Part III

Second order differential equations

Chapter 12

Constant coefficients

We now turn our attention to **second order linear ordinary differential equations**, i.e. equations involving second derivatives of the dependent variables. In general, we are not guaranteed to be able to solve such equations in closed form, but there are various methods available to us to understand the properties of solutions to such equations, which arise very commonly when considering physical problems...

12.1 General form with constant coefficients

First, we consider a particularly simple form of second order ODE:

$$a\frac{d^2}{dx^2}y + b\frac{d}{dx}y + cy = f(x),$$

where a, b and c are constants. Generalising the approach for first order differential equations, we solve equations of this form in two steps:

1. We find the **complementary functions** which satisfy the homogeneous (unforced) equation, i.e.

$$a\frac{d^2}{dx^2}y_c + b\frac{d}{dx}y_c + cy_c = 0.$$

2. We find a **particular integral** that satisfies the full equation.

12.2 Characteristic equation

Remember that $e^{\lambda x}$ is an eigenfunction of d/dx, and hence it is an eigenfunction of $d^2/dx^2 = d/dx[d/dx]$. Therefore, complementary functions have the form

$$y_c = e^{\lambda x}; \ \frac{d}{dx} y_c = \lambda e^{\lambda x}; \ \frac{d^2}{dx^2} y_c = \lambda^2 e^{\lambda x},$$

where the eigenvalue λ satisfies the **characteristic equation**:

$$a\lambda^2 + b\lambda + c = 0.$$

Since this equation is quadratic, there are **two** solutions of the characteristic equation, λ_1, λ_2, leading to two complementary functions

$$y_1 = e^{\lambda_1 x}; \ y_2 = e^{\lambda_2 x}.$$

If $\lambda_1 \neq \lambda_2$, then y_1 and y_2 are **linearly independent** and **complete**, i.e. they form a basis of the space of solutions of the homogeneous equation. Therefore the general complementary function is

$$y_c = Ae^{\lambda_1 x} + Be^{\lambda_2 x},$$

for arbitrary constants A and B. It is important to remember that λ_1 and λ_2 may be complex, and it is obvious to wonder what happens when the characteristic equation has a double root, and so $\lambda_1 = \lambda_2$.

Example 1

Consider the equation

$$y'' - 5y' + 6y = 0.$$

Substituting in the eigenfunction $y = e^{\lambda x}$, we obtain the characteristic equation

$$\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0 \rightarrow \lambda = 2, 3.$$

Therefore

$$y_c = Ae^{2x} + Be^{3x},$$

for arbitrary constants A and B.

Example 2: complex eigenvalues

Now consider

$$y'' + 4y = 0.$$

Substituting in the eigenfunction $y = e^{\lambda x}$, we obtain the characteristic equation

$$\lambda^2 + 4 \rightarrow \lambda = \pm 2i \rightarrow y_c = Ae^{2ix} + Be^{-2ix}.$$

Therefore

$$\begin{aligned} y_c &= A(\cos 2x + i\sin 2x) + B(\cos 2x - i\sin 2x), \\ &= \alpha\cos 2x + \beta\sin 2x;\ \alpha = A+B; \beta = i(A-B). \end{aligned}$$

Of course, α and β are still two arbitrary constants.

Example 3: degeneracy

Now consider

$$y'' - 4y' + 4y = 0. \tag{12.1}$$

Substituting in the eigenfunction $y = e^{\lambda x}$, we obtain the (degenerate) characteristic equation

$$\lambda^2 - 4\lambda + 4 = 0 \rightarrow (\lambda - 2)^2 = 0,$$

and so we have a repeated eigenvalue $\lambda = 2, 2$. Clearly, e^{2x} and e^{2x} are not linearly independent, so we have not got a complete set of eigenfunctions. How can we find a distinct linearly independent eigenfunction? Remembering the "radioactive example" of Chapter 7 might be useful at this stage ...

12.3 Detuning

Consider a similar, but **detuned** differential equation:

$$y'' - 4y' + (4 - \epsilon^2)y = 0.$$

Substituting in the eigenfunction $y = e^{\lambda x}$, we now obtain a non-degenerate characteristic equation

$$\lambda^2 - 4\lambda + 4 - \epsilon^2 = 0 \rightarrow \lambda = 2 \pm \epsilon,$$

and so the general form of the complementary function is

$$Ae^{(2+\epsilon)x} + Be^{(2-\epsilon)x}.$$

Rearrange this expression, and use the Taylor series expansions for the exponential function to obtain

$$y_c = e^{2x}[Ae^{\epsilon x} + Be^{-\epsilon x}] = e^{2x}\left[(A+B) + x\epsilon(A-B) + O(A\epsilon^2) + O(B\epsilon^2)\right].$$

Now define two new constants:

$$\alpha = A + B;\ \beta = \epsilon(A-B) \leftrightarrow A = \frac{1}{2}\left(\alpha + \frac{\beta}{\epsilon}\right); B = \frac{1}{2}\left(\alpha - \frac{\beta}{\epsilon}\right).$$

Therefore, because of the terms in A and B involving $1/\epsilon$,

$$y_c = e^{2x}\left[\alpha + \beta x + O(\epsilon)\right] \to e^{2x}(\alpha + \beta x) \text{ as } \epsilon \to 0,$$

as we are perfectly allowed to require α and β to remain constant in this limit. Therefore we have constructed a second linearly independent complementary function for the original degenerate equation (12.1). This complementary function is very reminiscent of the radioactive example of Chapter 7 isn't it?

It is also a demonstration of a general rule. For linear equations with constant coefficients where the characteristic equation has a **repeated** (eigenvalue) root, if $y_1(x)$ is a **degenerate** complementary function, then $y_2(x) = xy_1(x)$ is an **independent** complementary function. However, you may find this detuning technique a little magical, so we are now going to consider algorithms for determining second complementary functions.

Chapter 13

Complementary functions

In this chapter, we consider methods for finding second complementary functions for more general second order differential equations, not just those with constant coefficients.

13.1 Reduction of order

Consider a general homogeneous linear second order differential equation

$$\frac{d^2}{dx^2}y + p(x)\frac{d}{dx}y + q(x)y = 0, \tag{13.1}$$

for general functions $p(x)$ and $q(x)$. Assume that we know a first complementary function $y_1(x)$ which solves (13.1). Let us assume that the second complementary function $y_2(x) = v(x)y_1(x)$ for some as yet undetermined function $v(x)$. We apply the product rule:

$$y_2 = vy_1;\ y_2' = vy_1' + v'y_1;\ y_2'' = vy_1'' + 2v'y_1' + v''y_1,$$

and require y_2 to be a solution of (13.1). Therefore

$$\begin{aligned} y_2'' + py_2' + qy_2 &= 0, \\ v''y_1 + v'(2y_1' + py_1) + v(y_1'' + py_1' + qy_1) &= 0, \end{aligned}$$

collecting terms involving different derivatives of v. We now observe that the bracket multiplying $v(x)$ is zero, since y_1 is a solution of (13.1).

Therefore, we have a **first order** equation for the variable $v' = u$:

$$v''y_1 + v'(2y_1' + py_1) = 0 \leftrightarrow \frac{u'}{u} = -\left(\frac{2y_1'}{y_1} + p\right); v' \equiv u.$$

Since we know y_1 and p, in principle we can integrate this simple equation to determine $u = v'$ (remembering to include an arbitrary constant). Once $u(x)$ is known, we can integrate straightforwardly a further time to determine v, and hence $y_2 = vy_1$. Notice the arbitrary constant arising when integrating u to get v will just lead to y_2 containing arbitrary amounts of y_1. This technique is called **reduction of order**, because we have reduced a second order differential equation to a first order equation (for v') which we can solve since it is separable. This is actually a very general technique for higher order differential equations.

Example

To illustrate this general procedure, let's return to the "detuning" example considered in the previous chapter, i.e. consider (12.1):

$$y'' - 4y' + 4y = 0,$$

for which we know that $y_1 = e^{2x}$. Assume that $y_2 = v(x)e^{2x}$, and so

$$y_2' = (v' + 2v)e^{2x};\; y_2'' = (v'' + 4v' + 4v)e^{2x}.$$

Substituting into the differential equation, and dividing across by e^{2x}, which is guaranteed to be non-zero for all x, we obtain:

$$v'' + 4v' + 4v - 4(v' + 2v) + 4v = 0.$$

Unsurprisingly there is lots of cancellation, and so

$$v'' = 0 \to u = v' = A \to v = Ax + B,$$

for arbitrary constants A and B, and so

$$y_2(x) = (Ax + B)e^{2x}.$$

Note that the constant B just adds arbitrary amounts of $y_1(x)$ to $y_2(x)$, and that this method constructs (thankfully!) the xe^{2x} function which we found by detuning in the previous chapter.

Let us now define some useful, very general, concepts.

13.2 Phase space

An n$^{\text{th}}$ order differential equation essentially defines the n$^{\text{th}}$ derivative $y^{(n)}(x)$ (and indeed all higher derivatives) in terms of $y(x)$, $y'(x) \ldots y^{(n-1)}(x)$. Therefore, we can think of the properties of the system defined by the differential equation being entirely determined by an $n-$dimensional **solution vector** $\mathbf{Y}(x)$:

$$\mathbf{Y}(x) = \begin{pmatrix} y(x) \\ y'(x) \\ \vdots \\ y^{(n-1)}(x) \end{pmatrix}.$$

For every x, this vector (note the convention of writing it in bold font) defines a point in an $n-$dimensional **phase space**. As x varies $\mathbf{Y}(x)$ traces out a **trajectory** in phase space.

Example

Consider

$$y'' + 4y = 0.$$

We have already calculated that the two independent solutions are $y_1 = \cos 2x$ and $y_2 = \sin 2x$. Therefore the solution vectors are

$$\begin{aligned} \mathbf{Y}_1(x) &= \begin{pmatrix} y_1 \\ y_1' \end{pmatrix} = \begin{pmatrix} \cos 2x \\ -2\sin 2x \end{pmatrix}, \\ \mathbf{Y}_2(x) &= \begin{pmatrix} y_2 \\ y_2' \end{pmatrix} = \begin{pmatrix} \sin 2x \\ 2\cos 2x \end{pmatrix}. \end{aligned}$$

The two solution vectors thus trace out an elliptical trajectory in phase space, as shown in figure 13.1.

13.3 The Wronskian

In general, the solutions $y_i(x)$ $(i = 1, \ldots, n)$ of an n^{th} order differential equation are independent solutions if the associated solution vectors $\mathbf{Y}_i(x)$ are linearly independent. This is equivalent to the requirement that the **Wronskian** $W(x)$ is non-zero where the Wronskian is the determinant of the **fundamental matrix** constructed by writing the i^{th} solution vector $\mathbf{Y}_i$ in the i^{th} column.

