Discrete Mathematics for Part I CST 2016/17
Sets Exercises
WITH SOME ANSWERS

Marcelo Fiore Ohad Kammar

January 6, 2017

1 On sets

1.1 Basic exercises

1. Prove the following statements:

   (a) Reflexivity: $\forall$ sets $A.\, A \subseteq A$.

   (b) Transitivity: $\forall$ sets $A, B, C.\, (A \subseteq B \ \wedge\ B \subseteq C) \implies A \subseteq C$.

   (c) Antisymmetry: $\forall$ sets $A, B.\, (A \subseteq B \ \wedge\ B \subseteq A) \iff A = B$.

2. Prove the following statements:

   (a) $\forall$ set $S.\, \emptyset \subseteq S$.

   (b) $\forall$ set $S.\, (\forall x.\, x \notin S) \iff S = \emptyset$.

3. Find the union and intersection of:

   (a) $\{1, 2, 3, 4, 5\}$ and $\{-1, 1, 3, 5, 7\}$;

   (b) $\{x \in \mathbb{R} \mid x > 7\}$ and $\{x \in \mathbb{N} \mid x > 5\}$.

4. Find the product of $\{1, 2, 3, 4, 5\}$ and $\{-1, 1, 3, 5, 7\}$.

5. Let $I = \{2, 3, 4, 5\}$, and for each $i \in I$ let $A_i = \{i, i + 1, i - 1, 2 \cdot i\}$.

   (a) List the elements of all the sets A_i for $i \in I$.

   (b) Let $\{\, A_i \mid i \in I \,\}$ stand for $\{A_2, A_3, A_4, A_5\}$. Find $\bigcup\{A_i \mid i \in I\}$ and $\bigcap\{A_i \mid i \in I\}$.

6. Find the disjoint union of $\{1, 2, 3, 4, 5\}$ and $\{-1, 1, 3, 5, 7\}$.

7. Let U be a set. For all $A, B \in \mathcal{P}(U)$ prove that

   (a) $A^{\mathrm{c}} = B \iff (A \cup B = U \ \wedge\ A \cap B = \emptyset)$,

   (b) $(A^{\mathrm{c}})^{\mathrm{c}} = A$, and

   (c) the De Morgan's laws:

   $$(A \cup B)^{\mathrm{c}} = A^{\mathrm{c}} \cap B^{\mathrm{c}} \text{ and } (A \cap B)^{\mathrm{c}} = A^{\mathrm{c}} \cup B^{\mathrm{c}} \ .$$

ANSWER. Let U be a set and consider $A, B \in \mathcal{P}(U)$.

(a) $(\Rightarrow)$ holds by the complementation laws.
$(\Leftarrow)$ Since

$$\forall\, x \in U.\; x \in (A \cup B) \iff (x \notin A \Rightarrow x \in B) \quad ,$$

we have that

$$A \cup B = U \implies A^{\mathrm{c}} \subseteq B \quad .$$

Moreover, since

$$\forall\, x \in U.\; x \notin (A \cap B) \iff (x \in B \Rightarrow x \notin A) \quad ,$$

we also have that

$$A \cap B = \emptyset \implies B \subseteq A^{\mathrm{c}}$$

and we are done.

(b) By item (a), it is enough to show that $A^{\mathrm{c}} \cup A = U$ and $A^{\mathrm{c}} \cap A = \emptyset$; which follow from the commutativity law for unions and intersections, and the complementation laws.

(c) To show the De Morgan law $(A \cup B)^{\mathrm{c}} = A^{\mathrm{c}} \cap B^{\mathrm{c}}$, by the first item, it is enough to show that

$$(A \cup B) \cup (A^{\mathrm{c}} \cap B^{\mathrm{c}}) = U \quad \text{and} \quad (A \cup B) \cap (A^{\mathrm{c}} \cap B^{\mathrm{c}}) = \emptyset \quad ;$$

which follow from the associativity, commutativity, and distributivity laws for unions and intersections, and the complementation laws.

To show the other De Morgan law, one proceeds analogously or derives it from the previous De Morgan law and previous item (b):

$$\begin{aligned} A^{\mathrm{c}} \cup B^{\mathrm{c}} &= \big((A^{\mathrm{c}} \cup B^{\mathrm{c}})^{\mathrm{c}}\big)^{\mathrm{c}} \\ &= \big((A^{\mathrm{c}})^{\mathrm{c}} \cap (B^{\mathrm{c}})^{\mathrm{c}}\big)^{\mathrm{c}} \\ &= (A \cap B)^{\mathrm{c}} \end{aligned}$$

8. Establish the laws of the powerset Boolean algebra.

1.2 Core exercises

1. Either prove or disprove that, for all sets A and B,

   (a) $A \subseteq B \implies \mathcal{P}(A) \subseteq \mathcal{P}(B)$,

   (b) $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$,

   (c) $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

   (d) $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$,

   (e) $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$.

   ANSWER. Let A and B be sets.

   (a) Assume $A \subseteq B$ and let $X \in \mathcal{P}(A)$. Then, $X \subseteq A$ and $A \subseteq B$. Hence, $X \subseteq B$ and so $X \in \mathcal{P}(B)$.

   (b) One can disprove it by taking two different singleton sets A and B and noticing that $(A \cup B) \in \mathcal{P}(A \cup B)$ while it is not the case that $(A \cup B) \in \mathcal{P}(A) \cup \mathcal{P}(B)$.

   (c) Assume $X \in \mathcal{P}(A) \cup \mathcal{P}(B)$; that is, either ($i$) $X \in \mathcal{P}(A)$ or (ii) $X \in \mathcal{P}(B)$.
   In case (i), $X \subseteq A$ and since $A \subseteq (A \cup B)$ we have $X \subseteq (A \cup B)$; and hence $X \in \mathcal{P}(A \cup B)$.
   In case (ii), $X \subseteq B$ and since $B \subseteq (A \cup B)$ we have $X \subseteq (A \cup B)$; and hence $X \in \mathcal{P}(A \cup B)$.

