Continuous Time Stochastic Processes

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1 Preliminaries

1.1 Types of Processes

A **right continuous** stochastic process.

There are three types of right continuous processes

- **Normal**
- **Absorption**
- **Explosion**

The **jump times** are random variables

The **holding times** are random variables defined as

A **jump process**

Compute probabilities using **countable union**

A **counting process** is a stochastic process $\{N_t\}_{t\geq 0}$ satisfying

- $N_0 = 0$
- $\forall t \geq 0, N_t \in \mathbb{N}_0$
- (Non-decreasing) If $0 \leq s \leq t$, $N_s \leq N_t$
- (Counting) When $s < t$, $N_t - N_s$ equals the no. of events in $(s, t]$
- (Right continuous) The process is piecewise constant and has upward jumps (single step) of size 1, therefore

$$N_{t^-} = \lim_{s\uparrow t} N_s$$

A **counting process associated the sequence** $(J_n)_{n\in\mathbb{N}_0}$

1.2 Properties of random variables

The **exponential random variable** has

$$f_X(x) = \lambda e^{-\lambda x}$$

and c.d.f.

$$F_X(x) = 1 - e^{-\lambda x}$$

with a nonnegative support.

It has expectation

$$\mathbb{E}[X] = \frac{1}{\lambda}$$

and variance

$$\mathrm{Var}[X] = \frac{1}{\lambda^2}$$

The **memoryless property** of a random variable refers to the fact:

$$\Pr(X > x + y \mid X > x) = \Pr(X > y)$$

- A continuous random variable is memoryless iff it is $\mathrm{Exp}(\lambda)$
- A discrete random variable is memoryless iff it is $\mathrm{Geom}(p)$

The **sum of exponential** $\mathrm{Exp}(\lambda)$ is a $\mathrm{Gamma}(n, \lambda)$ distribution

$$f_{J_n}(t) = \frac{\lambda^n}{\Gamma(n)} t^{n-1} e^{-\lambda t}, \qquad t > 0$$

The **convergence for infinite sum of exponential** has the following criteria

- If $\sum \frac{1}{\lambda_i} < \infty$, then $\Pr(J_\infty < \infty) < 1$
- If $\sum \frac{1}{\lambda_i} = \infty$, then $\Pr(J_\infty = \infty) = 1$

The **minimum of exponential** is

$$H \sim \mathrm{Exp}(\sum_{i=1}^{n} \lambda_i)$$

and the probability of any of the k variables being the minimum is

$$\Pr(H = H_k) = \frac{\lambda_k}{\sum_{i=1}^{n} \lambda_i}$$

The **Laplace Transform** of a random variable X is given by

$$\mathcal{L}_X(u) = \mathbb{E}[e^{-uX}]$$

A list of transformations for common random variables:

- (Poisson) $\exp(\lambda t[e^{-u} - 1])$
- (Exponential) $\dfrac{\lambda}{\lambda + u}$

The **characteristic function** of a random variable X is given by

$$\phi_X(t) = \mathbb{E}[e^{itX}]$$

2 Poisson Processes

2.1 Definitions

A **Poisson process**, denoted $\{N_t\}_{t\geq 0}$, is a non-decreasing stochastic process with nonnegative values satisfying

- $N_0 = 0$
- The increments are independent, $0 \leq t_0 \leq t_1 \leq \ldots \leq t_n$, the random variables $N_{t_0}, N_{t_1} - N_{t_0}, \ldots, N_{t_n} - N_{t_{n-1}}$ are independent
- The increments are stationary

$$\Pr(N_t - N_s = k) = \Pr(N_{t-s} = k)$$

- There is a single arrival (only one arrives in a small interval), for all $t \geq 0$ and $\delta > 0$, $\delta \to 0$

$$\begin{aligned}
&\Pr(N_{t+\delta} - N_t = 1) = \lambda\delta + o(\delta) \\
&\Pr(N_{t+\delta} - N_t \geq 2) = o(\delta) \\
&\Pr(N_{t+\delta} - N_t = 0) = 1 - \lambda\delta + o(\delta)
\end{aligned}$$

An **equivalent definition** replaces the last condition with the variable being Poisson with rate N_t

$$\Pr(N_t = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

Another **equivalent definition** characterizes Poisson process $\{N_t\}_{t\geq 0}$ explicitly

- Let $H_1, H_2, \ldots$ denote i.i.d. $\text{Exp}(\lambda)$ random variables
- Let $J_0 = 0$ and $J_n = \sum_{i=1}^n H_i$
- We define

$$N_t = \sup\{n \in \mathbb{N}_0 : J_n \leq t\}$$

2.2 Properties of Poisson Process

2.2.1 Inter-arrival times

The inter-arrival times are **i.i.d.** $\text{Exp}(\lambda)$ random variables

2.2.2 Time to n^{th} event

The time to n^{th} event is defined as

$$J_n = \sum_{i=1}^n H_i$$

which follows a $\text{Gamma}(n, \lambda)$ distribution

$$f_{J_n}(t) = \frac{\lambda^n}{\Gamma(n)} t^{n-1} e^{-\lambda t}, \qquad t > 0$$

2.2.3 Conditional distribution of arrival times

The conditional joint density of $(J_1, \dots, J_n)$ is given by the order statistic

$$f_{(J_1,\dots,J_n)}(t_1, \dots, t_n \mid N_t = n) = \begin{cases} \dfrac{n!}{t^n} & 0 < t_1 < \dots < t_n \\ 0 & \text{otherwise} \end{cases}$$

The expectation of the k^{th} value of n uniformly distributed order statistics on $[0, t]$ is

$$\mathbb{E}[X_{(k)}] = \frac{tk}{n+1} = \mathbb{E}[J_k \mid N_t = n]$$