For a second order differential equation, as considered here, the Wronskian $W(x)$ takes the particularly simple form

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'. \tag{13.2}$$

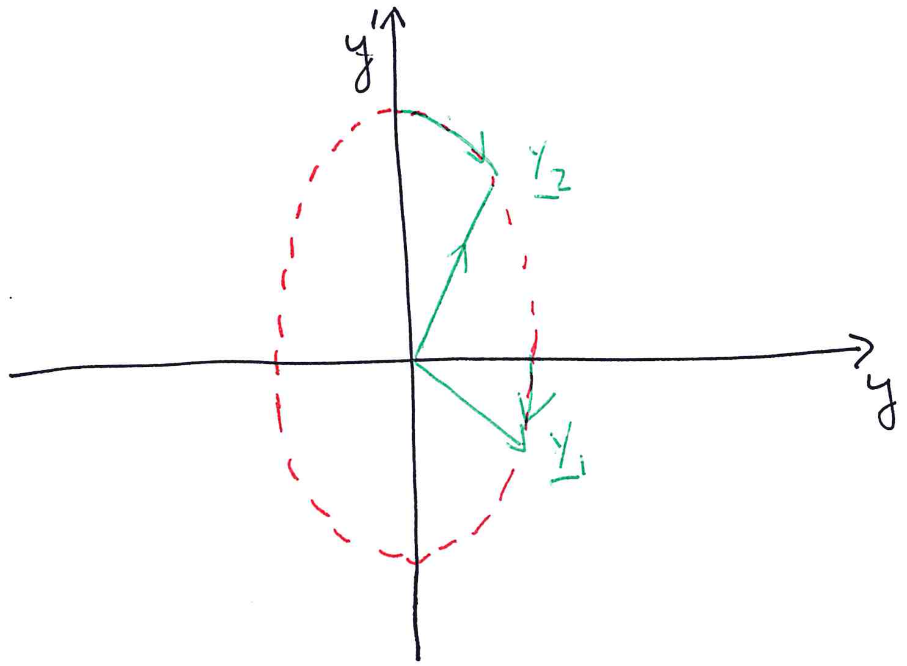

Figure 13.1: Schematic representation of the solution vectors $\mathbf{Y}_1$ and $\mathbf{Y}_2$ in phase space.

Example with complex eigenvalues

In this example, we have found that $y_1 = \cos 2x$ and $y_2 = \sin 2x$. Therefore

$$W(x) = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2(\cos^2 2x + \sin^2 2x) = 2 \neq 0,$$

as expected.

Example with degenerate eigenvalues

On the other hand, in this example, we have found that $y_1 = e^{2x}$ and $y_2 = xe^{2x}$. Therefore

$$W(x) = \begin{vmatrix} e^{2x} & xe^{2x} \\ e^{2x} & (1+2x)e^{2x} \end{vmatrix} = e^{4x}(1 + 2x - 2x) = e^{4x} \neq 0,$$

showing that W can indeed be a function of x, even when it is non-zero at all values of x.

13.4 Abel's theorem aka Abel's identity

Abel's theorem, leading to Abel's identity, is a particularly powerful (and generalisable) result which allows for the construction of the Wronskian, and hence a second independent solution y_2 from a first independent solution y_1 of an homogeneous second order linear differential equation.

Statement of Abel's Theorem

Consider the general form of second order unforced linear differential equation

$$y'' + p(x)y' + q(x)y = 0,$$

defined for all x. If p and q are continuous, then for two solutions y_1 and y_2 of the equation, the Wronskian is **either** $W \equiv 0$ **or** $W \neq 0$ for all values of x, where $W(x)$ is defined by (13.2).

Proof

By definition, y_1 and y_2 are two solutions of the differential equation. Therefore

$$\begin{aligned} y_1(y_2'' + p(x)y_2' + q(x)y_2) &= 0, \\ y_2(y_1'' + p(x)y_1' + q(x)y_1) &= 0. \end{aligned}$$

Subtracting the lower equation from the upper equation, we see that the term involving q cancels, and collecting terms involving p we obtain

$$y_1y_2'' - y_2y_1'' + p(y_1y_2' - y_2y_1') = 0.$$

From the definition of the Wronskian (13.2):

$$W' = y_1y_2'' + y_1'y_2' - y_2'y_1' - y_2y_1'' = y_1y_2'' - y_2y_1''.$$

Therefore $W(x)$ satisfies the differential equation

$$\begin{aligned} W' + p(x)W &= 0 \rightarrow W = W_0 \exp\left[-\int_{x_0}^{x} p(u)du\right], \qquad (13.3) \\ W_0 &= y_1(x_0)y_2'(x_0) - y_2(x_0)y_1'(x_0), \end{aligned}$$

for some x_0. Since the exponential function is never zero, if $W_0 = 0$, $W \equiv 0$. Conversely, if $W_0 \neq 0$, $W(x) \neq 0$ for all x, and the theorem is proved. Furthermore, since the differential equation is defined for all x, wlog $x_0 = 0$. Equation (13.3) is sometimes called **Abel's identity**.

Corollary

If $p(x) = 0$, then the Wronskian is a constant.

Application

Note that Abel's identity (13.3) can be written as

$$y_1 y_2' - y_2 y_1' = W_0 \exp\left[-\int_0^x p(u)du\right].$$

If y_1 is known, then this can be reposed as a first order differential equation for y_2:

$$y_2' - \frac{y_1'(x)}{y_1(x)} y_2 = \frac{W_0 \exp\left[-\int_0^x p(u)du\right]}{y_1(x)},$$

which in general can be solved using an integrating factor as described in section 7.2.1. Since the problem is linear, and so the solutions y_1 and y_2 can be arbitrarily scaled, W_0 can be rescaled to any convenient value.

Generalisation

As we discuss further in Part IV, any **linear** n$^{\text{th}}$ order differential equation can be written in the form

$$\mathbf{Y}' + \mathsf{A}(x)\mathbf{Y} = 0,$$

where $\mathsf{A}(x)$ is an $n \times n$ matrix, which may depend on x, and $\mathbf{Y}$ is an $n-$dimensional vector. It can be shown that

$$W' + Tr[\mathsf{A}(x)]W = 0 \to W = W_0 \exp\left[-\int_0^x Tr[\mathsf{A}(u)]du\right],$$

and so Abel's theorem still holds.

Chapter 14

Particular integrals

In this chapter we start to explore ways to determine particular integrals to the inhomogeneous or forced second order differential equation with constant coefficients, i.e.

$$a\frac{d^2}{dx^2}y + b\frac{d}{dx}y + cy = f(x).$$

14.1 Guesswork

For particular forms of the forcing function $f(x)$, the particular integral takes closely related forms:

$$\begin{aligned} f(x) &\rightarrow y_p(x) \\ e^{mx} &\rightarrow Ae^{mx} \\ \sin kx &\rightarrow A\sin kx + B\cos kx \\ \cos kx &\rightarrow A\sin kx + B\cos kx \\ p_n(x) &\rightarrow q_n(x) = a_n x^n + \cdots + a_1 x + a_0, \end{aligned}$$

where $p_n(x)$ is an n^{th} degree polynomial. The various arbitrary constants in the "guessed" particular integrals are determined by substitution in the governing equations. It is important to remember that the underlying equation is **linear**, and so we can superpose solutions and consider each forcing term separately.

Example

Just for a change, let's illustrate this guesswork technique by an example. Consider the equation

$$y'' - 5y' + 6y = 2x + e^{4x}.$$

From the list,

$$y_p = ax + b + ce^{4x};\ y_p' = a + 4ce^{4x};\ y_p'' = 16ce^{4x}.$$

Substituting these expressions into the equation, and collecting terms multiplied by the same functions of x, we obtain:

$$(16c - 20c + 6c)e^{4x} + (6a)x - 5a + 6b = 2x + e^{4x}.$$

Comparing coefficients, we find that $c = 1/2$, $a = 1/3$ and $b = 5/18$ and so the general solution of the equation is "clearly":

$$y = Ae^{3x} + Be^{2x} + \frac{e^{4x}}{2} + \frac{x}{3} + \frac{5}{18}.$$

14.2 Resonance

Now consider the forced equation

$$\ddot{y} + \omega_0^2 y = \sin \omega_0 t.$$

For this equation, the general complementary function, i.e. the solution of the unforced, homogeneous equation is

$$y_c = A \sin \omega_0 t + B \cos \omega_0 t,$$

for arbitrary constants A and B. In this case, the forcing term is linearly dependent on the complementary function.

Applying the suggested "guess" from the above list is no use, since if we try

$$y_p = C \sin \omega_0 t + D \cos \omega_0 t \to \ddot{y}_p + \omega_0^2 y_p = 0,$$

and so we cannot balance the forcing. This is an example of a simple harmonic oscillator being forced at its natural (resonant) frequency.

14.3 Detuning

In a somewhat similar fashion to the detuning technique we use in section 12.3 to determine a solution of an unforced equation with degenerate eigenvalues, let us consider a related equation where the frequency of the forcing function is **detuned** from the natural frequency and so

$$\ddot{y} + \omega_0^2 y = \sin \omega t;\ \omega \neq \omega_0. \tag{14.1}$$

Now, we guess y_p as

$$y_p = C(\sin\omega t - \sin\omega_0 t),$$

where C is a constant to be determined. (And well you might ask, where on earth did this "guess" come from?) Therefore

$$\ddot{y}_p = C(-\omega^2 \sin\omega t + \omega_0^2 \sin\omega_0 t),$$

and substituting these expressions back into the differential equation (14.1) we find

$$C(\omega_0^2 - \omega^2) = 1 \to y_p = \frac{\sin\omega t - \sin\omega_0 t}{\omega_0^2 - \omega^2}.$$

Now we remember our trigonometric addition formulae:

$$\begin{aligned}
\sin\omega t &= \sin\left[\left(\frac{\omega+\omega_0}{2}\right)t\right]\cos\left[\left(\frac{\omega-\omega_0}{2}\right)t\right] \\
&\quad + \cos\left[\left(\frac{\omega+\omega_0}{2}\right)t\right]\sin\left[\left(\frac{\omega-\omega_0}{2}\right)t\right], \\
\sin\omega_0 t &= \sin\left[\left(\frac{\omega+\omega_0}{2}\right)t\right]\cos\left[\left(\frac{\omega-\omega_0}{2}\right)t\right] \\
&\quad - \cos\left[\left(\frac{\omega+\omega_0}{2}\right)t\right]\sin\left[\left(\frac{\omega-\omega_0}{2}\right)t\right].
\end{aligned}$$

In the expression for y_p, the first terms cancel, and so

$$y_p = \frac{2}{\omega_0^2 - \omega^2}\left(\cos\left[\left(\frac{\omega+\omega_0}{2}\right)t\right]\sin\left[\left(\frac{\omega-\omega_0}{2}\right)t\right]\right).$$

If we now write $\omega_0 - \omega = \Delta\omega$,

$$\frac{\omega_0+\omega}{2} = \omega_0 - \frac{(\omega_0-\omega)}{2} = \omega_0 - \frac{\Delta\omega}{2},$$

we obtain

$$\begin{aligned}
y_p &= -\frac{2}{\Delta\omega(\omega+\omega_0)}\left(\cos\left[\left(\omega_0 - \frac{\Delta\omega}{2}\right)t\right]\sin\left[\left(\frac{\Delta\omega}{2}\right)t\right]\right), \\
&= -\left(\frac{\cos\left[\left(\omega_0 - \frac{\Delta\omega}{2}\right)t\right]}{(\omega+\omega_0)}\right)\left(\frac{\sin\left[\left(\frac{\Delta\omega}{2}\right)t\right]}{\frac{\Delta\omega}{2}}\right). \qquad (14.2)
\end{aligned}$$

If the forcing is at a frequency close to the natural frequency, then we observe beating, as shown in figure 14.1. As $\Delta\omega \to 0$, the wavelength of the

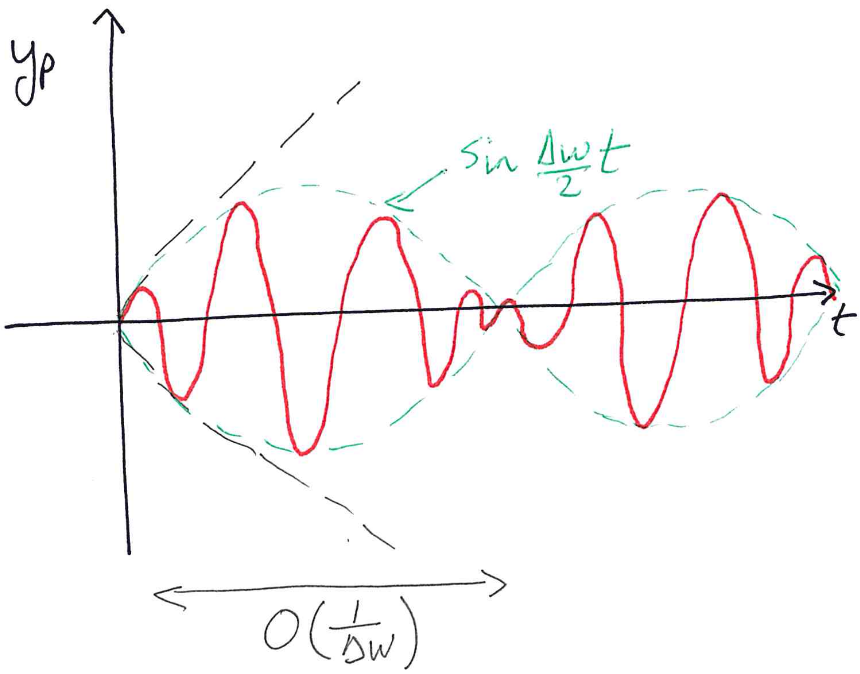

Figure 14.1: Schematic representation of beating. Notice that the oscillation with frequency close to ω_0 "beats" in amplitude with a much lower frequency $\Delta\omega/2$ of the amplitude envelope.

