Calculus Formulae

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1 Integrals

1.1 Integrating Hyperbolic Functions

The sinh related:

$$\int \sinh(ax)dx = \frac{1}{a}\cosh\ ax + C$$
$$\int \sinh^2(ax)dx = \frac{1}{4a}\sinh\ 2ax - \frac{x}{2} + C$$
$$\int x\sinh(ax)dx = \frac{1}{a}x\ \cosh\ ax - \frac{1}{a^2}\sinh\ ax + C$$
$$\int \sinh^n(ax)dx = \frac{1}{na}(\sinh^{n-1}\ 2ax)(\cosh ax) - \frac{n-1}{n}\int \sinh^{n-2}(ax)dx$$

The cosh related:

$$\int \cosh(ax)dx = \frac{1}{a}\sinh\ ax + C$$
$$\int \cosh^2(ax)dx = \frac{1}{4a}\sinh\ 2ax - \frac{x}{2} + C$$
$$\int x\ \cosh(ax)dx = \frac{1}{a}x\ \sinh\ ax - \frac{1}{a^2}\cosh\ ax + C$$
$$\int \cosh^n(ax)dx = \frac{1}{na}(\cosh^{n-1}\ 2ax)(\sinh ax) + \frac{n-1}{n}\int \cosh^{n-2}(ax)dx$$

Others:

$$\int \tanh(ax)dx = \frac{1}{a}\ln(\cosh\ ax) + C$$
$$\int \coth(ax)dx = \frac{1}{a}\ln(\sinh\ ax) + C$$

1.2 Integrating to Hyperbolic

$$\int \frac{1}{\sqrt{a^2+x^2}}dx = arcsinh(\frac{x}{a}) + C$$
$$\int \frac{1}{\sqrt{x^2-a^2}}dx = arccosh(\frac{x}{a}) + C$$
$$\int \frac{1}{a^2-x^2}dx = \frac{1}{a}\ arctanh(\frac{x}{a}) + C \qquad (x^2 < a^2)$$
$$\int \frac{1}{a^2-x^2}dx = \frac{1}{a}\ arccoth(\frac{x}{a}) + C \qquad (x^2 > a^2)$$
$$\int \frac{1}{x\sqrt{a^2-x^2}}dx = -\frac{1}{a}\ arcsech(\frac{x}{a}) + C$$
$$\int \frac{1}{x\sqrt{a^2+x^2}}dx = -\frac{1}{a}\ arcsech|\frac{x}{a}| + C$$

1.3 Integrating Trigs

The sin related:

$$\int \sin^2(ax)dx = \frac{x}{2} - \frac{1}{4a}\sin 2ax + C$$
$$\int x\ \sin(ax)dx = \frac{\sin ax}{a^2} - \frac{x\cos ax}{a} + C$$

The cos related:

$$\int \cos^2(ax)dx = \frac{x}{2} + \frac{1}{4a}\sin\ 2ax + C$$
$$\int x\ \cos(ax)dx = \frac{\cos ax}{a^2} - \frac{x\ \sin ax}{a} + C$$

Others:

$$\int \tan(x) = \ln\ |\sec x| + C$$

1.4 Integrating to Trigs

$$\int \frac{du}{\sqrt{a^2-u^2}} = arcsin\left(\frac{u}{a}\right) + C$$
$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\ arctan\left(\frac{u}{a}\right) + C$$
$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\ arcsec\left(\frac{|u|}{a}\right) + C$$

1.5 Weierstrass Sub

Set $t = \tan(\frac{x}{2})$, then

$$dx = \frac{2}{1+t^2}\ dt$$
$$\sin(x) = \frac{2}{1+t^2}$$
$$\cos(x) = \frac{1-t^2}{1+t^2}$$

2 Vector Calculus

2.1 Tensors

2.1.1 Levi-Civita

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$
$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

2.1.2 Common expressions

- $[A \times B]_i = \varepsilon_{ijk} A_j B_k$
- $A \cdot B = A_i B_i$
- (Divergence) $div A = \frac{\partial A_i}{\partial x_i}$, *solenoidal* if zero
- (Gradient) $[\nabla\phi]_i = \frac{\partial\phi}{\partial x_i}$
- (Curl) $[curl A]_i = \varepsilon_{ijk}\frac{\partial\phi}{\partial x_j} A_k$, *irrotational* if zero, and has a *potential*

