MATH50006 Week 1

18/02/2022

0.0.1 Algebra

X an arbitrary set. A family of sets $\mathcal{A} \subset 2^X$ is an algebra if: - $X \in \mathcal{A}$ - $A \in \mathcal{A} \implies A^C \in \mathcal{A}$ - $A_1, \cdots, A_m \in \mathcal{A} \implies \cup_{k=1}^m A_k \in \mathcal{A}$

$\mathcal{A}$ is $\sigma-$algebra, if last is changed to countable, and we also have countable intersection in the algebra

Intersection of sigma algebras is an algebra

$\sigma-$ algebra Generated by C

With $C \subset 2^X$, then the set $\sigma(C) := \cap_{\mathcal{A}:C\subset\mathcal{A}}\mathcal{A}$ is a sigma algebra generated by C and also the smallest one containing it. **e.g.** $\sigma() = \{, X\}, \sigma(\{A\}) = \{A, A^C, X, \}$, $\sigma(C) = C$ iff C is a $\sigma - algebra$
Borel $\sigma-$ algebra is defined as $\sigma(\tau)$ for topological space (X, τ)

Measurable Space Defined as a pair $(X, \mathcal{A})$ where $\mathcal{A}$ is a $\sigma - algebra$ over X, elements of the algebra are called measurable sets

Measure
A measure on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to [0, \infty]$ such that: - $\mu(\emptyset) = 0$ - (σ - additivity) For all pairwise disjoint sets, $\mu(\cup A_k) = \sum \mu(A_k)$

Measure Space A triple $(X, \mathcal{A}, \mu)$

Examples: The counting measure, the co-countable measure, Dirac Measure with fixed x,

$$\delta_x(A) = \begin{cases} 0 & x \in A \\ 1 & x \notin A \end{cases}$$

Properties of measure - Monotonicity $A \subset B \implies \mu(A) \leq \mu(B)$

- **Finite Additivity** $\mu(\cup_{k=1}^n A_k) = \sum_{k=1}^n \mu(A_k)$
- **Increasing Chain** $A_k \subset A_{k+1}, \mu(\cup_{k=1}^\infty A_k) = lim_{k\to\infty}\ \mu(A_k)$
  proved by taking $B_k = A_{k+1} \setminus A_k$
- **Decreasing Chain** All $\mu(A_k) < \infty$, $A_k \supset A_{k+1}, \mu(\cap_{k=1}^\infty A_k) = lim_{k\to\infty}\ \mu(A_k)$
  proved by taking $B_k = A_1 \setminus A_k$ and use above
  counter e.g. when not finite: counting measure with sets $A_k = \{k, k+1, \cdots\}$
- **Sigma-subadditivity** A covered by A_k's, $\mu(A) \leq \sum_{k=1}^\infty \mu(A_k)$
  proved by $B_1 = A \cap A_1, B_k = (A \cap A_k) \setminus \cup_{i=1}^{k-1} A_i$