Part II

First order differential equations

Chapter 6

Unforced linear differential equations

In this chapter, we consider methods to construct solutions to **unforced, linear first order** differential equations. We identify the key role played by **eigenfunctions**.

6.1 The exponential function

Consider the class of functions $f(x) = a^x$ where $a > 0$ is a constant. Applying the fundamental definition (1.1) of a derivative, we obtain:

$$\begin{aligned}\frac{df}{dx} &= \lim_{h\to 0}\frac{a^{x+h}-a^x}{h},\\ &= a^x\lim_{h\to 0}\frac{a^h-1}{h},\\ &= \mu(a)f(x),\end{aligned}$$

where

$$\mu(a) = \lim_{h\to 0}\frac{a^h-1}{h} = f'(0),$$

and we assume that this limit exists.

Definition

We now choose to **define** the function $f(x) = \exp(x) = e^x$: “the exponential function” as the solution to the differential equation

$$\frac{df}{dx} = f(x),$$

with the **initial condition** $f(0) = 1$. This function is uniquely defined, and has $\mu = 1$, $a = e = 2.718...$ One of the questions on the first example sheet leads to a proof that

$$e = \lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k .$$

Notation

Note that if $y = a^x = e^{x \ln a}$, then

$$\frac{dy}{dx} = (\ln a)e^{x \ln a} = (\ln a)a^x,$$

and so $\mu(a) = \ln a$, where $\ln z \equiv \log_e z \equiv \log z$, where ln is the **natural logarithm**.

6.2 First order linear differential equations

The exponential function plays a central role in the solution of linear differential equations with constant coefficients, since it is an **eigenfunction** of the **differential operator** d/dx. Pardon? Consider $e^{\lambda x}$, where λ is a constant. Then, by the definition of the exponential function, and the chain rule

$$\frac{d}{dx}\left(e^{\lambda x}\right) = \lambda e^{\lambda x}.$$

An eigenfunction of an operator is unchanged by the action of that operator, except for a multiplicative scaling factor, known as the **eigenvalue**. These quantities used to be called "proper" functions and "proper" values, until it was realised this was far too close to the French "propre", so the German "eigen" (loosely "own") has been substituted. Whatever, the concept is that these functions and values are in some sense "fundamental" characteristics of the underlying operator.

Any **linear**, **homogeneous**, ordinary differential equation with **constant** coefficients has solutions of the form $e^{\lambda x}$ where:

- **Linear** means that the dependent variable $y(x)$ (wlog) only appears linearly in the equation, i.e. to the first power only;

- **homogeneous** means in this context that $y = 0$ is a solution to the equation, which is equivalent to the equation being **unforced**;

- "**constant** coefficients" means that the independent variable (x) does not appear explicitly;

- "order" refers to the order of the derivatives in the equation.

As an example, consider the **first order** equation

$$5y' - 3y = 0. \tag{6.1}$$

Clearly, the coefficients are constant, and the equation is homogeneous: $y = 0$ is indeed a solution. For this equation, try $y = e^{\lambda x}$ in the equation. Therefore,

$$5\lambda e^{\lambda x} - 3e^{\lambda x} = 0.$$

Now, another attractive property of the exponential function is that $e^x \neq 0$ for all x. Therefore, we can cancel it across, to obtain the **characteristic equation**:

$$5\lambda - 3 = 0 \rightarrow \lambda = \frac{3}{5},$$

and so $y = e^{3x/5}$ is a solution to (6.1).

Indeed, since the equation is linear, any multiple of this solution is itself a solution, and $y = Ae^{3x/5}$ is a solution for all (constant) A. An n^{th} order linear differential equation has (precisely) n independent solutions, so $y = Ae^{3x/5}$ is the **general** solution of this first order equation. A specific unique solution is determined by having an **initial** or **boundary** condition for y, i.e. having a **well-posed problem**. So, if we require $y(0) = 1$, $y = e^{3x/5}$ is the unique solution to (6.1). (Indeed in general $y(0) = A$.)

6.2.1 Discrete equations: The interest rate "scam"

Have you ever wondered why banks **pay** interest yearly (or perhaps monthly) while they **charge** interest daily (on credit cards, loans, mortgages etc)? It's to do with approximations to the exponential function!

Consider equation (6.1), and imagine we want to approximate the solution at a set of specific values y_n, if we are given $y = y_0$ at $x = 0$. A natural (but actually not particularly good) approximation to use is **forward Euler** approximation where we approximate the derivative by the difference between the present (known) n^{th} value and the (unknown) future $n + 1^{th}$ value at a larger (by h) x-value, i.e.

$$5\left(\frac{y_{n+1} - y_n}{h}\right) - 3y_n = 0 \rightarrow y_{n+1} = \left(1 + \frac{3h}{5}\right) y_n.$$

This is very reminiscent (unsurprisingly) of the formula for compound interest. In particular, if we apply this **recurrence relation** repeatedly, we

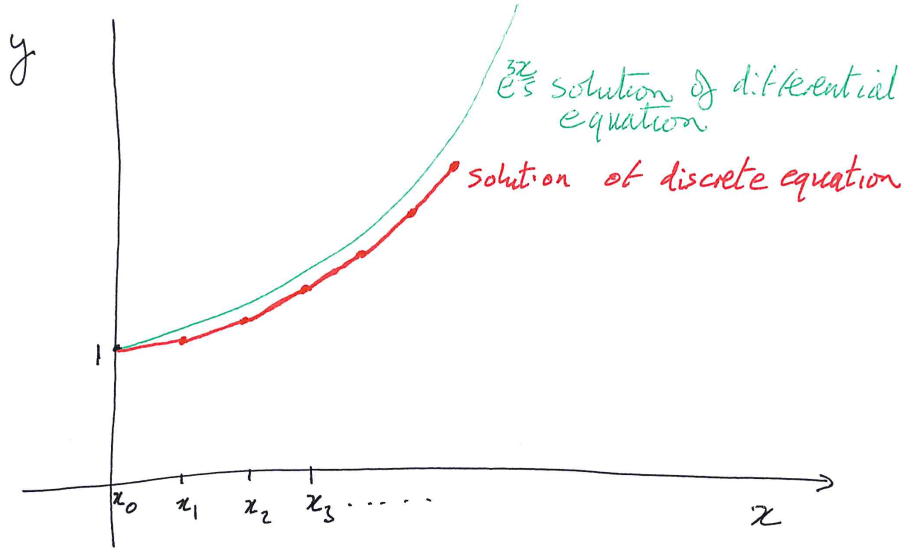

Figure 6.1: Schematic showing the difference between an exponential function and its discrete approximation.

obtain

$$y_n = \left(1 + \frac{3h}{5}\right) y_{n-1} = \left(1 + \frac{3h}{5}\right)^2 y_{n-2} = \left(1 + \frac{3h}{5}\right)^n y_0.$$

Therefore, if we consider y_n to be evaluated at $x = nh$,

$$y_n = \left(1 + \frac{3}{5}\frac{x}{n}\right)^n y_0.$$

As shown on the example sheet, in the limit as $n \to \infty$

$$y(x) = \lim_{n\to\infty} y_0 \left(1 + \frac{\frac{3x}{5}}{n}\right)^n = y_0 e^{3x/5}.$$

However, as shown in the figure, the larger n is, the larger the value of $y_n(x)$. So, if interest calculated and added daily at the **same** notional interest rate, will add up to a larger quantity than interest calculated and added yearly, as shown schematically in figure 6.1. Sneaky eh? Also, this shows that the forward Euler approximation is not particularly good, but that is a story for other (Numerical Analysis) courses ...

