Functional Analysis

2022

This set of notes is based on the lecture notes of Dr.Pierre François Rodriguez.

- The content in gray boxes like this are either unexaminable or included to facilitate understanding of the material.

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2 Preliminaries

This section aims to provide preliminary knowledge to functional analysis. This field of maths is decorated by ideas of both algebra and analysis, particularly linear algebra and real analysis. It is thus important to get familiar with the relevant ideas, as lack of either viewpoint stops you from getting the whole story. At some point, since we are talking about different spaces, topological concepts also comes in. Luckily, they're generally not complicated and presented here as preliminaries.

2.1 Linear space

Mathematicians usually talks about spaces. However they are simply sets with additional structure. Linear spaces, also called vector spaces, are those with linear structure. This means you have vector addition, scalar multiplication, commutativity and distributivity.

Definition 2.1 (Linear space)**.**
A linear space $(V, \oplus, (\mathbb{F}, +, \cdot), \odot)$ over a field $\mathbb{F}$, where

- $(V, \oplus)$ is an abelian group
- $(\mathbb{F}, +, \cdot)$ is a field

and multiplication by a scalar $\odot : \mathbb{F} \times V \to V$ satisfies for every $\alpha, \beta \in \mathbb{F}$ with $v, \omega \in V$

- $\alpha \odot (v \oplus \omega) = \alpha \odot v + \alpha \odot \omega$
- $(\alpha + \beta) \odot v = \alpha \odot v + \beta \odot v$
- $\alpha \odot (\beta \odot v) = (\alpha \cdot \beta) \odot v$
- $\mathbf{1} \cdot v = v$, where $\mathbf{1}$ is unit element in $\mathbb{F}$

2.1.1 Examples of linear spaces

In this section we have both example and counter example.

Example 2.2 (Vector space over field)**.**
$\mathbb{F}^n$ where $\mathbb{F}$ is a field, $n \in$
natu$isalinearspace.Thisincludes3or2-dimensionalvectorspaceover\mathrm{R}, or3-dimensionalvectorfieldover\mathrm{F}_p$.
For example, let's check

$$V = \mathbb{F}_2^2 = \{(v_1, v_2) | v_1, v_2 \in \mathbb{F}_2\}$$

with the natural definition of scalar multiplication and term-wise addition over $\mathbb{F}_2$. Note that this is indeed a space of four elements:

$$V = \{(0,0),(0,1),(1,0),(1,1)\}$$

With scalars only 1 or 0. Thus it's easy to closure under scalar multiplication. Now consider vector addition, one just have to check every pair of addition and see if still falls into V, like

$$(1,0)+(1,1)=(0,1)\in V$$

Example 2.3.

Consider set of convergent sequence:

$$V = \left\{\{x_n\}_1^\infty : x_i \in \mathbb{F}\, \forall i \in \mathbb{N},\, and \lim_{n\to\infty} x_n \to C\right\}$$

First we consider the addition to be term-wise and multiplication applied to whole sequence:

$$x+y=\{x_i+y_i\}_1^\infty,\, \alpha \odot x = \{\alpha \cdot x_i\},\ \forall x,y \in V,\, \alpha \in \mathbb{F}$$

When C is fixed, this is a vector space if and only if $C = 0$ When C is not fixed, this becomes a vector space.

Example 2.4 (Polynomials)**.**

V is set of all polynomials:

$$f(z)=\sum_{j=0}^{n} a_j z^j,\, n \in \mathbb{N}$$

with $z \in \mathbb{C}$ and $a_j \in \mathbb{Q}$ with addition and multiplication of polynomials. Unfortunately this is not a vector space, since the field is set to be $\mathbb{C}$. When we multiply an element in V by a complex number, say $1+2i$, we could end up in some polynomials with complex coefficient. But this would be a vector space if we change to $a_j \in \mathbb{C}$ or $z \in \mathbb{Q}$

Example 2.5 (Analytic functions)**.**

Consider set of all analytic functions $f : \mathbb{C} \to \mathbb{C}$ satisfying:

$$\frac{d^2}{dz^2} f - \frac{d}{dz} f - 2z = 0$$

This is not a vector space. Take a non-trivial f, consider $g = 2f$:

$$\begin{aligned}&\frac{d^2}{dz^2}g - \frac{d}{dz}g - 2z\\=&\frac{d^2}{dz^2}(2f) - \frac{d}{dz}(2f) - 2z\\=&2\frac{d^2}{dz^2}f - 2\frac{d}{dz}f - 2z\\=&4z - 2z = 2z \neq 0\end{aligned}$$

However, removing $-2z$ will make this a vector space.

Example 2.6 (Weak L^p space)**.**

Define $D_f(t) = \{\lambda x \in \mathbb{R} : |f(x)| > t\}$, where λ is Lebesgue measure on $\mathbb{R}$. Consider:

$$L^{p,w} = \left\{ f : \mathbb{R} \to \mathbb{R} : f,\ measurable,\ and\ \exists C > 0,\ s.t.\ D_f(t) < \frac{C^p}{t_p}\ \forall t > 0 \right\}$$

This is a vector space for $p > 0$ and $L^p(\mathbb{R}) \subset L^{p,w}(\mathbb{R})$. Distributivity and commutativity of scalar operation follows immediately from their definition. The point is to check closure under scalar multiplication and addition.

$\mathbf{f + g \in L^{p,w}}$:

Let $f, g \in L^{p,w}$, then we can find $C_1 > 0$ and $C_2 > 0$ with

$$C_1^p > t^p \cdot \lambda\{x : |f(x)| > t\},\ \forall t > 0$$
$$C_2^p > t^p \cdot \lambda\{x : |g(x)| > t\},\ \forall t > 0$$

Now by triangular inequality we have $|f(x)| + |g(x)| \geq |f(x) + g(x))|$.

Thus $|(f+g)(x)| > t \implies |f(x)| + |g(x)| > t \implies 2 \cdot max(|f(x)|, |g(x)|) > t$. So consider set A, B, C:

$$\begin{aligned}&X := \{x \in \mathbb{R} : |f(x) + g(x)| > t\}\\&Y := \{x \in \mathbb{R} : |f(x)| + |g(x)| > t\}\\&Z := \{x \in \mathbb{R} : 2max(|f(x)|, |g(x)|) > t\} = \{x \in \mathbb{R} : max(|f(x)|, |g(x)|) > t/2\}\end{aligned}$$

We have $X \subseteq Y \subseteq Z$, thus $\lambda(X) \leq \lambda(Y) \leq \lambda(Z)$.

Thus $t^p \cdot \lambda(A) \leq 2^p (t/2)^p \lambda\{x : max(|f(x)|, |g(x)|) > t/2\} \leq 2^p \cdot max(C_1^p, C_2^p) < \infty$ for all $t > 0$.

Showing closure under scalar multiplication is a bit easier. Take any $t > 0$, we have

$$\begin{aligned} & t^p \cdot \lambda\{x \in \mathbb{R} : |2f(x)| > t\} \\ =& 2^p \left(\frac{t}{2}\right)^p \cdot \lambda\{x \in \mathbb{R} : |f(x)| > t/2\} \\ <& 2^p C_1^p \end{aligned}$$

Which completes the proof.

2.2 Metric Linear space

Definition 2.7 (Metric linear space)**.**

A metric space (V, d) is called metric linear space if its vector addition and scalar multiplication $\oplus$, $\odot$ are continuous

Remark 2.8 (Equivalence of metrics)**.**

$\oplus$ can be considered as a function: $\oplus : V \times V \to V$, endowed with metric $\rho_1 : V \times V \to V$ defined as $\rho_1(x_1+y_1, x_2+y_2) = max(d(x_1, x_2), d(y_1, y_2))$ or sum $\rho_2 : V \times V \to V$ defined as $\rho_2(x_1+y_1, x_2+y_2) = d(x_1, x_2) + d(y_1, y_2)$. The two metrics are topologically equivalent, i.e. they induces same topology. Similarly, $\odot$ can be considered as a function: $\odot : \mathbb{F} \times V \to V$, endowed with metric $\rho_1 : \mathbb{F} \times V \to V$ defined as $\rho_1(k_1x_1, k_2x_2) = max(|k_1 - k_2|, d(x_1, x_2))$ or sum $\rho_2 : V \times V \to V$ defined as $\rho_2(x_1 + y_1, x_2 + y_2) = |k_1 - k_2| + d(x_1, x_2)$. Again, the two metrics are topologically equivalent.

Definition 2.9 (Translation invariant)**.**

A metric ρ is translation invariant if for all $x, y, z \in V$, we have $\rho(x, y) = \rho(x - z, y - z)$

Proposition 2.10.

Addition is continuous with respect to translation invariant metic.

Proposition 2.11 (Metric induced by norm)**.**

Let $\|\cdot\|$ be a norm on V, then its induced metric $\rho(x, y) = \|x - y\|$ is translation invariant. See definition of norm here.

2.3 Topology

2.3.1 Separability

Definition 2.12 (dense)**.** A set in S metric space (V, d) is dense if there it intersects with any open subset of V. Equivalently this is $\forall x \in V, \forall \varepsilon > 0$, $D \cap B_{x,\varepsilon} \neq \emptyset$.

Definition 2.13 (separable)**.** A metric space (V, d) is separable if it contains a countable dense subset.

Example 2.14 (separable space example)**.**

2.3.2 Schauder Basis and Hamel basis

Definition 2.15 (Schauder basis)**.** A Schauder basis of Normed vector space $(v, \|\cdot\|)$ is a set $B \subset V$ that is linearly independent, and that $\forall x \in V$, we have a sequence $\{a_n\}$ with $\lim_{n\to\infty} \sum_{k=1}^{n} a_k b_k \to x$. In plain language, this means that every element of the set can be expressed as a infinite linear combination of the basis.

Proposition 2.16 (Schauder implies separability)**.**

If a normed vector space has a Schauder basis, then it's separable.

Definition 2.17 (Hamel basis)**.**

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Remark 2.18 (Hamel basis and Schauder basis)**.**

2.3.3 Compactness

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Definition 2.19 (Compactness)**.**

Let (X, d) be a metric space. A subset $S \subset X$ is **compact** if any sequence (x_n) in S has a convergent subsequence converges in to some $x \in S$

Remark 2.20.

The definition is "sequential compactness". There're many versions of definition of compactness. One shall really pay attention to the definition in linear algebra or real analysis that compactness is equivalence to closedness and boundedness. We'll see later that this is guaranteed to true only for finite-dimensional spaces. However, it is true that compactness always implies closedness and boundedness. Following examples we prove the implications in detail and present the fact that compactness of space can studied by looking at closed unit ball.

Proposition 2.21 (Compact $\implies$ closed and bounded)**.**

If a set K is compact, then it's closed and bounded. Here, bounded means $\exists L > 0$ such that for all $x, y \in K$, we have $d(x, y) \leq L$.

Proof

First we show that compact implies closed. Let (x_n) be a convergent sequence in K, then by compactness, there is a subsequence of (x_n), name it (y_n) which converges to $y \in K$, but by uniqueness of limit (x_n) also converges to $y \in K$.

Now we show boundedness by contradiction. Assume K is not bounded. Then we fix $a \in K$, and by unboundedness we have that $\forall n \in \mathbb{N}$, $\exists x \in K$ with $d(x, a) > n$. Now consider a sequence (x_n) where $d(x_n, a) > n \ \forall n > 0$. This sequence has no convergent subsequence.