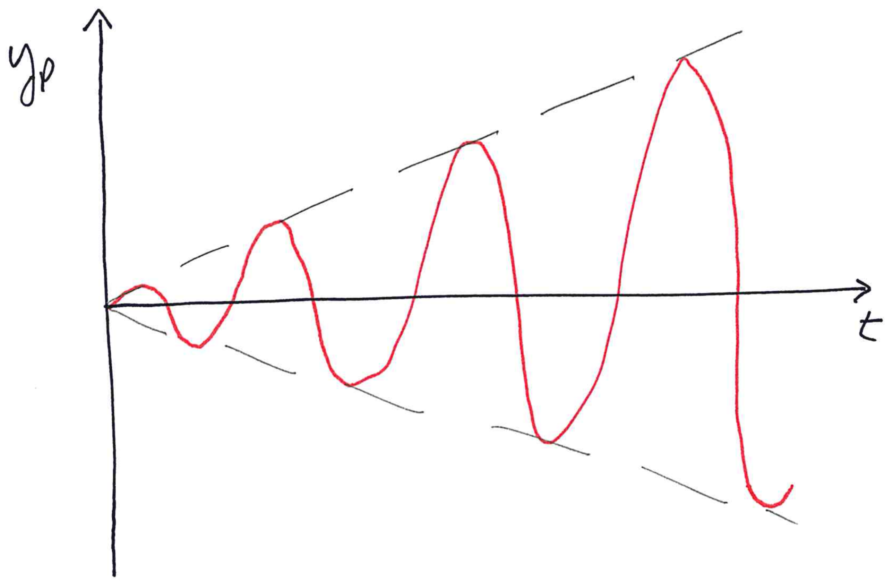

Figure 14.2: Schematic representation of resonance, when the forcing is at the eigenfrequency of the homogeneous equation.

envelope tends to infinity, and we obtain initially linear growth, as shown in figure 14.2. Mathematically, taking the limit of $\Delta\omega \to 0$ in (14.2), due to the properties of the sine function with small argument, and the fact that $\omega \to \omega_0$

$$y_p \to -\frac{t}{2\omega_0}\cos\omega t, \tag{14.3}$$

showing the "secular" linear growth due to resonance. Indeed, as a general rule, if the forcing is a constant linear combination of complementary functions, then the particular integral is proportional to t (i.e. the independent variable) times the non-resonant guess. Once again, this is very reminiscent of the "radioactive example".

14.4 Variation of parameters

Guesswork is by its very nature a bit "hit and miss". Sometimes it is very quick, but other times it can lead nowhere, rather like my taste in music... It is of course preferable to have a reliable algorithm. Fortunately, such an algorithm exists, known as "variation of parameters".

Let $y_1(x)$ and $y_2(x)$ be linearly independent complementary functions of the forced (inhomogeneous) ordinary differential equation:

$$\frac{d^2}{dx^2}y + p(x)\frac{d}{dx}y + q(x)y = f(x), \tag{14.4}$$

Consider the solution vectors:

$$\mathbf{Y}_1(x) = \begin{pmatrix} y_1 \\ y_1' \end{pmatrix}, \; \mathbf{Y}_2(x) = \begin{pmatrix} y_2 \\ y_2' \end{pmatrix}.$$

Now we assume that we can write a particular integral as

$$\mathbf{Y}_p(x) = u(x)\mathbf{Y}_1(x) + v(x)\mathbf{Y}_2(x),$$

with the functions $u(x)$ and $v(x)$ "to be determined". So, how do we determine them?

Well, from the first row:

$$y_p = uy_1 + vy_2, \tag{14.5}$$

while from the second row

$$y_p' = uy_1' + vy_2', \tag{14.6}$$

Differentiating (14.6), we obtain

$$y_p'' = uy_1'' + u'y_1' + vy_2'' + v'y_2'. \tag{14.7}$$

Constructing (14.7) $+p(x)\times$(14.6) $+q(x)\times$(14.7), remembering that y_1 and y_2 are complementary functions and that we want y_p to be a particular integral, we obtain

$$y_1'u' + y_2'v' = f(x). \tag{14.8}$$

Now, if we differentiate (14.5), we obtain

$$y_p' = uy_1' + y_1u' + vy_2' + y_2v'.$$

Therefore, by comparison with (14.6), we require

$$y_1u' + y_2v' = 0. \tag{14.9}$$

Combining (14.8) and (14.9), we obtain

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix},$$

from which we can determine u' and v'.

Note how the determination relies on Abel's theorem, since the fact that the Wronskian W is always non-zero **guarantees** that the matrix on the left-hand side is invertible, and thus that we are able to solve this equation... how cool is that! In particular the assumption of the structure of the $\mathbf{Y}_p$ is shown to be a perfectly reasonable assumption after all.

Specifically,

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \frac{1}{W}\begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix}\begin{pmatrix} 0 \\ f \end{pmatrix}.$$

Therefore,

$$u' = -\frac{y_2}{W}f;\ v' = \frac{y_1}{W}f,$$

and so, remembering the original assumption for the structure of y_p:

$$y_p = y_2(x)\int^x \frac{y_1(\xi)f(\xi)}{W(\xi)}d\xi - y_1(x)\int^x \frac{y_2(\xi)f(\xi)}{W(\xi)}d\xi.$$

There are three observations to make.

1. It is definitely recommended that you remember the method rather than the formulae, which are a bit fiddly.

2. The lower limit on the integrals has deliberately been left unspecified, as such a constant will just add some multiple of the complementary function to the general solution. Initial or boundary conditions will determine the arbitrary constants in the complementary function $Ay_1 + By_2$.

3. Speaking of boundary conditions, the integral expression suggests there might be a clever way to work out particular integrals for a wide range of forcing functions $f(x)$ without going through the whole variation of parameters technique. Remember that suggestion in Methods in Part IB, when the wonderful world of "Green's functions" (named after a chap who lived in a windmill near Nottingham and "succumbed to alcohol" when he came to Caius as a mature student...you can't make this stuff up) will be considered.

Example

As usual, let's fix ideas by considering an example. Let's return to a variant of the detuning problem considered above, compared with the previous example for complex eigenvalues. Consider the equation

$$y'' + 4y = \sin 2x.$$

For this equation,

$$y_1 = \sin 2x,\ y_2 = \cos 2x \to W = -2.$$

If we write

$$\begin{pmatrix} y_p \\ y_p' \end{pmatrix} = \begin{pmatrix} u \sin 2x + v \cos 2x \\ 2u \cos 2x - 2v \sin 2x \end{pmatrix},$$

then

$$u' = \frac{\cos 2x \sin 2x}{2} = \frac{\sin 4x}{4},$$

and

$$v' = -\frac{\sin^2 2x}{2} = \frac{\cos 4x - 1}{4},$$

remembering trigonometric addition formulae. Therefore

$$u = -\frac{\cos 4x}{16},\ v = \frac{\sin 4x}{16} - \frac{x}{4},$$

and so

$$\begin{aligned} y_p &= \frac{1}{16}\left(-\cos 4x \sin 2x + \sin 4x \cos 2x\right) - \frac{x}{4}\cos 2x, \\ &= \frac{1}{16}\sin 2x - \frac{1}{4}x \cos 2x. \end{aligned}$$

The first term is clearly a multiple of one of the complementary functions, while identifying $\omega_0 = 2$ and t with x, we see the second term is exactly the same as the particular integral (14.3) we identified above using detuning. Don't you love it when a plan comes together?

Chapter 15

Special equations

We now have several methods (e.g. reduction of order, Abel's identity, and the method on the third example sheet, which for second order equations is entirely equivalent to reduction of order) to construct a second linearly independent solution $y_2(x)$ if we have a first solution $y_1(x)$ to a second order linear homogeneous ODE with general coefficients of the form (13.1). We also have variation of parameters as a reliable technique to find a particular integral to the general inhomogeneous equation (14.4).

Unfortunately, it is not guaranteed that it is possible to find the first solution $y_1(x)$ to (13.1) in a nice neat form. Before considering general techniques to construct solutions in the form of infinite series, let's consider some special cases where nice closed form solutions **can** be found, which also start to lead towards ideas useful for the series solutions.

15.1 Equidimensional equations aka homogeneous equations

Consider the specific choices of $p(x)$ and $q(x)$ such that (14.4) takes the form

$$ax^2 \frac{d^2}{dx^2} y(x) + bx \frac{d}{dx} y(x) + cy(x) = f(x), \tag{15.1}$$

where a, b and c are constants. Such equations are called "equidimensional" since the equation doesn't change upon rescaling x, i.e. if $X = \alpha x$ for constant non-zero α, then the right-hand side of (15.1) has exactly the same form as a function of X, i.e.

$$aX^2 \frac{d^2}{dX^2} y(X) + bX \frac{d}{dX} y(X) + cy(X) = f(x) = f(X/\alpha),$$

by application of the chain rule. Confusingly, such equations are also sometimes referred to as homogeneous equations, even with non-zero $f(x)$ on the right-hand side. I guess it all depends on punctuation and context, like "let's eat, grandma...", but I prefer calling them equidimensional equations, as that name is more Ronsealesque.

15.1.1 Complementary functions

Whatever, we can note that $y = x^k$ is an eigenfunction of the differential operator xd/dx since:

$$x\frac{d}{dx}y = x\frac{d}{dx}(x^k) = x(kx^{k-1}) = kx^k = ky.$$

Therefore to solve the unforced homogeneous[2] equation:

$$ax^2y'' + bxy' + cy = 0, \tag{15.2}$$

we make the ansatz $y = x^k$, and so

$$y' = kx^{k-1} \to xy' = kx^k; y'' = k(k-1)x^{k-2} \to x^2y'' = k(k-1)x^k.$$

Therefore, all the terms in (15.2) are proportional to x^k, and so, cancelling across we obtain a quadratic equation for k:

$$ak(k-1) + bk + c = ak^2 + (b-a)k + c = 0.$$

This equation has two roots k_1, k_2, and so, provided they are distinct, the general complementary function is

$$y_c = Ax^{k_1} + Bx^{k_2}.$$

This is really rather reminiscent of the constant coefficient equation considered in chapter 12 isn't it?

Exercise

The relationship can be made completely clear by a substitution. Use the chain rule and the substitution $z = \ln x$ to show that (15.1) can be transformed into

$$a\frac{d^2}{dz^2}y + (b-a)\frac{d}{dz}y + cy = f(e^z).$$

All the techniques we have at our disposal for constant coefficient equations therefore can be used here. For example, with the ansatz $y = e^{\lambda z}$, we recover the expected characteristic equation

$$a\lambda^2 + (b - a)\lambda + c = 0 \to \lambda_{1,2} = k_{1,2}.$$

Therefore, in the generic case when $k_1 \neq k_2$,

$$y_c = Ae^{k_1 z} + Be^{k_2 z} = Ax^{k_1} + Bx^{k_2},$$

as before, since $z = \ln x$.