6.2.2 Series solution

We can also construct directly a power series for the function y which is a solution of (6.1). Assume that

$$y = \sum_{n=0}^{\infty} a_n x^n \to y' = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Therefore,

$$5y' - 3y = 0 \to 5(xy') - (3x)y = 0.$$

Substituting in the series form of the two terms, and being careful about matching up the powers of x consistently:

$$\sum_{n=1}^{\infty} (5na_n - 3a_{n-1})x^n = 0.$$

This actually defines an infinite number of conditions relating a_n to a_{n-1} as the coefficient of x^n:

$$\begin{aligned} 5na_n - 3a_{n-1} &= 0, \\ \to a_n &= \frac{3}{5n} a_{n-1}, \\ &= \left(\frac{3}{5}\right)^2 \frac{a_{n-2}}{n(n-1)}, \\ &= \left(\frac{3}{5}\right)^n \frac{a_0}{n!}. \end{aligned}$$

Therefore

$$y = a_0 \sum_{n=0}^{\infty} \frac{(3x/5)^n}{n!} = a_0 e^{3x/5}.$$

Chapter 7

Forced equations

Of course, differential equations can also involve terms that are explicit functions of the independent variables. Such equations are called **forced** or **inhomogeneous** equations, as the homogeneous (and trivial) function $y = 0$ is **not** a solution. In this chapter we consider different methods for solving such equations.

7.1 Simple forcing

There are two particularly simple, yet important, forms of forcing.

7.1.1 Constant forcing

The simplest form of forcing is constant forcing. As an example, consider this modified form of the differential equation (6.1) considered in the previous chapter:

$$5y' - 3y = 10.$$

Clearly this equation is inhomogeneous, since $y = 0$ is no longer a solution. We might be able to "spot" a **steady** or **equilibrium** solution, or more generally, a **particular integral** y_p:

$$y_p = -\frac{10}{3}, \; y_p' = 0.$$

Since the differential equation is linear we can now construct the general solution y as the sum of a particular integral y_p and the **complementary function**, i.e. the general solution to the associated **homogeneous** equation (with no forcing):

$$y = y_c + y_p,$$

where y_c is the general solution of (6.1) i.e.

$$5y_c' - 3y_c = 0 \to y_c = Ae^{3x/5},$$

where A is an arbitrary constant. Therefore, the general solution to the forced equation is

$$y = -\frac{10}{3} + Ae^{3x/5}.$$

For an **unique** solution, we need some initial or boundary condition giving the value of $y(x_0)$ at some $x = x_0$ to determine A. For example if $y(0) = 0$, $A = 10/3$.

7.1.2 Eigenfunction forcing

A second particularly simple form of forcing is when the forcing is an eigenfunction of the underlying differential operator. A classic example is radioactive decay. Consider a radioactive rock, in which isotope A decays into isotope B at a rate proportional to the number $a(t)$ of remaining nuclei of A, and B decays into C at a rate proportional to the number $b(t)$ of remaining nuclei of B. How do we determine the function $b(t)$?

The problem can be expressed mathematically as the two differential equations

$$\frac{da}{dt} = -k_a a; \ \frac{db}{dt} = k_a a - k_b b,$$

where k_a and k_b are the appropriate rate constants. We know the solution to the left hand equation is

$$a(t) = a_0 e^{-k_a t},$$

and so the right hand equation is

$$\frac{db}{dt} + k_b b = k_a a_0 e^{-k_a t}. \tag{7.1}$$

Since the forcing on the right hand side of this equation is an eigenfunction of the differential operator on the left hand side, we know that a particular integral is

$$b_p = Ce^{-k_a t},$$

for some suitable choice of the coefficient C.

Substituting b_p into the differential equation, and cancelling across by $e^{-k_a t}$, which is non-zero for all t, we obtain an equation for C:

$$-k_a C + k_b C = k_a a_0 \to C = \frac{k_a}{k_b - k_a} a_0,$$

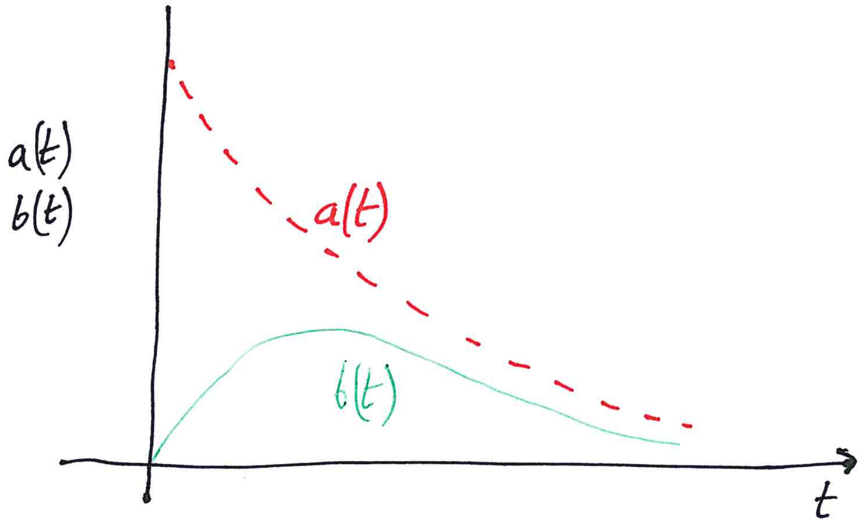

Figure 7.1: Schematic representation of the time variation of $b(t)$ as defined by (7.2) and $a(t) = a_0 e^{-k_a t}$.

provided $k_a \neq k_b$. As before, we then consider the general solution $b = b_c + b_p$, where b_c is the solution of the homogeneous equation:

$$b'_c + k_b b_c = 0 \to b_c = De^{-k_b t},$$

and so

$$b(t) = \frac{k_a}{k_b - k_a} a_0 e^{-k_a t} + De^{-k_b t}.$$

A particular situation of interest is when $b = 0$ at $t = 0$, i.e. the isotope B only appears due to decay of isotope A. In this situation,

$$\begin{aligned} b(t) &= \frac{k_a}{k_b - k_a} a_0 (e^{-k_a t} - e^{-k_b t}), \qquad (7.2) \\ \frac{b(t)}{a(t)} &= \frac{k_a}{k_b - k_a} \left[1 - e^{(k_a - k_b)t}\right]. \end{aligned}$$

Typical time variations of $a(t)$ and $b(t)$ are shown in figure 7.1. Therefore a rock can be dated by determining the ratio of certain isotopes (often those of carbon) present. Unfortunately, there is an obvious question: how can we solve the problem when $k_a = k_b$? We will return to this question below.

7.2 Non-constant coefficients

Consider the general form of a first order linear differential equation:

$$a(x)\frac{dy}{dx} + b(x)y = c(x).$$

Then we divide across by $a(x)$ to obtain the **standard form**:

$$\frac{dy}{dx} + p(x)y = f(x). \tag{7.3}$$

There are subtleties concerning whether or not $a(x) = 0$ in or on the boundaries of the domain of interest which we will try not to worry about here...