2.2 Grad,Div,Curl

2.2.1 Directional Derivative

Surface ϕ in direction of $\hat{s} := \vec{PQ}$ with length $s = |PQ|$.

$$\frac{\partial\phi}{\partial s} = \frac{\partial\phi}{\partial n}(\hat{n} \cdot \hat{s})$$
$$= \hat{\mathbf{s}} \cdot \nabla\phi$$

For Cartesian,

$$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

2.2.2 Laplacian

$$\nabla^2\phi = div(\nabla\phi)$$
$$= \frac{\partial^2\phi}{\partial x_i^2}$$

2.2.3 Some Results

- (Linearity) $\nabla(\phi_1 + \phi_2) = \nabla\phi_1 + \nabla\phi_2$, the same holds for div and curl
- (Chain-like ∇) $\nabla(\phi\psi) = \psi\ \nabla(\phi) + \phi\ \nabla(\psi)$
- (Chain-like div) $div(\phi\mathbf{A}) = \phi\ div(\mathbf{A}) + \nabla\phi \cdot \mathbf{A}$, with $\mathbf{A}$ being a vector field
- (Chain-like $curl$) $curl(\phi\mathbf{A}) = \phi curl(\mathbf{A}) + \nabla\phi \times \mathbf{A}$, with $\mathbf{A}$ being a vector field
- $div(curl\ \mathbf{A}) = 0$
- $curl(curl(\mathbf{A})) = \nabla(div(\mathbf{A})) - \nabla^2\mathbf{A}$

2.3 Green, Divergence, Gauss, Stokes

2.3.1 Projection

Choose plane of projection Σ, e.g. $z = 0$ with normal $\mathbf{k}$, only valid when the plane S with normal $\hat{n}$ is not orthogonal to plane of projection

$$\int_S f(P)dS = \int_\Sigma f(P)\frac{dxdy}{|\hat{n} \cdot \mathbf{k}|}$$

2.3.2 Green

Closed curve C, L and M are continuously differentiable over R.

$$\oint_C (L\ dx + M\ dy) = \int_R (\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y})\ dxdy$$
$$\oint_C \mathbf{F} \cdot \hat{\mathbf{n}}\ ds = \int_R \mathrm{div}\mathbf{F}\ dxdy$$

The second identity is *Divergence Thm.* in $2 - D$.

2.3.3 Divergence

Volume τ, closed surface S, outward normal $\hat{\mathbf{n}}$ continuous derivatives.

$$\int_S \mathbf{A} \cdot \hat{\mathbf{n}}\ dS = \int_\tau \mathrm{div}\mathbf{A}\ d\tau$$

2.3.4 Gauss

S closed surface with outward unit normal $\hat{\mathbf{n}}$, O is the origin of the coordinate system.

$$\int_S \frac{\mathbf{r} \cdot \hat{\mathbf{n}}}{r^3}\ dS = \begin{cases} 0, \text{ if } O \text{ is outside} \\ 4\pi,\ otherwise \end{cases}$$

2.3.5 Stokes

S an open surface, simple closed curve γ, $\mathbf{A}$ with continuous partial derivatives, outward normal $\hat{\mathbf{n}}$ determined by right-hand rule

$$\oint_\gamma \mathbf{A} \cdot \mathbf{dr} = \int_S curl\mathbf{A} \cdot \hat{\mathbf{n}}\; dS$$

3 Curvilinear Coordinates

3.1 Basics

For transformation $u_i = u_i(x_1, x_2, x_3)$

$$h_i \mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial u_i}$$

Scale factor is the norm of $\frac{\partial \mathbf{r}}{\partial u_i}$.

3.2 Cylindrical

The Jacobian determinant is r.

$$x = rcos\phi \qquad y = rsin\phi \qquad z = z$$

The **scale factors**:

$$\begin{aligned} h_1 &= 1 \\ h_2 &= r \\ h_3 &= 1 \end{aligned}$$

The **graident**

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\phi}}{r}\frac{\partial}{\partial \phi} + \hat{k}\frac{\partial}{\partial z}$$

The **divergence**:

$$\mathrm{div}\mathbf{A} = \frac{\partial A_1}{\partial r} + \frac{A_1}{r} + \frac{1}{r}\frac{\partial A_2}{\partial \phi} + \frac{\partial A_3}{\partial z}$$

