Chapter 1 : Fundamental concepts

We leave in a changing world
we often want to understand those changes
mathematically:

Ex: Body with a temperature $\Theta(t)$

Θ is the DEPENDENT VARIABLE

t is the INDEPENDENT VARIABLE

Θ is a <u>Function</u> of its argument t

In general Θ can change with t so
we need notation for RATE OF CHANGE

(ordinary) derivative : $\dfrac{d\Theta}{dt} = -k(\Theta - \Theta_0)$ $\quad k > 0$

EMPIRICALLY

3 obvious Q!

How is $\frac{d}{dt}$ defined?

How we got enough information to <u>solve</u> the equation

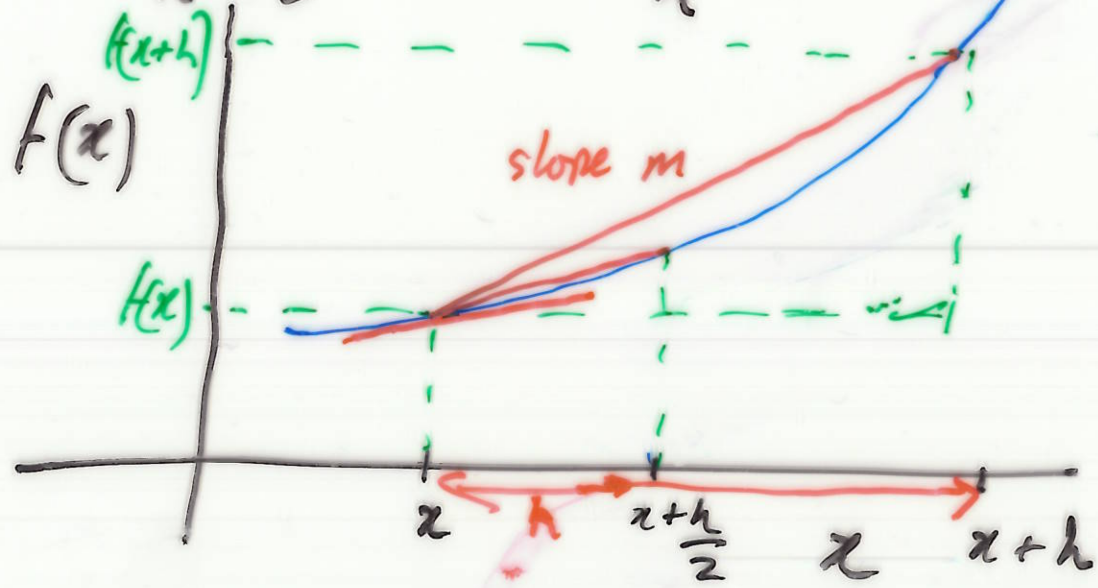
How would we solve <u>it</u>?

1.2 Preliminary definitions

Definition of a derivative.

Derivative of a function $f(x)$ w.r.t x is the function defined by the limit:

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \underline{1.1}$$



$f(x+h) = f(x) + \underline{m}\,h$

For the purposes of this discussion, a function is an object that takes an input (here a real number) and returns an output (another real number)

Essentially if we know x we can uniquely determine $f(x)$

Similarly if we know x, $f(x)$ we can use (1.1) to determine $\frac{df}{dx}$ at x.

ASSUMING THE LIMIT EXISTS.

NOTATION:

WE WISH TO distinguish between the function, defined over all its possible inputs (ie its DOMAIN) and the specific value for a particular input $x = x_0$

$$\left.\frac{df}{dx}\right|_{x=x_0} = \frac{df}{dx}(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} \qquad (1.2)$$

DIFFERENTIABILITY:

For the function $f(x)$ to be DIFFERENTIABLE.
and so for the function $\frac{df}{dx}$ to be well-defined
@ $x = x_0$