7.2.1 Integrating factor

There is a robust algorithm to solve (7.3). We multiply across by a further function $\mu(x)$ called the **integrating factor** so that

$$(\mu)y' + (\mu p)y = \mu f. \tag{7.4}$$

The left hand side is the expression of the product rule for $(\mu y)'$ **if** μ satisfies the differential equation

$$\frac{d\mu}{dx} = \mu' = \mu p,$$

and there is no free constant. Rearranging, integrating with respect to x, using the chain rule and ensuring there is no free constant:

$$\int p dx = \int \frac{1}{\mu}\frac{\partial \mu}{\partial x} dx = \int \frac{d\mu}{\mu} = \ln \mu.$$

Therefore, the integrating factor is

$$\mu(x) = \exp\left[\int^x p(u)du\right]. \tag{7.5}$$

Note that this integral is a fool-proof method to construct the integrating factor. Using this expression, (7.4) becomes

$$\frac{d}{dx}(\mu y) = \mu f,$$

and so

$$\mu(x)y(x) = \int^x \mu(u)f(u)du,$$

from which $y(x)$ can be determined straightforwardly.

Example

As ever, an example is helpful. Consider

$$x\frac{dy}{dx} + (1-x)y = 1 \to y' + \left(\frac{1}{x} - 1\right) y = \frac{1}{x}. \tag{7.6}$$

Here $p(x) = (1/x - 1)$ and so from (7.5),

$$\mu(x) = \exp\left[\int^x p(u)du\right] = \exp\left[\int^x \left(\frac{1}{x} - 1\right) du\right] = e^{\ln x - x} = xe^{-x}.$$

Therefore

$$\frac{d}{dx}\left(xe^{-x}y\right) = e^{-x} \to xe^{-x}y = -e^{-x} + C \to y = \frac{C}{x}e^{x} - \frac{1}{x},$$

for C a constant to be determined from initial or boundary conditions. In particular, if we require $y(x)$ to be finite at $x = 0$ (note this is a point where the coefficient $a(x)$ of y' in the general form of the equation is zero ...) we obtain $C = 1$ by application of L'Hôpital's rule, and so

$$y = \frac{e^x - 1}{x}.$$

Return to the radioactive example

The use of an integrating factor both takes the guesswork out of the radioactive example discussed above, and also allows the case $k_b = k_a$ to be handled seamlessly. From the key equation (7.1), we can see that

$$p(t) = k_b \to \mu(t) = \exp\left[\int^t k_b du\right] = e^{k_b t},$$

and so multiplying across (7.1), we obtain

$$\frac{d}{dt}\left(e^{k_b t}b\right) = k_a a_0 e^{(k_b - k_a)t}. \tag{7.7}$$

Now we can treat the two cases separately.

1. If $k_b \neq k_a$, the right hand side of (7.7) still varies with t, and so we integrate

$$\begin{aligned} e^{k_b t}b &= \frac{k_a}{k_b - k_a} a_0 e^{(k_b - k_a)t} + D, \\ b(t) &= \frac{k_a}{k_b - k_a} a_0 e^{-k_a t} + De^{-k_b t}, \end{aligned}$$

exactly as before, and in particular we recover (7.2) if we require $b(0) = 0$.

2. On the other hand, if $k_b = k_a = k$, the right hand side of (7.7) is just a constant, and so the integration is straightforward:

$$\begin{aligned} e^{kt} b &= k a_0 t + D, \\ b(t) &= k a_0 t e^{-kt} + D e^{-kt}. \end{aligned} \tag{7.8}$$

   In particular, if $b(0) = 0$, $D = 0$, and $b(t)$ exhibits the same qualitative structure as shown in figure 7.1.

Chapter 8

Nonlinear first order equations

In this chapter we start to consider the beautiful world of **nonlinear** equations, i.e. equations where the dependent variable (e.g. $y(x)$) does not only appear to the first power. The general form for a first order nonlinear ordinary differential equation is thus

$$Q(x,y)\frac{dy}{dx} + P(x,y) = 0, \tag{8.1}$$

for general, nontrivial functions P and Q.

8.1 Separable equations

Equation (8.1) is said to be **separable** if it can be manipulated into the form

$$q(y)dy = p(x)dx,$$

and so all the terms involving y explicitly can be collected to one side of the equation, and all the terms involving x explicitly can be collected to the other. In such a case the solution can be found directly by integration:

$$\int^y q(u)du = \int q(y)dy = \int p(x)dx = \int^x p(u)du.$$

Example

Consider

$$(x^2y - 3y)\frac{dy}{dx} - 2xy^2 = 4x.$$

Rearranging,

$$\frac{dy}{dx} = \frac{4x + 2xy^2}{x^2y - 3y} = \left(\frac{2x}{x^2-3}\right)\left(\frac{2+y^2}{y}\right).$$

Therefore

$$\begin{aligned}
\frac{2}{2+y^2}dy &= \frac{2x}{x^2-3}dx, \\
\frac{1}{2}\ln(y^2+2) &= \ln A(x^2-3), \\
(y^2+2)^{1/2} &= A(x^2-3),
\end{aligned}$$

where A is an arbitrary constant.

8.2 Exact equations

Equation (8.1) is said to be an **exact equation** if and only if there exists a function $f(x,y)$ whose differential df (as in the chain rule in differential form (4.5)) is equal to

$$df = Q(x,y)dy + P(x,y)dx. \tag{8.2}$$

Therefore, by comparison with (8.1), $df = 0$ and so $f(x,y) = C$ (where C is a constant) is the solution, in the sense that $f(x,y) = C$ defines an implicit relation betwen x and y which satisfies (8.1).

From the chain rule in differential form (4.5),

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy,$$

and so, by comparison with (8.2), for (8.2) to be an exact equation we require that

$$\frac{\partial f}{\partial x} = P; \qquad \frac{\partial f}{\partial y} = Q, \tag{8.3}$$

$$\rightarrow \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial P}{\partial y}; \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x};$$

$$\rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

8.2.1 Reverse implication

Such exact equations are very important in many physical problems. If

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

throughout a simply connected domain $\mathcal{D}$, then $Pdx + Qdy$ is an exact differential of a single-valued function $f(x,y)$ in $\mathcal{D}$, i.e. there exists $f(x,y)$ such that

$$df = Pdx + Qdy.$$

Example

Algorithmically, if an equation is exact, then the solution $f(x, y) = C$ a constant can be determined by integration of the two equations (8.3). Let's see how this works by considering an example, which shows a key aspect of "integrating" a partial differential equation. Consider

$$\begin{aligned} 6y(y-x)\frac{dy}{dx} + 2x - 3y^2 &= 0, \qquad (8.4)\\ \rightarrow (2x - 3y^2)dx + 6y(y-x)dy &= 0. \end{aligned}$$

Therefore

$$P(x, y) = 2x - 3y^2; \; Q(x, y) = 6y^2 - 6xy,$$

and so

$$\frac{\partial P}{\partial y} = -6y = \frac{\partial Q}{\partial x},$$

and so by definition the equation (8.4) is indeed **exact**.

Furthermore, the solution $f(x, y) = C$ a constant must satisfy the two equations

$$\frac{\partial f}{\partial x} = 2x - 3y^2 = P; \; \frac{\partial f}{\partial y} = 6y^2 - 6xy = Q. \qquad (8.5)$$

Now, remembering that y is being kept constant in the left hand equation in (8.5) we integrate to obtain

$$f(x, y) = x^2 - 3xy^2 + h(y),$$

for some **function** $h(y)$. This term in general is a function of y, since if we take a partial derivative with respect to x keeping y constant this term will make no contribution.