Furthermore, all the special cases carry over straightforwardly.

1. If the roots of the characteristic equation are equal, then

$$y_c = Ae^{kz} + Bze^{kz} \to y_c = Ax^k + Bx^k \ln x. \tag{15.3}$$

   We may well find this observation helpful when we consider more general series solutions. . .

2. For the determination of a particular integral of the forced equation (15.1), if there is a resonant forcing proportional to x^{k_1} or x^{k_2}, then there is a particular integral of the form $x^{k_1} \ln x$ or $x^{k_2} \ln x$ respectively.

15.2 Difference equations

Consider a recursion relation, known as a "difference equation" of the form

$$ay_{n+2} + by_{n+1} + cy_n = f_n. \tag{15.4}$$

Such equations are actually closely related to differential equations, can be solved in a similar way by exploiting linearity and eigenfunctions. We observe that the **difference operator**

$$\mathcal{D}[y_n] = y_{n+1},$$

has eigenfunction $y_n = k^n$ for some constant k since

$$\mathcal{D}y_n = \mathcal{D}[k^n] = k^{n+1} = ky_n.$$

We can use this fact to solve (15.4) in the usual two stage process. We first look for the complementary function solving the (unforced) homoegeneous equation

$$ay_{n+2} + by_{n+1} + cy_n = 0.$$

Trying the ansatz $y_n = k^n$, we find that k can be determined:

$$ak^{n+2} + bk^{n+1} + ck^n = 0 \to ak^2 + bk + c = 0 \to k = k_1,\ k_2.$$

Therefore, the general complementary function is

$$y_n^{(c)} = Ak_1^n + Bk_2^n \text{ for } \ k_1 \neq k_2;\ y_n^{(c)} = (A + Bn)k_1^n \text{ for } \ k_1 = k_2.$$

Analogously to differential equations, there are also natural particular "integrals" for different forcing functions:

$$\begin{array}{lcl} f_n & \to & y_n^{(p)} \\ k^n & \to & Ak^n,\ k \neq k_1, k_2 \\ k_1^n & \to & Ank_1^n \\ k_2^n & \to & Bnk_2^n \\ n^p & \to & An^p + Bn^{p-1} + \cdots + Cn + D. \end{array}$$

Example: The Fibonacci sequence

As usual, let's consider an example. A particularly beautiful example is the Fibonacci sequence, i.e. the sequence defined by the equation

$$y_n = y_{n-1} + y_{n-2},\ y_0 = y_1 = 1.$$

This sequence comes up everywhere, not least in biological systems describing the number of petals, the shape of broccoli, and the reproductive enthusiasms of rabbits, and other fascinating topics. The first few elements in the sequence for $n = 0, 1, 2, 3, 4, 5$ are of course $y_n = 1, 1, 2, 3, 5, 8$.

Let us now solve the equation:

$$y_{n+2} - y_{n+1} - y_n = 0.$$

With the ansatz $y_n = k^n$, we obtain the beautiful and classic equation

$$k^2 - k - 1 = 0 \to k = \frac{1 \pm \sqrt{5}}{2},$$

which we "immediately" (right ...) recognise as the "golden ratio" or "golden mean" and (the negative of) its inverse:

$$\phi_1 = \frac{1 + \sqrt{5}}{2};\ \phi_2 = \frac{1 - \sqrt{5}}{2} = \frac{-1}{\phi_1}.$$

For reasons that have still not been thoroughly resolved by neuroscientists, we are conditioned to find aesthetic appeal in structures with aspect ratios close to the golden ratio, a fact known empirically to artists and engineers from ancient times. One everyday example is the proportion of pieces of paper.

Therefore,

$$y_n = A\phi_1^n + B\phi_2^n,$$

where A and B are given by the initial conditions

$$y_0 = 1 = A + B;\; y_1 = 1 = A\phi_1 + B\phi_2 = \frac{\sqrt{5}}{2}(A - B) + \frac{1}{2}(A + B).$$

Solving these equations we obtain

$$A = \frac{\phi_1}{\sqrt{5}};\; B = -\frac{\phi_2}{\sqrt{5}} \to y_n = \frac{\phi_1^{n+1} - \phi_2^{n+1}}{\sqrt{5}}.$$

I find this one of the most amazing expressions in the entire Tripos, and its definitely "from the book": this is an expression for a sequence of **integers** in terms of differences in the powers of the golden mean, which can be argued in a certain sense to be the **most** irrational number of all! (The sense is in terms of the convergence properties of its continued fraction representation for those of you who are interested.) Furthermore, noting that $\phi_1 > 1$, the ratio of consecutive members of the Fibonacci sequence can now straightforwardly be shown to converge to the golden mean as $n \to \infty$ since:

$$\frac{y_{n+1}}{y_n} = \frac{\phi_1^{n+2} - \left(\frac{-1}{\phi_1}\right)^{n+2}}{\phi_1^{n+1} - \left(\frac{-1}{\phi_1}\right)^{n+1}} \to \phi_1 \quad \text{as} \quad n \to \infty.$$

And if you don't find that cool, I would argue you're studying the wrong subject at university ...

Chapter 16

Transients and damping

In many physical systems, there is competition between a restoring force and damping. In this chapter we consider the rich (and important) **responses** which can occur in such systems, even with a purely linear model.

16.1 Model set-up

A classic example is of course a car suspension system, where a shock absorber is designed to counteract the effects of a spring (with spring constant k) as shown schematically in figure 16.1. Applying Newton's second law to the mass M, subject to an externally imposed force $F(t)$, we find that

$$M\ddot{x} = F - kx - l\dot{x}. \tag{16.1}$$

Naturally, for this equation to be dimensionally consistent, the dimensions of k and l must be

$$[k] = \mathcal{M}\mathcal{T}^{-2};\ [l] = \mathcal{M}\mathcal{T}^{-1}.$$

Therefore, if we define the characteristic time scale of the system as

$$t_c = \sqrt{\frac{M}{k}},$$

we can define a **nondimensional** time

$$\tau = \frac{t}{t_c}.$$

Therefore, applying the chain rule to (16.1) we obtain

$$\frac{d^2}{d\tau^2}x(\tau) + 2\kappa\frac{d}{d\tau}x(\tau) + x(\tau) = f(\tau) = \frac{F(t)}{k};\ \kappa = \frac{l}{2\sqrt{kM}}, \tag{16.2}$$

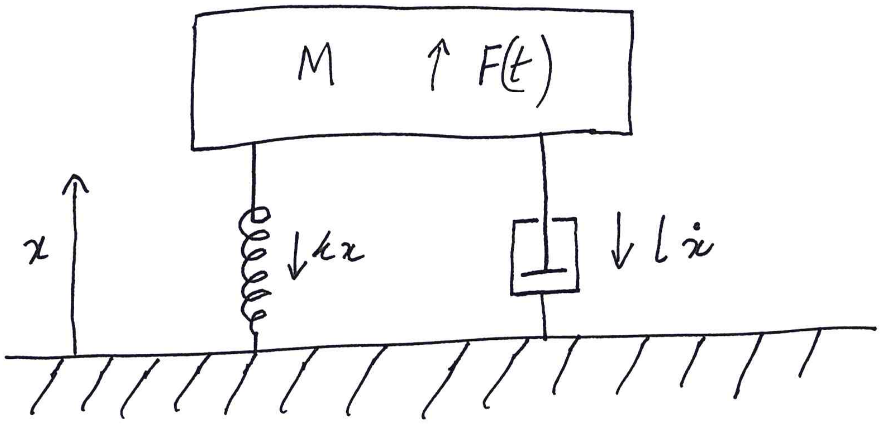

Figure 16.1: Schematic representation of the combined effect of a spring (with spring constant k for a displacement x, defined to be positive upwards) and a "shock absorber" dash-pot (with response proportional to the speed $\dot{x}$) on a mass M subject to a time-dependent force $F(t)$.

where both sides of the equation now have dimensions of length, and it is now clear that the single parameter κ determines the characteristics of the evolution of the system over the time scale t_c. As we shall see, the factor of 2 in the definition of κ is for convenience.

16.2 Free (natural) response

Let us first consider the **free** (i.e. unforced or **natural**) response of the system (16.2) in the absence of forcing, and so with $f(\tau) = 0$. Since equation (16.2) has constant coefficients we can straightforwardly construct the characteristic equation using the conventional ansatz $x = e^{\lambda\tau}$:

$$\lambda^2 + 2\kappa\lambda + 1 = 0 \to \lambda = \lambda_{1,2} = -\kappa \pm \sqrt{\kappa^2 - 1}. \tag{16.3}$$

There are then three qualitatively different responses, depending on the magnitude of κ.

16.2.1 $\kappa < 1$: Underdamping

If $\kappa < 1$, the square root in (16.3) is purely imaginary, and so the general form of the response is

$$x = e^{-\kappa\tau}\left(A \sin\sqrt{1-\kappa^2}t + B\cos\sqrt{1-\kappa^2}t\right).$$

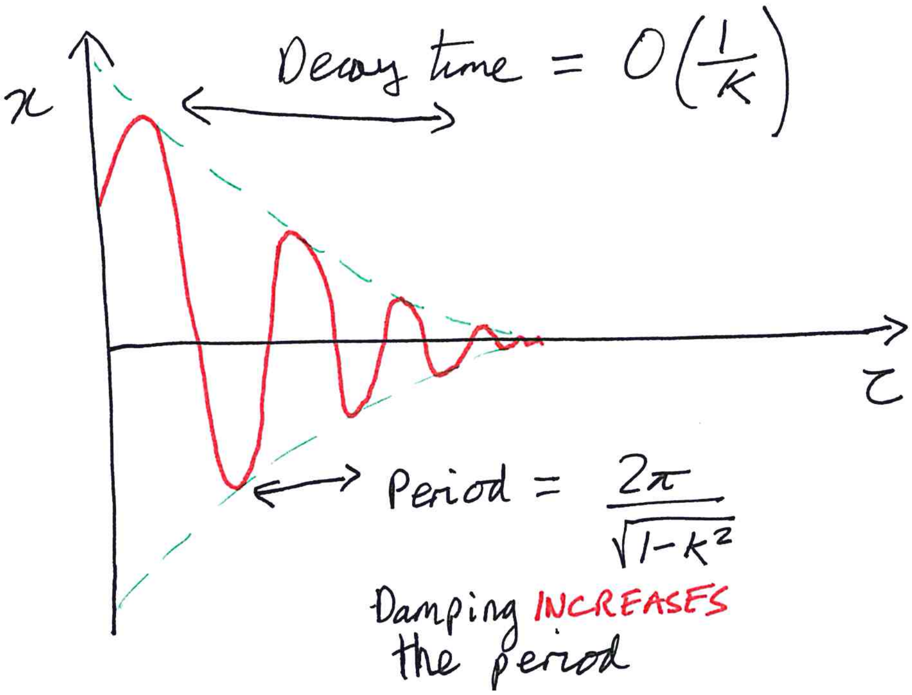

Figure 16.2: Schematic representation of the underdamped response when $\kappa < 1$.

The response is shown schematically in figure 16.2, and the system behaves as a **damped** oscillator, and the response is said to be **underdamped**. The system oscillates with period $2\pi/\sqrt{1-\kappa^2}$, while the amplitude decays on a characteristic e-folding nondimensional time $1/\kappa$. As $\kappa \to 1$, the oscillation period diverges to infinity.