Theorem 2.22 (F.Riesz)**.**

Let $(X, \|\cdot\|)$ be a normed vector space. Following are equivalent:

- $dim(X) \leq \infty$
- Closed unit ball $B_U = \{x \in X : \|x\| \leq 1\}$ is compact.

proof: Proof uses a lemma which is shown later

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Lemma 2.23 (F.Riesz)**.**

Let $(X, \|\cdot\|$ be a normed linear space with a subspace $Y \subset X$,$Y \neq X$. Then for all $\varepsilon \in (0, 1)$, there exists $x \in X$ with $\|x\| = 1$ and $d(x, Y) \equiv \inf_{y \in Y} \|x - y\| > 1 - \varepsilon$ *proof*:

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3 Banach Space

Banach space is the one of the core concepts of functional analysis. It is a special type of **vector space**, with a **norm** working on the space as well as the property that every **Cauchy sequence converges** in the space.

One should pay attention that a considerable portion of content in this chapter is not based on completeness but only requires a norm on the space. However, it is obvious that they can be applied to Banach spaces and most importantly, these results do relate themselves to Banach spaces.

3.1 Definitions

Definition 3.1 (Normed vector space).

A normed vector space $\mathbf{X}$ is a vector space (over $\mathbb{F}$, usually $\mathbb{C}$ or $\mathbb{R}$), equipped with a norm function $||\cdot|| : \mathbf{X} \times \mathbf{X} \to \mathbb{R}$ satisfying following:

- $||x|| \geq 0$, $\forall x \in \mathbf{X}$
- $||x|| = 0$ if and only if $x = 0$
- $||ax|| = a||x||$, $\forall x \in \mathbf{X}$, $\forall a \in \mathbb{F}$
- $||x + y|| \leq ||x|| + ||y||, \forall x, y \in \mathbf{X}$

A normed vector space is also called a normed linear space.

Definition 3.2 (Cauchy Sequence).

A Cauchy sequence in a normed vector space V is a sequence $\{a_n\}_1^\infty$, where each $a_n \in V$, satisfying the following: $\epsilon > 0, \exists\, N \in \mathbb{N}\ s.t.\ \forall\, m, n > N, ||a_m - a_n|| < \epsilon$

Definition 3.3 (Convergence).

A sequence $\{a_n\}_1^\infty$in a normed vector space V is convergent if $\exists a \in V$ s.t. $\forall \epsilon > 0, \exists N > 0 s.t. \forall n > N, ||a_n - a|| < \epsilon$.

Definition 3.4 (Completeness).

A normed vector space is complete if every Cauchy Sequence converges to a point in the space.

Theorem 3.5 (Every metric space can be completed).

Result is put here in appendix.

Definition 3.6 (Banach space).

A Banach space is a complete normed vector space.

3.2 Examples

Example 3.7 ($\mathbb{R}^n$)**.**

Our acquainted three-dimensional vector space over $\mathbb{R}$ is a Banach space under the standard vector norm, this is the case when $n = 3$. This is a trivial result, since such norm gives the "length" of a vector, and a Cauchy sequence of vectors indicates that 'endpoints' of vectors come arbitrarily close. It's easy to prove that such a sequence converges to a three-dimensional vector. In fact, any n-dimensional vector space over $\mathbb{R}$ is a Banach space.

Example 3.8 (Real valued functions)**.**

The set of all real-valued function on [0,1] with norm $\|f\| = \max_{t\in[0,1]} |f(t)|$ is a Banach space. To verify this (thoroughly) we shall first show that this is a vector space. This is trivial since sum and and scalar multiplication of a real-valued function is also real-valued. Then we should show that this function is indeed a norm. Finally, we should check that every Cauchy sequence under this norm is convergent. Intuitively, this norm gives the 'maximal pointwise difference' between two functions, hence if a sequence is Cauchy, the 'maximal pointwise difference' converges to zero. Rigorous proof is left to readers.

Example 3.9 (Continuous function under supremum norm)**.**

Consider $C[0,1]$, set of continuous real-valued defined on $[0,1]$ with supremum norm:

$$\|f\|_\infty = \sup_{t\in[0,1]} |f(t)|$$

This is a Banach space (why?). Now a relevant example is $C^1[0,1]$,set of complex-valued function defined on $[0,1]$ with continuous first derivative. Unfortunately this is not a Banach space under supremum norm, however, if we equip the space with a new norm: $\|f\| = \|f\|_\infty + \|f'\|_\infty$, we end up having a Banach space.

Example 3.10 (L-p spaces)**.**

L-p spaces are function spaces with finite $p-norm$. Let (S, Σ, μ) be measure space, $p \in [1, +\infty]$. L-p space consists of functions $S \to \mathbb{C}$ with $p - norm$:

$$\|f\|_p \equiv \left(\int_S |f|^p \, d\mu\right)^{\frac{1}{p}} < \infty$$

It is not obvious how $p - norm$ really gives a norm, the difficulties lie in the part of proving the triangular inequality. In the specific background of L-p spaces, the inequality is precisely *Minkowski's inequality.* Detailed material on Hölder's inequality and Minkowski's inequality are put in appendix.

3.3 Finite dimensional Normed space

Bare in mind that in this section we do not assume that spaces are complete, so please pay attention which results are based on completeness. However, we shall see that all finite dimensional vector space over complete fields are complete.

Some main theorems to focus on in this section, presented in plain language:

- All finite dimensional vector space over a complete field are complete
- All norms on finite dimensional vector spaces are equivalent
- $Bounded + Closed = Compact$ if and only if dimension is finite.

Reference: norm-equivalence note

3.3.1 Equivalence of Norms and topology

There are many norms. Some acts similarly, some acts differently. One may have seen different types of matrix norm, for example, Frobinius norm and operator norm. Computational mathematicians don't seem to care about this, why is it? Maybe they don't make much difference!

Definition 3.11 (Equivalence of Norm)**.**

Let X be a vector space. Two norms $\|\cdot\|_a$, $\|\cdot\|_b$ on X are **equivalent** if their exists $m, n > 0$ satisfying following equation:

$$m\,\|x\|_a \leq \|x\|_b \leq M\,\|x\|_a\,,\ \forall x \in X$$

Theorem 3.12 (Equivalence of finite dimensional norms)**.**

Let X be a finite dimensional vector space, then any two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent.

proof: The proof are divided into four steps:

- Showing that equivalence of norm is transitive
- Showing that equivalence of equivalence on unit sphere implies equivalence on X
- Showing that any norm is continuous with respect to $\|\cdot\|_1$
- Showing that any norm is equivalent to $\|x\|_1 \equiv \sum_{s=1}^{n} |x_s|$

Proof : STEP I

Let $\|\cdot\|_a$ be equivalent to $\|\cdot\|_b$ and $\|\cdot\|_b$ equivalent to $\|\cdot\|_c$. Then $\exists m_1, m_2, M_1, M_2 > 0$ with

$$m_1 \|x\|_a \leq \|x\|_b \leq M_1 \|x\|_a , \ \forall x \in X$$

$$m_2 \|x\|_b \leq \|x\|_c \leq M_2 \|x\|_b , \ \forall x \in X$$

Then

$$m_1 m_2 \|x\|_a \leq m_2 \|x\|_b \leq \|x\|_c , \ \forall x \in X$$

$$\|x\|_c \leq M_2 \|x\|_b \leq M_1 M_2 \|x\|_c \ \ \forall x \in X$$

Which gives $m_1 m_2 \|x\|_a \leq \|c\| \leq M_1 M_2 \|x\|_a$ for arbitrary x. Hence $\|\cdot\|_a$ and$\|\cdot\|_c$ are equivalent.

Proof : STEP II

Now let us assume that $\|\cdot\|_a$ is equivalent to $\|\cdot\|_b$ on $U_a = \{s \in X : \|s\|_a = 1\}$. Then let $x \in X$ be non-zero. Then we have

$$m \left\| \frac{x}{\|x\|_a} \right\|_a \leq \left\| \frac{x}{\|x\|_a} \right\|_b \leq M \left\| \frac{x}{\|x\|_a} \right\|_a$$

So

$$m \|x\|_a \frac{1}{\|x\|_a} \leq \|x\|_b \frac{1}{\|x\|_a} \leq M \|x\|_a \frac{1}{\|x\|_a}$$

And since $\|x\|_a$ is non-zero,we have that

$$m\|x\|_a \leq \|x\|_b \leq M \|x\|_a$$

Now since x is arbitrary, the proof is completed.

Proof : STEP III

Now we shall proof continuity of any norm under $\|\cdot\|_1$. This can be done by showing for a sequence (x_n) converging to x under the metric induced by $\|\cdot\|_1$, the norm if its terms under $\|\cdot\|_a$ converges to $\|x\|_a$. So let (x_n) be a sequence in X with $x_n \xrightarrow{n\to\infty} x$. We have

$$| \|x_n\|_a - \|x\|_a | \leq \|x_n - x\|_a \leq M \|x_n - x\|_1 \to 0 \quad \text{when } n \to \infty$$

Thus

$$\lim_{n\to\infty} | \|x_n\|_a - \|x\|_a | = 0$$

Proof : STEP IV

Now we shall apply extreme value theorem to obtain our final result here. Using the theorem requires unit sphere to be a compact set. Proof of this fact is given later, one should realise that the proof does not depend on equivalence of norms, as we only require compactness in $X, \|\cdot\|_1$. However, it is true that unit sphere is compact under any norm in finite dimensional cases. So we have that $U_1 = \{s \in X : \|s\|_1 = 1\}$ is compact with a function $\|\cdot\|_a$ continuous on it, so by extreme value theorem it attains maximum $M_U = max\{\|x\|_a : x \in U_1\}$ and minimum $m_u = min\{\|x\|_a : x \in U_1\}$ on U_1, thus for any $x \in U_1$ we have

$$m_u \|x\|_1 = m_u \leq \|x\|_a \leq M_u = M_u \|x\|_1$$

Hence we show that any norm is equivalent to $\|\cdot\|_1$ on unit sphere.

By combining the results of the four steps, we finish the proof of the theorem.

Remark 3.13.

The proof is not unique. We can also choose other norms to be the "bridging" norm, say supremum norm which in finite dimensional case becomes the max norm: $\|x\| \equiv max\{|x_i|\}$. About the meaning of equivalence here, in fact, equivalent norms are equivalent in the sense that they induces same topology, so it is also called "topologically"equivalent. Generally speaking, this means that topological properties such as open, close, compact, convergence, continuity which holds for one norm will hold in its equivalent norms.

Following results are simple exercises to check statements above.

Proposition 3.14 (Equivalence of openness)**.**

Open sets in $(X, \|\cdot\|_a)$ are open in $(X, \|\cdot\|_b)$ (Following notation in definition of equivalent norms).

Proof

It suffices to check open balls. Let $B_x^a(r) \equiv \{s \in X : \|x - s\|_a < r\}$ be open balls with radius $r > 0$ centered at x, which is an open ball in $(X, \|\cdot\|_a)$. Choose $p \in B_x^a(r)$, we should show that $\exists \varepsilon > 0$ with $B_p^b(\varepsilon) \equiv \{s \in X : \|p - s\|_b < \varepsilon\} \subset B_x^a(r)$.