Taking the partial derivative of this expression with respect to y and comparing to the right hand equation in (8.5), we obtain an equation for $h(y)$:

$$-6xy + \frac{dh}{dy} = 6y^2 - 6xy \rightarrow \frac{dh}{dy} = 6y^2 \rightarrow h(y) = 2y^3 + C_1,$$

for some constant C_1. Therefore the solution to the equation (8.4) is

$$f(x, y) = x^2 - 3xy^2 + 2y^3 = C,$$

for some arbitrary constant. This can of course be verified by direct substitution.

Chapter 9

Isoclines and solution curves

It is not always possible to solve nonlinear equations explicitly, but much insight into the "flow" of solutions can be gained by various graphical methods. In this chapter we introduce some of these methods through consideration of a specific example, where we can work out the explicit solution, and so we can see directly how useful the graphical methods are.

Example

Consider the (nonlinear) equation

$$\frac{dy}{dt} = t(1-y^2) = f(y,t). \tag{9.1}$$

This equation is separable:

$$\frac{dy}{1-y^2} = tdt,$$

which can be integrated to obtain

$$\frac{1}{2}\ln\left|\frac{1+y}{1-y}\right| = \frac{1}{2}t^2 + C,$$

where C is a constant. Therefore

$$\frac{1+y}{1-y} = Ae^{t^2} \to y = \frac{A-e^{-t^2}}{A+e^{-t^2}}, \tag{9.2}$$

for A also a constant, $A = e^{2C}$ for $|y| < 1$, and $A = -e^{2C}$ for $|y| > 1$. If we have an initial condition, i.e. we know $y(0)$, we can determine A since

$$y(0) = \frac{A-1}{A+1} \to A = \frac{1+y(0)}{1-y(0)}.$$

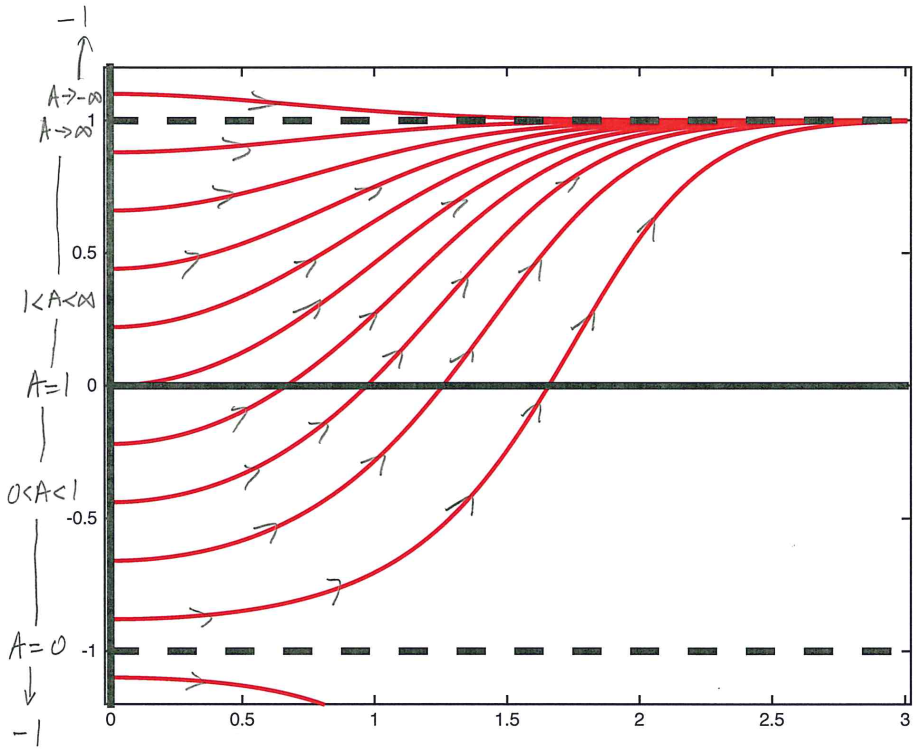

Figure 9.1: Schematic representation of the solution curves $y(t)$ defined by (9.2) of the differential equation (9.1). Arrows show the flow of the solutions from the initial conditions on the $y-$axis.

9.1 Solution curves

As shown on the $y-$axis of figure 9.1, as $y(0)$ increases from $-\infty$ to 1, A increases from -1 to ∞ with $A = 0$ when $y(0) = -1$ and $A = 1$ when $y(0) = 0$. A is naturally discontinuous at $y(0) = 1$, and for $y(0)$ increasing further from 1 to ∞, A increases again from $-\infty$ to -1.

As time increases from $t = 0$, we see that the solutions then "flow" (in the direction of the arrows) along **solution curves**. There is a family of these solution curves parameterised by A or equivalently by the initial values $y(0)$ of the dependent variable. Here, because we can solve the equation (9.1) explicitly, we can draw these solution curves, and hence understand the behaviour of solutions to (9.1) for given initial conditions (i.e. the starting point on the $y-$axis). The really interesting question is whether we can understand the key properties of the family of the solutions to this differential equation **without** explicitly solving it? This is important, since we are not guaranteed to be able to solve an arbitrary nonlinear differential equation.

9.2 Isoclines

The example (9.1) is of course just an example of a general first order nonlinear differential equation:

$$\frac{dy}{dt} = f(y,t), \tag{9.3}$$

where here of course $f(y,t) = t(1-y^2)$.

Considering figure 9.1, and indeed the right hand side of (9.1), the lines $y = \pm 1$ are special. Indeed, $\dot{y} = 0$ on $y = \pm 1$, and also when $t = 0$, as is apparent from the fact that the solution curves are horizontal in the vicinity of the y axis. Furthermore, for $t > 0$, $\dot{y} < 0$ for $|y| > 1$ and $\dot{y} > 0$ for $|y| < 1$.

Knowing the right hand side of the equation $f(y,t)$, it is in general possible to construct **isoclines** ("constant slopes" for those whose ancient Greek is a bit rusty) i.e. curves for which f, and hence the "slope" $\dot{y}$ is constant. For the particular example, we thus want to identify the class of curves for various constants D where

$$t(1-y^2) = D \to y^2 = 1 - \frac{D}{t}. \tag{9.4}$$

In figure 9.2, several of these lines are plotted. Along such isoclines, the rate of variation of y with respect to t is constant (for each line), and so we also draw arrows of constant slope along each of these lines. (This is a way to construct what is known as the **slope field** of the function.) The key insight is that if we draw lines connecting these arrows from different isoclines, respecting the direction in which each of the arrows are pointing, such sketched lines construct solution curves, as shown in figure 9.2. Therefore, combining the slope field and the isoclines, which we can identify straightforwardly from the right hand side of the equation (9.3), we can construct approximations to the solution curves even if we cannot calculate their functional form explicitly. We can note further if $f(y,t)$ is a single-valued function, the solution curves cannot cross. Cool!