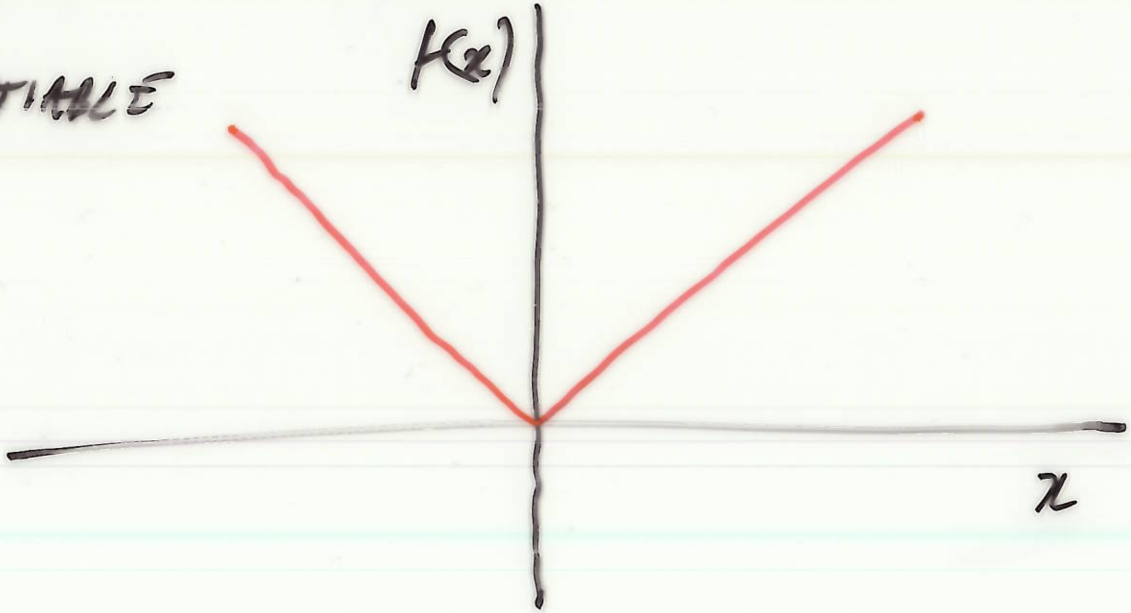
the left hand limit (ie $h<0$ approaching 0 from below)
= the right hand limit (ie $h>0$ approaching 0 from above)

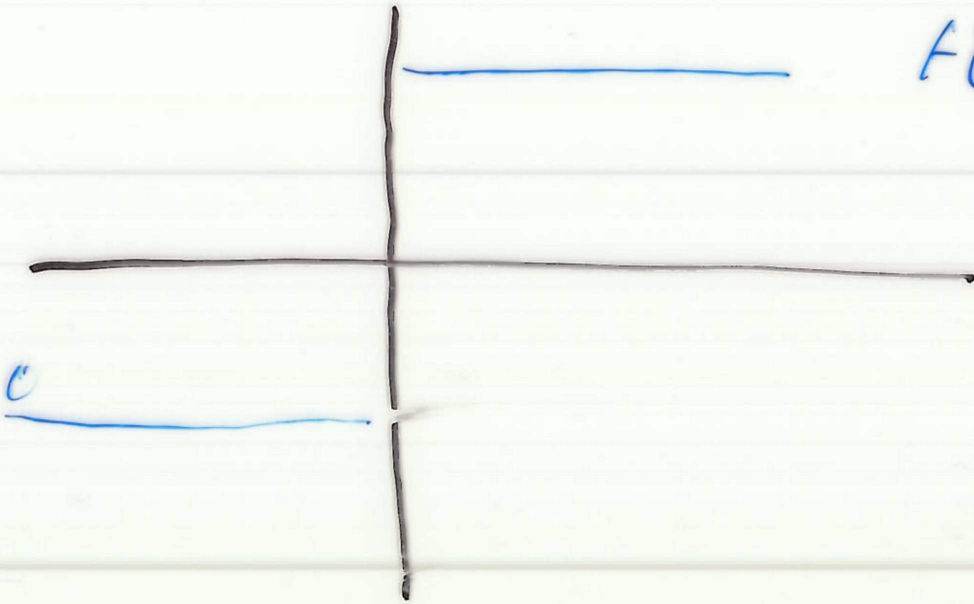
$$\lim_{h \to 0^-} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \to 0^+} \frac{f(x_0+h) - f(x_0)}{h}$$

This is actually a strong condition on the smoothness of a function.