By openness of $B_x^a(r)$, we have that $\exists \varepsilon_a > 0$ with

$$\begin{aligned} B_p^a(\varepsilon_a) &\equiv \{s \in X : \|p - s\|_a < \varepsilon_a\} \\ &= \{s \in X : m\|p - s\|_a < m\varepsilon_a\} \\ &\supseteq \{s \in X : \|p - s\|_b < m\varepsilon_a\} \\ &= B_p^b(m\varepsilon_a) \end{aligned}$$

Note that $B_p^b(m\varepsilon_a) \subseteq B_p^a(\varepsilon_a) \subset B_x^a(r)$, hence $\varepsilon = m\varepsilon_a$.

3.4 Subspace of Normed linear space

As shown previously, finite dimensional vector spaces are all complete (we will assume that the fields of vector spaces are complete, $\mathbb{R}$ or $\mathbb{C}$). This is also true for finite dimensional subspace of normed linear spaces. Note that we don't need the space to be complete when asserting completeness of their finite dimensional subspace.

Proposition 3.15 (Completeness of finite-dimensional subspace)**.**
Let $(x, \|\cdot\|)$ be a normed linear space. Then a linear subspace $Y \subset X$ with $dim(Y) < \infty$ endowed with $\|\cdot\|$ induced in Y is a Banach space.

Also, we have closedness of finite-dimensional subspace.

Proposition 3.16 (Closedness of finite-dimensional subspace)**.**
Let $(x, \|\cdot\|)$ be a normed linear space. Then a linear subspace $Y \subset X$ with $dim(Y) < \infty$ endowed with $\|\cdot\|$ induced in Y is closed.

Proof of this two proposition is simple use of the results of finite dimensional normed linear spaces begin Banach, which is left to readers. We shall now take a look at few examples showing different subspaces of normed vector spaces.

Example 3.17 (Finite-dim subspace being complete)**.**

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Example 3.18 (Infinite-dim subspace not complete)**.**
Consider $X = C[0, 2] \subset L^1[0, 2]$, set of all real-valued continuous function on $[0, 1$, endowed with

1-norm: $\|f\|_1 = \int_0^1 |f(t)|dt$. Consider sequence of function (f_n):

$$f_n(t) = \begin{cases} t^n & 0 \leq t < 1 \\ 1 & 1 \leq t \leq 2 \end{cases}$$

Now f_n is Cauchy in 1-norm, but its limit is not in $C[0,1]$:

$$f(t) = \begin{cases} 0 & 0 \leq t < 1 \\ 1 & 1 \leq t \leq 2 \end{cases}$$

Thus X is not a complete subspace of $L^1[0,2$, as we have a Cauchy sequence that does not converge to a point in X.

Example 3.19 (Completeness depends on choice of norm)**.**

We continue considering the settings in our last example, but endow X with supremum norm: $\|\cdot\|_\infty$. Now our (f_n) is no longer Cauchy. One can prove that $(X, \|\cdot\|_\infty$ is actually complete. Intuitively, supremum norm is sensitive to "discontinuity at points", so convergence in supremum norm ensures no "jump" of value at any point.

3.5 Fixed point

Contraction mapping theorem, sometimes called Banach fixed point theorem, guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces, and provides a constructive method to find those fixed points.

Theorem 3.20 (Fixed point)**.**

Let X be Banach space, $f : X \to X$ is a contraction mapping, i.e. a mapping satisfying $d(f(x), f(y)) \leq qd(x, y)$, $\forall x, y \in X$, where $q < 1$ is a fixed constant. Then f admits a unique fixed point $x^* \in X$, which can be found by defining sequence x_n in X, with $x_{n+1} = f(x_n)$, then $\lim_{n\to\infty} x_n = x^*$

proof:

By assumption, we have $\|x_n + 1 - x_n\| = \|f(x_n) - f(x_{n+1})\| \leq q\,\|x_n - x_{n+1}\|$.

So if we let $k = \|x_2 - x_1\|/q$, we have that $\|x_{n+1} - x_n\| \leq kq^n$ with $q < 1$. This sequence is clearly Cauchy. To see this, choose $m, n \in \mathbb{N}$ with $m < n$

$$\begin{aligned} \|x_m - x_n\| &\leq \sum_{i=m}^{n-1} \|x_{i+1} - x_i\| \\ &\leq \sum_{i=m}^{n-1} kq^i \\ &= \frac{kq^m(1 - q^{(n-m)})}{1-q} \\ &< \frac{kq^m}{1-q} \end{aligned} \tag{1}$$

Fix $\varepsilon > 0$, we can find large N so that

$$q^N \leq \frac{\varepsilon(1-q)}{k}$$

Then for all $m, n > N$ we have

$$\|x_m - x_n\| < \frac{kq^m}{1-q} < \frac{\varepsilon(1-q)}{k}\frac{k}{1-q} = \epsilon$$

Hence x_n is Cauchy, thus convergent to a unique point $x^* \in X$

Remark 3.21.

This theorem is easy to prove, however it's very useful. We may see examples of fixed point by playing with calculator: choose any real number and calculate its cos value, and continue input the result to *cos*, we may find that the result become stable around 0.7390....... This is a fixed point of *cos*. *sin* also has fixed point, which is zero. It can be used to give sufficient condition where Newton method for finding root converges. In study of ODE contraction mapping can be used to guarantee that Picard iteration converges to a certain function. See Picard–Lindelöf theorem on Wikipedia.

3.6 Exam

3.6.1 Proof of completeness

Generally there are three steps in proving completeness.

- Find a candidate for the "limit" of a Cauchy sequence.
- Show that it's indeed the "limit'.
- Show that it's still in the space.

4 Hilbert Space

Hilbert space is a special class of Banach space. Apart from completeness and norm, it is also equipped with an additional structure, **inner product**. This allow us to explore nice geometric properties of the space,like orthogonality and angle. We'll see later that this structure resemble Euclidean space in many ways. A Hilbert space is naturally Banach, while the reverse may not be true.

In this section we work with linear space H over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

Definition 4.1. (Bilinear Map) Let X be a vector space over $\mathbb{C}$. An **bilinear map** is a function $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$ satisfying following: $\forall x, y, z \in H, \alpha$ a scalar,

1 $\langle x, y \rangle = \overline{\langle y, x \rangle}, \forall x, y \in H$

2 $\langle x, x \rangle \geq 0$

3 $\langle x, x \rangle = 0$ iff $x = 0$

4 $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

5 $\langle ax, z \rangle = a \langle x, z \rangle$

1 is complex conjugation. 2 and 3 is positive-definiteness. 4 and 5 is left-linearity.

Theorem 4.2. *If $\langle \cdot, \cdot \rangle$ is an inner product on X, define $\|x\| \stackrel{def}{=} \sqrt{\langle x, x \rangle}$.*

i) (Cauchy-Schwarz) $\forall x, y \in X$,

$$| \langle x, y \rangle |^2 \leq \langle x, y \rangle \cdot \langle y, y \rangle$$

ii) $\|x\|$ is a norm

Proof. i) If $x = 0$ or $y = 0$, the inequality holds. Else, let $\xi = \frac{x}{\|x\|}, \eta = \frac{y}{\|y\|}$, so $\|\xi\| = \|\eta\| = 1$. Hence

$$0 \leq \|\eta - \langle \xi, \eta \rangle \xi\|^2 = \|\eta\|^2 - | \langle \xi, \eta \rangle |^2 = 1 - | \langle \xi, \eta \rangle |^2$$

so $| \langle \xi, \eta \rangle | \leq 1$

ii) Positivity and homogeneity follows from definition of $\langle \cdot, \cdot \rangle$; and triangle inequality follows from i)

$$\|x + y\|^2$$

□

Definition 4.3. (Hilbert space) An inner product space $(H, \langle\cdot,\cdot\rangle)$ which is complete w.r.t. the metric induced by $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$ is called a **Hilbert space**

Example 4.4 (Euclidean Space over $\mathbb{R}$)**.**

It happens that 2-D or 3-D vector space over $\mathbb{R}$ is an example of Hilbert space, under the standard definition of vector inner product.

Example 4.5 (Euclidean Space over $\mathbb{C}$)**.**

$\mathbb{C}^n$ with inner product

$$\langle x, y\rangle = \sum_{i=1}^{n} \overline{x_i}\, y_i$$

is a Hilbert space. This is a generalization of last example. Still, this is a finite-dimensional Hilbert space.

Example 4.6 (Sequence space)**.**

Complex sequence space:

$$\ell^2 = \left\{ \{x_n\}_1^\infty : \sum_{k=1}^{\infty} |x_k|^2 < \infty \right\}$$

with inner product,denoting $x = \{x_n\}_1^\infty$ and $y = \{y_n\}_1^\infty$

$$\langle x, y\rangle = \sum_{k=1}^{\infty} \overline{x_k}\, y_k$$

Structure of inner product allows discussion for nice geometric property of Hilbert spaces. This includes orthogonality, angles and nearest distance etc.

Orthogonality is the generalization of two lines being perpendicular. In euclidean geometry, we have Pythagorean theorem closely related to such property. Results on orthogonality in Hilbert spaces in many ways resemble their Euclidean version.

Proposition 4.7. *(Nearest Point Property) Let $\mathscr{H}$ be a Hilbert space, $K \subset \mathscr{H}$ be a closed, convex subset, then $\forall y \in \mathscr{H}$ there exists a* **unique** *$x_0 \in K$ such that*

$$\delta \overset{\text{def}}{=} \inf_{x\in K} \|x - y\| = \|x_0 - y\|$$

Proof. By considering the set $K - y = \{x - y : x \in K\}$ (still closed and convex), we can assume $y = 0$.

<u>**Existence:**</u> By definition of δ, $\exists (x_n)_{n\in\mathbb{N}}$, $x_n \in K$ such that $\lim_{n\to\infty} \|x_n\| = \delta$. We show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $\epsilon > 0$. Pick $N \in \mathbb{N}$ such that

$$\forall n \geq N \qquad \|x_n\|^2 < \delta^2 + \frac{\varepsilon^2}{4}$$

K being convex implies that $\frac{x_n+x_n}{2} \in K, \forall n, m \in \mathbb{N}$, which implies by definition of δ, $\|x_n + x_m\| \geq 2\delta$.

It follows that for all $n, m \geq N$,

$$\|x_n - x_m\|^2 = \underbrace{2(\|x_n\|^2 + \|x_m\|^2)}_{<2\delta^2+\varepsilon^2/2} \underbrace{- \|x_n + x_m\|^2}_{\leq 4\delta^2} < \varepsilon^2$$

where we have used the Parallelogram law.

By completeness, $\exists x_0$ s.t. $x_k \to x_0$ as $k \to \infty$. Since K is closed, the limit $x_0 \in K$, and $\|x_0\| = \delta$ by continuity of the norm $\|\cdot\|$.

<u>**Uniqueness:**</u> □ uniqueness

Corollary 4.8. *(Orthogonal Decomposition) Let H be a Hilbert space and $E \subset H$ be a closed subspace. Then*

$$H = E \oplus E^{\perp}$$

(i.e. $E \cap E^{\perp} = \{0\}$ and $H = E + E^{\perp}$, that is $\forall x \in H, x = e + e^{\perp}$ for some $e \in E, e^{\perp} \in E^{\perp}$)

Proof. If $x \in E \cap E^{\perp}$, then $\langle x, x \rangle = 0$, so $\|x\| = 0$, $x = 0$.