16.2.2 $\kappa = 1$: Critical damping

If $\kappa = 1$ precisely, the square root in (16.3) is zero, there are degenerate eigenvalues, and so the general form of the response is

$$x = e^{-\kappa\tau}\left(A + B\tau\right) = e^{-\tau}\left(A + B\tau\right),$$

since $\kappa = 1$ here. The response is shown schematically in figure 16.3. Typically, the response grows over a characteristic time $O(1/\kappa) = O(1)$, and then decays over a similar time scale. The response is said to be **critically damped.**

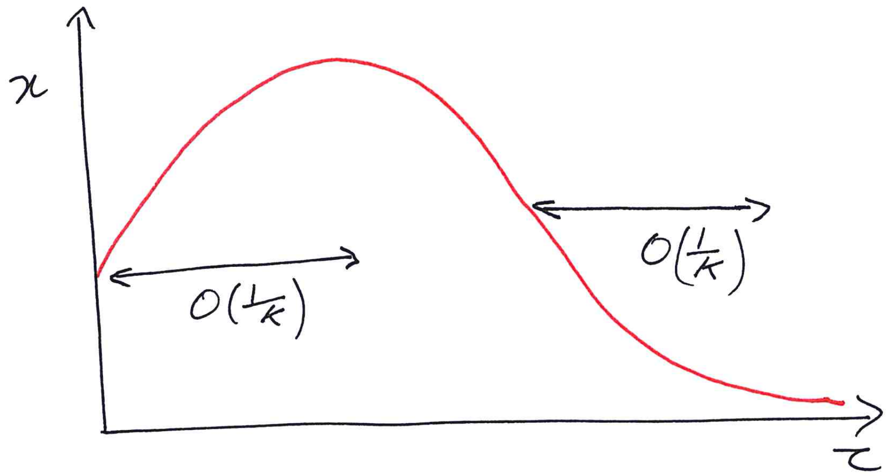

Figure 16.3: Schematic representation of the critically damped response when $\kappa = 1$.

16.2.3 $\kappa > 1$: Overdamping

Finally, if $\kappa > 1$, the square root in (16.3) is real, and there are two distinct negative eigenvalues, and so the general form of the response is

$$x = Ae^{\lambda_1 \tau} + Be^{\lambda_2 \tau}, \ \lambda_{1,2} = -\kappa \pm \sqrt{\kappa^2 - 1}.$$

Since

$$\left. \frac{dx}{d\tau} \right|_{t=0} = \lambda_1 A + \lambda_2 B,$$

if $B < 0$ and $|B| \simeq A$, then it is possible to have an early substantial **increase** in the response, as the magnitude of the term involving B will rapidly decay, on a time scale $O(1/\lambda_2)$. This response is shown schematically in figure 16.4. Typically, the response grows over a characteristic time $O(1/\lambda_2)$, and then decays over a longer time scale $O(1/\lambda_1)$ due to the ultimate decay of the term involving A. The response is said to be **overdamped.**

16.3 Forcing

In a forced system, the complementary function determines the short-time transient response, while the particular integral determines the long-time, asymptotic response.

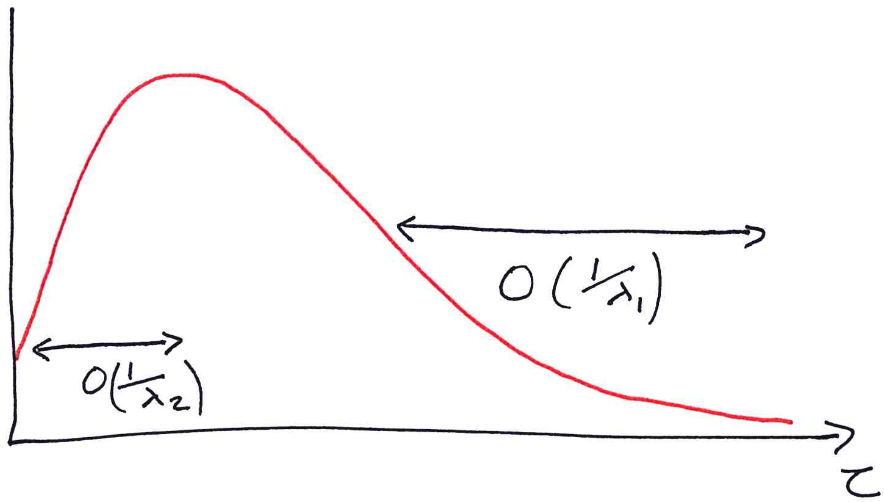

Figure 16.4: Schematic representation of the overdamped response when $\kappa > 1$.

Example

As an example consider

$$\ddot{x} + 2\kappa\dot{x} + x = \sin\tau, \;\; \kappa \neq 0.$$

As a particular integral, if we guess $x_p = C\sin\tau + D\cos\tau$, we find that $C = 0$ and $D = -1/(2\kappa)$. Therefore, since the real parts of both λ_1 and λ_2 are less than zero,

$$x = Ae^{\lambda_1\tau} + Be^{\lambda_2\tau} - \frac{1}{2\kappa}\cos\tau \to -\frac{1}{2\kappa}\cos\tau \to \text{ as } \;\; \tau \to \infty.$$

Note that the forced response is out of phase with the imposed forcing.

Chapter 17

Impulses and point forces

In this chapter, we introduce, in an informal way, the concept of **generalised functions** or more precisely **generalised functionals**, which prove very useful for description of impulsive or point forces, i.e. forces acting over very short time periods, or highly localised in space.

17.1 Impulse

Balls just can't help acting on impulse. Consider the situation shown in figure 17.1. A ball bounces on the ground, and is contact with the ground for a short time interval $T - \epsilon \leq T \leq T + \epsilon$, where ϵ is "small" relative to the total time interval of interest, i.e. $\epsilon \ll t_2 - t_1$. The force $F(t)$ exerted by the ground is thus a strongly peaked function centred on $t = T$.

We often are not particularly interested in the details of the structure of $F(t)$, but note that it only acts for a time interval of $O(\epsilon)$ which is much less than the overall time interval of interest, which is $O(t_2 - t_1)$. It is at least convenient mathematically to imagine that the force is **impulsive**, i.e. it is modelled as acting instantaneously at $t = T$, which naturally corresponds to taking the limit $\epsilon \to 0$.

Newton's second law for a ball of mass m bouncing purely vertically, where x is the positive vertical direction, takes the form:

$$m\ddot{x} = F(t) - mg.$$

$F(t)$ is only non-zero in the interval $[T - \epsilon, T + \epsilon]$, and so integrating from $T - \epsilon$ to $T + \epsilon$, we obtain

$$\int_{T-\epsilon}^{T+\epsilon} m\frac{d^2x}{dt^2}dt = \int_{T-\epsilon}^{T+\epsilon} F(t)dt - \int_{T-\epsilon}^{T+\epsilon} mgdt \to \left[m\frac{dx}{dt}\right]_{T-\epsilon}^{T+\epsilon} = I - 2mg\epsilon.$$

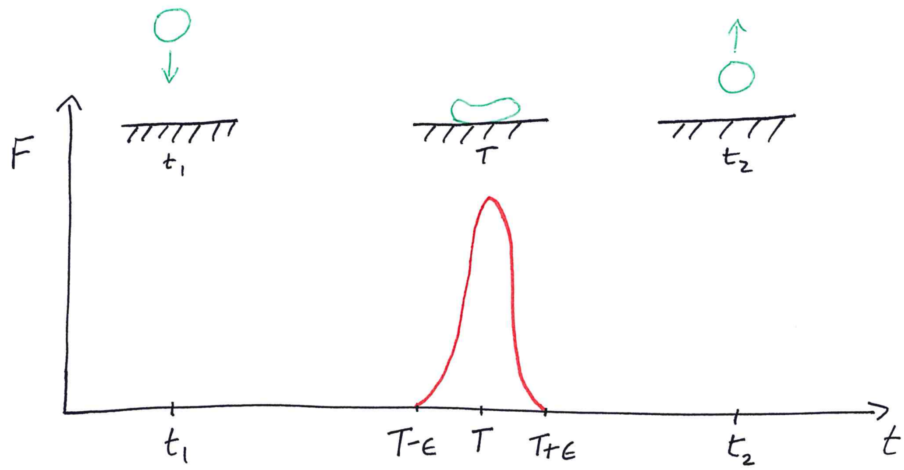

Figure 17.1: Schematic representation of: a ball bouncing on the ground; and the force $F(t)$ exerted on the ball by the ground when it is in contact with the ground for a short period centred on $t = T$.

The integral I, defined as

$$I = \int_{T-\epsilon}^{T+\epsilon} F(t)dt,$$

is the **impulse** of the force $F(t)$. The impulse is the area under the force curve, and is the only property of $F(t)$ that influences the macroscopic behaviour of the system.

If the contact time 2ϵ is small, we can neglect it, and then we obtain

$$\lim_{t\to T^+} m\frac{dx}{dt} - \lim_{t\to T^-} m\frac{dx}{dt} = \left[m\frac{dx}{dt}\right]_{T-}^{T+} = I,$$

The key concept to appreciate is that the only feature of the whole class of forces parameterised by ϵ $F(t;\epsilon)$ that is of interest is the area under the force curve, i.e. the **integral** of the force.

17.2 The Dirac $\delta-$function

Mathematically, we consider a family of functions $D(t;\epsilon)$ which have two key properties:

1. $$\lim_{\epsilon\to 0} D(t;\epsilon) = 0 \ \forall \ t \neq 0;$$

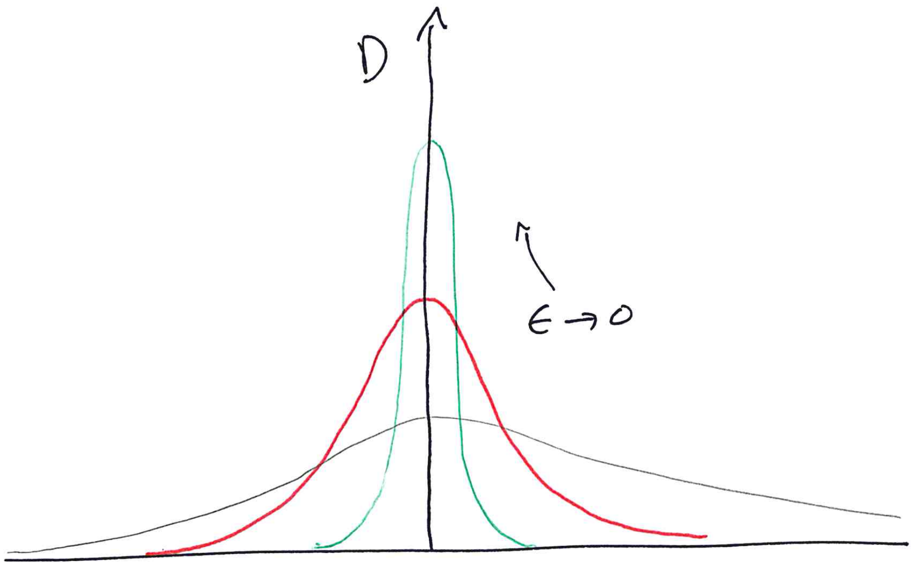

Figure 17.2: Schematic representation of family of functions $D(t;\epsilon) = \exp[-t^2/\epsilon^2]/(\epsilon\sqrt{\pi})$ as ϵ varies. Note how the width of the curve narrows, while its peak value increases.

2.
$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} D(t;\epsilon)dt = 1.$$

An example family:
$$D(t;\epsilon) = \frac{1}{\epsilon\sqrt{\pi}}e^{-t^2/\epsilon^2},$$
is shown schematically in figure 17.2. Note as ϵ decreases, the standard deviation or "width" of the gaussian gets narrower while the peak value gets larger in just such a way that the integral under the curve remains constant (and equal to one). is shown in figure 17.2.

