Complex #'s [2]

Vectors [5]

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Eigenvalues + eigenvectors [9]

I Complex Numbers

1.1 Basic properties

A complex number $z \in \mathbb{C}$ is of the form $z = a + ib$,

$a, b \in \mathbb{R}$, $i = \sqrt{-1}$.

$a = \mathrm{Re}(z)$, $b = \mathrm{Im}(z)$

clearly $\mathbb{R} \subset \mathbb{C}$ $(z = a + i0)$
and $z = 0 + ib$ are imaginary.

Algebraic manipulation

$$z_1 \pm z_2 = (a_1 + ib_1) \pm (a_2 + ib_2)$$
$$= a_1 \pm a_2 + i(b_1 \pm b_2)$$

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2)$$

$$= a_1 a_2 - b_1 b_2 + i(b_1 a_2 + a_1 b_2).$$

so $\mathbb{C}$ is closed under addition + multiplication.

<u>Inverse</u>: $z^{-1} = \dfrac{1}{a+ib} = \dfrac{a-ib}{a^2+b^2}$ (1.1)

<u>Complex conjugate</u> of $z = a+ib$ is

$$\bar{z} = z^* = a - ib.$$

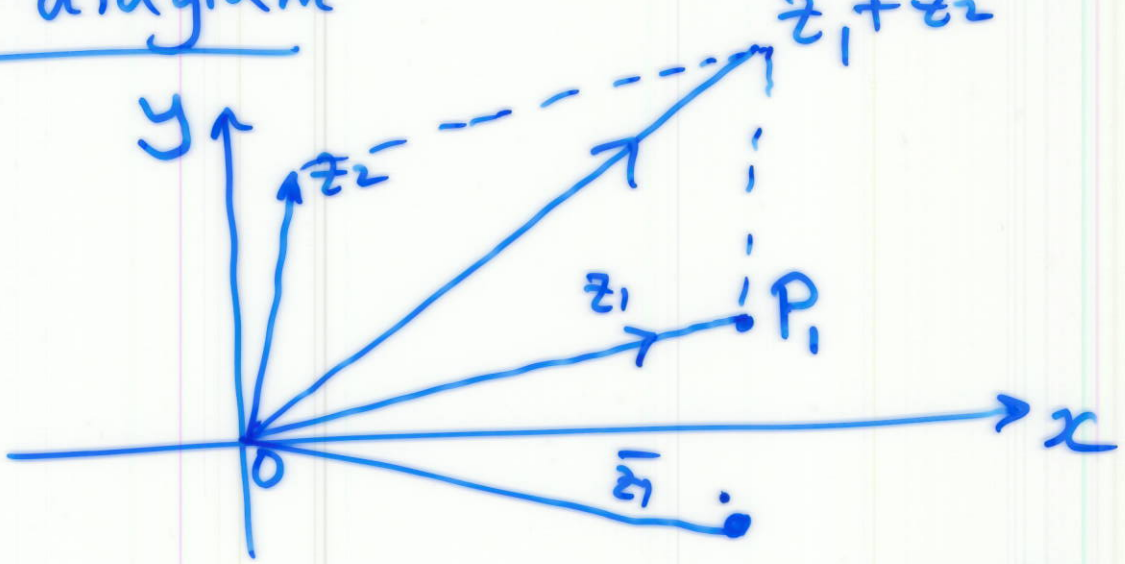
NB $\bar{\bar{z}} = z.$

<u>Modulus</u> $|z| = |a+ib| = (a^2+b^2)^{1/2}$

$$z\bar{z} = a^2 + b^2 = |z|^2.$$

$$\Rightarrow \quad z^{-1} = \frac{\bar{z}}{|z|^2}, \text{ i.e. } (1.1)$$

Argand diagram



$z_1 = x_1 + iy_1$, P_1, represented by vector $\overrightarrow{OP_1}$

$\bar{z}_1$ = reflection in x axis.

Addition $z_1 + z_2$ – "complete parallelogram".

For $z_1, z_2 \in \mathbb{C}$

$$|z_1 + z_2| \le |z_1| + |z_2| \qquad (1.2)$$

Triangle Inequality

Alternative form of (1.2)

$$|z_1 - z_2| \geqslant \Big|\, |z_1| - |z_2| \,\Big| \qquad (1.3)$$

Proof: write $z_1 = z_1' - z_2'$, $z_2 = z_2'$

$(1.2) \rightarrow \quad |z_1'| \leqslant |z_1' - z_2'| + |z_2'|$

$\rightarrow \quad |z_1' - z_2'| \geqslant |z_1'| - |z_2'| \qquad (a)$

swap 1 and 2 in (a)

LHS is unaffected, RHS $\rightarrow |z_2'| - |z_1'|$ (b)

Hence, $|z_1' - z_2'| \geqslant$ (a) and (b) [one of which is negative]

so modulus sign appears on RHS of (1.3).

