$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad (1.5)$$

Similarly, define

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$(1.6)$$

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \quad \text{all } z \in \mathbb{C}$$

From (1.5)

$$\exp(iz) = \sum_{n=0}^{\infty} \frac{i^n z^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{i^{2n+1} z^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{i^{2n} z^{2n}}{(2n)!}$$

$$= i \sin z + \cos z.$$

$$e^{iz} = \cos z + i\sin z \qquad (1.7).$$

Consider $\theta \in \mathbb{R}$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

(1.8).

1.3 Roots of Unity.

Solve $z^n = 1$, n integer >0.

polynomial of order n, $\therefore$ n roots

$$z^n = 1 = \exp(2\pi i k), \quad k = 0, 1, 2, \ldots$$

$$z = \exp\left(\frac{2\pi i k}{n}\right), \quad k = 0, 1, 2, \ldots$$

~~But~~,

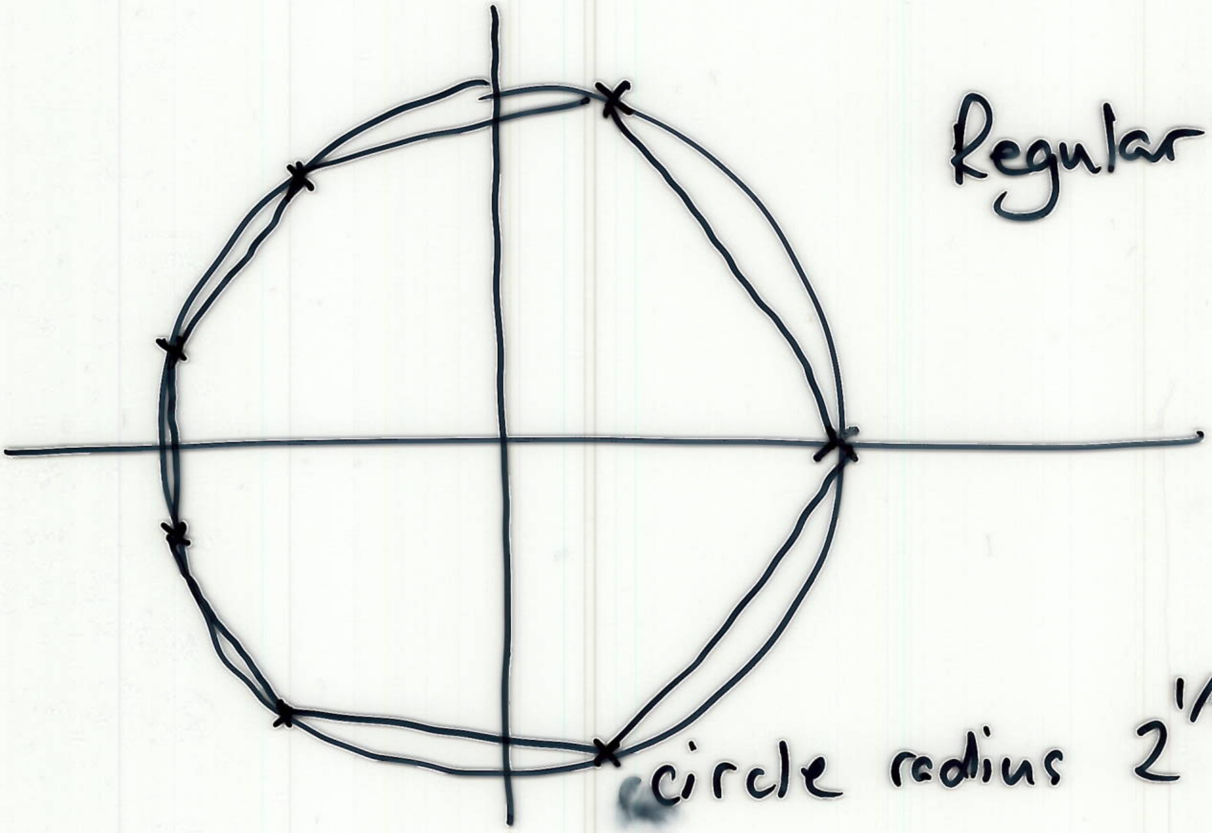
But, roots repeat after $k = n-1$

$$\therefore \quad z = \exp\left(\frac{2\pi i k}{n}\right) \quad k = 0, 1, \ldots n-1$$

<u>Example</u>

Solve $z^7 = 2$.

$$z = 2^{1/7} \exp\left(\frac{2\pi i k}{7}\right) \quad k = 0, 1, 2, 3, 4, 5, 6$$



