MATH50006 Week2 and 3

2/1/2022

0.1 Constructing Measure

0.1.1 Pre-measure

- Maps from an **algebra** to $[0, \infty]$
- $\mu(\emptyset) = 0$
- Countably additive whenever that countable union is in the algebra

0.1.2 Outer-measure

- Maps from $\mathbf{2^X}$ to $[0, \infty]$
- $\mu(\emptyset) = 0$
- Sigma-sub-additive

0.1.3 Cover

A family in $\mathbf{2^X}$ wrt X - Has empty set - Countably many sets' union equals X - An algebra is easily a cover - **e.g.** the union of all intervals cover R^n

0.1.4 Extend Pre to Outer

$\mathcal{K}$ is a cover, $\tilde{\mu} : \mathcal{K} \to [0, \infty]$ function on the sets in this cover with $\tilde{\mu}() = 0$, outer measure for a set is the infimum of the sum of all possible coverings of that set

$$\mu^*(A) = inf \left\{ \sum_{j=1}^{\infty} \tilde{\mu}(K_j) : \qquad K_j \in \mathcal{K}, A \subset \cup_{j=1}^{\infty} K_j \right\} \qquad A \in 2^X$$

This is well-defined, as $A \subset X$ is always covered by some sequence of sets, to show sigma sub-additivity, $A \subset \cup_{k=1}^{\infty} A_k$, and $A_k \subset \cup_{j=1}^{\infty} K_{k,j}$, by infimum $\sum_{j=1}^{\infty} \tilde{\mu}(K_{k,j}) \leq \mu^*(A_k) + 2^{-k}\epsilon$, then sum over k

0.1.5 Generate σ-algebra from outer-measure

From outer measure μ^* define:

$$\Sigma = \{A \subset X : \qquad \mu^*(B) = \mu^*(B \cap A) + \mu^*(B \setminus A), \ \forall B \subset X\}$$

The equality can be replaced by $\geq$ by sigma sub-additivity.

0.1.6 Hahn-Caratheodory Extension

Extending a pre-measure to a real measure given an algebra.

Conditions

- X with $\mathcal{A}$ an algebra over it
- $\tilde{\mu} : \mathcal{A} \mapsto [0, \infty]$ a pre-measure
- Define μ^* with $\mathcal{A}$ being the cover, and Σ by previous construction
- Limit μ^* to Σ, then $\mu = \mu^*|_\Sigma$ is a measure

Results

- (X, Σ, μ) is a measure space
- $\mathcal{A} \subset \Sigma$
- $\mu(A) = \mu^*(A) = \tilde{\mu}(A), \quad \forall A \in \mathcal{A}$

Proof

- Verify μ is a measure. Already satisfy $\mu(\emptyset) = 0$ and range $[0, \infty]$.
  - To show **finite additivity**, use definition of Σ.
  - To show **sigma additivity**, use sub-additivity of μ^* and other direction of inequality by finite additivity.

- Show $\mathcal{A} \subset \Sigma$.
  - $\forall A \in \mathcal{A}$, show $A \in \Sigma$, equiv to $\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \setminus A)$
  - Use $\mu^*(K) \leq \tilde{\mu}(K)$
  - Use inf definition, and take cover of $B \subset \cup K_i$, $\sum \tilde{\mu}(K_i) \leq \mu^*(B) + \epsilon$
  - by additivity of pre-measure.
  - $$\mu^*(B \cap A) + \mu^*(B \setminus A) \leq \sum \tilde{\mu}(K_i \cap A) + \tilde{\mu}(K_i \setminus A) = \sum \tilde{\mu}(K_i)$$

- Show $\mu^*(A) = \tilde{\mu}(A)$ in $\mathcal{A}$.
  - Only need $\mu^*(A) \geq \tilde{\mu}(A)$
  - Consider cover of A, K_i's, made into disjoint $\tilde{K}_i$'s, intersected with A, $\tilde{\tilde{K}}_i$'s.

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$$\tilde{\mu}(A) = \sum \tilde{\mu}(\tilde{\tilde{K}}_i) \leq \sum \tilde{\mu}(\tilde{K}_i) \leq \sum \tilde{\mu}$$

– Finish by taking infimum of rightmost to $\mu(A)$.

Proof of uniqueness

Assume there is another pre-measure with $\nu|_{\mathcal{A}} = \tilde{\mu}$, then we show their extensions are equal, namely on the same sigma algebra, $\nu|_{\Sigma} = \mu$

- $\nu(A) \leq \mu(A)$

Taking cover as usual,
$$\nu(A) \leq \sum \nu(K_i) = \sum \tilde{\mu}(K_i)$$
Taking infimum, obtain $\nu(A) \leq \mu^*(A) = \mu(A)$

- $\nu(A) \geq \mu(A)$

Suppose first S finite

$$\nu(A) + \nu(S \setminus A) \leq \mu(A) + \mu(S \setminus A) = \mu(S) = \tilde{\mu}(S) = \nu(S) \leq \nu(A) + \nu(S \setminus A)$$
So
$$\mu(A) = \nu(A) + [\nu(S \setminus A) - \mu(S \setminus A)] \leq \nu(A)$$

For case S infinite, use previous result with disjoint covering sets of A

$$\nu(A) \geq lim_{m \to \infty}\ \nu(\cup_{i=1}^m K_i) = lim\ \mu(\cup_{i=1}^m K_i) = \mu(A)$$