

Continuous Time Stochastic Processes

11 December, 2022

1 Preliminaries

1.1 Types of Processes

A **right continuous** stochastic process.

There are three types of right continuous processes

- **Normal**
- **Absorption**
- **Explosion**

The **jump times** are random variables

The **holding times** are random variables defined as

A **jump process**

Compute probabilities using **countable union**

A **counting process** is a stochastic process $\{N_t\}_{t \geq 0}$ satisfying

- $N_0 = 0$
- $\forall t \geq 0, N_t \in \mathbb{N}_0$
- (Non-decreasing) If $0 \leq s \leq t, N_s \leq N_t$
- (Counting) When $s < t, N_t - N_s$ equals the no. of events in $(s, t]$
- (Right continuous) The process is piecewise constant and has upward jumps (single step) of size 1, therefore

$$N_{t-} = \lim_{s \uparrow t} N_s$$

A **counting process associated the sequence** $(J_n)_{n \in \mathbb{N}_0}$

1.2 Properties of random variables

The **exponential random variable** has

$$f_X(x) = \lambda e^{-\lambda x}$$

and c.d.f.

$$F_X(x) = 1 - e^{-\lambda x}$$

with a nonnegative support.

It has expectation

$$\mathbb{E}[X] = \frac{1}{\lambda}$$

and variance

$$\text{Var}[X] = \frac{1}{\lambda^2}$$

The **memoryless property** of a random variable refers to the fact:

$$\Pr(X > x + y \mid X > x) = \Pr(X > y)$$

- A continuous random variable is memoryless iff it is $\text{Exp}(\lambda)$
- A discrete random variable is memoryless iff it is $\text{Geom}(p)$

The **sum of exponential** $\text{Exp}(\lambda)$ is a $\text{Gamma}(n, \lambda)$ distribution

$$f_{J_n}(t) = \frac{\lambda^n}{\Gamma(n)} t^{n-1} e^{-\lambda t}, \quad t > 0$$

The **convergence for infinite sum of exponential** has the following criteria

- If $\sum \frac{1}{\lambda_i} < \infty$, then $\Pr(J_\infty < \infty) < 1$
- If $\sum \frac{1}{\lambda_i} = \infty$, then $\Pr(J_\infty = \infty) = 1$

The **minimum of exponential** is

$$H \sim \text{Exp}\left(\sum_{i=1}^n \lambda_i\right)$$

and the probability of any of the k variables being the minimum is

$$\Pr(H = H_k) = \frac{\lambda_k}{\sum_{i=1}^n \lambda_i}$$

The **Laplace Transform** of a random variable X is given by

$$\mathcal{L}_X(u) = \mathbb{E}[e^{-uX}]$$

A list of transformations for common random variables:

- (Poisson) $\exp(\lambda t[e^{-u} - 1])$
- (Exponential) $\frac{\lambda}{\lambda + u}$

The **characteristic function** of a random variable X is given by

$$\phi_X(t) = \mathbb{E}[e^{itX}]$$

2 Poisson Processes

2.1 Definitions

A **Poisson process**, denoted $\{N_t\}_{t \geq 0}$, is a non-decreasing stochastic process with nonnegative values satisfying

- $N_0 = 0$
- The increments are independent, $0 \leq t_0 \leq t_1 \leq \dots \leq t_n$, the random variables $N_{t_0}, N_{t_1} - N_{t_0}, \dots, N_{t_n} - N_{t_{n-1}}$ are independent
- The increments are stationary

$$\Pr(N_t - N_s = k) = \Pr(N_{t-s} = k)$$

- There is a single arrival (only one arrives in a small interval), for all $t \geq 0$ and $\delta > 0$, $\delta \rightarrow 0$

$$\Pr(N_{t+\delta} - N_t = 1) = \lambda\delta + o(\delta)$$

$$\Pr(N_{t+\delta} - N_t \geq 2) = o(\delta)$$

$$\Pr(N_{t+\delta} - N_t = 0) = 1 - \lambda\delta + o(\delta)$$

An **equivalent definition** replaces the last condition with the variable being Poisson with rate N_t

$$\Pr(N_t = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

Another **equivalent definition** characterizes Poisson process $\{N_t\}_{t \geq 0}$ explicitly