   (d) Assume $X \in \mathcal{P}(A \cap B)$; that is $X \subseteq (A \cap B)$ or, equivalently, $X \subseteq A$ and $X \subseteq B$. Hence, $X \in \mathcal{P}(A)$ and $X \in \mathcal{P}(B)$; so that $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$.

   (e) Assume $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$; that is, $X \in \mathcal{P}(A)$ and $X \in \mathcal{P}(B)$. Then, $X \subseteq A$ and $X \subseteq B$; so that $X \subseteq (A \cap B)$ and hence $X \in \mathcal{P}(A \cap B)$.

NB The crucial lemma used in items (d) and (e) is that:

$$\forall \text{ set } X.\ X \subseteq (A \cap B) \iff X \subseteq A \ \wedge\ X \subseteq B \ . \tag{1}$$

Btw, we also have the following analogous property for unions:

$$\forall \text{ sets } X.\ (A \cup B) \subseteq X \iff A \subseteq X \ \vee\ B \subseteq X \ . \tag{2}$$

2. Let U be a set. For all $A, B \in \mathcal{P}(U)$ prove that the following statements are equivalent.

   (a) $A \cup B = B$.

   (b) $A \subseteq B$.

   (c) $A \cap B = A$.

   (d) $B^{\mathrm{c}} \subseteq A^{\mathrm{c}}$.

   ANSWER. Let U be a set and consider $A, B \in \mathcal{P}(U)$.

   $(a) \Rightarrow (b)$ Assume $A \cup B = B$. Then, $A \subseteq (A \cup B) = B$ and we are done.

   $(b) \Rightarrow (c)$ Assume $A \subseteq B$. Since, $(A \cap B) \subseteq A$ we need only show $A \subseteq (A \cap B)$ or, equivalently by (1), that $A \subseteq A$ and $A \subseteq B$; which are respectively the case by reflexivity of $\subseteq$ and assumption.

   $(c) \Rightarrow (d)$ Assume $(A \cap B) = A$ and let $x \in U$. Then, $x \notin B$ implies $x \notin (A \cap B) = A$ and we are done.

   $(d) \Rightarrow (b)$ Because $B^{\mathrm{c}} \subseteq A^{\mathrm{c}}$ stands for $x \notin B \implies x \notin A$ for all $x \in U$ which is equivalent to $x \in A \implies x \in B$ for all $x \in U$.

   $(b) \Rightarrow (a)$ Assume $A \subseteq B$. Since also $B \subseteq B$, by (2) above, we have $(A \cup B) \subseteq B$; and as $B \subseteq (A \cup B)$ we are done.

3. For sets A, B, C, D, either prove or disprove the following statements.

   (a) $(A \subseteq B \ \wedge\ C \subseteq D) \implies A \times C \subseteq B \times D$.

   (b) $(A \cup C) \times (B \cup D) \subseteq (A \times B) \cup (C \times D)$.

   (c) $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

   (d) $A \times (B \cup D) \subseteq (A \times B) \cup (A \times D)$.

   (e) $(A \times B) \cup (A \times D) \subseteq A \times (B \cup D)$.

4. Prove or disprove the following statements for all sets A, B, C, D:

   (a) $(A \subseteq B \ \wedge\ C \subseteq D) \implies A \uplus C \subseteq B \uplus D$,

   (b) $(A \cup B) \uplus C \subseteq (A \uplus C) \cup (B \uplus C)$,

   (c) $(A \uplus C) \cup (B \uplus C) \subseteq (A \cup B) \uplus C$,

   (d) $(A \cap B) \uplus C \subseteq (A \uplus C) \cap (B \uplus C)$,

   (e) $(A \uplus C) \cap (B \uplus C) \subseteq (A \cap B) \uplus C$.

5. For $\mathcal{F} \subseteq \mathcal{P}(A)$, let $\mathcal{U} = \{ X \subseteq A \mid \forall S \in \mathcal{F}.\ S \subseteq X \} \subseteq \mathcal{P}(A)$. Prove that $\bigcup \mathcal{F} = \bigcap \mathcal{U}$. Analogously, define $\mathcal{L} \subseteq \mathcal{P}(A)$ such that $\bigcap \mathcal{F} = \bigcup \mathcal{L}$. Also prove this statement.

   ANSWER.

- $\star$ For a set A let $\mathcal{F} \subseteq \mathcal{P}(A)$.

  (i) $\boxed{\forall U \in \mathcal{P}(A).\ \bigcup \mathcal{F} \subseteq U \iff (\forall X \in \mathcal{F}.\ X \subseteq U).}$

  Let $U \in \mathcal{P}(A)$.

  ($\Rightarrow$) Assume $\bigcup \mathcal{F} \subseteq U$ and let $X \in \mathcal{F}$. Then, since $X \subseteq \bigcup \mathcal{F}$ we also have $X \subseteq U$ as required.

  ($\Leftarrow$) Assume $\forall X \in \mathcal{F}.\ X \subseteq U$. Let $x \in \bigcup \mathcal{F}$, so that there exists $X_0 \in \mathcal{F}$ such that $x \in X_0$. Specialising the assumption to X_0, we have $X_0 \subseteq U$ and thus also $x \in U$.

  (ii) $\boxed{\forall U \in \mathcal{P}(A).\ U \subseteq \bigcap \mathcal{F} \iff (\forall X \in \mathcal{F}.\ U \subseteq X).}$

  Let $U \in \mathcal{P}(A)$.

  ($\Rightarrow$) Assume $U \subseteq \bigcap \mathcal{F}$ and let $X \in \mathcal{F}$. Then, since $\bigcap \mathcal{F} \subseteq X$ we also have $U \subseteq X$.

  ($\Leftarrow$) Assume $(\forall X \in \mathcal{F}.\ U \subseteq X)$ and let $x \in U$. Then, $x \in X$ for all $X \in \mathcal{F}$; that is, $x \in \bigcap \mathcal{F}$ and we are done.

- Let A be a set and $\mathcal{F} \subseteq \mathcal{P}(A)$.

