Chapter 3

Integration

It might be surprising, but integration can be thought of as the solution of a particular kind of differential equation.

3.1 Integrals as sums

Here, we are going to take the applied mathematician's cop-out of considering "appropriately well-behaved functions". Rigorously established calculus a.k.a. analysis is one of the intellectual triumphs of humanity, which is something to look forward to in other courses. Here, we have to remember that there are many things which are true that are not proven.

3.1.1 Integrals as Riemann sums

Let us **define** a definite integral as an infinite sum, and so

$$\int_a^b f(x)dx \equiv \lim_{\Delta x \to 0} \sum_{n=0}^{N-1} f(x_n)\Delta x = \lim_{N \to \infty} \sum_{n=0}^{N-1} f(x_n)\Delta x, \tag{3.1}$$

where $\Delta x = (b-a)/N$ and $x_n = a + n\Delta x$, as shown in figure 3.1.

An interesting question to address is how the difference between the "area under the curve" concept for integration and the above definition in terms of a sum of rectangles changes as $N \to \infty$. So let us consider a particular finite value of N, and consider the magnitude of the difference between the area under the curve, and the area of the sum of rectangles. Clearly, this difference is made up of a sum of such differences associated with each rectangle, so let us consider one such rectangle, wlog between x_n and x_{n+1}, as shown schematically in figure 3.1. How can we estimate the error $|\epsilon|$ for this particular rectangle?

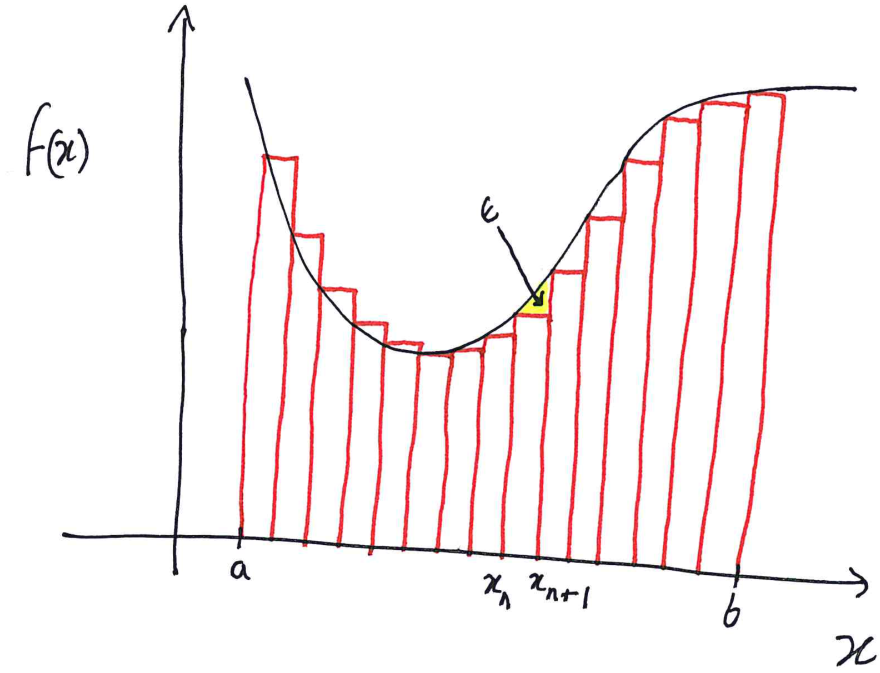

Figure 3.1: Pictorial representation of the Riemann sum approximation to an integral, showing the error ϵ between the area under the curve and the rectangle with base $x_n \leq x \leq x_{n+1}$.

