A1e **Vectors and Matrices: Study Sheet 1** Michaelmas 2010

Exercises on Complex Numbers and an Exercise on Summation

These examples are designed to dust off a few cobwebs and check familiarity with material that the lecturer will assume. The examples are not intended as part of the Examples Sheets for supervision.

1. For $z = a + ib$ and $z^{-1} = \frac{a-ib}{a^2+b^2}$ confirm that
$$z\,z^{-1} = 1 + i.0\,.$$

2. Confirm that

   (a)
$$\overline{\overline{z}} = z\,;$$

   (b)
$$\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}\,;$$

   (c)
$$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}\,;$$

   (d)
$$\overline{(z^{-1})} = (\overline{z})^{-1}\,.$$

3. Confirm that

   (a)
$$|z|^2 = z\,\overline{z}\,;$$

   (b)
$$z^{-1} = \frac{\overline{z}}{|z|^2}\,.$$

4. For $x, y \in \mathbb{C}$ give a geometric interpretation of x/y.

5. For $x, y \in \mathbb{C}$ and n a positive integer, show that
$$\log(xy) \quad = \quad \log(x) + \log(y)\,,$$
and
$$\log(x^n) \quad = \quad n\log(x)\,.$$
You may assume any property of the complex exponential function required, and that $\exp(\log(x)) = x$.

6. For non-zero $w, z \in \mathbb{C}$ show that if $z\bar{w} - \bar{z}w = 0$, then $z = \gamma w$ for some $\gamma \in \mathbb{R}$.

7. For positive integers m and n and coefficients $a_{ij} \in \mathbb{C}$, with $0 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant n$, show that
$$\sum_{p=0}^{m}\sum_{q=0}^{n} a_{pq} = \sum_{q=0}^{n}\sum_{p=0}^{m} a_{pq} = \sum_{r=0}^{m+n}\ \sum_{p=\max(0,r-n)}^{\min(r,m)} a_{p\ r-p} = \sum_{r=0}^{m+n}\ \sum_{q=\max(0,r-m)}^{\min(r,n)} a_{r-q\ q}\,. \tag{1}$$
By taking
$$a_{pq} = \frac{x^p y^q}{p!q!}\,, \quad \text{for } x, y \in \mathbb{C},$$
show that $\exp(x)\exp(y) = \exp(x+y)$ on the *assumption* that, for this choice of a_{pq}, (1) remains valid as $m \to \infty$ and $n \to \infty$.