  (i) $\boxed{\text{For } \mathcal{U} = \{X \subseteq A \mid \forall S \in \mathcal{F}.\ S \subseteq X\} \subseteq \mathcal{P}(A),\ \bigcup \mathcal{F} = \bigcap \mathcal{U}.}$

  ($\subseteq$) Note that, by item ($\star$), $\bigcup \mathcal{F} \subseteq \bigcap \mathcal{U} \iff (\forall X \in \mathcal{U}.\ \forall S \in \mathcal{F}.\ S \subseteq X)$ and that the latter statement holds by definition of $\mathcal{U}$.

  ($\supseteq$) Since $\forall S \in \mathcal{F}.\ S \subseteq \bigcup \mathcal{F}$, we have $\bigcup \mathcal{F} \in \mathcal{U}$; and since $\bigcap \mathcal{U} \subseteq X$ for all $X \in \mathcal{U}$; in particular, $\bigcap \mathcal{U} \subseteq \bigcup \mathcal{F}$.

  (ii) $\boxed{\text{For } \mathcal{L} = \{X \subseteq A \mid \forall S \in \mathcal{F}.\ X \subseteq S\} \subseteq \mathcal{P}(A),\ \bigcap \mathcal{F} = \bigcup \mathcal{L}.}$

  ($\subseteq$) Since $\forall S \in \mathcal{F}.\ \bigcap \mathcal{F} \subseteq S$, we have $\bigcap \mathcal{F} \in \mathcal{L}$; and since $X \subseteq \bigcup \mathcal{L}$ for all $X \in \mathcal{L}$; in particular, $\bigcap \mathcal{F} \subseteq \bigcup \mathcal{L}$.

  ($\supseteq$) Note that, by item ($\star$), $\bigcup \mathcal{L} \subseteq \bigcap \mathcal{F} \iff (\forall X \in \mathcal{L}.\ \forall S \in \mathcal{F}.\ X \subseteq S)$ and that the latter statement holds by definition of $\mathcal{L}$.

6. Prove that, for all collections of sets $\mathcal{F}$, it holds that

$$\forall \text{ set } U.\ \bigcup \mathcal{F} \subseteq U \iff \big(\forall X \in \mathcal{F}.\, X \subseteq U\big) \quad .$$

1.3 Optional advanced exercises

Prove that for all collections of sets $\mathcal{F}_1$ and $\mathcal{F}_2$,

$$\big(\bigcup \mathcal{F}_1\big) \cup \big(\bigcup \mathcal{F}_2\big) = \bigcup(\mathcal{F}_1 \cup \mathcal{F}_2) \quad .$$

State and prove the analogous property for intersections of non-empty collections of sets.

ANSWER. The stated identity for unions is a special case of the associativity law for big unions, so let us just consider the case of intersections; that is: for non-emtpy collections of sets $\mathcal{F}_1$ and $\mathcal{F}_2$,

$$\big(\bigcap \mathcal{F}_1\big) \cap \big(\bigcap \mathcal{F}_2\big) = \bigcap \big(\mathcal{F}_1 \cup \mathcal{F}_2\big) \quad .$$

Indeed, for all x, we have

$$\begin{aligned}
x \in \big(\bigcap \mathcal{F}_1\big) \cap \big(\bigcap \mathcal{F}_2\big) &\iff \big(x \in \bigcap \mathcal{F}_1\big) \wedge \big(x \in \bigcap \mathcal{F}_2\big) \\
&\iff \big(\forall X.\ X \in \mathcal{F}_1 \Rightarrow x \in X\big) \wedge \big(\forall X.\ X \in \mathcal{F}_2 \Rightarrow x \in X\big) \\
&\iff \forall X.\ \big(X \in \mathcal{F}_1 \Rightarrow x \in X\big) \wedge \big(X \in \mathcal{F}_2 \Rightarrow x \in X\big) \\
&\iff \forall X.\ \big(X \in \mathcal{F}_1 \vee X \in \mathcal{F}_2\big) \Rightarrow x \in X \\
&\iff \forall X.\ X \in \big(\mathcal{F}_1 \cup \mathcal{F}_2\big) \Rightarrow x \in X \\
&\iff x \in \bigcap \big(\mathcal{F}_1 \cup \mathcal{F}_2\big)
\end{aligned}$$

2 On relations

2.1 Basic exercises

1. Let $A = \{1,2,3,4\}$, $B = \{a,b,c,d\}$, and $C = \{x,y,z\}$.
   Let $R = \big\{(1,a),(2,d),(3,a),(3,b),(3,d)\big\} : A \mathrel{\longrightarrow\!\!\!\!\!\!\mapsto} B$ and $S = \big\{(b,x),(b,x),(c,y),(d,z)\big\} : B \mathrel{\longrightarrow\!\!\!\!\!\!\mapsto} C$.
   What is the composition $S \circ R : A \mathrel{\longrightarrow\!\!\!\!\!\!\mapsto} C$?

2. Prove that relational composition is associative and has the identity relation as neutral element.

3. For a relation $R : A \mathrel{\longrightarrow\!\!\!\!\!\!\mapsto} B$, let its *opposite*, or *dual*, $R^{\mathrm{op}} : B \mathrel{\longrightarrow\!\!\!\!\!\!\mapsto} A$ be defined by

$$b \, R^{\mathrm{op}} \, a \iff a \, R \, b \quad .$$

   For $R, S : A \mathrel{\longrightarrow\!\!\!\!\!\!\mapsto} B$, prove that

   (a) $R \subseteq S \implies R^{\mathrm{op}} \subseteq S^{\mathrm{op}}$.
   (b) $(R \cap S)^{\mathrm{op}} = R^{\mathrm{op}} \cap S^{\mathrm{op}}$.
   (c) $(R \cup S)^{\mathrm{op}} = R^{\mathrm{op}} \cup S^{\mathrm{op}}$.