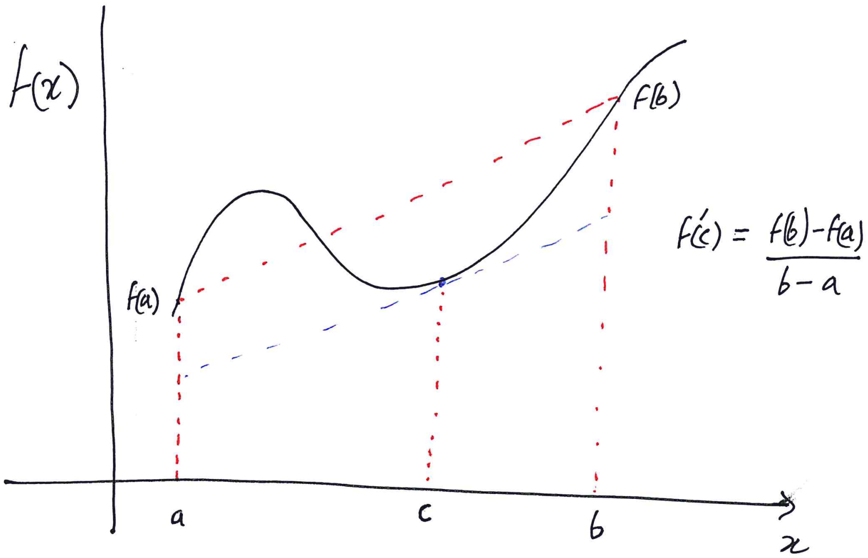

Figure 3.2: Schematic representation of the plausibility of the Mean Value Theorem. A point c can always be found such that the (dashed) tangent line to the curve at c is parallel to the (dashed) line connecting $f(a)$ and $f(b)$.

Well, the first thing we need to do is get an estimate for $|f(x)-f(x_n)|$ for $x_n \leq x \leq x_{n+1}$. Fortunately, we've got a theorem for that. We can appeal to the **Mean Value Theorem**, (MVT) which states for appropriately well-behaved functions, that there exists a point $a < c < b$ such that

$$f(b) = f(a) + (b-a)f'(c).$$

This is eminently plausible whenever you draw a curve representing a function, as shown in figure 3.2 but the rigorous proof is beyond the scope of this course. Therefore, for every x, there is going to be point $x_n < x_c < x$ such that

$$f(x) = f(x_n) + (x - x_n)f'(x_c).$$

Therefore,

$$|f(x) - f(x_n)| \leq \max_{x \in [a,b]} |f'(x)||x - x_n|.$$

So, provided $|f'(x)|$ remains bounded in this interval (related to the function being appropriately well-behaved of course) the absolute magnitude of the difference between the function and the height of the rectangle throughout the interval is

$$|f(x) - f(x_n)| \leq M|x - x_n| \leq M(x_{n+1} - x_n) = M\Delta x,$$

where M is a finite positive number.

M could be quite large if the function oscillates wildly like Morrissey, but all we care about is that it remains finite, and so we know that the difference in height between the Riemann sum rectangle, and a rectangle definitely big enough to contain the area under the curve is proportional to Δx, and so is $O(\Delta x)$. Since the **width** of this interval is $x_{n+1} - x_n = \Delta x$, and remembering the definiton of big Oh, we see that

$$|\epsilon| = O(\Delta x^2) = O([b-a]^2/N^2). \tag{3.2}$$

So, the area under the graph from x_n to x_{n+1} is $f(x_n)\Delta x + O([b-a]^2/N^2)$. Now if we add up N of such intervals, we have to add up N of such differences, and so the area under the curve from a to b is

$$\begin{aligned} A &= \lim_{N\to\infty}\left[\sum_{n=0}^{N-1} f(x_n)\Delta x + N \times O([b-a]^2/N^2)\right], \\ &= \lim_{N\to\infty}\left[\sum_{n=0}^{N-1} f(x_n)\Delta x + O([b-a]^2/N)\right], \\ &= \int_a^b f(x)dx, \end{aligned}$$

by definition, as the total error goes to zero as $N \to \infty$. Therefore, the definition of an integral as the limit of a sum of rectangles is consistent with the integral being the area under the curve (for functions with bounded derivatives which satisfy the conditions of the MVT...blah blah blah)

3.2 Fundamental Theorem of Calculus

The MVT is also central to the **Fundamental Theorem of Calculus**, (FTC ...I love TLAs ...) which essentially establishes that integration is “anti-differentiation”. The (first part of the) FTC states that if $F(x)$ is **defined** as the function

$$F(x) = \int_a^x f(t)dt + F(a), \tag{3.3}$$

then

$$\frac{dF}{dx} = f(x). \tag{3.4}$$

Another way to interpret the FTC is that an integral is the solution of a particularly simple differential equation, since $F(x)$ is the solution of the differential equation (3.4) with the initial condition $F(a)$ being given.