$\square$

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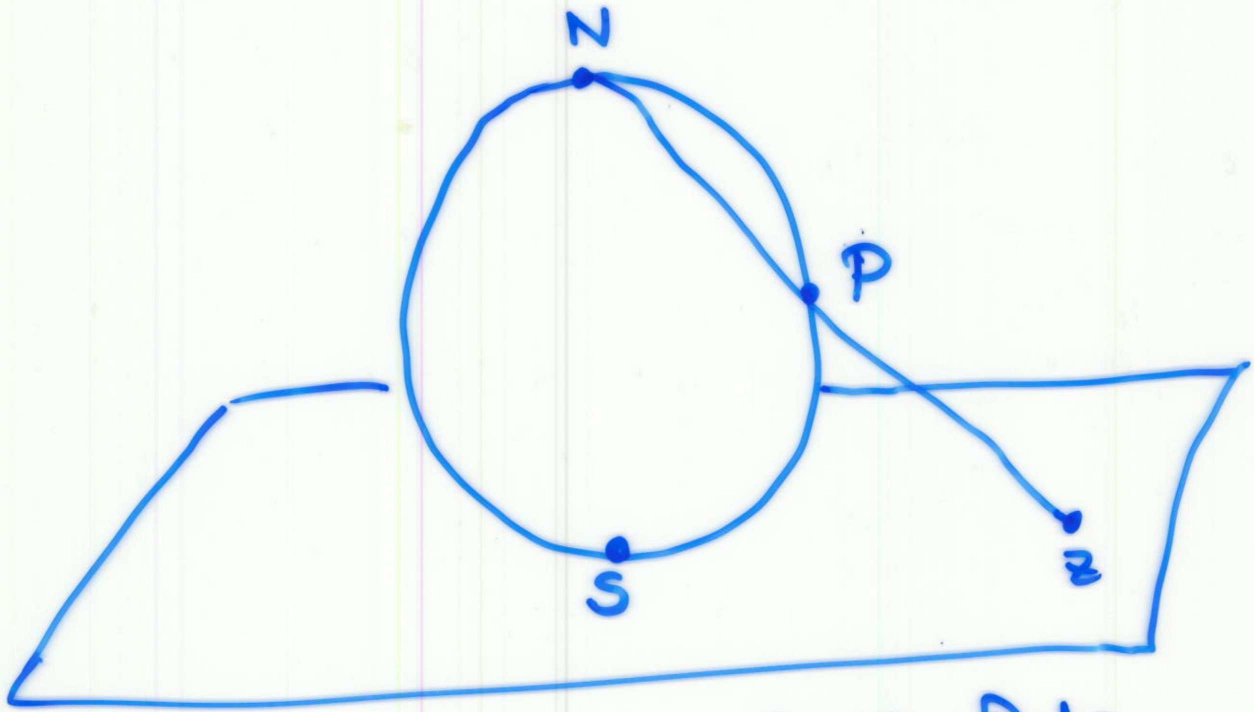
* Riemann sphere

often (not here) think about extended complex plane $\mathbb{C} \cup \{\infty\}$



Identify $z=0$ as South Pole
identify point P on sphere with z by projection
The North pole is identified with ∞.

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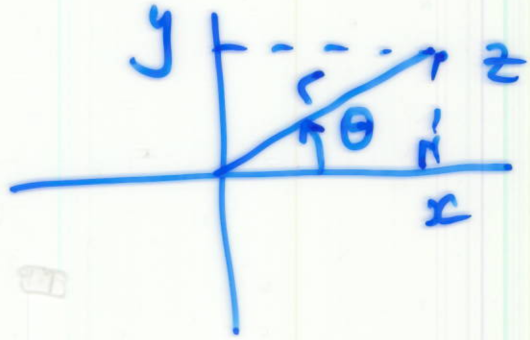
Polar representation



$z = x + iy$

$= r\cos\theta + ir\sin\theta$

$|z| = r$

If $r \neq 0$, $\theta = \tan^{-1}(y/x) = \arg(z)$

Argument.

Clearly $(r, \theta) \longrightarrow$ unique z

but $z \longrightarrow (r, \theta)$ is not because

can add $2n\pi$ to θ without changing z. (i.e rotate n times about 0).

