

# Calculus Formulae

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## 1 Integrals

### 1.1 Integrating Hyperbolic Functions

The *sinh* related:

$$\begin{aligned}\int \sinh(ax)dx &= \frac{1}{a} \cosh ax + C \\ \int \sinh^2(ax)dx &= \frac{1}{4a} \sinh 2ax - \frac{x}{2} + C \\ \int x \sinh(ax)dx &= \frac{1}{a} x \cosh ax - \frac{1}{a^2} \sinh ax + C \\ \int \sinh^n(ax)dx &= \frac{1}{na} (\sinh^{n-1} 2ax)(\cosh ax) - \frac{n-1}{n} \int \sinh^{n-2}(ax)dx\end{aligned}$$

The *cosh* related:

$$\begin{aligned}\int \cosh(ax)dx &= \frac{1}{a} \sinh ax + C \\ \int \cosh^2(ax)dx &= \frac{1}{4a} \sinh 2ax - \frac{x}{2} + C \\ \int x \cosh(ax)dx &= \frac{1}{a} x \sinh ax - \frac{1}{a^2} \cosh ax + C \\ \int \cosh^n(ax)dx &= \frac{1}{na} (\cosh^{n-1} 2ax)(\sinh ax) + \frac{n-1}{n} \int \cosh^{n-2}(ax)dx\end{aligned}$$

Others:

$$\begin{aligned}\int \tanh(ax)dx &= \frac{1}{a} \ln(\cosh ax) + C \\ \int \coth(ax)dx &= \frac{1}{a} \ln(\sinh ax) + C\end{aligned}$$

## 1.2 Integrating to Hyperbolic

$$\begin{aligned}\int \frac{1}{\sqrt{a^2 + x^2}} dx &= \operatorname{arcsinh}\left(\frac{x}{a}\right) + C \\ \int \frac{1}{\sqrt{x^2 - a^2}} dx &= \operatorname{arccosh}\left(\frac{x}{a}\right) + C \\ \int \frac{1}{a^2 - x^2} dx &= \frac{1}{a} \operatorname{arctanh}\left(\frac{x}{a}\right) + C \quad (x^2 < a^2) \\ \int \frac{1}{a^2 - x^2} dx &= \frac{1}{a} \operatorname{arccoth}\left(\frac{x}{a}\right) + C \quad (x^2 > a^2) \\ \int \frac{1}{x\sqrt{a^2 - x^2}} dx &= -\frac{1}{a} \operatorname{arcsech}\left(\frac{x}{a}\right) + C \\ \int \frac{1}{x\sqrt{a^2 + x^2}} dx &= -\frac{1}{a} \operatorname{arcsech}\left|\frac{x}{a}\right| + C\end{aligned}$$

## 1.3 Integrating Trigs

The *sin* related:

$$\begin{aligned}\int \sin^2(ax) dx &= \frac{x}{2} - \frac{1}{4a} \sin 2ax + C \\ \int x \sin(ax) dx &= \frac{\sin ax}{a^2} - \frac{x \cos ax}{a} + C\end{aligned}$$

The *cos* related:

$$\begin{aligned}\int \cos^2(ax) dx &= \frac{x}{2} + \frac{1}{4a} \sin 2ax + C \\ \int x \cos(ax) dx &= \frac{\cos ax}{a^2} - \frac{x \sin ax}{a} + C\end{aligned}$$

Others:

$$\int \tan(x) = \ln |\sec x| + C$$

## 1.4 Integrating to Trigs

$$\begin{aligned}\int \frac{du}{\sqrt{a^2 - u^2}} &= \operatorname{arcsin}\left(\frac{u}{a}\right) + C \\ \int \frac{du}{a^2 + u^2} &= \frac{1}{a} \operatorname{arctan}\left(\frac{u}{a}\right) + C \\ \int \frac{du}{u\sqrt{u^2 - a^2}} &= \frac{1}{a} \operatorname{arcsec}\left(\frac{|u|}{a}\right) + C\end{aligned}$$

