Markov Chains

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1 Basics

1.1 Chapman-Kolmogorov (CK) equations

The **n-step** transition probability is

$$p_{ij}(n) = \Pr(X_{m+n} = j | X_m = i)$$

1.2 First passage and hitting times

The **first passage time** is

$$T_j = \min\{n \in \mathbb{N} : X_n = j\}$$

The **first passage probability** is

$$f_{ij}(n) = \Pr(T_j = n | X_0 = i)$$

from which the hitting probability follows

$$f_{ij} = \Pr(T_j < \infty | X_0 = i)$$

1.3 Generating Functions of Markov Chain

Recall the **probability generating function**

$$G_X(s) = \sum_{x=0}^{\infty} s^x \Pr(X = x)$$

where this holds on the support

$$\mathcal{S}_{\mathcal{X}} = \left\{ s \in \mathbb{R} : \sum_{x=0}^{\infty} |s|^x \Pr(X = x) < \infty \right\}$$

The generating functions here are

$$G_{p_{ij}(n)} = \sum_{n=0}^{\infty} p_{ij}(n)s^n$$
$$G_{f_{ij}(n)} = \sum_{n=0}^{\infty} f_{ij}(n)s^n$$

By arguing using equating coefficients and an identity, we have a **theorem**

$$G_{p_{ij}(n)} = \delta_{ij} + G_{f_{ij}(n)}(s)G_{p_{ij}(n)}$$

The identity used is

$$p_{ij}(n) = \sum_{l=1}^{n} f_{ij}(l)p_{jj}(n-l)$$

2 Recurrence and Transience

proof using generating functions:

Taking $s \to 1$ and using Abel's theorem, we can deduce

2.1 Equivalent conditions for recurrence

2.2 Properties of recurrent/transient states

Examples of transient, irreducible chains

2.3 Mean recurrence time, null and positive recurrence

Theorem When state space is finite, at least one state is *recurrent* and all *recurrent* states are positive

Remark This combined with later results on stationarity makes a chain with finite state space particularly nice.

2.4 Examples

3 Aperiodicity and Ergodicity

3.1 Communicating classes

3.2 Properties preserved

- Same period
- Same transience/recurrence
- Null recurrence

3.3 Decomposition of Chains

3.4 Finite State Space

3.5 Gambler's Ruin

4 Staionarity

We are interested in the equilibrium states of a chain

4.1 Distribution

- Distribution is a row vector λ with $\Sigma_j \lambda_j = 1$
- If $\lambda P = \lambda$ then it is called *invariant*

4.2 Stationary distributions of irreducible chains

Theorem Every irreducible chain has a **stationary distribution** π if and only if all states are **positive recurrent** - π is unique - $\pi = \mu_i^{-1}$ the inverse of mean recurrence time

We first have some lemmas:

$$l_{ji}(n) = \Pr(X_n = i, T_j \geq n | X_0 = j)$$

being the probability that the chain reaches i in n steps without returning to j
Lemma

$$f_{jj}(m+n) = \sum_{i \in E, i \neq j} l_{ji}(m) f_{ij}(n)$$

from which $f_{jj}(m+n) \geq l_{ji}(m) f_{ij}(n)$ follows

Lemma We also have the following recurrence relation

Lemma: A positive recurrent chain has a stationary distribution.

Proof: (constructive)

- **(Step1 Construction)** Let $N_i(j)$ be the number of visits to state i before state j ; the sum of such numbers over i is equal to the hitting time T_j
- Define $\rho_i(j)$ to be the expected number of visits to the state i between two successive visits to state j (in this step the **recurrence** of the chain is used, as the T_j is finite with probability 1)

$$\begin{aligned} \rho_i(j) &= \mathbb{E}[N_i(j) | X_0 = j] \\ &= \sum_n \Pr(X_n = i, T_j \geq n | X_0 = j) \\ &= \sum_n l_{ij}(n) \end{aligned}$$

- Now the mean hitting time can be computed as

$$\begin{aligned}\mu_j &= \mathbb{E}\big[\sum_i N_i(j)|X_0 = j\big] \\ &= \sum_i \rho_i(j)\end{aligned}$$

- which can be written as sum of $\rho_i(j)$ by Tonelli and linearity of conditional expectation
- **(Step2 Finiteness)** Use a lemma to bound $\rho_i(j)$ so it's finite
- Namely write $\rho_i(j) = \sum_n l_{ji}(n)$ and bound using the fact that the chain is irreducible, so there exists $f_{ij}(n^*) > 0$, so $f_{jj}(m + n^*) \geq l_{ji}(m) f_{ij}(n^*)$
- **(Step3 Stationarity)** Use a recurrence to show

$$\begin{aligned}\rho_i(j) &= \sum_n l_{ji}(n) \\ &= p_{ji} + \sum_{n=2} \sum_{r, r\neq j} p_{ri} l_{jr}(n-1) \\ &= p_{ji}\rho_i(j) + \sum_{n=1} \sum_{r, r\neq j} p_{ri} l_{jr}(n) \\ &= p_{ji}\rho_i(j) + \sum_{r, r\neq j} p_{ri} \sum_{n=1} l_{jr}(n) \\ &= \sum_r \rho_r(j) p_{ri}\end{aligned}$$

- This $\rho_i(j)$ does not necessarily give a probability vector when the chain is not positive recurrent.
- Now if the chain is positive recurrent, we have μ_j finite for every j, we have

$$\pi_i = \frac{\rho_i(j)}{\mu_j}$$

Lemma If a stationary distribution exists, then the chain is positive recurrent and the distribution must be given by $\pi_i = \mu_i^{-1}$

proof: ...

4.3 Limiting Distribution

4.4 Ergodic Theorem

4.5 Summary of properties of irreducible chains

5 Time reversibility