For all $x \in H$, E □

Definition 4.9 (Orthogonality)**.**

Let $\mathscr{H}$ be a Hilbert space and $f \in \mathscr{H}$. Say g is orthogonal to f if $\langle f, g \rangle = 0$, writes $f \perp g$. For two sets $A, B \in \mathscr{H}$, write $A \perp B$ if $a \perp b$ for all $a \in A$ and $b \in B$

Definition 4.10 (Orthogonal Complement)**.**

Let A be a set in Hilbert space $\mathscr{H}$. Its orthogonal complement, $A^{\perp}$ is the set of all vectors $f \in \mathscr{H}$ such that $f \perp g, \forall g \in A$. $A^{\perp}$ is always a subset of $\mathscr{H}$. Moreover, $A^{\perp} = \cup_{a \in A}(a)^{\perp}$, and we can prove that $A^{\perp}$ is always a closed subset of $\mathscr{H}$.

Proposition 4.11 (Pythagorean Theorem)**.**

Let f_1, f_2, ..., f_n be pairwise orthogonal vectors in Hilbert space $\mathscr{H}$. Then

$$\left\| \sum_{k=1}^{n} f_k \right\|^2 = \sum_{k=1}^{n} \|f_k\|^2$$

proof:

It suffices to show for n=2, and proceed with induction. Consider $a, b \in \mathscr{H}$ with $a \perp b$. Then

$$\begin{aligned} \|a+b\|^2 &= \langle a+b, a+b\rangle \\ &= \langle a,a\rangle + \langle a,b\rangle + \langle b,a\rangle + \langle b,b\rangle \\ &= \langle a,a\rangle + 0 + 0 + \langle b,b\rangle \\ &= \|a\|^2 + \|b\|^2 \end{aligned} \tag{2}$$

Remark 4.12 (Parallelogram equality)**.**

Let $a, b \in \mathscr{H}$ be arbitrary vectors. Then

$$\begin{aligned} \|a+b\|^2 &= \langle a+b, a+b\rangle \\ &= \langle a,a\rangle + \langle a,b\rangle + \langle b,a\rangle + \langle b,b\rangle \\ &= \langle a,a\rangle + \langle a,b\rangle + \overline{\langle a,b\rangle} + \langle b,b\rangle \\ &= \|a\|^2 + 2Re(\langle a,b\rangle) + \|b\|^2 \end{aligned} \tag{3}$$

Similarly,

$$\begin{aligned} \|a-b\|^2 &= \langle a-b, a-b\rangle \\ &= \langle a,a\rangle - \langle a,b\rangle - \langle b,a\rangle + \langle b,b\rangle \\ &= \langle a,a\rangle - \langle a,b\rangle - \overline{\langle a,b\rangle} + \langle b,b\rangle \\ &= \|a\|^2 - 2Re(\langle a,b\rangle) + \|b\|^2 \end{aligned} \tag{4}$$

Adding up, we obtain ***parallelogram equality***:

$$\|a+b\|^2 + \|a-b\|^2 = 2\|a\|^2 + 2\|b\|^2$$

This equation holds for all Hilbert spaces including 2-d vector space over $\mathbb{R}$. Readers may find this form identical to the parallelogram equality in that vector space. It is also where the name comes from.

4.0.1 Nearest Point Property

In Euclidean geometry, choosing a point and a line we can find the minimal distance form the point to any point on the line, defined as the distance from the point to the line. This can be generalized, with line extending to a ***closed convex set*** and space becoming a Hilbert space.

Definition 4.13 (Convexity).

A set $S \subset \mathscr{H}$ is convex if $\forall f, g \in S, \forall t \in [0,1]$, we have $(tf + (1-t)g) \in S$.

Intuitively, this means that given any two points in the a convex set, the "segment" connecting the points stays inside the set. Hence a closed convex set is such a set with the property that every convergent sequence in the set converges to a point in the set.

Definition 4.14 (Set-point distance).

Let $\mathscr{H}$ be a Hilbert space, $S \in \mathscr{H}$ be a closed subset, $x \in \mathscr{H}$ be arbitrary vector. We define the **distance** from x to S to be the infimum of distance between x and element of S:

$$\operatorname{dist}(S, x) \equiv \inf_{s \in S}(\|s - x\|)$$

4.0.2 Projection Theorem

Definition 4.15 (Projection mapping).

Let $\mathscr{H}$ be a Hilbert space. A mapping $P : \mathscr{H} \to \mathscr{H}$ is a projection mapping if

$$P(Px) = Px, \forall x \in \mathscr{H}$$

Proposition 4.16 (Projection Theorem).

Let $\mathscr{H}$ be a Hilbert space and M a closed subspace. Then there exists unique pair of projection mapping $P : \mathscr{H} \to M$ and $Q : \mathscr{H} \to M^{\perp}$ satisfying $x = Px + Qx$ for any $x \in \mathscr{H}$, with the following property:

(1) $x \in M$ if and only if $Px = x$, $Qx = 0$

(2) $x \in M^{\perp}$ if and only if $Px = 0$, $Qx = x$

(3) $\|Px\|^2 + \|Qx\|^2 = \|x\|^2$

(4) P and Q are linear maps

(5) Px is the closest vector in M to x.

(6) Qx is the closest vector in $M^{\perp}$ to x.

proof:

We shall use nearest point property. Since M is a closed subspace, it is clearly a convex subset of $\mathscr{H}$. Thus for each $x \in \mathscr{H}$ there exists a unique point in M that is closest to x. So we define Px to be the unique nearest point in M for each x. Uniqueness of nearest point gives the uniqueness of mapping P. Q is then defined as $x - Px$. We should show that this definition of Q indeed gives a mapping from $\mathscr{H}$ to $M^{\perp}$.

Fix x, let $m \in M$ be such that $\|m\| = 1$. We must have, for all $a \in \mathbb{C}$:

$$\begin{aligned} \|Qx\|^2 &\leq \|Qx + am\|^2 \\ &= \|Qx\|^2 + |a|^2 \|m\|^2 + \langle Qx, m \rangle + \langle m, Qx \rangle \end{aligned} \tag{5}$$

Then we choose $a = -\langle Qx, m \rangle$ and simplify the equation, we have

$$\begin{aligned} 0 &\leq |a|^2 \|m\|^2 + \langle Qx, am \rangle + \langle am, Qx \rangle \\ &= |\langle Qx, m \rangle|^2 \|m\|^2 + \langle Qx, -\langle Qx, m \rangle m \rangle + \langle -\langle Qx, m \rangle m, Qx \rangle \\ &= |\langle Qx, m \rangle|^2 - \overline{\langle Qx, m \rangle} \langle Qx, m \rangle - \overline{\langle m, Qx \rangle} \langle m, Qx \rangle \\ &= -|\langle Qx, m \rangle|^2 \end{aligned} \tag{6}$$

Thus we must have $\langle Qx, m \rangle = 0$, so $Qx \in M^{\perp}$.

We proceed to prove (1) and (2).

First $x \in M$. We have $Qx \in M^{\perp}$. Note that $x \in M$ and $Px \in M$ we have $Qx = x - Px \in M$. Hence $Qx \in (M \cap M^{\perp}) = 0$. Thus $Qx = 0$ and $Px = x - 0 = x$.

To see other direction, we simply notice that $x = Px \in M$ by definition.

Proof of (2) is similar.

Now let's prove (3). We shall use the fact that $Px \perp Qx$, giving $\langle Px, Qx \rangle = 0$

$$\begin{aligned} \|x\|^2 &= \langle x, x \rangle \\ &= \langle Px + Qx, Px + Qx \rangle \\ &= \langle Px, Px \rangle + \langle Qx, Px \rangle + \langle Px, Qx \rangle + \langle Qx, Qx \rangle \\ &= \langle Px, Px \rangle + \langle Qx, Qx \rangle \\ &= \|Px\|^2 + \|Qx\|^2 \end{aligned} \tag{7}$$

Proof of (4) is a routine work verifying linearity. Left as an exercise.

Now let's prove (5) and (6).

(3) is given by the construction of P. To see (4), consider $y \in M^{\perp}$, we have:

$$\|x - y\| = \|Px + Qx - y\| = \|Px\| + \|Qx - y\| \geq \|Px\|$$

So minimal distance is $\|Px\|$, Obtained at $y = Qx$

5 Duality

Recall that the space of all bounded linear operators is defined as

$$\mathcal{L}(X,Y) = \{A : X \to Y, A \text{ bounded, linear}\}$$

$\mathcal{L}(X,Y)$ is Banach if Y is Banach and it has norm

$$\|A\| = \|A\|_{\mathcal{L}(X,Y)} = \sup_{\|x\|_X \leq 1} \|Ax\|$$

An Important special case is

$$X^* \stackrel{\text{def}}{=} \mathcal{L}(X,\mathbb{R})$$

which is the *dual space* of X.

Definition 5.1. (Dual Spaces) The space of all continuous linear operators $\mathcal{L}(X,\mathbb{R})$ is called the **dual space** of X and is denoted as X^*

Remark 5.2. X^* is always Banach (eventhough X may not be). We often abbreviate $\|\cdot\|_* = \|\cdot\|_{X^*}$. The elements of X^* are called (bounded, linear) **functionals**.

Dual spaces play a central role in functional analysis. They are easiest to grasp in the following contexts.

5.1 Duality in Hilbert Spaces

In this section, let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space over $\mathbb{R}$. For $y \in H$, we define the map

$$\Lambda_y : X \to \mathbb{R}, \qquad x \mapsto \langle y, x\rangle_H$$

We note that this is an injective map from H to its dual H^* and we will show that this is in fact a bijective isometry.

Lemma 5.3. *(Mapping to dual space)*

i) $\Lambda_y \in H^*$

ii) The map $\Lambda : H \to H^*$ *is a linear isometry with* $||\Lambda_y||_* = ||y||$

Proof. i) We need to check the linearity and boundedness of the operator Λ_y^*. The former follows from that of the inner product and the latter is proven by applying Cauchy-Schwarz

$$||\Lambda_y||_* = \sup_{x \in H, ||x|| \leq 1} | \langle y, x \rangle_H | \leq \|y\|_H$$

which implies $\Lambda_y \in H^*$

ii) Choose $x = \frac{y}{\|y\|_H}$ to attain the equality in the equation above , whence we have $\|\Lambda_y\|_* = \|y\|$.

□

Theorem 5.4. *(Riesz Representation) For every $\ell \in H^*$, there is a unique $y \in H$, such that $\ell = \Lambda_y$*

Proof. We show the existence and uniqueness of such a linear operator.

- **(Existence)** If $\ell(x) \equiv 0$, then take $y = 0$. Otherwise, assume $\|\ell\|_* = 1$ (as we can replace $\ell(\cdot)$ by $\frac{\ell(\cdot)}{\|\ell\|_*}$). By the definition of $\|\cdot\|_*$, there is a sequence of $(y_n)_{n\in\mathbb{N}} \subset H$ with

$$|\ell(y_n)| \to \|\ell\|_* , \qquad \|y_n\| = 1, \forall n \in \mathbb{N}$$

We will show that the limit of this sequence is the desired y.