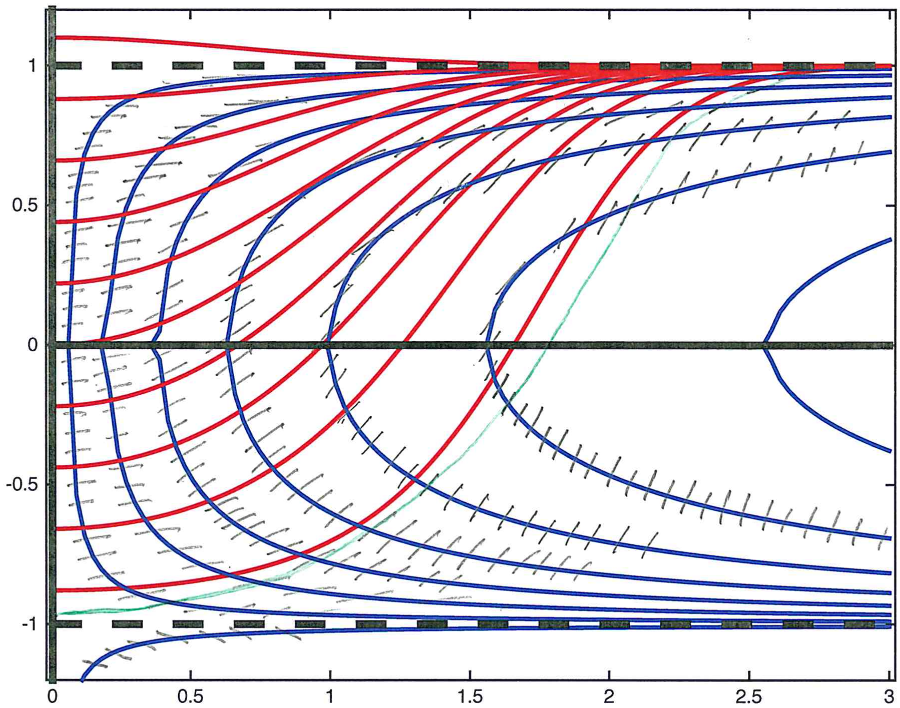

Figure 9.2: Schematic representation of the isoclines (blue lines) defined by (9.2) of the differential equation (9.1). Connecting the grey sloped lines (with constant slope on each blue isocline) representing the slope field constructs good approximations (e.g. the green line) to the solution curves (red lines) shown in figure 9.1.

Chapter 10

Fixed points

In this chapter, we consider the properties of **fixed points** or **equilibrium points**, the analysis of which typically reveals many important properties of the solution of a differential equation.

Definition

For general equations of the form (9.3) **fixed points** or **equilibrium points** are points where $dy/dt = 0$ for all t, or equivalently $f(y,t) = 0$ for all t.

In the specific example (9.1), there are fixed points at $y_f^{\pm} = \pm 1$. From consideration of the solution curves shown in figure 9.1, it is clear that these two points have qualitatively different character.

10.1 Stability of fixed points

Specifically, the "flow" along the solution curves converges towards $y = 1$ for all initial conditions $y(0) > -1$, and in particular for $y(t)$ close to the fixed point value $y_f^+ = 1$. Conversely for $y(t)$ close to $y_f^- = -1$, the flow of the solutions diverges away from the fixed point. Because of these properties, the fixed point $y_f^+ = 1$ is said to be a **stable fixed point**, while $y_f^- = -1$ is said to be an **unstable fixed point**.

10.1.1 Perturbation analysis

To determine the stability of a fixed point, from the above description we clearly need to consider the properties of the solution $y(t)$ near to the fixed point. We can do this by "perturbation analysis". So for the general equation (9.3), let us suppose that $y = y_f$ is a fixed point and so $f(y_f, t) = 0$.

Let us write $y(t) = y_f + \epsilon(t)$, where ϵ is assumed to be small, i.e. $y(t)$ is "close" to y_f. Since y_f is a constant $dy/dt = d\epsilon/dt$, and so

$$\begin{aligned} \frac{d\epsilon}{dt} &= f(y_f + \epsilon, t), && (10.1) \\ &= f(y_f, t) + \epsilon \frac{\partial f}{\partial y}(y_f, t) + o(\epsilon). && (10.2) \end{aligned}$$

Therefore, near y_f, the rate of change of ϵ (and so the **perturbation** of y from y_f) with time must be well-approximated by

$$\frac{d\epsilon}{dt} \simeq \left[\frac{\partial f}{\partial y}(y_f, t) \right] \epsilon,$$

which has the **huge** attraction that this is a **linear** equation in ϵ, and so is much more straightforward to solve than the underlying nonlinear equation.

Example

For our specific example (9.1), since $f(y,t) = t(1 - y^2)$,

$$\frac{\partial f}{\partial y} = -2yt,$$

and in particular

$$\frac{\partial f}{\partial y}(1,t) = -2t, \ \frac{\partial f}{\partial y}(-1,t) = 2t.$$

Therefore, near $y_f^+ = 1$,

$$\frac{d\epsilon}{dt} = -2t\epsilon \to \epsilon = \epsilon_0 e^{-t^2} \to 0 \text{ as } \ t \to \infty,$$

and so $y \to 1 = y_f^+$ as $t \to \infty$. Since the perturbation ϵ near $= y_f^+$ decays as $t \to \infty$, this fixed point is said to be **stable**.

Conversely, near $y_f^- = -1$,

$$\frac{d\epsilon}{dt} = 2t\epsilon \to \epsilon = \epsilon_0 e^{t^2} \to \infty \text{ as } \ t \to \infty,$$

and so the perturbation ϵ near $= y_f^-$ increases as $t \to \infty$. Therefore, this fixed point is said to be **unstable**.

10.2 Autonomous systems

The analysis of the stability of the fixed points is more straightforward when the system (9.3) is **autonomous**, i.e. the function $f = f(y)$, with no explicit dependence on t. In this case, near a fixed point y_f where $f(y_f) = 0$ by definition, we once again write $y = y_f + \epsilon(t)$. The evolution equation for ϵ then takes the particularly simple form

$$\frac{d\epsilon}{dt} = \left[\frac{df}{dy}(y_f)\right]\epsilon = k\epsilon \to \epsilon = \epsilon_o e^{kt}. \tag{10.3}$$

Therefore the stability of the fixed point is effectively determined by the sign of k, the constant value the derivative of f takes at the fixed point y_f. The fixed point y_f is stable if

$$k \equiv \frac{df}{dy}(y_f) < 0,$$

while the fixed point y_f is unstable if

$$k \equiv \frac{df}{dy}(y_f) > 0.$$

10.3 Phase portraits

Another way to analyse the behaviour of the **dynamical system** described by a differential equation of the form (9.3) is through a geometrical representation of the evolution of the solutions, i.e. through a **phase portrait**. As is becoming usual in this course, we will show the utility of this approach by considering an example.

Example

Let us consider a highly idealised mode of a chemical reaction, where the rate of growth of the concentration c of species C is proportional to the product of the concentrations a and b of two reactant species A and B, and so

$$\frac{dc}{dt} = \lambda ab = \lambda(a_0 - c)(b_0 - c) = f(c), \tag{10.4}$$

where λ, a_0 and b_0 are constants. Without loss of generality, we can assume that $a_0 < b_0$.

In figure 10.1(a), we plot f against c. Clearly, a_0 and b_0 are fixed points. From this figure, it is apparent that if $c < a_0$, then c increases, while if c is slightly larger than a_0, then c decreases, and so a_0 is a **stable** fixed point. On

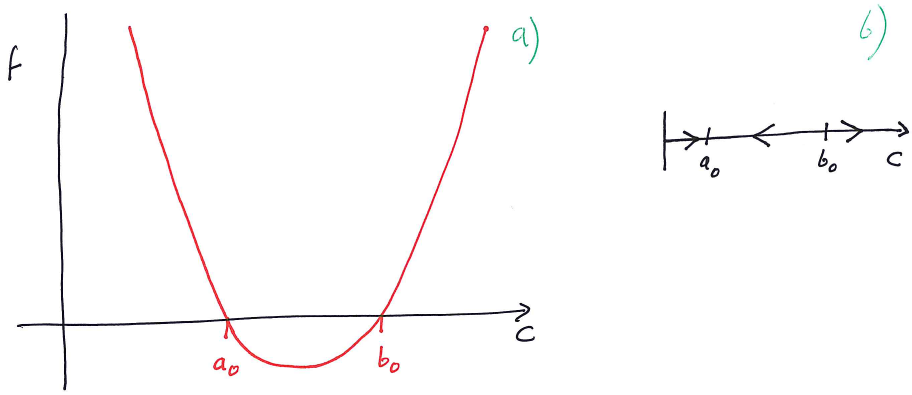

Figure 10.1: (a) Variation with c of the right hand side $f(c)$ of the differential equation (10.4). (b) The phase portrait of this system showing the evolution of the concentration $c(t)$ relative to the fixed points a_0 and b_0.

the other hand, if $c > b_0$, then c increases, while if c is slightly less than b_0, c decreases, and so b_0 is an **unstable** fixed point. This stability is represented by the **phase portrait** shown in figure 10.1(b), with the arrows showing how $c(t)$ will move away from b_0 and towards a_0.