4. For a relation R on a set A, prove that R is antisymmetric iff $R \cap R^{\mathrm{op}} \subseteq \mathrm{id}_A$.

2.2 Core exercises

1. Let $\mathcal{F} \subseteq \mathcal{P}(A \times B)$ be a collection of relations from A to B. Prove that,

   (a) for all $R : X \mathrel{\longrightarrow\!\!\!\!\!\!\mapsto} A$,
$$\big(\bigcup \mathcal{F}\big) \circ R \;=\; \bigcup \big\{\, S \circ R \mid S \in \mathcal{F} \,\big\} \;: X \mathrel{\longrightarrow\!\!\!\!\!\!\mapsto} B \quad ,$$
   and that,

   (b) for all $R : B \mathrel{\longrightarrow\!\!\!\!\!\!\mapsto} Y$,
$$R \circ \big(\bigcup \mathcal{F}\big) \;=\; \bigcup \big\{\, R \circ S \mid S \in \mathcal{F} \,\big\} \;: A \mathrel{\longrightarrow\!\!\!\!\!\!\mapsto} Y \quad .$$

ANSWER.

- ⋆ Let $S \subseteq S'$ in $\mathcal{P}(A \times B)$.
  (a) $\boxed{\text{For } R \text{ in } \mathcal{P}(X \times A),\ S \circ R \subseteq S' \circ R \text{ in } \mathcal{P}(X \times B).}$
  Assume $(x,b) \in (S \circ R)$. Hence, there exists $a \in A$ such that $(x,a) \in R$ and $(a,b) \in S$; let a_0 be such an element. Then, since $(a_0,b) \in S$ and $S \subseteq S'$, we have that $(a_0,b) \in S'$ and so, since $(x,a_0) \in R$, also that $(x,b) \in (S' \circ R)$.
  (b) $\boxed{\text{For } R \text{ in } \mathcal{P}(B \times Y),\ R \circ S \subseteq R \circ S' \text{ in } \mathcal{P}(A \times Y).}$
  Assume $(a,y) \in (R \circ S)$. Hence, there exists $b \in B$ such that $(a,b) \in S$ and $(b,y) \in R$; let b_0 be such an element. Then, since $(a,b_0) \in S$ and $S \subseteq S'$, we have that $(a,b_0) \in S'$ and so, since $(b_0,y) \in R$, also that $(a,y) \in (R \circ S')$.
- Let $\mathcal{F} \subseteq \mathcal{P}(A \times B)$.
  (a) Let $R \in \mathcal{P}(X \times A)$.
  Remark: Note that the notation $\big\{S \circ R \in \mathcal{P}(X \times B) \mid S \in \mathcal{F}\big\}$ stands for the formal definition $\big\{T \in \mathcal{P}(X \times B) \mid \exists\, S \in \mathcal{F}.\ T = S \circ R\big\}$. Hence,
$$T \in \big\{S \circ R \in \mathcal{P}(X \times B) \mid S \in \mathcal{F}\big\} \iff \exists\, S \in \mathcal{F}.\ T = S \circ R \ .$$

($\subseteq$) We show: $(\bigcup\mathcal{F})\circ R \subseteq \bigcup\{S\circ R \in \mathcal{P}(X\times B) \mid S\in\mathcal{F}\}$.
Assume $(x,b)\in\big((\bigcup\mathcal{F})\circ R\big)$. Hence, there exists $a\in A$ such that $(x,a)\in R$ and $(a,b)\in\bigcup\mathcal{F}$; let a_0 be such an element. Then, we have $(a_0,b)\in S$ for some $S\in\mathcal{F}$ and so, since $(x,a_0)\in R$, it follows that $(x,b)\in S\circ R$ for some $S\in\mathcal{F}$. That is, $(x,b)\in\bigcup\{S\circ R\mid S\in\mathcal{F}\}$.

($\supseteq$) We have that $\bigcup\{S\circ R\in\mathcal{P}(X\times B)\mid S\in\mathcal{F}\}\subseteq(\bigcup\mathcal{F})\circ R$ iff $S\circ R\subseteq\big(\bigcup\mathcal{F}\big)\circ R$ for all $S\in\mathcal{F}$, which follows from the previous item because $S\subseteq\bigcup\mathcal{F}$ for all $S\in\mathcal{F}$.

(b) The proof that, for $R\in\mathcal{P}(B\times Y)$, $R\circ\big(\bigcup\mathcal{F}\big)=\bigcup\big\{R\circ S\mid S\in\mathcal{F}\big\}$ is analogous to the above and left as an exercise.

What happens in the case of big intersections?

2. For a relation R on a set A, let

$$\mathcal{T}_R \;=\; \big\{\, Q\subseteq A\times A \;\mid\; R\subseteq Q \,\wedge\, Q \text{ is transitive} \,\big\} \quad .$$

For $R^{\circ+}=R\circ R^{\circ*}$, prove that (i) $R^{\circ+}\in\mathcal{T}_R$ and (ii) $R^{\circ+}\subseteq\bigcap\mathcal{T}_R$. Hence, $R^{\circ+}=\bigcap\mathcal{T}_R$.

ANSWER. Let $R\subseteq A\times A$.

Observe first that by the previous item, $R^{\circ+}=\bigcup\big\{R^{\circ n}\mid n\in\mathbb{N}^+\big\}=\bigcup_{n\in\mathbb{N}^+}R^{\circ n}$, where $\mathbb{N}^+$ denotes the set of positive integers.

(i) We show that $R^{\circ+}$ is transitive by establishing the following equivalent property:

$$R^{\circ+}\circ R^{\circ+}\subseteq R^{\circ+} \quad .$$

We start by calculating as follows:

$$\begin{aligned}
R^{\circ+}\circ R^{\circ+} &= \big(\textstyle\bigcup_{n\in\mathbb{N}^+}R^{\circ n}\big)\circ R^{\circ+}\\
&= \textstyle\bigcup_{n\in\mathbb{N}^+}\big(R^{\circ n}\circ\bigcup_{m\in\mathbb{N}^+}R^{\circ m}\big)\\
&= \textstyle\bigcup_{n\in\mathbb{N}^+}\big(\bigcup_{m\in\mathbb{N}^+}\big(R^{\circ n}\circ R^{\circ m}\big)\big)\\
&= \textstyle\bigcup_{n\in\mathbb{N}^+}\big(\bigcup_{m\in\mathbb{N}^+}R^{\circ(n+m)}\big)
\end{aligned}$$

and notice that

$$\begin{aligned}
&\textstyle\bigcup_{n\in\mathbb{N}^+}\big(\bigcup_{m\in\mathbb{N}^+}R^{\circ(n+m)}\big)\subseteq R^{\circ+}\\
&\iff \forall\, n\in\mathbb{N}^+.\ \textstyle\bigcup_{m\in\mathbb{N}^+}R^{\circ(n+m)}\subseteq R^{\circ+}\\
&\iff \forall\, n\in\mathbb{N}^+.\ \forall\, m\in\mathbb{N}^+.\ R^{\circ(n+m)}\subseteq R^{\circ+}
\end{aligned}$$

which is indeed the case because $R^{\circ(n+m)}\in\big\{R^{\circ\ell}\mid\ell\in\mathbb{N}^+\big\}$ for all $n,m\in\mathbb{N}^+$.