Claim 1: The sequence $(y_n)_{n\in\mathbb{N}}$ is Cauchy

Apply the parallelogram identity to $x = \frac{y_n}{2}$ and $y = \frac{y_m}{2}$, so we have

$$\forall n, m \geq 1, \qquad \left\| \frac{y_n - y_m}{2} \right\|^2 = 1 - \left\| \frac{y_n + y_m}{2} \right\|^2$$

Using linearity and boundedeness of ℓ,

$$\frac{1}{2}|l(y_n) + l(y_m)| = |l(\frac{y_n + y_m}{2})| \leq \|l\|_* \left\| \frac{y_n + y_m}{2} \right\|$$

The LHS of the equation above converges to 1 by assumption on $(y_n)_{n\in\mathbb{N}}$, which implies and $(y_n)_{n\in\mathbb{N}}$ is Cauchy. Since H is complete, there is a unique y, such that $y_n \to y$

Claim 2: $\ell = \Lambda_y$

Since span$\{y\}$ is closed, we can consider the orthogonal decomposition $H = \text{span}\{y\} \oplus (\text{span}\{y\})^\perp$. It suffices to show:

(1) $\ell(y) = \Lambda_y(y), \forall y \in \text{span}\{y\}$

(2) $\ell(x) = \Lambda_y(x), \forall x \in (\text{span}\{y\})^\perp$

To show (1), assume wlog $\|y\| = 1$, we note by continuity of ℓ

$$\ell(y) = \lim_{n\to\infty} |\ell(y_n)| = \|\ell\|_* = 1$$

and

$$\|y\|_H^2 = \langle y, y\rangle_H = \Lambda_y(y) = 1$$

So $\Lambda_y(y) = \ell(y)$.

To show (2), we need to argue that

$$\ell(x) = 0, \qquad \forall x \in (\text{span}\{y\})^\perp$$

Now take $y_a = \frac{y+ax}{\sqrt{1+a^2}}$ and $\|y_a\| = 1$, where $y \in \text{span}\{y\}, a \in \mathbb{R}$ and $x \in (\text{span}\{y\})^\perp$. By definition of the norm $\|\cdot\|_*$ and (1),

$$\ell(y_a) \le |\ell(y_a)| \le 1 = \ell(y)$$

So $\ell(y_a)$ has a global maximum at $a = 0$ ($y_0 = y$). Therefore,

$$0 = \frac{d}{da}\ell(y_a)\Big|_{a=0} = \frac{d}{da}\frac{1}{\sqrt{1+a^2}}(\ell(y) + a\ell(x)) = \ell(x)$$

So $\ell(x) = \Lambda_y(x), \forall x \in (\text{span}\{y\})^\perp$.

- **(Uniqueness)** If $\ell = \Lambda_y = \Lambda_z$ for some $y, z \in H$, then

$$\forall x \in H \qquad \Lambda_y(x) = \langle y, x\rangle_H = \langle z, x\rangle_H = \Lambda_z(x)$$

Pick $x = y - z$, then $\langle y, z - x\rangle_H = \langle z, z - x\rangle_H \implies \|y - z\|_H^2 = 0$. Hence $y = z$.

□

Corollary 5.5. *All Hilbert spaces H are isomorphic to their duals H^*.*

Example 5.6. Some of the examples are:

- $(l^2)^* \cong l^2$
- $(L^2(\mu))^* \cong L^2(\mu)$

5.2 Duality in Banach Spaces

Theorem 5.7. *(Conjugates are duals) For all $p \in (1, \infty)$, $(\ell^p)^* \cong \ell^q$, where $\frac{1}{p} + \frac{1}{q} = 1$*

Proof. For $y \in \ell^q$ define

$$\Lambda_y : \ell^p \to \mathbb{R}$$
$$x \mapsto \Lambda_y(x) = \sum_{n \in \mathbb{N}} y_n x_n$$

Lemma 5.8. *(cf.)* what lemma

i) $\Lambda_y \in (\ell^p)^*$

ii) $\Lambda : \ell^q \to (\ell^p)^*, y \mapsto \Lambda_y$ *is a linear isometry*

i) By Hölder's inequality,

$$|\Lambda_y(x)| \leq \sum_{n=1}^{\infty} |y_n x_n| \leq \|y\|_q \|x\|_p$$

in particular Λ_y is well-defined (i.e. maps to $\mathbb{R}$). The inequality implies $\|\Lambda_y\|_* \leq \|y\|_q$. In fact, one has equality. Let $x = (x_n)_n$ with

$$x_n = \text{sign}(y_n)|y_n|^{q-1}$$

where the sign function is

$$\text{sign}(t) = \begin{cases} 1 & t \geq 0 \\ -1 & t < 0 \end{cases}$$

with $\text{sign}(t)t = |t|, \forall t \in \mathbb{R}$. Then $x \in \ell^p$ as $|x_n|^p = |y_n|^{p(q-1)} = |y_n|^q$ so

$$\|x\|_p = \left(\sum_{n \in \mathbb{N}} |y_n|^q \right)^{\frac{1}{p}} = \|y\|_q^{\frac{q}{p}} = \|y\|_q^{q-1}.$$

and

$$|\Lambda_y(x)| = |\sum_{n \in \mathbb{N}} x_n y_n| = \sum_{n \in \mathbb{N}} |y_n|^q = \|y\|_q^q = \|y\|_q \|x\|_p$$

which implies

$$\|\Lambda_y\|_* \geq \frac{|\Lambda_y(x)|}{\|x\|_p} = \frac{\Lambda_y(x)}{\|x\|_p} = \|y\|_q$$

from which ii) follows.

To complete the proof of the theorem, we have to show the following analogue of Riesz representation theorem (which applies only when $p = q = 2$).

Lemma 5.9. *$\Lambda : \ell^q \to (\ell^p)^*$ given by $y \mapsto \Lambda_y$ is surjective (onto).*

Let $e_n = (0, \ldots, 0, 1, 0, \ldots) \in \ell^p$ and define $y_n = \ell(e_n)$ for $\ell \in (\ell^p)^*$

Claim:

i) $y = (y_n) \in \ell^q$

ii) $\ell = \Lambda_y$ for some y

Consider the "truncated y": $y^{(n)} = (y_1, \ldots, y_n, 0, \ldots) = \sum_{i=1}^n y_i e_i \in \ell^q$ and

$$x^{(n)} = \sum_{i=1}^n |y_i|^{q-1} \operatorname{sign}(y_i) e_i \in \ell^p$$

Then as before: $\left\|x^{(n)}\right\|_p = \left\|y^{(n)}\right\|_q^{q-1}$ with

$$\ell(x^{(n)}) = \sum_{i=1}^n |y_i|^{q-1} \operatorname{sign}(y_i) \ell(e_i) = \sum_{i=1}^n |y_i|^q = \left\|y^{(n)}\right\|_q^q$$

where $y_i = \ell(e_i)$. Hence

$$\left(\sum_{i=1}^n |y_i|^q\right)^{\frac{1}{q}} = \left\|y^{(n)}\right\|_q = \frac{\ell(x^{(n)})}{\left\|y^{(n)}\right\|_q^{q-1}} = \frac{\ell(x^{(n)})}{\left\|x^{(n)}\right\|_p} \leq \|\ell\|_* < \infty$$

and letting $n \to \infty$, Claim (i) follows.

For ii), let $x \in \ell^p$ and $\varepsilon > 0$. Since e_n's form a Schauder basis, we know that

$$\left\|x^{(n)} - x\right\|_p \overset{n \to \infty}{\to} 0$$

(since by definition of Schauder basis, we have unique rep. of $x = \sum_{n \in \mathbb{N}} x_n e_n$)

By choosing n large, we can ensure that

$$\left\|l(x) - l(x^{(n)})\right\| < \varepsilon/2, \qquad |\Lambda_y(x) - \Lambda_y(x^{(n)})| < \frac{\varepsilon}{2}$$

using continuity of $\ell(\cdot)$ and $\Lambda_y(\cdot)$, where the latter is a consequence of **??**.

But writing

$$|\ell(x) - \Lambda_y(x)| \leq |\ell(x) - \ell(x^{(n)})| + |\ell(x^{(n)}) - \Lambda_y(x^{(n)})| + |\Lambda_y(x^{(n)}) - \Lambda_y(x)| \qquad (8)$$

and observing that $\ell(x^{(n)}) = \sum_{i=1}^n x_i \ell(e_i) = \sum_{i=1}^n x_i y_i = \Lambda_y(x^{(n)})$, it follows that

$$|\ell(x) - \Lambda_y(x)| \leq \varepsilon \quad \text{and} \quad \ell = \Lambda_{\mathrm{y}}$$

by letting $\varepsilon \downarrow 0$ $\square$

Remark 5.10. 1) The proof extends to $p = 1$, so $(\ell^1)^* = \ell^\infty$. In fact, for $(X, \mathcal{A}, \mu)$, one has

$$L^p(\mu)^* \cong L^q(\mu) \qquad \forall p \in [1, \infty)$$

2) For $p = \infty$, one can still define

$$\Lambda : \ell^1 \to (\ell^\infty)^* \qquad y \mapsto \Lambda_y$$

as before and check that Λ is a linear isometry between Banach spaces.

However, it is **not** surjective.

To see this, consider

$$c_0 = \{(x_n) : \lim_{n\to\infty} = 0\} \subset \ell^\infty$$

Definition 5.11. (Dual Operators)

Definition 5.12. (Adjoint Operators)

Example 5.13. (Adjoint operators as matrices)

Remark 5.14.

6 Hahn-Banach Theorem

Definition 6.1. (Sublinear functional) Let X be a vector space. $p : X \to \mathbb{R}$ is called **sublinear** if the following holds

i) $p(\alpha x) = \alpha p(x)$, $\forall x \in X$ and $\alpha \geq 0$

ii) $p(x+y) \leq p(x) + p(y), \ \forall x, y \in X$

Example 6.2. Any linear functional is also sublinear. Also, the norms on X are sublinear.

Theorem 6.3. *(Hahn-Banach) Let $M \subset X$ be a linear subspace, $p : X \to \mathbb{R}$ is sublinear, and $f : M \to \mathbb{R}$ is linear with*

$$f(x) \leq p(x) \qquad \forall x \in M$$

Then, there exists a linear map $F : X \to \mathbb{R}$ with $F|_M = f$ and

$$F(x) \leq p(x) \qquad \forall x \in X$$

Lemma 6.4 (one-dimensional extension)**.** *Let X be a normed vector space over $\mathbb{F}$ ($\mathbb{C}$ or $\mathbb{R}$), Y_n is a proper subspace of X. Let $v \in X \backslash Y$, $X_{n+1} = \{x + hv : x \in X_n,\, h \in \mathbb{C}\}$. If $T_n : Y \to \mathbb{F}$ is a bounded linear functional, then there exists a bounded linear functional $T_{n+1} : X_{n+1} \to \mathbb{F}$ satisfying:*

- $T_{n+1}(x) = T_n(x)$ *for all* $x \in X_n$
- $\|T_{n+1}\| = \|T_n\|$

Proof : One-dimensional extension

Define linear functional $P : X_{n+1} \to \mathbb{R}$ by

$$P(x + kv) = T_n(x) - Ck, \; \forall\, x \in X_n, \; k \in \mathbb{R}$$

where C is a constant to be determined. First we shall check linearity, which is left as an exercise. Then we shall show that we can find a proper constant C so that $\|P\| = \|T_n\|$. Note that $X_n \subset X_{n+1}$, so we have

$$\begin{aligned} \|P\| &= \sup_{x \in X_{n+1}} (\{|Px| : \|x\| = 1\}) \\ &\geq \sup_{x \in X_n} (\{|Px| : \|x\| = 1\}) \\ &= \sup_{x \in X_n} (\{|T_n x| : \|x\| = 1\}) \\ &= 1 \end{aligned} \tag{9}$$

So by choosing C such that $P(x + kv) \leq \|x + kv\|$ for any $x \in X_n$ and $k \in \mathbb{R}$, we will have that $\|P\| \leq 1$, giving $\|P\| = 1$. Thus it remains to show that we can find such a constant C. We aim to find C such that

$$|P(x + kv)| = |T_n(x) - Ck| \leq \|x + kv\|, \; \forall x \in X_n, \; \forall k \in \mathbb{R}$$

Hence,

$$T_n(x) - \|x + kv\| \leq Ck \leq T_n(x) + \|x + kv\|, \; \forall x \in X_n, \; \forall k \in \mathbb{R}$$

Note that for all $x, y \in X_n$ we have:

$$\begin{aligned} T_n x - T_n y &= T_n(x - y) \\ &\leq \|x - y\| \\ &= \|(x + kv) - (kv + y)\| \\ &\leq \|x + kv\| + \|y + kv\| \end{aligned} \tag{10}$$

Thus

$$l^- = \sup_{x \in X_n, k \in \mathbb{R}} (T_n(x) - \|x + kv\|) \leq \inf_{x \in X_n, k \in \mathbb{R}} (T_n(x) + \|x + kv\|) = l^+$$

Hence we can always find a C such that

$$T_n(x) - \|x + kv\| \leq l^- \leq Ck \leq l^+ \leq T_n(x) + \|x + kv\|, \; \forall x \in X_n, \; \forall k \in \mathbb{R}$$

Which finishes the proof.

