

8 Discrete Mathematics (MPF)

- (a) Let $\#X$ denote the cardinality of a set X .

Define a unary predicate P for which the statement

$$\forall \text{ sets } X. [P(\#X) \iff (\forall \text{ sets } A, B. A \times X = B \times X \implies A = B)]$$

holds.

[1 mark]

Prove the statement for your given predicate P .

[4 marks]

- (b) For sets X, Y, Z , let $(X \Rightarrow Y)$ denote the set of functions from X to Y and let $\mathcal{P}(Z)$ be the powerset of Z .

For sets A and B , consider the function

$$(\cdot)^\sharp : (A \Rightarrow \mathcal{P}(B)) \longrightarrow (\mathcal{P}(A) \Rightarrow \mathcal{P}(B))$$

given, for all $f \in (A \Rightarrow \mathcal{P}(B))$ and $X \in \mathcal{P}(A)$, by

$$f^\sharp(X) = \bigcup_{a \in X} f(a)$$

Show that for all $g \in (\mathcal{P}(A) \Rightarrow \mathcal{P}(B))$ there exists $f \in (A \Rightarrow \mathcal{P}(B))$ such that $f^\sharp = g$ iff, for all $\mathcal{F} \subseteq \mathcal{P}(A)$, $g(\bigcup_{X \in \mathcal{F}} X) = \bigcup_{X \in \mathcal{F}} g(X)$. [6 marks]

- (c) For sets S and A , let $\text{Bij}(S, S)$ be the set of bijections from S to S , let $\text{Inj}(S, A)$ be the set of injections from S to A , and let $\mathcal{P}_S(A) = \{X \subseteq A \mid X \cong S\}$ be the set of subsets of A that are in bijection with S .

- (i) Prove that the relation $\approx \subseteq \text{Inj}(S, A) \times \text{Inj}(S, A)$ defined, for all $f, g \in \text{Inj}(S, A)$, by

$$f \approx g \iff \exists h \in \text{Bij}(S, S). f = g \circ h$$

is an equivalence relation.

[3 marks]

- (ii) Define a bijection

$$\text{Inj}(S, A) / \approx \longrightarrow \mathcal{P}_S(A)$$

You need not prove your function is bijective, but you should explain why your mapping is well defined. [6 marks]