(ii) Let

$$\mathcal{T}_R=\big\{Q\subseteq A\times A\mid R\subseteq Q\,\wedge\,Q\text{ is transitive}\big\} \quad .$$

Noting that

$$\textstyle\bigcup_{n\in\mathbb{N}^+}R^{\circ n}\subseteq\bigcap\mathcal{T}_R \iff \forall\, n\in\mathbb{N}^+.\ \forall\, Q\in\mathcal{T}_R.\ R^{\circ n}\subseteq Q$$

we prove the latter statement by induction on $n\in\mathbb{N}^+$.

The base case amounts to showing that, for all $Q\in\mathcal{T}_R$, $R^{\circ1}\subseteq Q$; which holds because $R^{\circ1}=R$ and $R\subseteq Q$.

As for the inductive step, for $n\in\mathbb{N}^+$, assume that $\forall\, Q\in\mathcal{T}_R.\ R^{\circ n}\subseteq Q$.

For the purpose of proving $\forall\, Q\in\mathcal{T}_R.\ R^{\circ(n+1)}\subseteq Q$, consider an arbitrary $Q\in\mathcal{T}_R$. Then, $R\subseteq Q$ and since by induction hypothesis $R^{\circ n}\subseteq Q$ it follows, using item (2^-), that $R^{\circ(n+1)}=R\circ R^{\circ n}\subseteq Q\circ Q$. But, as Q is transitive, $Q\circ Q\subseteq Q$ and so $R^{\circ(n+1)}\subseteq Q$ as required.

3 On partial functions

3.1 Basic exercises

1. Let $A_2 = \{1, 2\}$ and $A_3 = \{a, b, c\}$. List the elements of the four sets $(A_i \rightharpoonup A_j)$ for $i, j \in \{2, 3\}$.

2. Prove that a relation $R : A \mathrel{-\!\!\!+\!\!\!\to} B$ is a partial function iff $R \circ R^{\mathrm{op}} \subseteq \mathrm{id}_B$.

3. Prove that the identity relation is a partial function, and that the composition of partial functions yields a partial function.

3.2 Core exercises

1. Show that $\big(\mathrm{PFun}(A, B), \subseteq\big)$ is a partial order.

2. Show that the intersection of a non-empty collection of partial functions in $\mathrm{PFun}(A, B)$ is a partial function in $\mathrm{PFun}(A, B)$.

3. Show that the union of two partial functions in $\mathrm{PFun}(A, B)$ is a relation that need not be a partial function; but that for $f, g \in \mathrm{PFun}(A, B)$ such that $f \subseteq h \supseteq g$ for some $h \in \mathrm{PFun}(A, B)$, the union $f \cup g$ is a partial function in $\mathrm{PFun}(A, B)$.

4 On functions

4.1 Basic exercises

1. Let $A_2 = \{1, 2\}$ and $A_3 = \{a, b, c\}$. List the elements of the four sets $(A_i \Rightarrow A_j)$ for $i, j \in \{2, 3\}$.

2. A relation $R : A \mathrel{-\!\!\!+\!\!\!\to} B$ is said to be total whenever $\forall\, a \in A.\, \exists\, b \in B.\, a\, R\, b$. Prove that this is equivalent to $\mathrm{id}_A \subseteq R^{\mathrm{op}} \circ R$.

   Conclude that a relation $R : A \mathrel{-\!\!\!+\!\!\!\to} B$ is a function iff $R \circ R^{\mathrm{op}} \subseteq \mathrm{id}_B$ and $\mathrm{id}_A \subseteq R^{\mathrm{op}} \circ R$.

3. Prove that the identity partial function is a function, and that the composition of functions yields a function.

4.2 Core exercises

1. Find endofunctions $f, g : A \to A$ such that $f \circ g \neq g \circ f$. Prove your claim.

2. Let $\chi : \mathcal{P}(U) \to (U \Rightarrow [2])$ be the function mapping subsets S of U to their characteristic (or indicator) functions $\chi_S : U \to [2]$.

   (a) Prove that, for all $x \in U$,
   - $\chi_{A \cup B}(x) = \big(\chi_A(x) \text{ OR } \chi_B(x)\big) = \max\big(\chi_A(x), \chi_B(x)\big)$,
   - $\chi_{A \cap B}(x) = \big(\chi_A(x) \text{ AND } \chi_B(x)\big) = \min\big(\chi_A(x), \chi_B(x)\big)$,
   - $\chi_{A^c}(x) = \text{NOT}\big(\chi_A(x)\big) = \big(1 - \chi_A(x)\big)$.

   (b) For what construction $A?B$ on sets A and B it holds that

   $$\chi_{A?B}(x) = \big(\chi_A(x) \text{ XOR } \chi_B(x)\big) = \big(\chi_A(x) +_2 \chi_B(x)\big)$$

   for all $x \in U$? Prove your claim.

4.3 Optional advanced exercises

Consider a set A together with an element $a \in A$ and an endofunction $f : A \to A$.

Say that a relation $R \subseteq \mathbb{N} \times A$ is (a,f)-*closed* whenever

$$(0,a) \in R \quad \text{and} \quad \forall\,(n,x) \in \mathbb{N} \times A.\,(n,x) \in R \implies \big(n+1, f(x)\big) \in R \ .$$

Define the relation $F \subseteq \mathbb{N} \times A$ as

$$F \;=\; \bigcap \{\, R \subseteq \mathbb{N} \times A \mid R \text{ is } (a,f)\text{-closed} \,\} \ .$$

(a) Prove that the relation F is (a,f)-closed.