This geometric picture is of course entirely consistent with the perturbation analysis presented above.

1. If we assume that $c = a_0 + \epsilon$, then

$$\frac{d\epsilon}{dt} = \lambda(-\epsilon)(b_0 - a_0 - \epsilon) \simeq -\lambda(b_0 - a_0)\epsilon \rightarrow \epsilon = \epsilon_0 \exp[-\lambda(b_0 - a_0)t].$$

This is also consistent with the approach described above in section 10.2, since this is an autonomous system as the right hand side $f(c)$ of the governing equation does not depend on t. The derivative of $f(c)$ with respect to c is

$$\frac{df}{dc} = \lambda(2c - [b_0 + a_0]) \rightarrow \frac{df}{dc}(a_0) = -\lambda(b_0 - a_0) < 0,$$

unsurprisingly showing that $c = a_0$ is a stable fixed point.

2. For the other fixed point, if we assume that $c = b_0 + \epsilon$, then

$$\frac{d\epsilon}{dt} = \lambda(-\epsilon)(a_0 - b_0 - \epsilon) \simeq \lambda(b_0 - a_0)\epsilon \rightarrow \epsilon = \epsilon_0 \exp[\lambda(b_0 - a_0)t].$$

This is also consistent with the approach described above using (10.3), as

$$\frac{df}{dc}(b_0) = \lambda(b_0 - a_0) > 0,$$

unsurprisingly showing that $c = b_0$ is an unstable fixed point.

Exercise

Assume that $c(0) = 0$. Show that

$$c(t) = \frac{a_0 b_0 \left[1 - e^{-(b_0 - a_0)\lambda t}\right]}{b_0 - a_0 e^{-(b_0 - a_0)\lambda t}}. \tag{10.5}$$

Chapter 11

The logistic equation & map

In this chapter, we will consider a beautiful and important nonlinear differential equation and its associated "map" which illustrate several of the important properties of nonlinear **dynamical systems**.

11.1 The logistic equation

The logistic equation is a simple (and famous) model of population dynamics.

11.1.1 A linear model

Let us assume that there is a population $y(t)$, which experiences a birth rate αy, and a death rate βy. Therefore

$$\frac{dy}{dt} = (\alpha - \beta)y \to y = y_0 e^{(\alpha-\beta)t},$$

and so the population increases or decreases exponentially depending on whether the birth rate is greater than or less than the death rate. This linear model is thus fundamentally flawed.

11.1.2 Limited resources

Now imagine that the environment is finite, and that there is a need for food. The probability of some food being found by some individual member of the population may be modelled as being proportional to the population size y. Therefore, the probability that the same source of food is found by two individuals is proportional to the square of the population size, i.e. y^2. If food is scarce (e.g. the very last slice of delivered pizza being squabbled over by flatmates...) then there will be a fight to the death at a rate γ, and so

$$\frac{dy}{dt} = (\alpha-\beta)y - \gamma y^2 \to \dot{y} = \lambda y \left(1 - \frac{y}{Y}\right) = f(y);\ \lambda = \alpha - \beta;\ Y = \frac{\lambda}{\gamma}. \quad (11.1)$$

This is called the **(differential) logistic equation**.

In figure 11.1(a), we plot $f(y)$ as defined in (11.1) against y, while in figure 11.1(b), we plot the associated phase portrait. When the population is small $\dot{y} \simeq \lambda y$, and so the population grows essentially exponentially. Eventually however, members of the population start fighting over the last cup cake, and so a stable equilibrium at $y = Y$ is reached.

11.2 The logistic map

The evolution of the population may occur discretely (e.g. births in the spring, deaths in the winter etc) rather than continuously, or alternatively we may be more interested in measuring the population at discrete intervals (e.g. once a year for taxes or whatever). Therefore, it might be better to evaluate y at different instants or **iterates** x_i. Similarly to the "interest rate" discussed above, we thus approximate the derivative as a difference over a time interval Δt:

$$\frac{x_{n+1} - x_n}{\Delta t} = \lambda x_n - \gamma x_n^2 \to x_{n+1} = (1 + \lambda \Delta t)x_n - \gamma \Delta t x_n^2. \quad (11.2)$$

The simplest (yet still richly interesting) case is when the time interval is chosen so that $r = 1 + \lambda \Delta t = \gamma \Delta t$, when this iteration becomes the **(discrete) logistic equation** or the **logistic map**:

$$x_{n+1} = r x_n (1 - x_n). \quad (11.3)$$

11.3 Behaviour of the logistic map

From an initial condition x_0 this expression "maps" this initial condition to x_1, then to x_2, and so on. It is a member of a general class of maps

$$x_{n+1} = f(x_n),$$

and it exhibits a rich variety of behaviour as r varies from zero to its maximum value 4, since $0 \leq x_n \leq 1$. The logistic maps associated with four different values of r are shown in figure 11.2. In each panel, the thick black