Remark 6.5.

Proof. $\square$

Definition 6.6. (Partial Order) A partial order on set X, is a binary relation, written generically $\leq$, satisfying following property.

- transitivity: if $a \leq b$ and $b \leq c$ then $a \leq c$
- reflexivity: $a \leq a$
- anti-symmetry: if $a \leq b$ and $b \leq a$ then $a = b$

If we also have that for any a and b, either $a \leq b$ or $b \leq a$, then we say $\leq$ is a total order.

Definition 6.7. (Upper bound) Let X be a set partially ordered by $\leq$ and $Y \subset X$, we say an element $x \in X$ is an **upper bound** of Y if $y \leq x$ for all $y \in Y$

Definition 6.8. (Maximal element) Let X be a set partially ordered by $\leq$ and $Y \subset X$. say $x \in X$ is a **maximal element** of X if $x \leq m$ implies $m = x$.

Lemma 6.9. *(Zorn's lemma) If X is a nonempty partially ordered set with the property that every totally ordered subset of X has an upper bound in X, then X has a maximal element.*

6.1 Applications of Hahn-Banach (H-B)

Let $(X, \|\cdot\|_X)$ be a normed vector space, we have the following corollaries.

Corollary 6.10. *(Extending a linear functional) Let $M \subset X$ be a linear subspace. Then $\exists F \in X^*$ with*

$$F|_M = f \text{ and } \|\mathrm{F}\|_{\mathrm{X}^*} = \|\mathrm{f}\|_{\mathrm{M}^*}$$

Proof. Define $p : X \to \mathbb{R}$ via

$$p(x) = \|x\|_X \|f\|_{M^*}$$

Note that p is sublinear and $\forall x \in M$ and,

$$f(x) \leq |f(x)| = \|x\|_X \frac{|f(x)|}{\|x\|_X} \leq \|x\|_X \|f\|_{M^*} = p(x)$$

Now apply Hahn-Banach to obtain $F : X \to \mathbb{R}$, with

$$\|F\|_{X^*}(x) \le \|x\|_X \|f\|_{M^*} \implies \|F\|_{X^*} \le \|f\|_{X^*}$$

and the other direction of the inequality follows as $F|_M = f$

□

Theorem 6.11. *(Dual functional) Let X be a normed linear space. $\forall x \in X$, $\exists x^* \in X^*$ s.t.*

$$\langle x^*, x\rangle \equiv x^*(x) = \|x\|_X^2 = \|x^*\|_{X^*}^2$$

Proof. Let $M = \text{span}(x)$. Define $f : M \to \mathbb{R}$

$$f(tx) = t\|x\|_X^2 \qquad \forall t \in \mathbb{R}$$

Then f is linear, and

$$\|f\|_{M^*} = \sup_{\|tx\|_X \le 1} |f(tx)| = \|x\|_X$$

Then we apply **??** to extend f to $x^* = F \in X^*$, with $\|x^*\|_{X^*} = \|f\|_M = \|x\|_X$ and $\langle x^*|_M, x\rangle = f(x) = \|x\|_X^2$ □

Remark 6.12. **??** gives dual characterisation of the norm later.

When the space is a Hilbert space, this theorem becomes Riesz representation theorem. (without changing notation! That's why bracket is a good notation here) In short, this theorem says that you can always find a linear functional such that for its value for a chosen element is precisely the norm of this element.

Using Hahn-Banach, one can "separate" all sorts of things. Two examples:

1) Separating points

Proposition 6.13. *$\forall x, y \in X$, $x \neq y$, there exists $\ell \in X^*$, such that $\ell(x) \neq \ell(y)$*

Proof. Choose $\ell \in X^*$ according to **??** with $y - x$ in place of x. Then

$$\ell(x-y) = \ell(x) - \ell(y) = \|y-x\|_X^2 > 0$$

Thus, $\ell(x) \neq \ell(y)$. □

(Question here is do we need X to be normed? or even Banach?)

2) Separating points from closed subspaces (Urysohn-type result)

Theorem 6.14. *$M \subset X$ linear, closed. Assume $x_0 \in M$, such that*

$$d = \mathrm{dist}(\mathrm{x_0}, \mathrm{M}) = \inf_{\mathrm{x} \in \mathrm{M}} \|\mathrm{x_0} - \mathrm{x}\|_{\mathrm{M}} > 0$$

Then $\exists \ell \in X^*$ with $\ell|_M = 0$ and

$$\|\ell\|_{X^*} = 1, \ \ell(x_0) = d$$

Proof. Let $M_0 = \{x + tx_0 : x \in M\}$. Define a linear functional $f : M_0 \to \mathbb{R}$, $f(x + tx_0) = td$. □

From **??**, one gets a dual characterization of the norm:

Corollary 6.15.

i) $\forall x \in X$: $\|x\|_X = \sup_{\|x^*\|_{X^*} \leq 1} |\langle x^*, x\rangle|$

ii) $\forall x^* \in X^*$: $\|x^*\|_{X^*} = \sup_{\|x\|_X \leq 1} |\langle x^*, x\rangle|$

The supremum in i) is always achieved.

Proof. For $x = 0$, the RHS of i) is 0 by linearity. Let $x \neq 0$, we show two directions of inequality.

- "$\geq$": By homogeneity, we can assume $\|x\|_X = 1$. If $x^* \in X^*$ satisfies $\|x^*\|_{X^*} \leq 1$, then

$$|\langle x^*, x\rangle| \leq \|x^*\|_{X^*} \|x\|_X \leq \|x\|_X$$

- "$\leq$": By **??**, $\exists x^* \in X^*$, such that $|\langle x^*, x\rangle| = \|x\|_X^2 = 1$. So the supremum is achieved.

For ii), note that this is the definition of operator norm. □

Another consequence

NOOOOOOT COM-PLEEEEEEET! COOOOOOOM BAAAAAACK!!

Theorem 6.16. *Let X, Y be normed linear spaces and $A \in \mathcal{L}(X, Y)$. The dual operator $A^* : Y^* \to X^*$ is bounded and $\|A^*\|_{\mathcal{L}(Y^*, X^*)} = \|A\|_{\mathcal{L}(X,Y)}$*

Proof.

$$
\begin{aligned}
\|A^*\| &\overset{\text{def of } \|\cdot\|}{=} \sup_{\|y^*\|_{Y^*}} \|A^* y^*\|_{X^*} \\
&\overset{\text{def of } \|\cdot\|_{X^*}}{=} \sup_{\|y^*\|_{Y^*}} \sup_{\|x\|_X=1} |\langle A^* y^*, x\rangle| \\
&\overset{\text{def of } A^*}{=} \sup_{\|x\|_X=1} \sup_{\|y^*\|_{Y^*}} |\langle y^*, Ax\rangle| \\
&= \sup_{\|x\|_X=1} \|Ax\|_Y
\end{aligned}
$$

where in the last step we used the "$\leq$" direction in the proof of **??** holds and the supremum over y^* is attained. $\square$

7 Uniform Boundedness principle

Uniform boundedness principle is sometimes called Banach–Steinhaus theorem. In its basic form, it asserts that for a family of bounded linear operators whose domain is a Banach space, pointwise boundedness is equivalent to uniform boundedness in operator norm.

Theorem 7.1 (Uniform Boundedness principle)**.**

Let X be a Banach space, Y a normed vector space. Let F be a collection of bounded linear operators from X to Y. Then if

$$\sup_{f\in F} \|fx\| < \infty, , \forall x \in X$$

Then

$$\sup_{f\in F} \|f\| < \infty$$

To prove this theorem we shall use **Baire category theorem**.

Definition 7.2 (Nowhere dense)**.**

A set S in metric space X is **nowhere dense** if its closure has empty interior. i.e. $\overset{\circ}{\overline{S}} = \emptyset$.

Theorem 7.3 (Baire Category Theorem)**.**

A complete metric space is not countable union of nowhere dense set.

Proof : Baire Category Theorem

The idea of proof is to construct a Cauchy sequence in the space with no limit point, giving contradiction. First let M be a complete metric space. Assume

$$M = \bigcup_{n=1}^{\infty} A_n$$

We know that A_1 is nowhere dense, which means $\overset{\circ}{\overline{A_1}} = \emptyset$. Since $\overline{A_1}$ is closed, $M \backslash \overline{A_1}$ is open, we may find open ball B_1 with radius $r_1 < 1$ such that $B_1 \cap \overline{A_1} = \emptyset$. Clearly we have that $B_1 \not\subset \overline{A_2}$ otherwise $\overline{A_2}$ has non-empty interior. So we have that $(M \backslash \overline{A_2}) \cap B_1$ is open and non empty. Now we may choose open ball $B_2 \subset ((M \backslash \overline{A_2}) \cap B_1) \subset B_1$ with radius $r_2 < 1/2$.
We repeat this process, so that $B_n \subset ((M \backslash \overline{A_n}) \cap B_{n-1}) \subset B_{n-1}$ is an open ball with radius $r_n < 2^{1-n}$. and name the center of the open ball B_i to be x_i. Clearly $\{x_i\}$ gives a Cauchy sequence (why?). Thus it converges to a point x. Since x is a limit point in open ball B_j, it has the property that

$$x \in \overline{B_{j+1}} \subset B_j \subset (M \backslash \overline{A_j}) \subset (M \backslash A_j), \forall j \in \mathbb{N}$$

So we have that $x \notin A_j$, $\forall j \in \mathbb{N}$. So

$$x \notin \bigcup_{j=1}^{\infty} A_j = M$$

Which leads to contradiction.

Proof : Uniform Boundedness principle

Let $A_n = \{x \in X, \|fx\| \leq n, \forall f \in F\}$, $n \in \mathbb{N}$. By assumption we have $\bigcap_{n=1}^{\infty} A_n = X$.