## 1.5 Weierstrass Sub

Set  $t = \tan\left(\frac{x}{2}\right)$ , then

$$dx = \frac{2}{1+t^2} dt$$

$$\sin(x) = \frac{2}{1+t^2}$$

$$\cos(x) = \frac{1-t^2}{1+t^2}$$

## 2 Vector Calculus

### 2.1 Tensors

#### 2.1.1 Levi-Civita

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

### Common expressions

### 2.2 Grad, Div, Curl

#### 2.2.1 Directional Derivative

$$\frac{\partial \phi}{\partial s} = \frac{\partial \phi}{\partial n} (\hat{n} \cdot \hat{s})$$

$$= \hat{s} \cdot \nabla \phi$$

#### 2.2.2 Laplacian

$$\nabla^2 \phi = \text{div}(\nabla \phi)$$

$$= \frac{\partial^2 \phi}{\partial x_i^2}$$

### 2.3 Green, Divergence, Gauss, Stokes

#### 2.3.1 Green

Closed curve  $C$ ,  $L$  and  $M$  are continuously differentiable over  $R$ .

$$\oint_C (L \, dx + M \, dy) = \int_R \left( \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) \, dx \, dy$$

$$\oint_C \mathbf{F} \cdot \hat{\mathbf{n}} \, ds = \int_R \text{div} \mathbf{F} \, dx \, dy$$

#### 2.3.2 Divergence

Volume  $\tau$ , closed surface  $S$ , outward normal  $\hat{\mathbf{n}}$  continuous derivatives.

$$\int_S \mathbf{A} \cdot \hat{\mathbf{n}} \, dS = \int_\tau \text{div} \mathbf{A} \, d\tau$$

### 2.3.3 Gauss

$S$  closed surface with outward unit normal  $\hat{\mathbf{n}}$ ,  $O$  is the origin of the coordinate system.

$$\int_S \frac{\mathbf{r} \cdot \hat{\mathbf{n}}}{r^3} dS = \begin{cases} 0, & \text{if } O \text{ is outside} \\ 4\pi, & \text{otherwise} \end{cases}$$

### 2.3.4 Stokes

$S$  an open surface, simple closed curve  $\gamma$ ,  $\mathbf{A}$  with continuous partial derivatives, outward normal  $\hat{\mathbf{n}}$  determined by right-hand rule

$$\oint_{\gamma} \mathbf{A} \cdot d\mathbf{r} = \int_S \text{curl} \mathbf{A} \cdot \hat{\mathbf{n}} dS$$

## 3 Curvilinear Coordinates

### 3.1 Cylindrical

The Jacobian determinant is  $r$ .

$$x = r \cos \phi \quad y = r \sin \phi \quad z = z$$

The scale factors:

$$\begin{aligned} h_1 &= 1 \\ h_2 &= r \\ h_3 &= 1 \end{aligned}$$

The gradient

$$\nabla = \hat{r} \frac{\partial}{\partial r} + \frac{\hat{\phi}}{r} \frac{\partial}{\partial \phi} + \hat{k} \frac{\partial}{\partial z}$$

The divergence:

$$\text{div} \mathbf{A} = \frac{\partial A_1}{\partial r} + \frac{A_1}{r} + \frac{1}{r} \frac{\partial A_2}{\partial \phi} + \frac{\partial A_3}{\partial z}$$

The curl:

$$\text{curl} \mathbf{A} = \frac{1}{r} \begin{vmatrix} \hat{r} & r\hat{\phi} & \hat{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_1 & rA_2 & A_3 \end{vmatrix}$$

The Laplacian:

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{1}{r} \frac{\partial^2 \Phi}{\partial z^2}$$

### 3.2 Spherical

The Jacobian determinant is  $r^2 \sin \theta$ .

$$x = r \sin \theta \cos \phi \quad y = r \sin \theta \sin \phi \quad z = r \cos \theta$$

The scale factors:

$$\begin{aligned} h_1 &= 1 \\ h_2 &= r \\ h_3 &= r \sin \theta \end{aligned}$$

The gradient

$$\nabla = \hat{\mathbf{r}} \frac{\partial}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\boldsymbol{\phi}}}{r \sin \theta} \frac{\partial}{\partial \phi}$$

The divergence:

$$\text{div} \mathbf{A} = \frac{1}{r^2 \sin \theta} \left\{ \frac{\partial}{\partial r} (r^2 \sin \theta A_1) + \frac{\partial}{\partial \theta} (r \sin \theta A_2) + \frac{\partial}{\partial \phi} (r A_3) \right\}$$