(b) Prove that the relation F is total; that is, $\forall\, n \in \mathbb{N}.\, \exists\, y \in A.\, (n,y) \in F$.

(c) Prove that the relation F is a (total) function $\mathbb{N} \to A$; that is,

$$\forall\, n \in \mathbb{N}.\, \exists!\, y \in A.\, (n,y) \in F \ .$$

Hint: Proceed by induction. Observe that, in view of the previous item, to show that $\exists!\, y \in A.\, (\ell,y) \in F$ it suffices to exhibit an (a,f)-closed relation R_ℓ such that $\exists!\, y \in A.\, (\ell,y) \in R_\ell$. (Why?) For instance, as the relation $R_0 = \{\, (m,y) \in \mathbb{N} \times A \mid m = 0 \implies y = a \,\}$ is (a,f)-closed one has that $(0,y) \in F \implies (0,y) \in R_0 \implies y = a$.

(d) Show that if h is a function $\mathbb{N} \to A$ such that $h(0) = a$ and $\forall\, n \in \mathbb{N}.\, h(n+1) = f\big(h(n)\big)$ then $h = F$.

Thus, for every set A together with an element $a \in A$ and an endofunction $f : A \to A$ there exists a unique function $F : \mathbb{N} \to A$, typically said to be *inductively defined*, satisfying the recurrence relation

$$F(n) = \begin{cases} a & \text{, for } n = 0 \\ f\big(F(n-1)\big) & \text{, for } n \geq 1 \end{cases}$$

5 On bijections

5.1 Basic exercises

1. (a) Give examples of functions that have (i) none, (ii) exactly one, and (iii) more than one retraction.

   (b) Give examples of functions that have (i) none, (ii) exactly one, and (iii) more than one section.

2. Let n be an integer.

   (a) How many sections are there for the absolute-value map $[-n..n] \to [0..n] : x \mapsto |\,x\,|$?

   (b) How many retractions are there for the exponential map $[0..n] \to [0..2^n] : x \mapsto 2^x$?

3. Give an example of two sets A and B and a function $f : A \to B$ satisfying both:

   (i) there is a retraction for f, and

   (ii) there is no section for f.

   Explain how you know that f has these two properties.

4. Prove that the identity function is a bijection, and that the composition of bijections yields a bijection.

5. For $f : A \to B$, prove that if there are $g, h : B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ h = \mathrm{id}_B$ then $g = h$.

   Conclude as a corollary that, whenever it exists, the inverse of a function is unique.

5.2 Core exercises

1. We say that two functions $s : A \to B$ and $r : B \to A$ are a *section-retraction* pair whenever $r \circ s = \mathrm{id}_A$; and that a function $e : B \to B$ is an *idempotent* whenever $e \circ e = e$.

   (a) Show that if $s : A \to B$ and $r : B \to A$ are a section-retraction pair then the composite $s \circ r : B \to B$ is an idempotent.

   (b) Prove that for every idempotent $e : B \to B$ there exists a set A and a section-retraction pair $s : A \to B$ and $r : B \to A$ such that $s \circ r = e$.

   (c) Let $p : C \to D$ and $q : D \to C$ be functions such that $p \circ q \circ p = p$. Can you conclude that

   - $p \circ q$ is idempotent? If so, how?
   - $q \circ p$ is idempotent? If so, how?

2. Prove the isomorphisms of the *Calculus of Bijections, I.*

3. Prove that, for all $m, n \in \mathbb{N}$,

   (a) $\mathcal{P}\big([n]\big) \cong [2^n]$

   (b) $[m] \times [n] \cong [m \cdot n]$

   (c) $[m] \uplus [n] \cong [m + n]$

   (d) $\big([m] \rightharpoonup [n]\big) \cong [(n+1)^m]$

   (e) $\big([m] \Rightarrow [n]\big) \cong [n^m]$

   (f) $\mathrm{Bij}\big([n], [n]\big) \cong [n!]$

6 On equivalence relations

6.1 Basic exercises

1. For a relation R on a set A, prove that

   - R is reflexive iff $\mathrm{id}_A \subseteq R$,
   - R is symmetric iff $R \subseteq R^{\mathrm{op}}$,
   - R is transitive iff $R \circ R \subseteq R$.

2. Prove that the isomorphism relation $\cong$ between sets is an equivalence relation.

3. Prove that the identity relation id_A on a set A is an equivalence relation and that $A/_{\mathrm{id}_A} \cong A$.

4. Show that, for a positive integer m, the relation $\equiv_m$ on $\mathbb{Z}$ given by

   $$x \equiv_m y \iff x \equiv y \pmod{m} \quad .$$

   is an equivalence relation.

5. Show that the relation $\equiv$ on $\mathbb{Z} \times \mathbb{N}^+$ given by

   $$(a, b) \equiv (x, y) \iff a \cdot y = x \cdot b$$

   is an equivalence relation.

   ANSWER. Let $\equiv \subseteq (\mathbb{Z} \times \mathbb{N}^+) \times (\mathbb{Z} \times \mathbb{N}^+)$ be given by

   $$(a, b) \equiv (x, y) \iff a \cdot y = x \cdot b$$

for all $(a,b),(x,y) \in \mathbb{Z} \times \mathbb{N}^+$

The relation $\equiv$ is an equivalence:

Reflexivity.

For every $(a,b) \in \mathbb{Z} \times \mathbb{N}^+$, $a \cdot b = a \cdot b$ and so $(a,b) \equiv (a,b)$.

Symmetry.

Let $(a,b),(x,y) \in \mathbb{Z} \times \mathbb{N}^+$ and assume $(a,b) \equiv (x,y)$; that is $a \cdot y = x \cdot b$. Then also $(x,y) \equiv (a,b)$ as $x \cdot b = a \cdot y$.

Transitivity.