Of course, the family of $D(t;\epsilon)$ having these two defining characteristics is not unique. Just two other examples are:

$$\begin{aligned} D(t;\epsilon) &= \frac{\sin\left(\frac{t}{\epsilon}\right)}{\pi t}, \\ D(t;\epsilon) &= \left\{\begin{array}{ll} \frac{1}{2\epsilon} & |x| < \epsilon \\ 0 & |x| \geq \epsilon \end{array}\right. \end{aligned}$$

Note that, for all these examples, as $\epsilon \to 0$, $D(0;\epsilon) \to \infty$, and so

$$\lim_{\epsilon\to 0} D(t;\epsilon)$$

is undefined.

Nevertheless, we crack on regardless and define the **Dirac $\delta-$function** as the "function"

$$\delta(x) \text{“=”} \lim_{\epsilon\to 0} D(x;\epsilon), \tag{17.1}$$

on the strict understanding that we only use its integral properties, i.e. when the $\delta-$function is multiplied by an appropriately well-behaved **test function** and integrated over an appropriate interval. Paul Dirac was a Lucasian professor initially from Bristol, and one of the greatest theoretical physicists of the 20th century, when there was a lot of competition around ... As an example, consider the **sampling property**. For all functions $g(x)$ which are continuous at $x = 0$,

$$\int_{-\infty}^{\infty} g(x)\delta(x)dx = \lim_{\epsilon\to 0} g(x)D(x;\epsilon)dx = g(0)\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} D(x;\epsilon)dx = g(0).$$

For any continuous test function, the $\delta-$function $\delta(x)$ samples its value at $x = 0$. Loosely (oh very loosely) the $\delta-$function skewers the value of g on a spike.

This property can actually be put on a rigorous and beautiful mathematical footing using the theory of **distributions**, as you will have the opportunity to discover in other courses. It also demonstrates that the $\delta-$function is actually a **functional** or a function of a function, as it effectively takes as input a function (continuous at the key point $x = 0$ for the unshifted $\delta-$function $\delta(x)$) and outputs a number, i.e. the value $g(0)$ of that function at the key point.

The concept of a $\delta-$function gives us a convenient way of representing and making calculations involving impulsive or point forces, particularly as $\delta-$functions can be manipulated using the various techniques of calculus, such as integration by parts etc. For example, the problem involving a bouncing ball starting from rest at $x = x_0 > 0$ at $t = 0$ can now be written in the compact form

$$m\ddot{x} = -mg + I\delta(t - T).$$

Exercise

1. Show that, for $t > T$:

$$x = \frac{g(T^2 - t^2)}{2} + \frac{I}{m}(t - T).$$

2. If the bounce is completely elastic, and so the **speed** immediately before and after the impact is the same, show that $I = 2mgT$, and thus that the ball returns to its initial height.

17.2.1 Properties of the $\delta-$function

The key properties of the $\delta-$function which we use in this course (always remembering that we only apply them when integrating with respect to test-functions) are:

1. $\delta(x) = 0 \ \forall x \neq 0$.

2. Using the first property and simple calculus substitutions involving changes of variable, we can obtain a more general from of the sampling property. For all test functions $g(x)$ which are continuous at $x = x_0$:

$$\begin{aligned} \int_a^b g(x)\delta(x - x_0)dx &= g(x_0) \ \text{ if } a < x_0 < b; \\ &= 0 \ \text{ if } x_0 < a \text{ or } x_0 > b. \end{aligned}$$

3. Therefore, for the special constant test function $g(x) = 1$, we obtain:

$$\int_{-\infty}^{\infty} \delta(x) = 1.$$

17.3 $\delta-$function forcing

For continuous functions $p(x)$ and $q(x)$, if $y(x)$ satisfies

$$y'' + p(x)y' + q(x)y = I\delta(x - x_0),$$

then $y(x)$ actually satisfies, for **both** $x < x_0$ and $x > x_0$,

$$y'' + p(x)y' + q(x)y = 0,$$

with the **jump conditions** that $y(x)$ is continuous at $x = x_0$ and $y'(x)$ has a "jump" of I, i.e.

$$[y]_{x_0^-}^{x_0^+} = 0; [y']_{x_0^-}^{x_0^+} = I.$$

The fact that $y(x)$ has to be continuous at $x = x_0$ can be established by contradiction. If it were discontinuous there, then the discontinuity in its second derivative would have to be "worse" than the $\delta-$function, (actually at least as bad as the derivative of the $\delta-$function, which is scary indeed) which is not possible from the differential equation.

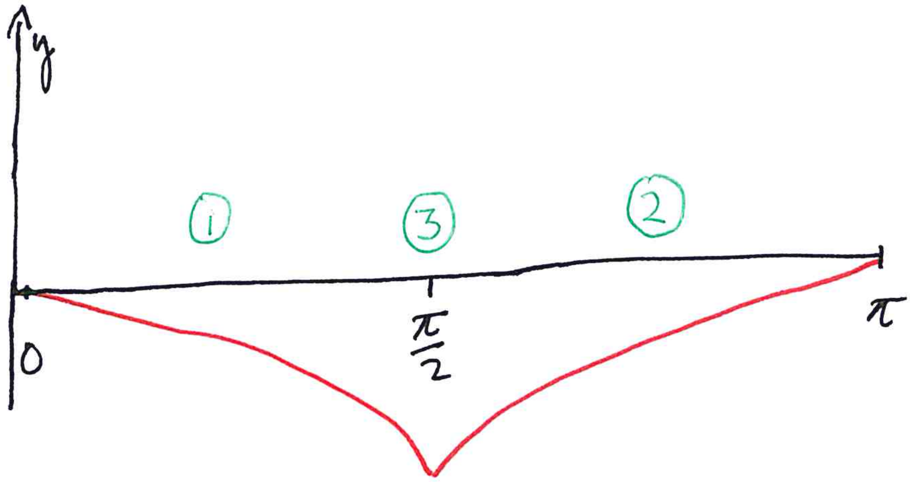

Figure 17.3: Schematic representation of the solution to the boundary value problem (17.2) showing the regions (1) and (2) and the point (3) where the jump conditions are applied. Note that $y(x)$ is continuous there, but there is a finite jump in the derivative.

Example

As ever, let us set ideas by considering an example, involving just such a point force. Consider the **boundary value problem** defined on $[-\pi/2, \pi/2]$:

$$y'' - y = 3\delta\left(x - \frac{\pi}{2}\right);\ y(0) = y(\pi) = 0. \tag{17.2}$$

This is a simple model of the tension in a surface, fixed at two ends and supported by springs subjected to a point force at the middle.

From the properties of the $\delta-$function listed above, it is sensible to solve the equation in regions (1) and (2) shown in figure 17.3, and then apply the jump conditions at (3), i.e. at $x = \pi/2$.

- Region (1): $0 \leq x < \pi/2$. Here

  $$y'' - y = 0.$$

  This is straightforward to solve in the form $y(x) = A_1 e^x + B_1 e^{-x}$ right? Well, yes, but it might have been better to stop and think about the boundary condition which has to be applied at $y = 0$, which is $y(0) = 0$. An entirely equivalent form for the solution in this region is $y = A \sinh x + B \cosh x$, and applying the boundary condition now implies that $B = 0$.

- Region (2): $\pi/2 < x \leq \pi$. Here, once again

  $$y'' - y = 0.$$

  In this case, it really is a good idea to choose the appropriate form for the solution to satisfy the boundary condition, and so we write the two linearly independent solutions as $y(x) = C\sinh(\pi - x) + D\cosh(\pi - x)$. Applying the boundary condition $y(\pi) = 0$ now reduces simply to $D = 0$. Using solutions of such forms is a **really** good idea for boundary value problems of this type, as it minimises risks of calculational slips.

- Matching conditions at (3). Requiring y to be continuous at $x = \pi/2$ implies that $A = C$. Applying the jump condition on the derivative, we obtain

  $$\left[\frac{dy}{dx}\right]_{\frac{\pi}{2}^-}^{\frac{\pi}{2}^+} = -A\cosh\frac{\pi}{2} - \left(A\cosh\frac{\pi}{2}\right) = 3 \rightarrow A = \frac{-3}{2\cosh\frac{\pi}{2}}.$$

Therefore, we have the complete solution

$$y = \begin{cases} -\frac{3\sinh x}{2\cosh\frac{\pi}{2}} & 0 \leq x < \frac{\pi}{2}, \\ -\frac{3\sinh(\pi - x)}{2\cosh\frac{\pi}{2}} & \frac{\pi}{2} < x \leq \pi, \end{cases}$$

as shown in figure 17.3. The distinctive discontinuity in the slope of the solution at $x = \pi/2$ is apparent, and the solution really looks like it is being poked by a point force at $x = \pi/2$. Once again, it might be good to remember this next year when considering Green's functions as a generic method for solving such forced boundary value problems.

17.4 Heaviside step function $H(x)$

The **Heaviside step function** (also a generalised function in truth) is defined as the **integral** of the $\delta-$function:

$$H(x) = \int_{-\infty}^{x} \delta(t)dt, \tag{17.3}$$

$$= \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \\ \text{undefined} & \text{at } x = 0. \end{cases} \tag{17.4}$$

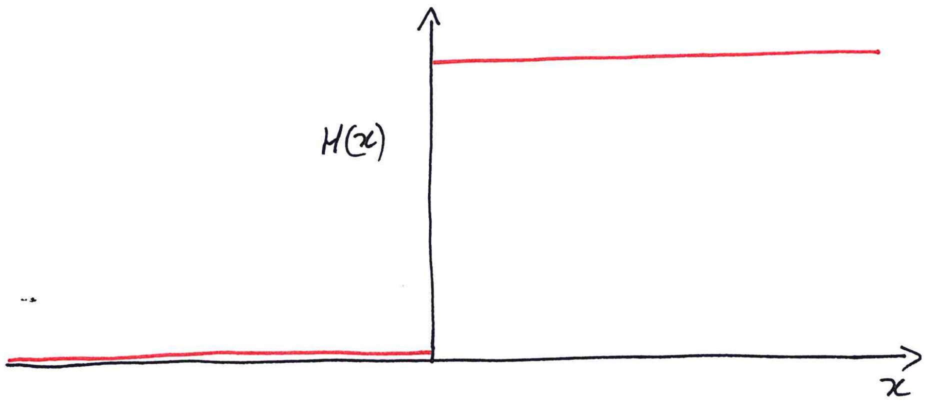

Figure 17.4: Schematic representation of the Heaviside step function $H(x)$. Note that the value at $x = 0$ is undefined.

The Heaviside function is shown schematically in figure 17.4. Note that the value at $x = 0$ is undefined. Furthermore, the fundamental theorem of calculus can be used to yield

$$\frac{dH}{dx} = \delta(x), \tag{17.5}$$

though it is always important to remember that the functions and relationships can only be used within integrals. This relationship between $H(x)$ and $\delta(x)$ illustrates nicely the smoothing effect of integration and the sharpening effect of differentiation. Indeed, the integral of the Heaviside step function is continuous at the origin, and just has a discontinuous first derivative. On the other hand, the derivative of the $\delta-$function gets seriously scary.

Example

The Heaviside step function is useful for switching problems. Consider the circuit showing schematically in figure 17.5. From Ohm's law, the voltage V_R across the resistor is the product of the current I and the resistance R. The voltage across the capacitor is the ratio of the charge Q to the capacitance C. Remembering that current is rate of change of charge by definition, throwing the switch at $t = 0$ we obtain

$$VH(t) = IR + \frac{Q}{C} = R\frac{dQ}{dt} + \frac{Q}{C} \to \dot{Q} + \frac{Q}{RC} = \frac{V}{R}H(t).$$

With an appropriate initial condition, this equation can be solved straightforwardly, noting that Q is continuous, but $\dot{Q}$ jumps by an amount V/R at t=0.