- Let H_1, H_2, \dots denote i.i.d. $\text{Exp}(\lambda)$ random variables
- Let $J_0 = 0$ and $J_n = \sum_{i=1}^n H_i$
- We define

$$N_t = \sup\{n \in \mathbb{N}_0 : J_n \leq t\}$$

2.2 Properties of Poisson Process

2.2.1 Inter-arrival times

The inter-arrival times are **i.i.d.** $\text{Exp}(\lambda)$ random variables

2.2.2 Time to n^{th} event

The time to n^{th} event is defined as

$$J_n = \sum_{i=1}^n H_i$$

which follows a $\text{Gamma}(n, \lambda)$ distribution

$$f_{J_n}(t) = \frac{\lambda^n}{\Gamma(n)} t^{n-1} e^{-\lambda t}, \quad t > 0$$

2.2.3 Conditional distribution of arrival times

The conditional joint density of (J_1, \dots, J_n) is given by the order statistic

$$f_{(J_1, \dots, J_n)}(t_1, \dots, t_n \mid N_t = n) = \begin{cases} \frac{n!}{t^n} & 0 < t_1 < \dots < t_n \\ 0 & \text{otherwise} \end{cases}$$

The expectation of the k^{th} value of n uniformly distributed order statistics on $[0, t]$ is

$$\mathbb{E}[X_{(k)}] = \frac{tk}{n+1} = \mathbb{E}[J_k \mid N_t = n]$$

2.3 Extensions to Poisson Processes

2.3.1 Superposition

Given n independent Poisson processes, their sum is also a Poisson process

This is called a **superposition of Poisson processes**.

2.3.2 Thinning

Each arrival of a Poisson Process is marked as a type k event with probability p_k .

2.4 Non-homogenous Poisson processes

2.5 Compound Poisson processes

2.5.1 Cramer-Lundberg

2.6 Coalescent Process

3 Continuous-time Markov chains

A continuous-time process $\{X_t\}_{t \in [0, \infty)}$ satisfies the **Markov property** if

$$\Pr(X_{t_n} = j \mid X_{t_1} = i_1, \dots, X_{t_{n-1}} = i_{n-1}) = \Pr(X_{t_n} = j \mid X_{t_{n-1}} = i_{n-1})$$

for all $j, i_1, \dots, i_{n-1} \in E$ and for any sequence $0 \leq t_1 < \dots < t_n < \infty$.

The **transition probability**

The chain is **homogeneous**

Theorem The family is a **stochastic semigroup** if:

- $\mathbf{P}_0 = I_{K \times K}$
- \mathbf{P}_t is stochastic
- Chapman-Kolmogorov

The semigroup $\{P_t\}$ is called **standard** if

$$\lim_{t \downarrow 0} \mathbf{P}_t = \mathbf{I}$$

The Poisson process is a continuous time Markov chain.

3.1 Holding times

We define the **holding time at state i** as

$$H_i = \inf\{s \geq 0 : X_{t+s} \neq i\}$$

Theorem The holding time follows an **exponential distribution** (due to its memoryless property)

3.1.1 Exponential Alarm Clocks

3.2 The jump chain

3.3 The generator

3.4 Forward and backward equations

3.5 Irreducibility and stationarity

The chain is **irreducible**

A distribution is the **stationary distribution**

Theorem (find stationary distr)

Subject to regularity conditions, $\pi = \pi \mathbf{P}_t$ for all $t \geq 0$ if and only if $\pi \mathbf{G} = 0$.