Let $(a,b),(x,y),(p,q) \in \mathbb{Z} \times \mathbb{N}^+$, and assume $(a,b) \equiv (x,y)$ and $(x,y) \equiv (p,q)$; that is, $a \cdot y = x \cdot b$ and $x \cdot q = p \cdot y$. Then, $a \cdot q = \frac{x \cdot b}{y} \cdot \frac{p \cdot y}{x} = p \cdot b$ and so $(a,b) \equiv (p,q)$.

6. Let B be a subset of a set A. Define the relation E on $\mathcal{P}(A)$ by

$$\big(X,Y\big) \in E \iff X \cap B = Y \cap B \quad .$$

Show that E is an equivalence relation.

ANSWER. For $B \subseteq A$, define $E \subseteq \mathcal{P}(A) \times \mathcal{P}(A)$ by

$$(X,Y) \in E \iff X \cap B = Y \cap B$$

for all $X, Y \in \mathcal{P}(A)$.

The relation E is an equivalence:

Reflexivity.

For every $X \in \mathcal{P}(A)$, $X \mathrel{E} X$ because $X \cap B = X \cap B$.

Symmetry.

Let $X, Y \in \mathcal{P}(A)$ and assume $X \mathrel{E} Y$; that is, $X \cap B = Y \cap B$. Then, $Y \mathrel{E} X$ because $Y \cap B = X \cap B$.

Transitivity.

Let $X, Y, Z \in \mathcal{P}(A)$, and assume $X \mathrel{E} Y$ and $Y \mathrel{E} Z$; that is, $X \cap B = Y \cap B$ and $Y \cap B = Z \cap B$. Then, $X \cap B = Z \cap B$ and so $X \mathrel{E} Z$.

6.2 Core exercises

1. Let E_1 and E_2 be two equivalence relations on a set A. Either prove or disprove the following statements.

   (a) $E_1 \cup E_2$ is an equivalence relation on A.

   (b) $E_1 \cap E_2$ is an equivalence relation on A.

2. For an equivalence relation E on a set A, show that $[a_1]_E = [a_2]_E$ iff $a_1 \mathrel{E} a_2$, where $[a]_E = \{\, x \in A \mid x \mathrel{E} a \,\}$.

   ANSWER. Let E be an equivalence relation on a set A, and let $a_1, a_2 \in A$.

   ($\Rightarrow$) Assume $[a_1]_E = [a_2]_E$. Then, $a_1 \in [a_1]_E = [a_2]_E$ implies $a_1 \mathrel{E} a_2$.

   ($\Leftarrow$) Assume $a_1 \mathrel{E} a_2$.

   ($\subseteq$) Let $x \in [a_1]_E$ so that $x \mathrel{E} a_1$. By transitivity then $x \mathrel{E} a_2$ and hence $x \in [a_2]_E$.

   ($\supseteq$) Let $x \in [a_2]_E$ so that $x \mathrel{E} a_2$. By symmetry, $a_2 \mathrel{E} a_1$ and by transitivity then $x \mathrel{E} a_1$. Hence, $x \in [a_1]_E$.

3. For a function $f : A \to B$ define a relation $\equiv_f$ on A by the rule

$$a \equiv_f a' \iff f(a) = f(a')$$

for all $a, a' \in A$.

(a) Show that for every function $f : A \to B$, the relation $\equiv_f$ is an equivalence on A.

(b) Prove that every equivalence relation E on a set A is equal to $\equiv_q$ for q the quotient function $A \twoheadrightarrow A_{/E} : a \mapsto [a]_E$.

(c) Prove that for every surjection $f : A \twoheadrightarrow B$,

$$B \cong \left(A_{/\equiv_f}\right) \quad .$$

7 On surjections

7.1 Basic exercises

1. Give three examples of functions that are surjective and three examples of functions that are not.

2. Prove that the identity function is a surjection, and that the composition of surjections yields a surjection.

7.2 Core exercises

From surjections $A \twoheadrightarrow B$ and $X \twoheadrightarrow Y$ define, and prove surjective, functions $A \times X \twoheadrightarrow B \times Y$ and $A \uplus X \twoheadrightarrow B \uplus Y$.

8 On injections

8.1 Basic exercises

1. Give three examples of functions that are injective and three of functions that are not.

2. Prove that the identity function is an injection, and that the composition of injections yields an injection.

8.2 Core exercises

From injections $A \rightarrowtail B$ and $X \rightarrowtail Y$ define, and prove injective, functions $A \times X \rightarrowtail B \times Y$ and $A \uplus X \rightarrowtail B \uplus Y$.

9 On images

9.1 Basic exercises

1. What is the direct image of $\mathbb{N}$ under the integer square-root relation $R_2 = \{\, (m,n) \mid m = n^2 \,\} : \mathbb{N} \not\rightarrow \mathbb{Z}$? And the inverse image of $\mathbb{N}$?

2. For a relation $R : A \not\rightarrow B$, show that

(a) $\overrightarrow{R}(X) \;=\; \bigcup_{x \in X} \overrightarrow{R}\big(\{x\}\big)$ for all $X \subseteq A$, and

(b) $\overleftarrow{R}(Y) \;=\; \big\{ a \in A \mid \overrightarrow{R}\big(\{a\}\big) \subseteq Y \big\}$ for all $Y \subseteq B$.

ANSWER.