The **curl**:

$$curl\mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\phi} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_1 & rA_2 & A_3 \end{vmatrix}$$

The **Laplacian**:

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{1}{r}\frac{\partial^2 \Phi}{\partial z^2}$$

3.3 Spherical

The Jacobian determinant is $r^2 sin\theta$.

$$x = rsin\theta cos\phi \qquad y = rsin\theta sin\phi \qquad z = rcos\theta$$

The **scale factors**:

$$h_1 = 1$$
$$h_2 = r$$
$$h_3 = r\sin\theta$$

The **graident**

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin\ \theta}\frac{\partial}{\partial\phi}$$

The **divergence**:

$$\mathrm{div}\mathbf{A} = \frac{1}{r^2\ \sin\theta}\left\{\frac{\partial}{\partial r}(r^2\ \sin\theta\ A_1) + \frac{\partial}{\partial\theta}(r\ \sin\theta\ A_2) + \frac{\partial}{\partial\phi}(r\ A_3)\right\}$$

The **curl**:

$$curl\mathbf{A} = \frac{1}{r^2\ \sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\theta} & r\ \sin\theta\ \hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_1 & rA_2 & r\ \sin\theta\ A_3 \end{vmatrix}$$

The **Laplacian**:

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{2}{r}\frac{\partial\Phi}{\partial r} + \frac{cot\ \theta}{r^2}\frac{\partial\Phi}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{1}{r^2\sin^2\ \theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

3.4 General

The **graident**:

$$\nabla = \frac{\hat{e_1}}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{e_2}}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{e_3}}{h_3}\frac{\partial}{\partial u_3}$$

The **divergence**:

$$\mathrm{div}\mathbf{A} = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u_1}(A_1\ h_2\ h_3) + \frac{\partial}{\partial u_2}(A_2\ h_3\ h_1) + \frac{\partial}{\partial u_3}(A_3\ h_1\ h_2)\right\}$$

The **curl**:

$$curl\mathbf{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{\mathbf{e_1}} & h_2\hat{\mathbf{e_2}} & h_3\hat{\mathbf{e_3}} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1\ A_1 & h_2\ A_2 & h_3\ A_3 \end{vmatrix}$$

The **Laplacian**:

$$\nabla^2\Phi = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u_1}(\frac{h_2\ h_3}{h_1}\ \frac{\partial\Phi}{\partial u_1}) + \frac{\partial}{\partial u_2}(\frac{h_1\ h_3}{h_2}\ \frac{\partial\Phi}{\partial u_2}) + \frac{\partial}{\partial u_3}(\frac{h_2\ h_1}{h_3}\ \frac{\partial\Phi}{\partial u_3})\right\}$$

3.5 Vector Jacobian

S parametrised by u_1, u_2.

$$x = x(u_1, u_2) \qquad y = y(u_1, u_2) \qquad z = z(u_1, u_2)$$

$$dS \ = \ |\mathbf{J}| \ du_1 du_2$$

where $\mathbf{J}$ is $\frac{\partial \mathbf{r}}{\partial u_1} \times \frac{\partial \mathbf{r}}{\partial u_2}$, $\mathbf{r} = x \ \hat{\imath} + y \ \hat{\jmath} + z \ \hat{k}$

e.g. Parametrising a sphere's surface with θ, ϕ, and $|\det J| = r^2 \sin\theta$

4 Calculus of variations

Family of curves $y(x, \epsilon) = Y(x) + \epsilon\eta(x)$

With *functional* $L := L(x, y, y')$

4.1 One Dimensional

$$\frac{\partial L}{\partial y} - \frac{d}{dx}(\frac{\partial L}{\partial y'}) = 0$$

4.2 Independent of y

$$\frac{\partial L}{\partial y'} = K$$

4.3 Independent of y'

$$\frac{\partial L}{\partial y} = 0$$

4.4 Independent of x

$$L - y' \ \frac{\partial L}{\partial y'} = K$$

4.5 Multivariate

A system of **E-L** equations

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial x_i'} = 0$$

4.6 With constraint

Constraint $J_0 = \int_{x_1}^{x_2} g(x, y, y')dx$ being a fixed constant

$$\frac{\partial}{\partial y}(L + \lambda g) - \frac{d}{dx}\left(\frac{\partial}{\partial y'}(L + \lambda g)\right) = 0$$