Often restrict θ to principal value

$-\pi < \theta \leq \pi$

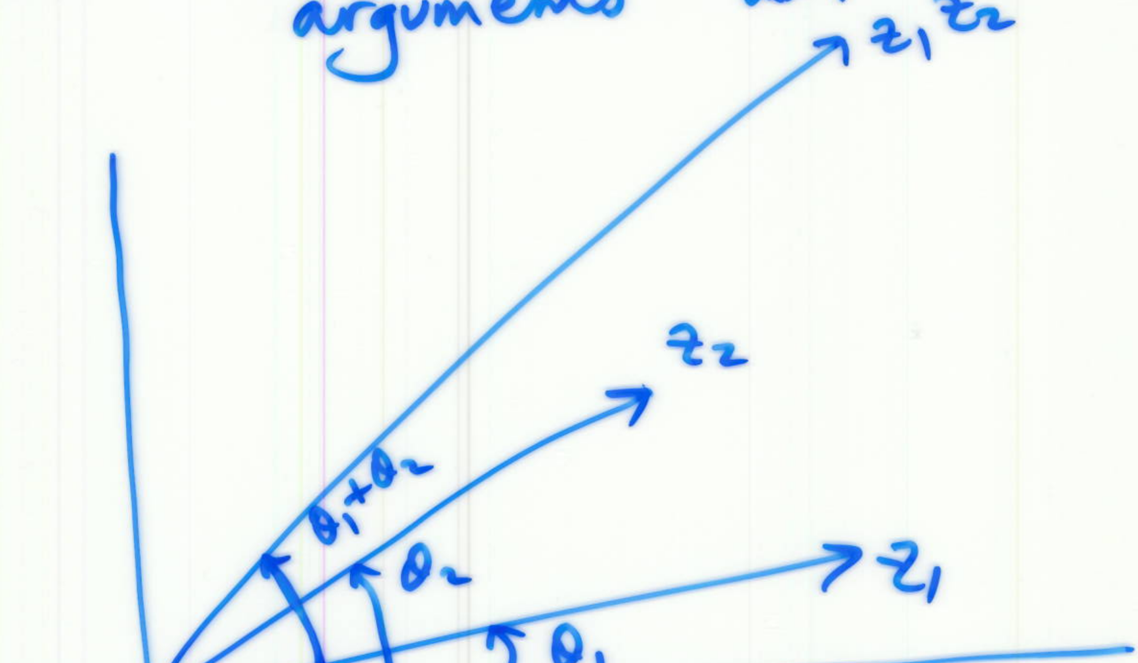
Geometrical representation of rotation

Have $\quad z_i = x_i + i y_i \quad i = 1, 2$

$$= r_i (\cos\theta_i + i \sin\theta_i)$$

$$z_1 z_2 = r_1 r_2 \left[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \right] \qquad (1.4)$$

i.e. moduli multiply, arguments add



1.2 Complex Exponential function

$$\exp(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad (1.5)$$

series converges for all $z \in \mathbb{C}$ (see Analysis I).

Multiplication:

$$\exp(z_1)\exp(z_2) = \exp(z_1 + z_2)$$

Proof: $\exp(z_1)\exp(z_2)$

$$= \sum_{m,n=0}^{\infty} \frac{z_1^m}{m!} \cdot \frac{z_2^n}{n!}$$

$$\begin{array}{lllll} & r=0 & r=1 & r=2 & r=3 \\ \displaystyle\sum_{m,n=0}^{\infty} = & a_{00} & + a_{01} & + a_{02} & + a_{03} + \cdots \\ & + a_{10} & + a_{11} & + a_{12} & + a_{13} + \cdots \\ & + a_{20} & + a_{21} & + a_{22} & + a_{23} + \cdots \\ & + a_{30} & + a_{31} & + a_{32} & + \cdots \end{array}$$

(convergence, see Analysis I)

$$= \sum_{r=0}^{\infty} \sum_{m=0}^{r} a_{r-m,m}$$

Hence, $\exp(z_1)\exp(z_2)$

$$= \sum_{r=0}^{\infty} \sum_{m=0}^{r} \frac{z_1^{r-m}}{(r-m)!} \frac{z_2^{m}}{m!}$$

$$= \sum_{r=0}^{\infty} \frac{1}{r!} \underbrace{\sum_{m=0}^{r} \frac{r!}{(r-m)!\,m!} z_1^{r-m} z_2^{m}}_{(z_1+z_2)^r,\ \text{Binomial}}$$

$$= \sum_{r=0}^{\infty} \frac{(z_1+z_2)^r}{r!}$$

$$= \exp(z_1+z_2). \quad \text{from (1.5).}$$