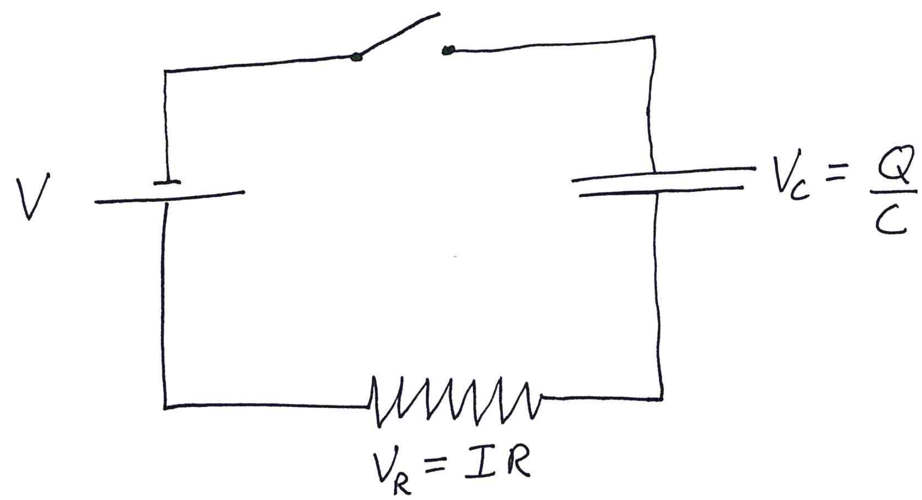

Figure 17.5: Schematic representation of a circuit with resistor with resistance R and capacitor with capacitance C. When the circuit is closed at $t = 0$, the imposed voltage V induces a current $I = dQ/dt$ where Q is the charge.

Chapter 18

Series solutions

In this chapter, we develop techniques to find series solutions to general second order differential equations, when we (perhaps) cannot find simple closed forms. We are always guaranteed that the general second order differential equation:

$$y'' + p(x)y' + q(x)y = 0,$$

has two linearly independent solutions. The exercise is to find them!

18.1 Classification of singular points

Construction of linearly independent solutions using series relies critically on the properties of so-called **singular** points. Consider an equation of the form

$$\alpha(x)y'' + \beta(x)y' + \gamma(x)y = 0. \tag{18.1}$$

The point $x = x_0$ is an **ordinary point** if

$$p(x) \equiv \frac{\beta(x)}{\alpha(x)} \text{ and } q(x) \equiv \frac{\gamma(x)}{\alpha(x)}$$

both have Taylor series about x_0. Otherwise x_0 is a **singular point**.

If x_0 is a singular point, but the equation can be written as

$$A(x)(x - x_0)^2 y'' + B(x)(x - x_0)y' + C(x)y = 0,$$

where

$$P(x) \equiv \frac{B(x)}{A(x)} \text{ and } Q(x) \equiv \frac{C(x)}{A(x)}$$

both have Taylor series about x_0 then x_0 is a **regular singular point**. Otherwise, it is a an **irregular singular point**. Loosely, for a regular singular point, the equation is singular, but not **too** singular, due to the properties of the derivative.

Example I

Consider

$$(1-x^2)y'' - 2xy' + 2y = 0.$$

For this equation, $x = 0$ is an ordinary point and $x = \pm 1$ are regular singular points.

Example II

Consider

$$\sin xy'' + \cos xy' + 2y = 0.$$

For this equation, $x = n\pi$ are regular singular points, while all other points are ordinary points.

18.1.1 Example III

Consider

$$\left(1+\sqrt{x}\right) y'' - 2xy' + 2y = 0.$$

For this equation $x = 0$ is an irregular singular point.

18.2 Fuchs' Theorem

This is the section that needs very careful pronunciation... If x_0 is an ordinary point of (18.1) then the equation has two linearly independent solutions of the form

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

i.e. in the form of a Taylor series, convergent in some neighbourhood of x_0.

Fuchs' theorem states that if x_0 is a regular singular point of the equation (18.1), then there exists at least one solution of the form

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+\sigma},$$

where σ is real, and $a_0 \neq 0$ by construction. The radius of convergence of this series is at least as large as the smaller radii of convergence of $p(x) = \beta(x)/\alpha(x)$ and $q(x) = \gamma(x)/\alpha(x)$.

18.3 Example: Ordinary point

Consider example 1:

$$(1-x^2)y'' - 2xy' + 2y = 0. \tag{18.2}$$

Write the equation as

$$(1-x^2)(x^2y'') - 2x^2(xy') + 2x^2y = 0.$$

The point $x = 0$ is an ordinary point, so try

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

Then, substituting this expression into the differential equation, we have:

$$\sum_{n=2}^{\infty} a_n \left[(1-x^2)n(n-1)\right] x^n - 2\sum_{n=1}^{\infty} a_n(x^2n)x^n + 2\sum_{n=0}^{\infty} a_n(x^2)x^n = 0.$$

Considering the general coefficient of x^n (for $n \geq 2$) yields the general recurrence relation

$$n(n-1)a_n + \left[-(n-2)(n-3) - 2(n-2) + 2\right] a_{n-2} = 0.$$

Note, because there were terms multiplied by x^2, that the term involving a_{n-2} can actually be multiplied by x^n. Also, since we are considering a Taylor series, we know that taking the second derivative definitely "kills" the terms a_0 and a_1, thus justifying the restriction to $n \geq 2$. We can now obtain the general recurrence relation (remember for $n \geq 2$) determining a_n in terms of a_{n-2}:

$$n(n-1)a_n = (n^2 - 3n)a_{n-2} \rightarrow a_n = \frac{(n-3)}{(n-1)}a_{n-2}.$$

We now deal with the special cases. The coefficients a_0 and a_1 are arbitrary, as they are not set by the recursion relation. Note from the recurrence relation $a_3 = 0$, and so $a_{2k+1} = 0$ for all $k \geq 1$. Considering the even coefficients in general:

$$a_n = \frac{(n-3)}{(n-1)}a_{n-2} = \frac{(n-3)}{(n-1)}\frac{(n-5)}{(n-3)}a_{n-4} \cdots = -\frac{1}{(2k-1)}a_0,$$

as terms alternately cancel. Therefore

$$\begin{aligned} y &= a_0\left[1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5}\dots\right] + a_1 x, \\ &= a_0\left[1 - \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)\right] + a_1 x, \end{aligned}$$

remembering that

$$\ln(1 \pm x) = \pm x - \frac{x^2}{2} \pm \frac{x^3}{3}\dots$$

Note that the logarithmic behaviour near $x = \pm 1$. We will return to that observation later.

18.4 Example: Generic indicial equation

Now consider the equation

$$4xy'' + 2(1 - x^2)y' - xy = 0. \tag{18.3}$$

For this equation, $x = 0$ is a regular singular point. Write the equation as

$$4(x^2 y'') + 2(1 - x^2)(xy') - x^2 y = 0,$$

and assume that

$$y = \sum_{n=0}^{\infty} a_n x^{n+\sigma},$$

with $a_0 \neq 0$ by construction.

Now, upon substitution, the equation becomes

$$\sum a_n\left[4(n+\sigma)(n+\sigma-1) + 2(1-x^2)(n+\sigma) - x^2\right]x^{n+\sigma} = 0, \tag{18.4}$$

where we have deliberately kept the lower limit of the summation arbitrary at this stage, since σ is as yet undetermined.

As before, considering the coefficient of $x^{n+\sigma}$, we obtain the general recurrence relation:

$$[4(n+\sigma)(n+\sigma-1) + 2(n+\sigma)]\,a_n + [-2(n+\sigma-2) - 1]\,a_{n-2} = 0, \tag{18.5}$$

which can be rearranged to obtain:

$$2(n+\sigma)(2n+2\sigma-1)a_n = (2n+2\sigma-3)a_{n-2}. \tag{18.6}$$

To determine σ, we consider the lowest power of x. In this case, that is the coefficient of $x^{\sigma-1}$ in the original equation (18.3) or equivalently x^σ in (18.4) after multiplication by x, corresponding to $n = 0$ for the first term on the left-hand side of the general recursion relation (18.5) involving a_0, which by construction is non-zero. Therefore, we obtain the **indicial equation**:

$$2\sigma(2\sigma - 1)a_0 = 0 \rightarrow \sigma = 0, 1/2.$$

In general the indicial equation is quadratic. If the roots are distinct, and do not differ by an integer, we are guaranteed that these two values of σ lead to two linearly independent solutions of the differential equation. Let us consider each of the solutions in turn.

18.4.1 Solution with $\sigma = 0$

For this specific case, the treatment of the solution with $\sigma = 0$ is a little subtle, since the lowest power of the equation surviving is clearly not x^{-1}. However, we still have by construction that $a_0 \neq 0$. In this case the lowest surviving power is x^0 from the term involving y', which implies

$$2a_1 = 0 \rightarrow a_1 = 0.$$

Since the recursion relation (18.6) relates a_n to a_{n-2}, we can then deduce that $a_n = 0$ for n odd. Considering finally the recursion relation for even $n = 2k$, we obtain

$$a_{2k} = \frac{4k-3}{4k(4k-1)} a_{2k-2},$$

and so

$$y_1 = a_0 \left[1 + \frac{x^2}{4.3} + \frac{5x^4}{8.7.4.3} + \ldots\right].$$

18.4.2 Solution with $\sigma = 1/2$

When $\sigma = 1/2$, the general recurrence relation (18.6) becomes

$$(2n+1)(2n)b_n = (2n-2)b_{n-2} \rightarrow b_n = \frac{(n-1)}{n(2n+1)} b_{n-2},$$

where we have relabelled the coefficients to avoid confusion between the two solutions.

For this solution the lowest power surviving is indeed $x^{-1/2}$ in the underlying equation (18.3), or equivalently $x^{1/2}$ in the "scaled" equation (18.4).

This corresponds in either case to $n = 0$ for the first term on the left-hand side of (18.5), and so b_0 is arbitrary (as of course expected). Considering now the term with $n = 1$, i.e. the coefficient of $x^{1/2}$ in (18.3) or the coefficient of $x^{3/2}$ in (18.4), we find that

$$2(1 + 1/2)(2 + 1 - 1)b_1 = 6b_1 = 0 \rightarrow b_1 = 0.$$

Once again therefore, only even coefficients are non-zero, and so

$$y_2 = b_0 x^{1/2} \left[1 + \frac{x^2}{2.5} + \frac{3x^4}{2.5.4.9} + \ldots \right].$$

So life is straightforward when the two roots of the indicial equation do not differ by an integer. If only that was always the case ...

Chapter 19

Second solutions

In this chapter we discuss the form of second series solutions for special cases in the indicial equation, where the roots of the indicial equation are either repeated, or differ by an integer. As ever, we will make liberal use of examples.

19.1 General behaviour near $x = x_0$

There are three different possible behaviours for different properties of the roots σ_1 and σ_2 of the indicial equation, where wlog $Re(\sigma_1) \geq Re(\sigma_2)$.

Case I: Non-integer $\sigma_1 - \sigma_2$

If $\sigma_1 - \sigma_2$ is not an integer, then there are two linearly independent solutions as considered in the previous chapter:

$$y = (x - x_0)^{\sigma_1} \sum_{n=0}^{\infty} a_n (x - x_0)^n + (x - x_0)^{\sigma_2} \sum_{n=0}^{\infty} b_n (x - x_0)^n,$$

where b_0 and a_0 are non-zero by construction.