Regular heptagon.

circle radius $2^{1/7}$

N.B. $z^n = 1$, write $\omega = \exp\left(\frac{2\pi i}{n}\right)$

roots $1, \omega, \omega^2, \ldots, \omega^{n-1}$

coeff. of z^{n-1} in polynomial is zero

$\therefore$ sum of roots $= 0$

$$\therefore \underline{1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0}$$

1.4 Complex logs and powers

The complex logarithm, $\log z$, is a solution w of

$$e^{w} = z, \quad \text{i.e. } w = \log z$$

so $e^{\log z} = z$, so $\exp(\cdot)$ and $\log(\cdot)$ are inverse functions.

Write $z = re^{i\theta}$ from (1.8)

$$\log z = \log(re^{i\theta}) = \log r + \underbrace{\log e^{i\theta}}_{i\theta}$$

$$\therefore \log z = \log|z| + i\arg(z)$$

But, recall can add $2n\pi$ onto θ

$$\therefore \log z = \log|z| + i\theta + 2in\pi \qquad \underline{\underline{(1.9)}}$$

$$n = 0, \pm 1, \pm 2, \ldots$$

Define a single-valued log by insisting $\arg(z)$ lies between $\pm\pi$

$$\underline{\log z = \log|z| + i\theta}, \quad -\pi < \theta \le \pi \qquad (1.10)$$

eg $\log(2i)$

$$= \log(2e^{i\pi/2})$$

$$= \underline{\log 2 + i\pi/2}.$$

The <u>complex power</u> z^α, with $z, \alpha \in \mathbb{C}$, is now defined by

$$z^\alpha = e^{\alpha \log z} \qquad (1.11)$$

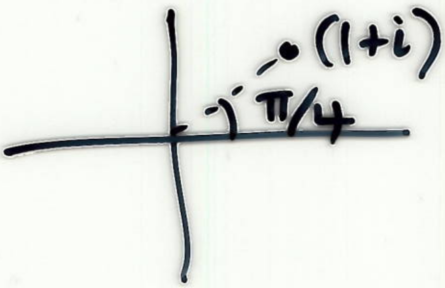
This is also multi-valued, from (1·9)

$$z^{\alpha} = e^{\alpha \log|z|} e^{i\alpha\theta} e^{2n\pi i\alpha}$$

$n = 0, \pm 1, \pm 2, \ldots$

If α is real and rational then only a finite number of different values.

<u>example</u> $\alpha = 1/2$, consider $(1+i)^{1/2}$



$|1+i| = \sqrt{2}$

$\arg(z) = \theta = \pi/4$

$$\therefore (1+i)^{1/2} = \exp\left(\tfrac{1}{2}\log\sqrt{2}\right) e^{i\pi/8} e^{n\pi i}$$

$n = 0, 1$
$n = 2$ repeats

$$\underline{2^{1/4} e^{i\pi/8} e^{n\pi i}}$$

But, if α is irrational, or complex, z^{α} can take infinitely many values.

Make z^{α} single-valued by insisting

$$-\pi < \arg(z) \leq \pi$$

as before.

<u>Example.</u>

$$i^{i} = \exp\left(i \log[e^{i\pi/2}]\right)$$

from (1.11)

$$= \exp\left(i \cdot i\pi/2\right)$$

$$\therefore \; i^{i} = e^{-\pi/2}$$

1.5 De Moivre's Theorem

$$\cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n \qquad (1.12).$$

Proof: for n integer, by induction.

$n=1$ obviously true.

Assume true for $n=k$,

$$(\cos\theta + i\sin\theta)^{k+1} = (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$$

$$= \cos(k+1)\theta + i\sin(k+1)\theta \quad \checkmark$$

for $n<0$ write $m=-n$, so $m>0$

$$(\cos\theta + i\sin\theta)^{-m} = (\cos m\theta + i\sin m\theta)^{-1}$$ by first part

$$= \cos m\theta - i\sin m\theta$$ using (1.1)

$$= \cos(-m\theta) + i\sin(-m\theta)$$

$$= \cos(n\theta) + i\sin(n\theta) \qquad \square$$

Example

$$\cos 5\theta + i\sin 5\theta$$

$$= (\cos\theta + i\sin\theta)^5$$

expand RHS using binomial,
equate real + imagenary parts
use $\cos^2\theta + \sin^2\theta = 1$

$\rightarrow$ ....

$$\cos 5\theta = 5\cos\theta - 20\cos^3\theta + 16\cos^5\theta$$

$$\sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$$