Calculus Formulae

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1 Some Integrals

1.1 Integrating Hyperbolic Functions

$$\int sinh(ax)dx = \frac{1}{a}cosh\ ax + C$$

$$\int cosh(ax)dx = \frac{1}{a}sinh\ ax + C$$

$$\int sinh^2(ax)dx = \frac{1}{4a}sinh\ 2ax - \frac{x}{2} + C$$

$$\int cosh^2(ax)dx = \frac{1}{4a}sinh\ 2ax - \frac{x}{2} + C$$

$$\int x\ sinh(ax)dx = \frac{1}{a}x\ cosh\ ax - \frac{1}{a^2}sinh\ ax + C$$

$$\int x\ cosh(ax)dx = \frac{1}{a}x\ sinh\ ax - \frac{1}{a^2}cosh\ ax + C$$

$$\int sinh^n(ax)dx = \frac{1}{na}(sinh^{n-1}\ 2ax)(coshax) - \frac{n-1}{n}\int sinh^{n-2}(ax)dx$$

$$\int cosh^n(ax)dx = \frac{1}{na}(cosh^{n-1}\ 2ax)(sinhax) + \frac{n-1}{n}\int cosh^{n-2}(ax)dx$$

$$\int tanh(ax)dx = \frac{1}{a}ln(cosh\ ax) + C$$

$$\int coth(ax)dx = \frac{1}{a}ln(sinh\ ax) + C$$

1.2 Integrating to Hyperbolic Functions

$$\int \frac{1}{\sqrt{a^2+x^2}}dx = arcsinh(\frac{x}{a}) + C$$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = arccosh(\frac{x}{a}) + C$$

$$\int \frac{1}{a^2-x^2}dx = \frac{1}{a}\ arctanh(\frac{x}{a}) + C \qquad (x^2 < a^2)$$

$$\int \frac{1}{a^2-x^2}dx = \frac{1}{a}\ arccoth(\frac{x}{a}) + C \qquad (x^2 > a^2)$$

$$\int \frac{1}{x\sqrt{a^2-x^2}}dx = -\frac{1}{a}\ arcsech(\frac{x}{a}) + C$$

$$\int \frac{1}{x\sqrt{a^2+x^2}}dx = -\frac{1}{a}\ arcsech|\frac{x}{a}| + C$$

Integrating Trigs

$$\int sin^2(ax)dx = \frac{x}{2} - \frac{1}{4a}sin\ 2ax + C$$

$$\int x\ sin(ax)dx = \frac{sin\ ax}{a^2} - \frac{x\ cos\ ax}{a} + C$$

$$\int cos^2(ax)dx = \frac{x}{2} + \frac{1}{4a}sin\ 2ax + C$$

$$\int x\ cos(ax)dx = \frac{cos\ ax}{a^2} - \frac{x\ sin\ ax}{a} + C$$

Integrating to Trigs

2 Some Vector Calculus

Tensors

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

Grad, Div, Curl

Directional Derivative

$$\begin{aligned}\frac{\partial\phi}{\partial s} &= \frac{\partial\phi}{\partial n}(\hat{n}\cdot\hat{s})\\ &= \hat{\mathbf{s}}\cdot\nabla\phi\end{aligned}$$

Laplacian

$$\begin{aligned}\nabla^2\phi &= div(\nabla\phi)\\ &= \frac{\partial^2\phi}{\partial x_i^2}\end{aligned}$$

Green's Theorem

Closed curve C, L and M are continuously differentiable over R.

$$\begin{aligned}\oint_C (L\ dx + M\ dy) &= \int_R (\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y})\ dxdy\\ \oint_C \mathbf{F}\cdot\hat{\mathbf{n}}\ ds &= \int_R div\mathbf{F}\ dxdy\end{aligned}$$

Divergence Theorem

Volume τ, closed surface S, outward normal $\hat{\mathbf{n}}$ continuous derivatives.

$$\int_S \mathbf{A}\cdot\hat{\mathbf{n}}\ dS = \int_\tau \mathrm{div}\mathbf{A}\ d\tau$$

Gauss' Flux Theorem

S closed surface with outward unit normal $\hat{\mathbf{n}}$, O is the origin of the coordinate system.

$$\int_S \frac{\mathbf{r}\cdot\hat{\mathbf{n}}}{r^3}\,dS = \begin{cases} 0, \text{ if } O \text{ is outside} \\ 4\pi, \ otherwise \end{cases} \tag{1}$$

Stokes Theorem

S an open surface, simple closed curve γ, $\mathbf{A}$ with continuous partial derivatives, outward normal $\hat{\mathbf{n}}$ determined by right-hand rule

$$\oint_\gamma \mathbf{A}\cdot\mathbf{dr} = \int_S curl\mathbf{A}\cdot\hat{\mathbf{n}}\,dS$$

Curvilinear coordinates

Cylindrical Coordinates

The Jacobian determinant is r.

$x = rcos\phi \qquad y = rsin\phi \qquad z = z$

$$h_1 = 1$$
$$h_2 = r$$
$$h_3 = 1$$

Gradient, Divergence, Curl, and Laplacian

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\phi}}{r}\frac{\partial}{\partial\phi} + \hat{k}\frac{\partial}{\partial z}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_1}{\partial r} + \frac{A_1}{r} + \frac{1}{r}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$curl\mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\phi} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ A_1 & rA_2 & A_3 \end{vmatrix}$$

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi} + \frac{1}{r}\frac{\partial^2\Phi}{\partial z}$$

Spherical Polar

The Jacobian determinant is $r^2sin\theta$.

$$x = rsin\theta cos\phi \qquad y = rsin\theta sin\phi \qquad z = rcos\theta$$

$$h_1 = 1$$
$$h_2 = r$$
$$h_3 = rsin\theta$$

Grad, Div, Curl and Laplacian in Curvilinear

$$\nabla = \frac{\hat{e_1}}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{e_2}}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{e_3}}{h_3}\frac{\partial}{\partial u_3}$$

3 Differential Equations

Euler-Cauchy

$$\mathcal{L}[y] = \beta_k x^k \frac{d^k y}{dx^k} + \cdots + \beta_1 x \frac{dy}{dx} = f(x)$$

Try substitution $x = e^z$:

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz}$$
$$\frac{d^2y}{dx^2} = \frac{1}{x^2}[\frac{d^2y}{dz^2} - \frac{dy}{dz}]$$
$$\frac{d^3y}{dx^3} = \frac{1}{x^3}[\frac{d^3y}{dz^3} - 3\ \frac{d^2y}{dz^2} + 2\ \frac{dy}{dz}]$$

Note the $a(a-1)(a-2)$ factorial like pattern here.