9 Discrete Mathematics (MPF)

(*a*) Let p and m be positive integers such that $p > m$.

(*i*) Prove that $\gcd(p, m) = \gcd(p, p - m)$. [3 marks]

(*ii*) Without using the Fundamental Theorem of Arithmetic, prove that if $\gcd(p, m) = 1$ then $p \mid \binom{p}{m}$. You may use any other standard results provided that you state them clearly. [3 marks]

(*b*) Let A^* denote the set of strings over a set A.

For a function $h : X \to Y$, let $\mathrm{map}_h : X^* \to Y^*$ be the function inductively defined by

$$\begin{aligned} \mathrm{map}_h(\varepsilon) &= \varepsilon \\ \mathrm{map}_h(x\,\omega) &= \big(h(x)\big)\,\big(\mathrm{map}_h(\omega)\big) \qquad (x \in X, \omega \in X^*) \end{aligned}$$

Prove that, for functions $f : A \to B$ and $g : B \to C$,

$$\mathrm{map}_g \circ \mathrm{map}_f = \mathrm{map}_{g \circ f}$$

Note: You may use the following Principle of Structural Induction for properties $P(\omega)$ of strings $\omega \in A^*$:

$$\big(P(\varepsilon) \wedge \forall\, \omega \in A^*.\, P(\omega) \Rightarrow \forall\, a \in A.\, P(a\,\omega)\big) \implies \forall\, \omega \in A^*.\, P(\omega)$$

[6 marks]

(*c*) We say that a relation $T \subseteq A \times B$ is a *total cover* whenever $\mathrm{id}_A \subseteq T^{\mathrm{op}} \circ T$ and $\mathrm{id}_B \subseteq T \circ T^{\mathrm{op}}$. (Recall that $T^{\mathrm{op}} \subseteq B \times A$ denotes the opposite, or dual, of the relation $T \subseteq A \times B$.)

For a relation $R \subseteq \{1, \ldots, m\} \times \{1, \ldots, n\}$ $(m, n \in \mathbb{N})$, we define a new relation $\overset{R}{\rightsquigarrow}$ between strings over a set X as follows: for all $u, v \in X^*$,

$$u \overset{R}{\rightsquigarrow} v \iff \begin{array}{l} R \text{ is a total cover and} \\ u = a_1 \ldots a_m,\ v = b_1 \ldots b_n, \text{ and } a_i = b_j \text{ for all } (i, j) \in R \end{array}$$

(*i*) Prove that for $R = \mathrm{id}_{\{1,\ldots,m\}}$, we have that $u \overset{R}{\rightsquigarrow} u$ for all $u = a_1 \ldots a_m$.

(*ii*) Prove that $u \overset{R}{\rightsquigarrow} v$ implies $v \overset{R^{\mathrm{op}}}{\rightsquigarrow} u$.

(*iii*) Prove that $u \overset{R}{\rightsquigarrow} v$ and $v \overset{S}{\rightsquigarrow} w$ imply $u \overset{S \circ R}{\rightsquigarrow} w$.

(*iv*) Prove that the further relation $\sim$ on X^* defined by

$$u \sim v \iff \exists\, R.\, u \overset{R}{\rightsquigarrow} v$$

is an equivalence relation.

[8 marks]