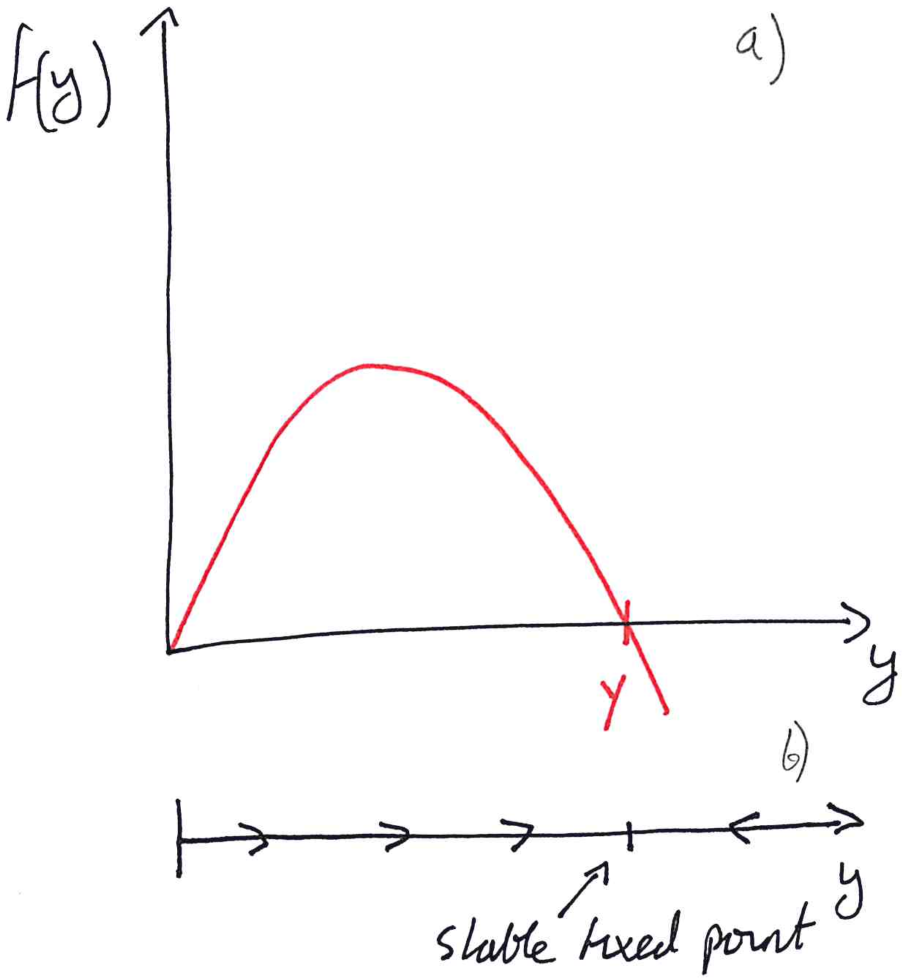

Figure 11.1: (a) Variation with y of the right hand side $f(y)$ of the differential logistic equation (11.1). (b) The phase portrait of this system showing the evolution of the population $y(t)$ away from the (unstable) fixed point $y = 0$ to the stable fixed points $y = Y = \lambda/\gamma$.

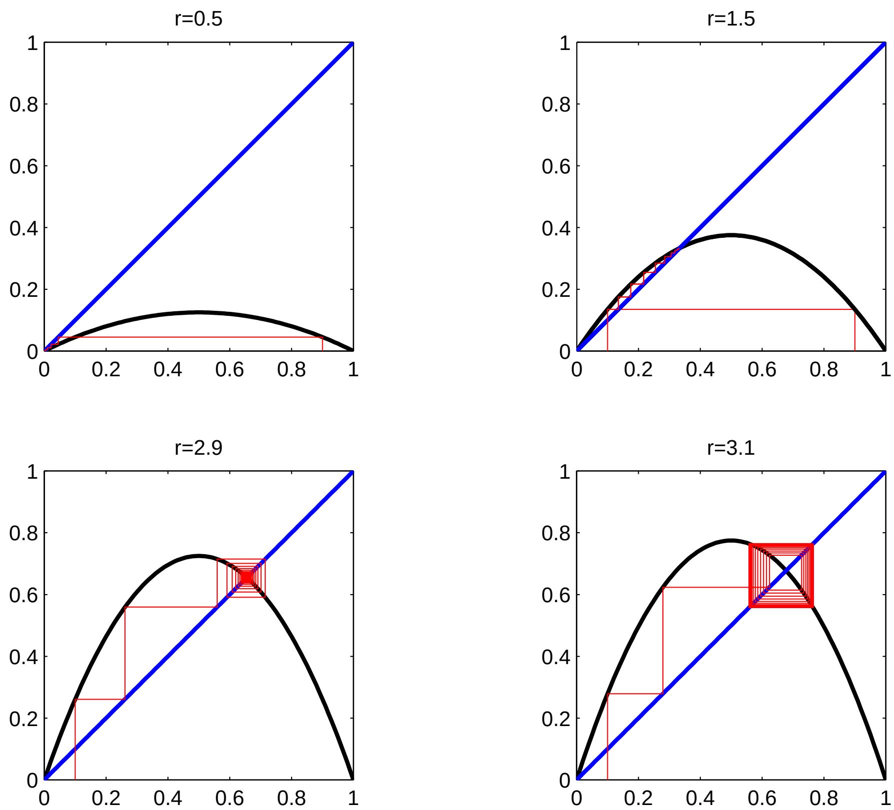

Figure 11.2: Evolution of the iterates of the logistic map for four illustrative choices of the parameter $r = 0.5, 1.5$ (where two initial conditions are shown) $r = 2.9, 3.1$.

line is a plot of $f(x) = rx(1-x)$ for each marked value of r. The thin blue line is the diagonal $y = x$, while the red line shows the evolution of the iterates.

The iterates of the population are determined as follows, producing what is called a **cobweb diagram**.

- Start from an initial value x_0 on the $x-$axis.
- Draw a **vertical** line up to the plot of $f(x)$. This determines x_1.
- Draw a **horizontal** line to the diagonal $y = x$. This is thus at the appropriate $x = x_1$ value.
- Draw a vertical line to the plot of $f(x)$ to determine $x_2 \ldots$
- Repeat this sequence of: vertical line to $f(x)$ then horizontal line to $y = x$ as often as required.

Let us now discuss each of the cases in turn.

$r = 0.5$**, i.e.** $r < 1$

From the picture, the only fixed point is at $x = 0$, and it appears to be stable (and indeed strongly **attracting**).

$r = 1.5$**, i.e.** $1 < r < 2$

For $r > 1$, the curve $f(x)$ intersects the line $y = x$ at two points, and so there are two fixed points for the iteration. This can be derived directly from the map (11.3) since for a fixed point

$$x_{n+1} = x_n = f(x_n) = rx_n(1-x_n) \to x_n = 0;\ x_n = 1 - \frac{1}{r}.$$

For $1 < r < 2$, $x = 0$ appears to be an unstable fixed point, and the new fixed point at $x_f = 1 - 1/r$ is stable, with the iterates converging directly towards x_f.

$r = 2.9$**, i.e.** $2 < r < 3$

The fixed point at $x_f = 1 - 1/r$ still appears to be stable, but the iterates appear to "spiral" in to the fixed point, exhibiting an oscillatory convergence.

$r = 3.1$**, i.e.** $r > 3$

For r slightly larger than three, the fixed point appears to become unstable, with the iterates converging to a period 2 **limit cycle** i.e. $x_{n+2} = x_n$. Can we understand this behaviour? And what happens as $r \to 4$?

11.4 Stability of fixed points

Let us try and understand the stability of the two fixed points.

11.4.1 Stability of $x = 0$

Consider first the fixed point at $x = 0$. Assume at the n^{th} iterate that the "error" of the iterate away from the fixed point is ϵ_n. Therefore, at the $(\mathrm{n}+1)^{\mathrm{th}}$ iterate, the error is

$$\epsilon_{n+1} = r\epsilon_n(1 - \epsilon_n) \simeq r\epsilon_n,$$

if ϵ_n is sufficiently small. Therefore the error will get smaller from iteration to iteration if $r < 1$ (and so the fixed point $x = 0$ is **stable**), while the error will get larger from iteration to iteration if $r > 1$ (and so the fixed point $x = 0$ is **unstable**), consistently with what is apparent in figure 11.2.

11.4.2 Stability of $x_f = 1 - 1/r$

For this fixed point, let us again consider how the error at the n^{th} iterate changes in the $(\mathrm{n}+1)^{\mathrm{th}}$ iterate. Application of (11.3), remembering that the iterates are "near" to the fixed point at $1 - 1/r$, implies

$$\begin{aligned}
x_f + \epsilon_{n+1} &= r(x_f + \epsilon_n)(1 - [x_f + \epsilon_n]), \\
1 - \frac{1}{r} + \epsilon_{n+1} &= r\left(1 - \frac{1}{r} + \epsilon_n\right)\left(1 - 1 + \frac{1}{r} - \epsilon_n\right), \\
&\simeq 1 - \frac{1}{r} + \epsilon_n - r\epsilon_n + \epsilon_n, \\
\to \epsilon_{n+1} &\simeq (2 - r)\epsilon_n.
\end{aligned}$$

Therefore, for $1 < r < 2$, $\epsilon_{n+1} < \epsilon_n$ and they have the same sign, while for $2 < r < 3$, $|\epsilon_{n+1}| < |\epsilon_n|$, and they have opposite signs, entirely consistently with the cobweb diagrams…unsurprisingly!