(a) Note that
$$\overrightarrow{R}(\{x\}) = \{\, b \in B \mid \exists\, a \in \{x\}.\ a\,R\,b \,\} = \{\, b \in B \mid x\,R\,b \,\} \tag{3}$$
so that, for all $b \in B$,
$$\begin{array}{rcl} b \in \bigcup_{x\in X} \overrightarrow{R}(\{x\}) & \iff & b \in \bigcup_{x\in X} \{\, b \in B \mid x\,R\,b \,\} \\ & \iff & \exists\, x \in X.\ x\,R\,b \\ & \iff & b \in \overrightarrow{R}(X) \end{array}$$

(b) For $R : A \nrightarrow B$ and $Y \subseteq B$, $\overleftarrow{R}(Y) = \{a \in A \mid \overrightarrow{R}(\{a\}) \subseteq Y\}$ because, for all $a \in A$, by the definition of inverse image and by (3) above,
$$a \in \overleftarrow{R}(Y) \iff (\forall\, b \in B.\ a\,R\,b \implies b \in Y) \iff \overrightarrow{R}(\{a\}) \subseteq Y \quad .$$
Btw, note that, for $f : A \to B$,
$$\begin{array}{rcl} \overleftarrow{f}(Y) & = & \{\, a \in A \mid \overrightarrow{f}(\{a\}) \subseteq Y \,\} \\ & = & \{\, a \in A \mid \{f(a)\} \subseteq Y \,\} \\ & = & \{\, a \in A \mid f(a) \in Y \,\} \end{array}$$

9.2 Core exercises

1. For $X \subseteq A$, prove that the direct image $\overrightarrow{f}(X) \subseteq B$ under an injective function $f : A \rightarrowtail B$ is in bijection with X; that is, $X \cong \overrightarrow{f}(X)$.

2. Prove that for a surjective function $f : A \twoheadrightarrow B$, the direct image function $\overrightarrow{f} : \mathcal{P}(A) \to \mathcal{P}(B)$ is surjective.

   ANSWER. We need show that for all $Y \subseteq B$ there exists $X \subseteq A$ such that $\overrightarrow{f}(X) = Y$. To this end consider an arbitrary $Y \subseteq B$, let $X = \overleftarrow{f}(Y)$, so that
$$x \in X \iff f(x) \in Y \quad ,$$
   and note that:
$$\begin{array}{rcll} \overrightarrow{f}(X) & = & \{\, b \in B \mid \exists\, x \in X.\ f(x) = b \,\} & \\ & = & \{\, b \in B \mid \exists\, x \in A.\ f(x) \in Y \ \wedge\ f(x) = b \,\} & \\ & = & \{\, b \in Y \mid \exists\, x \in A.\ f(x) = b \,\} & \\ & = & Y & \text{, whenever } f \text{ is surjective} \end{array}$$

3. Show that, by inverse image,
$$\text{every map } A \to B \text{ induces a Boolean algebra map } \mathcal{P}(B) \to \mathcal{P}(A) \ .$$
   That is, for every function $f : A \to B$,
   - $\overleftarrow{f}(\emptyset) = \emptyset$
   - $\overleftarrow{f}(X \cup Y) = \overleftarrow{f}(X) \cup \overleftarrow{f}(Y)$
   - $\overleftarrow{f}(B) = A$
   - $\overleftarrow{f}(X \cap Y) = \overleftarrow{f}(X) \cap \overleftarrow{f}(Y)$
   - $\overleftarrow{f}(X^{c}) = \big(\overleftarrow{f}(X)\big)^{c}$

for all $X, Y \subseteq B$.

ANSWER.

- $\overleftarrow{f}(\emptyset) \;=\; \{a \in A \mid f(a) \in \emptyset\} \;=\; \emptyset$
- $$\begin{aligned} \overleftarrow{f}(X \cup Y) &= \{a \in A \mid f(a) \in X \cup Y\} \\ &= \{a \in A \mid f(a) \in X \ \vee\ f(a) \in Y\} \\ &= \{a \in A \mid f(a) \in X\} \cup \{a \in A \mid f(a) \in Y\} \\ &= \overleftarrow{f}(X) \cup \overleftarrow{f}(Y) \end{aligned}$$
- $\overleftarrow{f}(B) \;=\; \{a \in A \mid f(a) \in B\} \;=\; A$
- $$\begin{aligned} \overleftarrow{f}(X \cap Y) &= \{a \in A \mid f(a) \in X \cap Y\} \\ &= \{a \in A \mid f(a) \in X \ \wedge\ f(a) \in Y\} \\ &= \{a \in A \mid f(a) \in X\} \cap \{a \in A \mid f(a) \in Y\} \\ &= \overleftarrow{f}(X) \cap \overleftarrow{f}(Y) \end{aligned}$$
- For all $a \in A$, $$\begin{aligned} a \in \big(\overleftarrow{f}(X)\big)^{\mathrm{c}} &\iff a \notin \overleftarrow{f}(X) \\ &\iff f(a) \notin X \\ &\iff f(a) \in X^{\mathrm{c}} \\ &\iff a \in \overleftarrow{f}(X^{\mathrm{c}}) \end{aligned}$$

9.3 Optional advanced exercises

For a relation $R : A \twoheadrightarrow B$, prove that

(a) $\overrightarrow{R}\big(\bigcup \mathcal{F}\big) = \bigcup \big\{ \overrightarrow{R}(X) \mid X \in \mathcal{F} \big\} \in \mathcal{P}(B)$ for all $\mathcal{F} \in \mathcal{P}(\mathcal{P}(A))$, and

(b) $\overleftarrow{R}\big(\bigcap \mathcal{G}\big) = \bigcap \big\{ \overleftarrow{R}(Y) \mid Y \in \mathcal{G} \big\} \in \mathcal{P}(A)$ for all $\mathcal{G} \in \mathcal{P}(\mathcal{P}(B))$.

10 On countability

10.1 Basic exercises

Prove that:

(a) Every finite set is countable.

(b) $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ are countable sets.

10.2 Core exercises

1. For an infinite set S, prove that if there is a surjection $\mathbb{N} \to S$ then there is a bijection $\mathbb{N} \to S$.

2. Prove that:

   (a) Every subset of a countable set is countable.

   (b) The product and disjoint union of countable sets is countable.

3. For an infinite set S, prove that the following are equivalent:

   (a) There is a bijection $\mathbb{N} \to S$.

(b) There is an injection $S \to \mathbb{N}$.

(c) There is a surjection $\mathbb{N} \to S$

4. For a set X, prove that there is no injection $\mathcal{P}(X) \to X$.

10.3 Optional advance exercises

Prove that if X and A are countable sets then so are A^*, $\mathcal{P}_{\text{fin}}(A)$, and $(X \rightharpoonup_{\text{fin}} A)$.

11 On indexed sets

11.1 Optional advanced exercises

Prove the isomorphisms of the *Calculus of Bijections, II*.