A sketch of the proof is as follows. By definition

$$\frac{dF}{dx} = \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right].$$

The two overlapping parts cancel, and so

$$\frac{dF}{dx} = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)dt.$$

By a similar argument to the one presented above using the MVT leading to (3.2), the integral is approximated by h multiplied by the value of the function at the lower limit, with an error of order $O(h^2)$:

$$\int_x^{x+h} f(t)dt = f(x)h + O(h^2), \tag{3.5}$$

and so

$$\frac{dF}{dx} = \lim_{h\to 0} \frac{1}{h}\left[f(x)h + O(h^2)\right] = f(x),$$

as required, since a term which is $(1/h)\times O(h^2)$ is $O(h)$ as $h \to 0$ and hence $\to 0$ as $h \to 0$.

Corollaries to the FTC

- It is (relatively) straightforward to establish that

$$\frac{d}{dx}\int_x^b f(t)dt = -f(x).$$

- Using the chain rule (2.4),

$$\frac{d}{dx}\int_a^{g(x)} f(t)dt = f[g(x)]g'(x).$$

Notation

Two alternative notations for an **indefinite** integral are

$$\int f(x)dx = \int^x f(t)dt.$$

3.3 Integration by substitution

Calculation of integrals can often (particularly in STEP ...) seem like an art involving recognition. If the integrand contains a function of a function, it can sometimes help with recognition to substitute for the inner function and remember the chain rule. For example:

$$\begin{aligned} I &= \int \frac{(1-2x)}{\sqrt{x-x^2}}dx; \quad u = x - x^2,\ du = (1-2x)dx; \\ &= \int \frac{du}{u^{1/2}} = 2u^{1/2} + C = 2\sqrt{x-x^2} + C, \end{aligned}$$

where C is a constant.

3.3.1 Trigonometric substitution

There are of course several identities, which are associated with particular expressions in the integrand leading to particular substitutions. Here are lists of identities I, expressions E and substitutions S:

- I: $\cos^2\theta + \sin^2\theta = 1$; E: $\sqrt{1-x^2}$; S: $x = \sin\theta$.
- I: $1 + \tan^2\theta = \sec^2\theta$; E: $1 + x^2$; S: $x = \tan\theta$.
- I: $\cosh^2 u - \sinh^2 u = 1$; E: $\sqrt{x^2-1}$; S: $x = \cosh u$.
- I: $\cosh^2 u - \sinh^2 u = 1$; E: $\sqrt{1+x^2}$; S: $x = \sinh u$.
- I: $1 - \tanh^2 u = \mathrm{sech}^2 u$; E: $1 - x^2$; S: $x = \tanh u$.

As an example/exercise, show that

$$I = \int \sqrt{2x - x^2}dx = \frac{1}{2}\arcsin(x-1) + \frac{1}{2}(x-1)\sqrt{1-(x-1)^2} + C,$$

for C a constant and arcsin being the inverse sine function.

3.4 Integration by parts

"Integration by parts" exploits the product rule (2.5). Applying this rule:

$$\int uv'dx = uv - \int vu'dx. \tag{3.6}$$

- Example 1:

$$\begin{aligned} I &= \int_0^\infty xe^{-x}dx; \quad u = x,\ v' = e^{-x}; \\ &= [-xe^{-x}]_0^\infty - \int_0^\infty (-e^{-x})dx; \quad u' = 1,\ v = -e^{-x}; \\ &= [-e^{-x}]_0^\infty = 1. \end{aligned}$$

- Example 2:

$$\begin{aligned} I &= \int \ln x dx; \quad u = \ln x,\ v' = 1; \\ &= x \ln x = \int x \left(\frac{1}{x}\right) dx; \quad u' = \frac{1}{x},\ v = x, \\ &= x \ln x - x + C. \end{aligned}$$