We claim that there exists some $j \in \mathbb{N}$ such that A_j is non-empty and closed. To see this, first by by Baire category theorem, there is some A_j such that $\overline{A_j}^{\circ} \neq \emptyset$. Then let $\{x_m\}$ be a Cauchy sequence in A_j with $x_n \to x$, then by continuity of f, $\|fx\| = \lim_{n\to\infty} \|fx_m\| \leq n, \forall f \in F$. So $x \in A_j$, hence A_j is closed, thus $\overline{A_j} = A_j$, $A_j^{\circ} = \overline{A_j}^{\circ} \neq \emptyset$. So we can choose a point p from interior of A_j, and $\varepsilon > 0$ such that open ball $B_\varepsilon(p) \subseteq A_j$.

Now for any $x < \|\varepsilon\|$ with any $T \in F$ we have

$$\|T(x)\| = \|T(x+p-p)\| = \|T(x+p) - T(p)\| \leq \|T(x+p)\| + \|T(p)\| \leq n + n = 2n$$

So for any non-zero vector $x \in X$, we have

$$\|T(x)\| = \frac{\|x\|}{\varepsilon} \left\| T(\varepsilon \frac{x}{\|x\|}) \right\| \leq \frac{2n}{\varepsilon} \|x\|$$

This holds for any $T \in F$, thus

$$\sup_{f \in F} \|f\| \leq \frac{2n}{\varepsilon} < \infty$$

A simple corollary of the theorem is Banach limit.

Corollary 7.4 (Banach Limit)**.**

Let $T_n : X \to Y$ be a sequence of operators, where X and Y are Banach spaces. Suppose $\{T_n\}$ converges pointwise, then these pointwise limits define a bounded linear operator T.

8 Open mapping theorem

Definition 8.1 (Open Ball)**.**

An open ball in normed linear space X with radius $r > 0$ centered at $x \in X$ is

$$B_X(x, r) = \{y \in X : \|y - x\|_X < r\}$$

Also, when $x = 0$ we write

$$B_X(0, r) \equiv B_x(r)$$

Definition 8.2 (Open map)**.**

Let X, Y be linear spaces. $A : X \to Y$ is <u>open</u> if $A(U) \subset Y$ is open.

Remark 8.3.

- A being continuous means $A^{-1}(V) \subset X$ open $\forall V \subset Y$ open.
- A being continuous need not be open. e.g. $Ax \overset{def}{=} 0 \in Y$

Theorem 8.4 (Open Mapping Theorem)**.**

Let X, Y be Banach, $A \subset \mathcal{L}(X, Y)$. Then:

i) if A is surjective, A is open.

ii) if A is bijective, then $A^{-1} \in \mathcal{L}(X, Y)$. (Inverse operator theorem)

Remark 8.5.

ii) important in application. If $A \in \mathcal{L}(X, Y)$ is bijective then $A^{-1} : X \to Y$ liner is easy (why?). The point is A^{-1} is also bounded, or equivalently continuous.

The main step of the proof is the following:

Lemma 8.6 (A as in i))**.**

$\exists r > 0$ s.t. $B_Y(r) \subset \overline{A(B_x(1))}$

Proof

Since A is suerjective

$$Y = \bigcup_{k=1}^{\infty} A(B_X(k))$$

Since Y is complete, by Baire Category theorem, $\exists k_0$ s.t.

$$int(\overline{A(B_X(1))}) \neq \emptyset$$

So by surjectivity of A, one can find $y_0 = Ax_0 \in Y$, $r_0 > 0$ s.t.

$$\underbrace{B_Y(y_0, r_0)}_{=Ax_0 + B_Y(r_0)} \subset \overline{A(B_X(k_0))}$$

By linearity of A,

$$\begin{aligned} B_Y(r_0) &\subset \overline{A(B_X(k_0))} - Ax_0 \\ &= \overline{A(B_X(k_0) - x_0)} \\ &\subset \overline{A(B_X(k_0 + M))} \\ &= (k_0 + M)\overline{A(B_X(1))} \end{aligned}$$

Where $M \overset{def}{=} \|x_0\|_X$. So pick $r = \frac{r_0}{k_0+M}$.

Proof of theorem:

Proof

i) Pick r as in Lemma.

Claim: $B_Y(r/2) \subset A(B_X(1)))$.

If claim holds, then for $U \subset X$ open, pick $x_0 \in U$, $s > 0$ small so that $B_X(x_0, s) \subset U$. Letting $y_0 \overset{def}{=} Ax_0$, get

$$B_Y(y_0, rs/2) = y_0 + sB_Y(r/2) \overset{claim}{\subset} Ax_0 + sA(B_X(1) \overset{lin.}{=} A(B_X(x_0, s)) \subset A(U)$$

which proves i). To see i) $\implies$ ii), it's enough to show that $B = A^{-1} : Y \to X$ is continuous; but for any $U \subset X$ open, $B^{-1}(U) = (A^{-1})^{-1}(U) = A(U)$ which is open by i). $\square$

Proof of claim:

Proof

Fix $y \in B_Y(r/2)$. Need to show: $y = Ax$ for some $x \in X$ with $\|x\|_X < 1$.

We construct a sequence $(x_k) \subset X$ with

$$\sum_{k=1}^{\infty} \|x_k\|_X < 1 \text{ and } \sum_{k=1}^{\infty} Ax_k \overset{wrt\|\cdot\|_Y}{\longrightarrow} y,\, n \to \infty$$

By completeness of X, $\sum_{k=1}^{\infty} x_k \overset{def.}{=} x$ exists, $x \in B_X(1)$ and by continuity of A,

$$Ax = \sum_{k=1}^{\infty} Ax_k = y$$

By lemma above,

$$\forall s > 0, B_Y(sr) \subset \overline{A(B_X(s))} \ (*)$$

$s = 1/2$. Pick $x_1 \in B_X(1/2)$ s.t. $\|Ax_1 - y\| < r/2$. Now set $y_1 = y - Ax(\in B_X(r/2)$. Iterate.

Assume that for some ≥ 1 have $x_1, ...x_k, y_1, ...y_k$ s.t.

$$\forall 1 \leq \tilde{k} \leq k: \ \|\tilde{x_k}\|_X < 2^{-k},\, y_{\tilde{k}} = y_{\tilde{k}-1} - Ax_{\tilde{k}} \in B_Y(2^{-\tilde{k}}r$$

Then using $(*)$ with $s = 2^{-(k+1)}$ find $x_{k+1} \in B_X(2^{-(k+1)})$ such that

$$y_{k+1} \overset{def}{=} y_k - Ax_{k+1} \in B_Y(2^{-(k+1)}r$$

This yields $\sum k = 1^{\infty} \|x_k\|_X < 1$ and

$$y - \sum_{k=1}^{n} Ax_k = y_1 - \sum_{k=2}^{n} Ax_k = ... = y_n \to 0\,(n \to \infty) \quad \square$$

Example 8.7 (Equivalence of Norm)**.**

Let $X = Y$, with norms $\|\cdot\|_1$ and $\|\cdot\|_2$ and assume $\exists C > 0$ s.t.

$$\|x\|_2 \leq C\,\|x\|_1\,, \forall x \in X \quad (1)$$

If X is complete, with respect to both $\|\cdot\|_1$ and $\|\cdot\|_2$ then consider $A = id : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ is open by Theorem (indeed thm applies b/c A is bounded by (1). Since A is bijective, ii) gives that $A^{-1} = id : (X, \|\cdot\|_2) \to (X, \|\cdot\|_1)$ is bounded, i.e.

$$\exists C' : \left\|A^{-1}\right\|_1 = \|x\|_1 \leq C'\,\|x\|_2$$

so $\|\cdot\|_1$ and $\|\cdot\|_2$ are actually equivalent.

Example 8.8 (Completeness of Y)**.**

Consider $X = C(= C^0[0,1])$ with $\|\cdot\|_1 = \|\cdot\|_\infty$, $\|\cdot\|_2 = \|\cdot\|_{L^1}$. Then $A = id : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ is continuous:

$$\|Af\|_2 = \|f\|_2 = \int_0^1 |f(t)|dt \le \|f\|_\infty = \|f\|_1$$

but not open. Else by 1), $\|\cdot\|_1$ and $\|\cdot\|_2$ would be equivalent. However consider counter example:

$$f_n(x) = \begin{cases} 2n^2x & x \in [0, \frac{1}{2n}] \\ -2n^2x + 2n & x \in (\frac{1}{2n}, \frac{1}{n}] \\ 0 & x \in (\frac{1}{n}, 1] \end{cases} \quad \text{satisfy} \quad \|f_n\|_2 = 1, \|f_n\|_1 = n \to \infty$$

This shows Y needs to be complete in theorem.

Example 8.9 (Completeness of X)**.**

This example shows completness of X is also required. Take

$$X = Y = \{(x_n) \in \ell^\infty : \exists N : x_n = 0 \, \forall m \ge N\} \subset \ell^\infty$$

with norm $\|\cdot\|_X = \|\cdot\|_Y = \|\cdot\|_\infty$. This is a linear normed space. It's not complete (Exercise: show directly $\overline{X} = c_0$). Another way: Define $A : X \to X$,

$$Ax = (x_1, \frac{x_2}{2}, \frac{x_3}{3} \underbrace{......}_{0\,eventually}) \quad if\ x = (x_1, x_2......)$$

Then A is linear, bijective with

$$A^{-1} : X \to, \;\; A^{-1}x = (x_1, 2x_2, 3x_3 \underbrace{......}_{0\,eventually})$$

and A is bounded.

$$\|Ax\|_\infty = \sup_{n\ge1} \frac{|x_n|}{n} \le \sup_{n\ge1} |x_n| = \|x\|_\infty$$

so $\|A\| \le 1$. But A^{-1} is unbounded. Pick $x^{(n)} = (\overbrace{1,1,1,1}^{n}, 0, ...)$ then $\left\|x^{(n)}\right\|_\infty = 1$ but $\left\|A^{-1}x^{(n)}\right\| = n$. Hence $A^{-1} \notin \mathcal{L}(X)$ and X cannot be complete, else by theorem i), A^{-1} would be bounded.

9 Closed Graph Theorem

Consider X, Y normed spaces. Often an operator A not defined on all of A but on a "domain" $D(A)$. So we assume that $D(A) \subset X$ is a linear subspace on which $A : D(A)(\subset X) \to Y$, linear is defined.

Example 9.1. Running Example

$Y = X = C = C^0[0,1]$ with $\|\cdot\|_X = \|\cdot\|_\infty$ and $A = \frac{d}{dt}$, with $D(A) \overset{eg}{=} C^1[0,1] \subset X$ or subspaces thereof. Prime example of (<u>unbounded</u>) operator with dense domain $D(A)$: indeed $\overline{C^1[0,1]} = C$ using e.g. Weierstrass Approximation Theorem (Polynomials are already $\|\cdot\|_\infty$-dense in C)

Definition 9.2 (Graph)**.**

Let X, Y be normed space, $A : D(A)(\subset X) \to Y$. Graph of A (really of $(A, D(A))$) is the linear (!) space

$$\Gamma_A = \{(x, Ax) : x \in D(A)\} \subset X \times Y$$

We endowed $X \times Y$ with the norm $\|(x, y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$, for all $x \in X$, $y \in Y$.

Definition 9.3 (Closed Operator)**.** A is called <u>closed</u> if Γ_A is closed in $(X \times Y, \|\cdot\|_{X\times Y})$

Example 9.4. Let $A \in \mathcal{L}(X, Y)$ with $D(A) = X$. Then A is closed.