$f(x) = |x|$

not DIFFERENTIABLE @ $x = 0$





$f'(x) = 1 \quad \forall x > 0$

$f'(x) = -1 \quad \forall x < 0$

NOTATION:

DERIVATIVE of $f(x)$ written as $\frac{df}{dx}$ is LEIBNIZ NOTATION

of $f(x)$ written as $f'(x)$ LAGRANGE'S NOTATION.

$f(t)$ written as $\dot{f}(t)$ NEWTON'S NOTATION.

FURTHERMORE, definitions can be written recursively provided of course that the higher derivatives are well-defined eg:

$$\frac{d}{dt}\left(\frac{df}{dt}\right) = \frac{d^2f}{dt^2} = f''(t) = \ddot{f}$$

Big O and little o notation.

These are ORDER parameters and are useful to give comparative scalings between functions close to some limiting point x_0

1 Definition o: $f(x)$ is $o[g(x)]$ as $x \to x_0$

if $$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$$

commonly written as $f(x) = o[g(x)]$

$f(x)$ belongs to the class of functions with this limiting property.

3 Definition of O

$F(x)$ is $O[G(x)]$ as $x \to x_0$ if

a) when $x_0 < \infty$ $\exists$ two positive finite constants M and δ s.t $\forall x$ with $|x - x_0| < \delta$

$$|F(x)| \leq M|G(x)|$$

b) when x_0 is ∞ $\exists$ 2 positive finite constants M and x_1 s.t $\forall x > x_1$

$$|F(x)| \leq M|G(x)|.$$

commonly written as $F(x) = O[G(x)]$

$F(x)$ belongs to the class of functions satisfying this limiting property

Observations:

1. $g(x) \to \infty$ as $x \to \infty$ (ie $x_0 = \infty$)

then $f(x)$ is definitely growing more slowly.

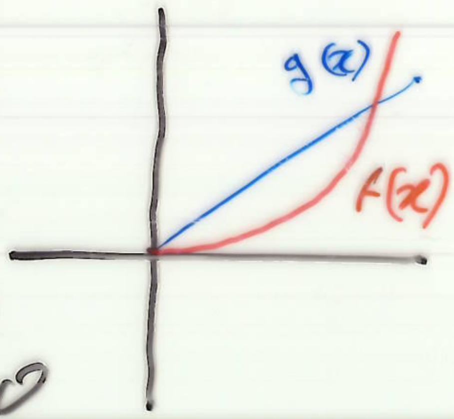
eg: $g(x) = x$, $f(x) = x^{\frac{1}{2}}$ $\Longrightarrow$ $f(x) = o[g(x)]$ as $x \to \infty$.

2. if $g(x) \to 0$ as $x \to 0$ (ie $x_0 = 0$)

then $f(x) \to 0$ more rapidly.

eg $g(x) = x$, $f(x) = x^2$ $\Longrightarrow$ $f(x) = o[g(x)]$

$f(x) = O[g(x)]$ as $x \to 0$



3 In the definition for O

if $G(x) \to \infty$ at some rate as $x \to x_0$

$F(x)$ can at most be going at a fixed multiple of that rate

$$G(x) = x \qquad\qquad F(x) = Mx \qquad \Rightarrow \qquad F(x) = O[G(x)] \text{ as } x \to \infty.$$

Note $f = o(g) \Longrightarrow f = O(g)$ as $x \to x_0$

But not vice versa.

4 ! if $G(x) \to 0$ as $x \to x_0$ $F(x)$ is definitely decaying to zero at least as quickly as a fixed multiple of $G(x)$

eg $G(x) = x$, $F(x) = Mx$ $\Rightarrow F = O[G(x)]$ as $x \to 0$

Note F IS <u>NOT</u> $o[G(x)]$ as $x \to 0$