4.7 Higher dimensions

$$\frac{\partial L}{\partial f} - \operatorname{div}(\nabla_{\nabla \mathbf{f}}\ L) = 0$$

where $\nabla_{\mathbf{p}} A = i\frac{\partial}{\partial p_1} + j\frac{\partial}{\partial p_2}$

5 Differential Equations

5.1 Euler-Cauchy

$$\mathcal{L}[y] = \beta_k x^k \frac{d^k y}{dx^k} + \cdots + \beta_1 x \frac{dy}{dx} = f(x)$$

Try substitution $x = e^z$:

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{x}\frac{dy}{dz} \\ \frac{d^2 y}{dx^2} &= \frac{1}{x^2}[\frac{d^2 y}{dz^2} - \frac{dy}{dz}] \\ \frac{d^3 y}{dx^3} &= \frac{1}{x^3}[\frac{d^3 y}{dz^3} - 3\ \frac{d^2 y}{dz^2} + 2\ \frac{dy}{dz}] \end{aligned}$$

Note the $a(a-1)(a-2)$ factorial like pattern here.

5.2 Linear 1st order ODE

$$\frac{dx}{dt} + a(t)x = f(t)$$

solved by

$$\lambda(t) = e^{-\int a(t)dt}\left[\int e^{\int a(t)dt} f(t)dt + c\right]$$

5.3 Trace and Det. Rule

- $tr < 0,\ \det > 0 \implies Re\lambda < 0, stable$
- $tr^2 - 4\det < 0 \implies Non-real$

- $tr > 0,\ \det > 0 \implies Re\lambda > 0, unstable$
- $\det < 0 \implies saddle$ (opposite signs)

5.4 Solution to Linear Systems

$$\dot{x} = Ax$$

is solved by

$$x(t) = \exp(At)x_0$$

Lyapunov Exponents
Defined for nontrivial solutions starting in E_j (cf. Pg.57&PS5)

$$\sigma_{Lyap}(\varphi(\cdot, x)) = \lim_{t\to\infty} \frac{\ln ||\varphi(t,x)||}{t}$$

Variation of Constants

General solution to $\dot{x} = Ax + g(t)$, g continuous.

$$\lambda(t, t_0, x) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-s)}g(s)ds \qquad \forall t, t_0 \in I, x_0 \in \mathbb{R}^d$$

5.5 Complex Jordan Forms

Eigenvalues $\lambda = a \pm ib$ and eigenvectors $u \pm iv$ resp.

$$J = \begin{pmatrix} a & b \\ -b & a \end{pmatrix},\ T = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$$

T has columns as basis vectors.

$$A = TJT^{-1}$$

Easier to see the sign of b by writing (either is fine)

$$\begin{aligned} Au &= au - bv \\ Av &= bu + av \end{aligned}$$

5.6 Matrix Exponentials

Real
d is dimension of this block

$$\exp(\begin{pmatrix} \lambda & 1 & \cdots & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}) = e^{\lambda t} \begin{pmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{d-1}}{(d-1)!} \\ 0 & 1 & t & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2} \\ 0 & & & 1 & t \\ 0 & 0 & 0 & & 1 \end{pmatrix}$$

In $\mathbb{R}^2$

$$\exp(t \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}) = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$$

Complex

Eigenvalues $\lambda = a \pm ib$ and $G(t) = \begin{pmatrix} cos(bt) & sin(bt) \\ -sin(bt) & cos(bt) \end{pmatrix}$

$$\exp(\begin{pmatrix} C & I_2 & \cdots & 0 \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ 0 & & & C \end{pmatrix}) = e^{at} \begin{pmatrix} G(t) & tG(t) & \frac{t^2}{2}G(t) & \cdots & \frac{t^{d-1}}{(d-1)!}G(t) \\ 0 & G(t) & tG(t) & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2}G(t) \\ 0 & & & G(t) & tG(t) \\ 0 & 0 & 0 & & G(t) \end{pmatrix}$$

In $\mathbb{R}^2$

$$\exp(t \begin{pmatrix} a & b \\ -b & a \end{pmatrix}) = e^{at} \begin{pmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{pmatrix}$$

5.7 Ansatz für Lyapunov Functions

$$V(x, y) = ax^2 + 2bxy + cy^2$$

with $a > 0$ and $ac - b^2 > 0$.