Relationship to slope of $f(x)$

Looking closely at the figure, the slope of the line $f(x)$ in the vicinity of the fixed point appears to be important. This can actually be understood directly from considering the Taylor series expansion of $f(x)$ about x_f for points $x = x_f + \epsilon$ "close" to x_f, rather similarly to the perturbation analysis leading to (10.2). Here,

$$f(x_f + \epsilon) = f(x_f) + \epsilon \frac{df}{dx}(x_f) + o(\epsilon) \simeq x_f + \epsilon \frac{df}{dx}(x_f),$$

for sufficiently small ϵ. Therefore, the iterates get closer to the fixed point depending on the magnitude of df/dx at x_f, i.e. the **slope** of $f(x)$ at the fixed point. Specifically:

- the fixed point x_f is **stable** if
$$\left| \frac{df}{dx} \right|_{x=x_f} < 1;$$
- the fixed point x_f is **unstable** if
$$\left| \frac{df}{dx} \right|_{x=x_f} > 1.$$

11.5 Higher iterates

The development of the stable 2-period limit cycle for $r = 3.1$ can now be understood by considering the second iterate of this map, i.e. considering the map

$$X_{n+1} = f[f(X_n)] = f(rX_n[1 - X_n]) = r^2 X_n(1 - X_n)(1 - rX_n[1 - X_n]).$$

This map, which defines a quartic function with three turning points, is shown in figure 11.3. The fixed point of the logistic map $x_f = 1 - 1/r$ is naturally a fixed point of the second iterate. The slope of the second iterate map at this fixed point x_f changes as r increases through 3, as two neigbouring fixed points appear, corresponding to the period 2 points of the first iterate logistic map.

Further bifurcations occur as r increases. Indeed, for $r > 1 + \sqrt{6} \simeq 3.449$, a period 4 limit cycle appears, as shown for $r = 3.52$ in figure 11.3, where the second iterate now has a period 2 limit cycle, and so the first iterate has a period 4 limit cycle.

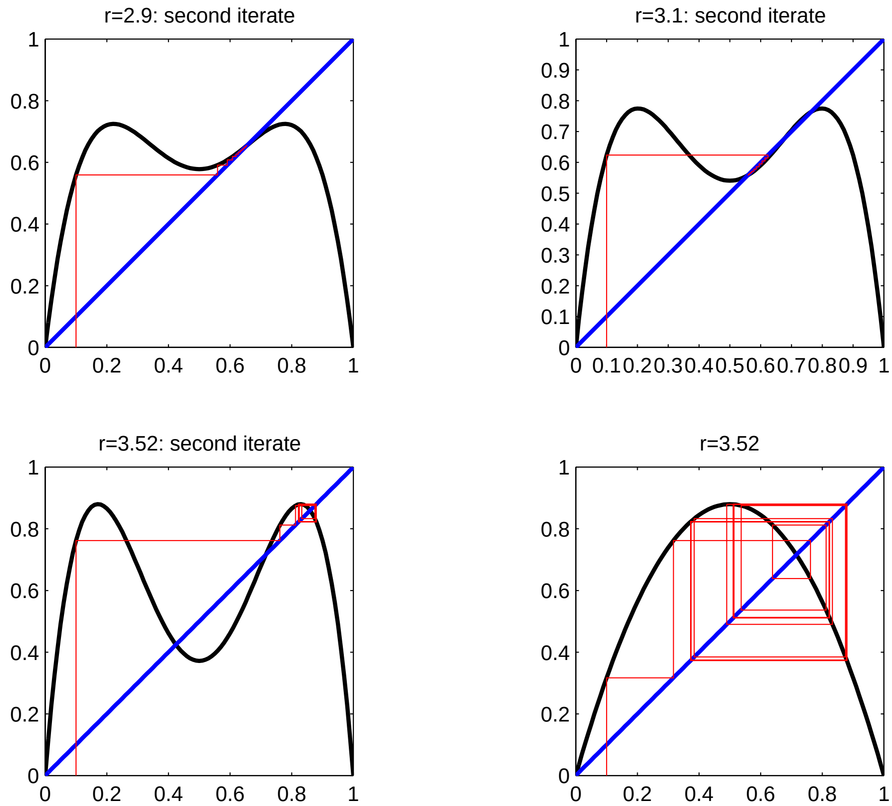

Figure 11.3: Evolution of the second iterates of the logistic map for three illustrative choices of the parameter $r = 2.9, 3.1$ and 3.52. The logistic map is also shown for 3.52, showing a period 4 limit cycle.

This **period doubling** continues as r increases, with the length of the intervals showing each class of periodicity decreasing rapidly, as shown in figure 11.4, which plots the values of x to which (almost) all initial conditions are attracted. Indeed, the ratio of the lengths of successive intervals converges to a particular value, a universal constant called the **Feigenbaum constant** $\delta \simeq 4.7$. The logistic map also exhibits **chaos**, i.e. very sensitive dependence on initial conditions, for most values $r > 3.57$, although there are also some **islands of stability**. All in all, this map has many fascinating characteristics, which you can learn about in any number of different ways...pretty groovy!

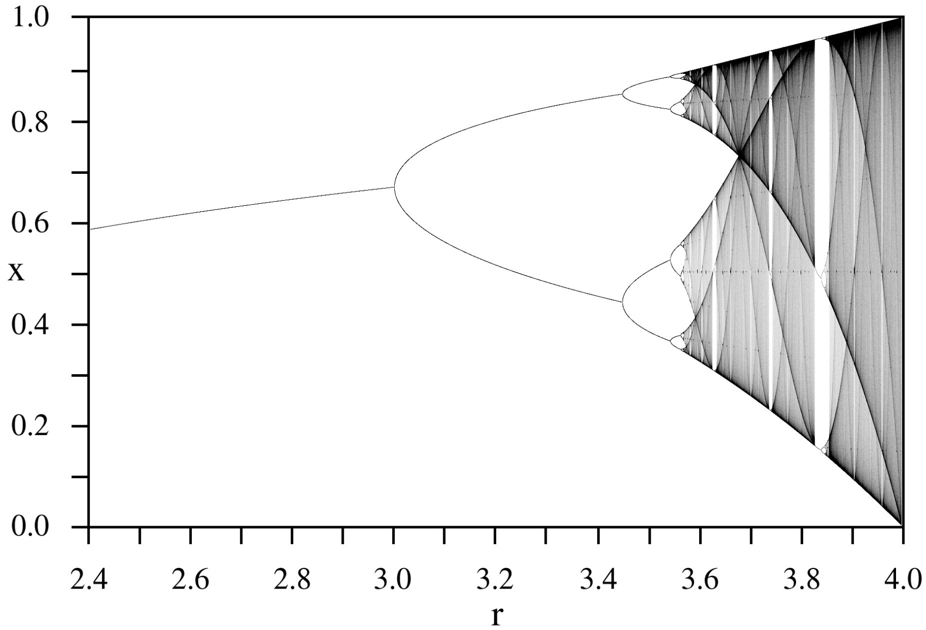

Figure 11.4: Variation with r of the attracting values of x for the logistic map (pinched from wikipedia).