Case II: Non-zero integer $\sigma_1 - \sigma_2$

If $\sigma_1 - \sigma_2$ is a non-zero integer, then there is one solution of the form

$$y_1 = (x - x_0)^{\sigma_1} \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

where $a_0 \neq 0$. Note this is for the **larger** root of the indicial equation. The other solution is of the form

$$\begin{aligned} y_2 &= (x-x_0)^{\sigma_2}\sum_{n=0}^{\infty} b_n(x-x_0)^n + c\ln(x-x_0)\sum_{n=0}^{\infty} a_n(x-x_0)^{n+\sigma_1} \\ &= cy_1\ln(x-x_0) + (x-x_0)^{\sigma_2}\sum_{n=0}^{\infty} b_n(x-x_0)^n, \end{aligned}$$

The coefficient c may or may not be zero depending on the particular properties of the recursion relation. Although there are apparently three arbitrary constants (a_0, b_0 and c) this is not actually the case. Provided c is nonzero, its value is set by the particular choice a_0 in the multiplying series solution y_1 and the recursion relation determining the coefficients $b_1, \ldots, b_{N-1}$ where $\sigma_1 - \sigma_2 = N$, effectively through the coefficient b_0, which is non-zero and arbitrary. Furthermore, in this specific case, varying the coefficient b_N, i.e. the coefficient of $(x-x_0)^{\sigma_2+N} = (x-x_0)^{\sigma_1}$, will vary this second solution y_2 by constant multiples of the first series solution y_1 associated with $\sigma = \sigma_1$. In summary, we effectively do have two free constants to scale the two linearly independent solutions.

Case III: $\sigma_2 = \sigma_1$

If the roots of the indicial equation are repeated, i.e. $\sigma_1 = \sigma_2 = \sigma$, then it is **guaranteed** that the constant c is nonzero, and wlog $c = 1$, and then there is one solution

$$y_1 = (x-x_0)^{\sigma}\sum_{n=0}^{\infty} a_n(x-x_0)^n,$$

where $a_0 \neq 0$, and there is a second solution

$$y_2 = y_1\ln(x-x_0) + (x-x_0)^{\sigma}\sum_{n=0}^{\infty} b_n(x-x_0)^n,$$

where (for interesting reasons beyond the remit of this course) $b_0 = 0$.

19.2 Example of Case II: Log solution

As ever, let's try and make the issues clearer by considering some examples. First, we consider an example where the roots of the indicial equation differ by an integer, and yet the recursion relation only allows one solution, therefore implying the existence of a logarithmic-type solution. Consider the equation

$$x^2y'' - xy = 0. \tag{19.1}$$

As usual, let us assume that

$$y = \sum_{n=0}^{\infty} a_n x^{n+\sigma},$$

where $a_0 \neq 0$ by construction. Therefore

$$\sum a_n x^{n+\sigma}[(n+\sigma)(n+\sigma-1) - x] = 0.$$

Considering the coefficient of $x^{n+\sigma}$, we require in general

$$(n+\sigma)(n+\sigma-1)a_n = a_{n-1}. \tag{19.2}$$

The lowest power of x is x^σ, whose coefficient yields the indicial equation:

$$\sigma(\sigma-1)a_0 = 0.$$

Since by construction $a_0 \neq 0$, we find two roots $\sigma_1 = 1$ and $\sigma_2 = 0$. We are guaranteed that there is a series solution for the root with larger real part, i.e. associated with $\sigma_1 = 1$. Remembering that a_0 is arbitrary, we then find for $n \geq 1$ substituting $\sigma = \sigma_1 = 1$ into (19.2):

$$a_n = \frac{1}{n(n+1)} a_{n-1} = \frac{a_0}{(n+1)(n!)^2},$$

and so

$$y_1 = a_0 x \left[1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144} \dots \right]. \tag{19.3}$$

Returning to (19.2) and substituting in $\sigma = \sigma_2 = 0$, we find that, for another series solution

$$n(n-1)b_n = b_{n-1},$$

with the requirement that $b_0 \neq 0$. This immediately runs into a contradiction for $n = 1$. When $n = 1$, b_0 must be zero, and so b_1 is arbitrary, i.e. the coefficient of x must be arbitrary, and so we just recover the first series solution y_1 defined by (19.3) associated with $\sigma_1 = 1$. Therefore, the second solution must take the form

$$y_2 = y_1 \ln x + \sum_{n=0}^{\infty} b_n x^n.$$

19.2.1 Construction of solution

NON-EXAMINABLE

Formally, we already have techniques to construct this solution, although there are other methods beyond the remit of this course which are typically better adapted to finding the solution. One in particular called the **derivative method** can often lead straightforwardly to a solution, but here we can rely on the statement of various theorems that the solution (up to a multiplicative constant) is unique, so whichever way we find the solution we have found it!

So, let's try the old warhorse of reduction of order, which is suspiciously of the right form since it involves constructing $y_2 = vy_1$ where v is a function to be determined, and hopefully will involve a logarithm ...

Assuming $y_2 = vy_1$, we obtain as before

$$y_2 = vy_1;\ y_2' = vy_1' + v'y_1;\ y_2'' = vy_1'' + 2v'y_1' + v''y_1.$$

Requiring y_2 to be a solution of (19.1) we find that

$$x^2(v''y_1 + 2v'y_1') + v(x^2y_1'' - xy_1) = 0.$$

Therefore, defining $u = v'$, we obtain

$$\frac{u'}{u} = -2\frac{y_1'}{y_1} \to \ln u = \ln y_1^{-2} + \ln B \to v' = \frac{B}{y_1^2},$$

for an arbitrary constant B. From the expression for y_1 (19.3),

$$\frac{dv}{dx} = \frac{B}{a_0^2x^2}\left[1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144}\dots\right]^{-2}. \tag{19.4}$$

There are three points we can immediately make.

1. Upon integrating this equation, an integration constant (i.e. $v = f(x) + B_v$) will just lead directly to adding multiples of the first series solution y_1 to y_2 since $y_2 = vy_1$ by construction.

2. Using the binomial theorem, (19.4) can be written as

$$\frac{dv}{dx} = \frac{B}{a_0^2}\left[\frac{1}{x^2} - \frac{1}{x} + \sum_{n=0}^{\infty} B_n x^n\right],$$

for some coefficients B_n which can be determined. Upon integration, we will thus obtain:

- a term of the form $1/x$ with an arbitrary constant $-B/a_0^2$, multiplying it;
- a term of the form $\ln x$ multiplied by a coefficient which is completely determined by fixing B;
- an infinite series of powers of x^n starting at $n = 1$.

3. When we multiply this form of v by y_1 we therefore have:

   - an arbitrary coefficient of x^0, i.e. $-B/a_0 = b_0$;
   - a term of the form $cy_1 \ln x$ where c is completely determined by the arbitrary constant b_0;
   - a series which we can label as

     $$\sum_{n=2}^{\infty} b_n x^n,$$

     since the integrated series is multiplied by y_1 which has a leading order term x.

Therefore, we have exactly the situation as described above, in that we have found a second series solution

$$y_2 = cy_1 \ln x + \sum_{n=0}^{\infty} b_n x^n, \tag{19.5}$$

where if $b_1 \neq 0$, then we have just added a multiple of y_1 to the infinite series in this expression, and c is not actually independent from b_0, but is fully determined by how we choose to scale y_1 and b_0. Isn't that pure dead brilliant?

19.3 Example of Case II: Two series solutions

In the above example, we saw that there was a contradiction in the recursion relation (19.2) when using the smaller value of σ, i.e. σ_2, to attempt to determine the coefficient of the power of x corresponding to the larger value σ_1. This contradiction does not necessarily occur if we get lucky with the recursion relation.

Consider the equation

$$xy'' - 2y' + xy = 0.$$

As usual, assuming that

$$y = \sum_{n=0}^{\infty} a_n x^{n+\sigma}, \;\; a_0 \neq 0,$$

we obtain

$$\sum \left[(n+\sigma)(n+\sigma-1) - 2(n+\sigma) + x^2\right] a_n x^{n+\sigma-1} = 0. \tag{19.6}$$

Considering the coefficient of the lowest power of x, i.e. $x^{\sigma-1}$, we require

$$\sigma(\sigma-1) - 2\sigma = 0 \rightarrow \sigma = 0, 3.$$

We are guaranteed to be able to find a series solution for the larger value of $\sigma = \sigma_1 = 3$, in which case the general recursion relation must take the form

$$\left[(n+3)(n+2) - 2(n+3)\right] a_n = -a_{n-2} \rightarrow a_n = -\frac{a_{n-2}}{n(n+3)}. \tag{19.7}$$

We have already considered the lowest power of x, establishing that $\sigma_1 = 3$ and a_0 is arbitrary. The other special case we need to consider is the coefficient of $x^{\sigma_1} = x^3$, corresponding to $n = 1$ in (19.6):

$$\left[(4)(3) - 2(4)\right] a_1 = 0 \rightarrow a_1 = 0. \tag{19.8}$$

Therefore all odd coefficients are zero, and so from the recursion relation (19.7), we can construct the first series solution:

$$y_1 = a_0 x^3 \left[1 - \frac{x^2}{10} + \ldots\right].$$

We now try to find a series solution for the smaller value of $\sigma = \sigma_2 = 0$, in which case the general recursion relation must take the form

$$\left[n(n-1) - 2n\right] b_n = -b_{n-2} \rightarrow [n(n-3)] b_n = -b_{n-2}. \tag{19.9}$$

From the indicial equation that established $\sigma_2 = 0$ we can assume that b_0 is arbitrary. The other special case we need to consider is the constant coefficient, i.e. the coefficient of x^0 corresponding to $n = 1$ in (19.6):

$$-2b_1 = 0 \rightarrow b_1 = 0.$$

However, we cannot immediately argue that all the odd coefficients are zero, because when $n = 3$, the left-hand side of (19.9) is **also** zero. This leads to no contradiction, but apparently b_3 is arbitrary. However, comparing (19.7) and (19.9), we see that there is an exact mapping possible of $b_3 \to a_0$, $b_5 \to a_2$ etc, as recursion relation (19.9) when shifted by three maps precisely to the recursion relation (19.7). This works perfectly since both b_3 and a_0 by construction are coefficients of x^3. Therefore, we have found a second linearly independent series solution,

$$y_2 = b_0 \left[1 + \frac{x^2}{2} + \dots \right],$$

precisely because we can avoid the "crisis" at $\sigma_2 + N = \sigma_1$. How cool is that!

19.4 Example of Case III: Double root

NON-EXAMINABLE

Now let us revisit equation (18.2) i.e.

$$(1 - x^2)y'' - 2xy' + 2y = 0,$$

where we found the general solution:

$$y = a_0 \left[1 - \frac{x}{2} \ln \left(\frac{1+x}{1-x} \right) \right] + a_1 x.$$

The logarithmic term $\ln(1 + x)$ is very suggestive, so let us make the substitution:

$$z = x + 1 \to x = z - 1, \; 1 - x = 2 - z, \; 1 - x^2 = z(2 - z),$$

and so we obtain the equation

$$z(2 - z)y'' - 2(z - 1)y' + 2y = 0.$$

Of course $z = 0$ is a regular singular point, and so making the usual assumption of the form of the series solution, we obtain

$$\sum a_n z^{n+\sigma-1} \left[(n + \sigma)(n + \sigma - 1)(2 - z) - 2(n + \sigma)(z - 1) + 2z \right] = 0.$$

Considering the lowest power of z with $n = 0$, i.e. $z^{\sigma-1}$, we find the indicial equation:

$$2\sigma(\sigma - 1) + 2\sigma = 0 \to \sigma^2 = 0,$$

which does indeed have a double root of $\sigma = 0$.

Exercise

Show that the series solution is $A_0(1 - z)$ for A_0 an arbitrary constant, and demonstrate that this is consistent with the discussion of this equation (18.2) in the previous chapter. If you are feeling brave, determine the other linearly independent solution, which clearly should involve a term of the form $(1 - z)\ln z$.