Proof

Let $(x_k, y_k)_k \subset \Gamma_A$ with $\|(x_k, y_k) - (x, y)\|_{X\times Y} \xrightarrow{k\to\infty} 0$ for some $(x, y) \in X \times Y$

NTS: $(x, y) \in \Gamma_A$ i.e. $y = Ax$. Know $y_k = Ax_k$ and $\|x_k - x\|_X \xrightarrow{k\to\infty} 0$, $\|Ax - y\|_Y \xrightarrow{k\to\infty} 0$

But $\forall k \geq 1$

$$\|y - Ax\|_Y \leq \|y - Ax\|_Y + \|Ax_k - ax\|_Y \leq \|y - Ax\|_Y \|A\| \|x_k - x\|_X$$

Thus

$$\lim_{k\to\infty} \|y - Ax\|_Y \leq \lim_{k\to\infty} \|y - Ax\|_Y \|A\| \|x_k - x\|_X = 0$$

Theorem 9.5 (Closed Graph)**.**

Let X, Y be Banach $A : X \to Y$ linear. The following are equivalent:

i) $A \in \mathcal{L}(X, Y)$

ii) A is closed

Proof

i) $\implies$ ii): see example

ii) $\implies$ i): If X, Y complete, then so is $(X \times Y, \|\cdot\|_{X\times Y})$ (exercise). A closed meas Γ_A is closed in $(X \times Y, \|\cdot\|_{X\times Y})$, so $(\Gamma_A, \|\cdot\|_{X\times Y})$ is complete. Consider:

$$\begin{aligned} \Pi_X :\ \Gamma_A &\to X & \qquad \Pi_Y : \Gamma_A &\to Y \\ (x, Ax) &\mapsto x & \qquad (x, Ax) &\mapsto Ax \end{aligned} \tag{11}$$

Π_X, Π_Y are continuous with $\|\Pi_X\|, \|\Pi_Y\| \leq 1$, Π_X is injective, and surjective. By OMT, ii), ${\Pi_X}^{-1} \in \mathcal{L}(X, \Gamma_A)$ and so

$$A = \Pi_Y \circ \Pi_X^{-1} \in \mathcal{L}(X, Y)$$

Remark 9.6. ii) is simpler than i), but equivalent.

i) says A is continuous, i.e. if $(x_n) \subset X$, $x \in X$

$$\|x_n - x\|_X \to n \to \infty 0 \implies \|Ax_n - Ax\|_Y \to n \to \infty 0$$

This contains two things to check: (Ax_n) converges and limit is Ax.

ii) says A is closed, i.e.

$$\begin{cases} \|x_n - x\|_X \to n \to \infty 0 \\ \|Ax_n - y\|_Y \to n \to \infty 0 \end{cases} \implies Ax = y \tag{12}$$

Which is only one condition to check.

Example 9.7 (running example continues)**.** $(D(A), \|\cdot\|_\infty)$ with $D(A) = C^1[0,1]$ is NOT Banach, and $A : D(A) \to C$ is an example of an operator which is:

claim:

i) closed, but

ii) not continuous

For ii), take $f_n(t) = t^n \in D(A)$, $Af_n = nf_{n-1}$ so $\|f_n\|_\infty = 1$, $\|Af_n\| \infty = n\|f_{n-1}\| \infty = n$. So

$$\sup_{f\in D(A), \|f\|\infty \leq 1} \|Af\|_\infty = \infty$$

For i), if $(f_n, f_n') \to (f, g)$ in $(D(A) \times C)$ then $\|f - f_n\|_\infty \to 0$, $\|f_n' - g\|_\infty \to 0$ but

$$\forall t \in (0,1], \quad \underbrace{f_n(t)}_{\to n \to \infty f(t)} = \underbrace{\int_0^t f_n'(x)dx}_{\to DCT \int_0^t g(x)dx} + f_n(0)$$

so $f' = g$ by fundamental theorem of calculus(FTC), i.e. $(f, g) = (f, f') \in \Gamma_A$.

Corollary 9.8 (Continuous Inverse)**.**

X, Y Banach, $A : (DA) \subset X \to Y$ linear, closed and bijective. Then $\exists B = A^{-1} \in \mathcal{L}(Y, X)$ with $AB = id_Y$ and $BA = id_{D(A)}$. Proof is left as an exercise. Hint: similar to CGT, consider $\Pi_Y : \Gamma_A \to Y$, $B \overset{def.}{=} \Pi_X \circ \Pi_Y^{-1}$

Example 9.9 (???)**.**

A is surjective: for $g \in C$ define $f(t) = \int_0^t g(s)ds$. Then by FTC, $Af = g$.
A is not injective: $Af = A\tilde{f} \implies f = \tilde{f} + c, c \in \mathbb{R}$. Let $D(A) \overset{def.}{=} C_0^1[0,1] = \{f \in C^1[0,1] : f(0) = 0\}$
Then $A : D(A) \to C$ is bijective and has continuous inverse $B = A^{-1}$ by corollary. In fact, $Bf(t) = \int_0^t f(s)ds$ with $Bf \in D(A)$.

10 Appendix

10.1 Young's Inequality

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10.2 Minkowski's inequality

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10.3 Hölder's Inequality

Theorem 10.1 (Hölder's inequality,)**.**

Let $p, q \in [1, \infty]$, satisfying:

$$\frac{1}{p} + \frac{1}{q} = 1$$

Then if $x \in \ell^p$ and $y \in \ell^q$ then

$$\|x\|_p \|y\|_q \geq \|xy\|_1 \iff \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} \left(\sum_{n=1}^{\infty} |y_n|^q\right)^{1/q} \geq \sum_{n=1}^{\infty} |x_n y_n|$$

Then if $g \in L^p(S)$ and $g \in L^q(S)$ then

$$\|f\|_p \|g\|_q \geq \|fg\|_1 \iff \left(\int_S |f|^p dx\right)^{1/p} \left(\int_S |g|^q dx\right)^{1/q} \geq \int_S |fg| dx$$

Remark 10.2 (Short version)**.**

Here's the key information of the above theorem:

$$\frac{1}{p} + \frac{1}{q} = 1 \implies \|x\|_p \|y\|_q \geq \|xy\|_1$$

Remark 10.3. When supremum occurs, which is one of p and q becomes infinity and another becomes 1, the inequality still holds. One should pay attention that for L^p spaces the supremum norm is actually essential supremum.

Remark 10.4 (Proof of Hölder's inequality)**.**

The proof uses young's inequality.

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10.4 Jensen's Inequality (convex function)

Let $\mathbb{I} \subset \mathbb{R}$. A function $f : \mathrm{I} \to \mathbb{R}$ is **convex** if the following holds for all $t \in [0, 1]$:

$$t \cdot f(x) + (1-t) \cdot f(y) \geq f(tx + (1-t)y), \ \forall x, \ y \in \mathbb{R} \tag{Jensen}$$

Similarly, a function $f : \mathrm{I} \to \mathbb{R}$ is **concave** if the following holds for all $t \in [0, 1]$:

$$t \cdot f(x) + (1-t) \cdot f(y) \leq f(tx + (1-t)y), \ \forall x, \ y \in \mathbb{R}$$

For convex function, the inequality may be rewritten as

$$\frac{1}{t}(f(y + t = (x-y)) - f(y)) \leq f(x) - f(y)$$

For concave function with $f(0) \geq 0$ we have **sub-additivity**,which is for $t \in [0, 1]$,

$$f(tx) = f(tx + (1-t)0) \geq tf(x) + (1-t)f(0) \geq tf(x)$$

and thus,for $a, b \in \mathbb{R}^+$

$$f(a) + f(b) = f(\frac{a}{a+b}(a+b)) + f(\frac{b}{a+b}(a+b)) \geq \frac{a}{a+b}f(a+b) + \frac{b}{a+b}f(a+b) = f(a+b)$$

Remark 10.5 (Generalised Jensen's Inequality)**.**

Jensen's inequality can be generalized to a sequence variables with weight. Consider a convex function evaluated at $x_1, x_2 ...,\ f(x_1), f(x_2)...f(x_n)$, with weight $w_1, w_2...$ with $\sum_{j\in\mathbb{N}} w_j = 1$, where

$$\sum_{j\in\mathbb{N}} w_j f(x_j) < \infty \quad \text{and} \quad \sum_{j\in\mathbb{N}} f(w_j x_j) < \infty$$

Then,

$$\sum_{j\in\mathbb{N}} f(w_j x_j) \leq \sum_{j\in\mathbb{N}} w_j f(x_j)$$

Remark 10.6 ($f'' > 0$)**.**

For single variable twice differentiable function, second derivative being non=negative implies convexity.

10.5 Completion of metric space

Theorem 10.7 (Completion of metric space)**.**

Every metric space has a completion.

Idea of proof: First construct a space of Cauchy sequence and define a metric on this space, then show that the original space can be embedded to the space of Cauchy sequence as a dense subset by an isometry. Proof given step by step.

Lemma 10.8 (Step I:Space of Cauchy sequence)**.**

Let (X, d) be a metric space. Let $C[X]$ denote set of all Cauchy sequence in X. Define equivalence relation $\sim$ on $C[X]$: $x \sim y \iff \lim_{n\to\infty} d(x_n, y_n) = 0$. Define set $X^* = \{[(x_n)], (x_n) \in C[X]\}$ and metric on this set $d^*((x_n), (y_n)) = \lim_{n\to\infty} d(x_n, y_n)$. One can check that d^* is indeed a well-defined metric on X^*

10.6 L^p space and l^p space

L^p space and ℓ^p space are very important for both Lebesgue measure and also functional analysis. We shall fist introduce ℓ^p space by introducing its $p - norm$. Here the definition is valid on both $\mathbb{R}$ and $\mathbb{C}$.

Definition 10.9 (p-norm,discrete)**.**

Let $p \in \mathbb{R}$ with $1 \le p \le \infty$.

For $p < \infty$,we define p-norm of sequence $(x_n)_1^\infty$ to be:

$$\|(x_n)_1^\infty\|_\infty \equiv \left(\sum_{s=1}^{\infty} |x_n|^p\right)^{1/p}$$

And for when "$p = \infty$" the p-norm becomes ∞-norm, defined as

$$\|(x_n)_1^\infty\|_p \equiv \sup_{n\in\mathbb{N}} |x_n|$$

And ℓ^p space is defined to be the set of all sequences with finite p-norm:

$$\ell^p \equiv \left\{(x_n)_1^\infty : \|(x_n)_1^\infty\|_p < \infty\right\}$$

Definition 10.10 (p-norm,continuous)**.**

Let $p \in \mathbb{R}$ with $1 \le p \le \infty$. Again,∞ gives supremum.(why?)

Consider measurable functions $f : [a, b] \to \mathbb{R}$.

For $p < \infty$,we define p-norm of function f to be:

$$\|f\|_p \equiv \left(\int_a^b |f(x)|^p dx \right)^{1/p}$$

And for when "$p = \infty$" the p-norm again becomes ∞-norm, defined as

$$\|f\|_\infty \equiv \sup_{n \in [a,b} |f(x)|$$

And L^p space is defined to be the set of measurable all functions with finite p-norm from $[a, b]$ to $\mathbb{R}$ or $\mathbb{C}$:

$$L^p \equiv \left\{ f : \|f\|_p < \infty \right\}$$

Remark 10.11.

There are very typical ℓ^p spaces, like

- ℓ^1, space of absolutely convergent sequences
- ℓ^2, gives a set of square-summable sequences, forms a Hilbert space
- ℓ^∞, set of all bounded sequences.