The curl:

$$\text{curl} \mathbf{A} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{\mathbf{r}} & r \hat{\boldsymbol{\theta}} & r \sin \theta \hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_1 & r A_2 & r \sin \theta A_3 \end{vmatrix}$$

The Laplacian:

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r} \frac{\partial \Phi}{\partial r} + \frac{\cot \theta}{r^2} \frac{\partial \Phi}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2}$$

### 3.3 General

The gradient

$$\nabla = \frac{\hat{\mathbf{e}}_1}{h_1} \frac{\partial}{\partial u_1} + \frac{\hat{\mathbf{e}}_2}{h_2} \frac{\partial}{\partial u_2} + \frac{\hat{\mathbf{e}}_3}{h_3} \frac{\partial}{\partial u_3}$$

The divergence:

$$\text{div} \mathbf{A} = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u_1} (A_1 h_2 h_3) + \frac{\partial}{\partial u_2} (A_2 h_3 h_1) + \frac{\partial}{\partial u_3} (A_3 h_1 h_2) \right\}$$

The curl:

$$\text{curl} \mathbf{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\mathbf{e}}_1 & h_2 \hat{\mathbf{e}}_2 & h_3 \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

The Laplacian:

$$\nabla^2 \Phi = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u_1} \left( \frac{h_2 h_3}{h_1} \frac{\partial \Phi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left( \frac{h_1 h_3}{h_2} \frac{\partial \Phi}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left( \frac{h_2 h_1}{h_3} \frac{\partial \Phi}{\partial u_3} \right) \right\}$$

### 3.4 Vector Jacobian

$S$  parametrised by  $u_1, u_2$ .

$$x = x(u_1, u_2) \quad y = y(u_1, u_2) \quad z = z(u_1, u_2)$$

$$dS = |\mathbf{J}| du_1 du_2$$

where  $\mathbf{J}$  is  $\frac{\partial \mathbf{r}}{\partial u_1} \times \frac{\partial \mathbf{r}}{\partial u_2}$ ,  $\mathbf{r} = x \hat{i} + y \hat{j} + z \hat{k}$

## 4 Calculus of variations

### 4.1 Independent of $y$

$$\frac{\partial L}{\partial y'} = K$$

### 4.2 Independent of $y'$

$$\frac{\partial L}{\partial y} = 0$$

### 4.3 Independent of $x$

$$L - y' \frac{\partial L}{\partial y'} = K$$

### 4.4 Multivariate

A system of **E-L** equations

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt} \frac{\partial L}{\partial x'_i} = 0$$

### 4.5 With constraint

$$\frac{\partial}{\partial y}(L + \lambda g) - \frac{d}{dx} \left( \frac{\partial}{\partial y'}(L + \lambda g) \right) = 0$$

### 4.6 Higher dimensions

$$\frac{\partial L}{\partial f} - \text{div}(\nabla_{\nabla \mathbf{f}} L) = 0$$

where  $\nabla_{\mathbf{p}} A = i \frac{\partial}{\partial p_1} + j \frac{\partial}{\partial p_2}$

## 5 Differential Equations

### 5.1 Euler-Cauchy

$$\mathcal{L}[y] = \beta_k x^k \frac{d^k y}{dx^k} + \cdots + \beta_1 x \frac{dy}{dx} = f(x)$$

Try substitution  $x = e^z$ :

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{x} \frac{dy}{dz} \\ \frac{d^2 y}{dx^2} &= \frac{1}{x^2} \left[ \frac{d^2 y}{dz^2} - \frac{dy}{dz} \right] \\ \frac{d^3 y}{dx^3} &= \frac{1}{x^3} \left[ \frac{d^3 y}{dz^3} - 3 \frac{d^2 y}{dz^2} + 2 \frac{dy}{dz} \right]\end{aligned}$$

Note the  $a(a-1)(a-2)$  factorial like pattern here.

### 5.2 Linear 1st order ODE

$$\frac{dx}{dt} + a(t)x = f(t)$$

solved by

$$\lambda(t) = e^{-\int a(t)dt} \left[ \int e^{\int a(t)dt} f(t)dt + c \right]$$

### 5.3 Trace and Det. Rule

- $tr < 0, det > 0 \implies Re\lambda < 0, stable$
- $tr^2 - 4det < 0 \implies Non - real$
- $tr > 0, det > 0 \implies Re\lambda > 0, unstable$
- $det < 0 \implies saddle$  (opposite signs)