A distribution π is the **limiting distribution** if for all $i, j \in E$

$$\lim_{t \rightarrow \infty} p_{ij}(t) = \pi_j$$

Theorem (Ergodicity in continuous time)

1. If there exists a stationary distribution, then it is *unique* and $\forall i, j \in E$

$$\lim_{t \rightarrow +\infty} p_{ij}(t) = \pi_j$$

2. If there is no stationary distribution then

$$\lim_{t \rightarrow +\infty} p_{ij}(t) = 0$$

3.6 Jump chain and explosion

3.6.1 From continuous to discrete

- J_n being the n^{th} change in value of the chain X , $J_0 = 0$
- Values right after the jump $Z_n = X_{J_n+}$ form a discrete time Markov chain
- Construct transition matrix $p_{ij}^Z = \frac{g_{ij}}{-g_{ii}}$ and 0 if absorption (all the diagonal entries are 0)
- $\{Z_n\}_{n \geq 0}$ is the **jump chain**

3.6.2 From discrete to continuous

- Let $p_{ii}^Z = 0$ to avoid jumps to itself in the discrete chain
- Construct generator matrix with arbitrary nonnegative g_i for each i

$$g_{ij} = \begin{cases} g_i p_{ij}^Z & i \neq j \\ -g_i & i = j \end{cases}$$

- Condition on Z_i , let $H_i \sim \text{Exp}(g_{Z_{i-1}})$ be the ‘holding times’
- Then at time t , check if between two jump times

$$X_t = \begin{cases} Z_n & J_n \leq t < J_{n+1} \\ \infty & \text{otherwise} \end{cases}$$

The chain explodes if $\Pr(J_\infty < \infty) > 0$.

3.7 Relation between common quantities

Notation	Element	Meaning and Conditions
q_{ij}	$q_i = \sum_{j \in E} q_{ij}$	The exponential rates $q_{ij} > 0$ when $i \neq j$ and $i \leftrightarrow j$, zero otherwise generator , $\mathbf{P}_t = \exp(t\mathbf{G})$, not stochastic, row sum is 0 the stochastic semigroup , transition matrix at time t , a stochastic matrix transition matrix of jump chain , a stochastic matrix
\mathbf{G}	$g_{ij} = q_{ij}$ and $g_{ii} = -q_{ii}$	
\mathbf{P}_t	$p_{ij}(t) = \exp(t\mathbf{G})_{ij}$	
\mathbf{P}^Z	$p_{ij}^Z = -g_{ij}/g_{ii} = q_{ij}/q_i$	

3.8 Birth Processes

A **birth process** with intensities $\lambda_1, \lambda_2, \dots$ is a continuous Markov chain $\{N_t\}_{t \geq 0}$ with nonnegative values such that

- It is non-decreasing
- There is ‘single arrival’

$$\Pr(N_{t+\delta} = n + m \mid N_t = n) = \begin{cases} 1 - \lambda_n \delta + o(\delta) & m = 0 \\ \lambda_n \delta + o(\delta) & m = 1 \\ o(\delta) & m > 1 \end{cases}$$

- Conditional on N_s , the increment $N_t - N_s$ is independent of all arrivals prior to time s , where $t > s$.

A birth process with constant intensity is a Poisson process.

3.8.1 Simple Birth Process

The **Forward & Backward** equations are given by

The equations have a unique solution

3.9 Birth-Death Processes

3.9.1 Stationary Distributions

The solution to $\pi G = 0$ is given by

3.9.2 Immigration

4 Brownian Motions

A real-valued stochastic process $B = \{B_t\}_{t \geq 0}$ is a **Brownian Motion** if

- $B_0 = 0$ almost surely
- B has independent increments
- B has stationary increments
- The increments are Gaussian, for $0 \leq s < t$

$$B_t - B_s \sim N(0, t - s)$$

- The sample paths are a.s. continuous. ($t \mapsto B_t$ is a.s. continuous)

4.1 Construction of Brownian Motion

Consider the random walk $X_n = \sum_{i=1}^n Y_i$ with $Y_i \in \{-1, 1\}$, from the central limit theorem, we have

$$\frac{X_n}{\sqrt{n}} \xrightarrow{d} N(0, 1)$$

We define the Brownian motion as a limit when $n \rightarrow \infty$

$$B_t^{(n)} = \frac{X_{\lfloor nt \rfloor}}{\sqrt{n}} \xrightarrow{d} N(0, t)$$

by Slutsky's Theorem. So $\frac{X_{\lfloor nt \rfloor}}{\sqrt{n}} \xrightarrow{d} B_t$

4.2 Properties