc(u))$ for some $f \in C^1$.

Now combining with initial condition $u(0,x) = f(x)$, we have $u(t, tc(f(y)) + y) = f(y)$.

The Rankine-Hugoniot condition becomes:

$$\frac{d\sigma}{dt} = \frac{C(u^-(t)) + C(u^+(t))}{u^-(t) - u^+(t)}$$

where $C(u) = \int c(u)du$ an antiderivative of $c(u)$.