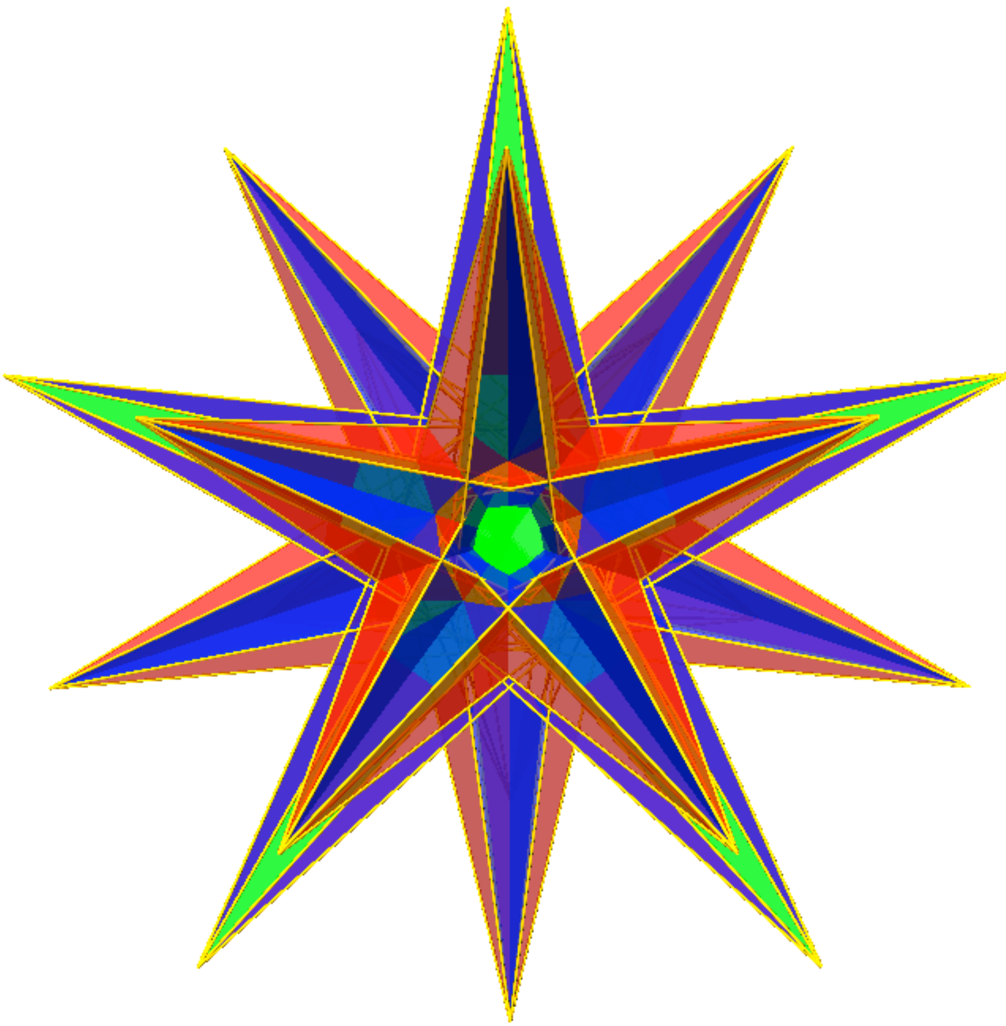
Edge-Transitive Polyhedra



Edge-Transitive Polyhedra

Gordon Collins

The nine regular polyhedra have been thoroughly studied. When we relax the conditions that define them, many more possibilities appear. Regular-faced vertex-transitive (uniform) ones have been completely enumerated. Face-transitive ones have been studied, but have not been enumerated. There are numerous infinite families of them.

There seems to be very little literature on edge-transitive (or "isotoxal") polyhedra, perhaps due to a misconception that they must also be face-transitive or vertex-transitive. This conclusion may be due to an implicit assumption that polyhedra must be topologically spherical (and thus satisfy Euler's formula), or that there cannot be equatorial faces, or perhaps simply that there cannot be star-shaped vertices. Yet the set of uniform polyhedra has been well known and accepted for decades and many of them violate all of these restrictions. As will be apparent below, accepting a definition that is sufficient to include the well-accepted uniform polyhedra results in a much richer set of isotoxal polyhedra, including some that appear to be new.

Many writers have treated polyhedra as solids and many more have been loose with such terminology. This, however, makes the Kepler-Poinsot objects irregular and eliminates all the non-convex uniform polyhedra as well. To retain those, it is necessary to consider polyhedra to be surfaces possibly with interpenetrating faces. Any enclosed region of space is not part of the object, just as a Klein Bottle, for example, has neither interior nor exterior.

Other writers take a very generous view, treating polyhedra as almost arbitrary collections of closed cycles of segments. Such "polypolygons" are interesting, but it is useful to consider them to be a separate category of mathematical objects.

The current work is a complete enumeration of edge-transitive polyhedra in Euclidean 3-space using a common and intuitive definition of "polyhedron", one that includes all of the commonly-accepted uniform ones. Briefly, a *polyhedron* is a finite, connected, closed surface, with no coincident elements (vertices, edge, or faces), decomposable into a finite set of faces. A *face* is a planar immersion of a disk (the question of holes does not arise in this context), whose interior is path-connected and whose boundary is a polygon having a finite number of sides and any rotation number[1]. Those sides meet in pairs at (linear) *edges*, where the normal to the surface changes. These are incident with exactly two faces and terminate at two *vertices*, which are shared with other edges. Vertices are distinct with valence at least 3. The faces incident with a vertex form a single circuit. This latter condition is not required of compounds. In general, more properties need to be considered in defining what to accept as a polyhedron. For this work, the constraints imposed by edge-transitivity eliminate many unconventional characteristics.

We find 47 individual polyhedra and 10 compounds that are edge-transitive, of which 11 are neither vertex-transitive nor face-transitive. They are listed below in Tables 8.1 through 8.4.

[1] Rotation number is often inaccurately referred to as "winding number". See [GS90] for a careful treatment of the difference.

1. The Enumeration

Transitivity of the edges under the symmetry group of a polyhedron constrains the vertices to be in at most two symmetry classes.

Suppose that all the vertices are in one class. This constrains them to be cospherical. The equal length of the edges makes the face angles equal and the faces regular. Therefore any such polyhedron will be uniform and at least quasi-regular. This set of polyhedra is well known and it is straightforward to determine the transitivity of the edges. All of the uniform polyhedra for which the vertices lie on rotation axes are isotoxal. They are listed in Table 8.3.

Suppose instead that the vertices are in two symmetry classes, A and B. Each edge must be incident with one vertex of each class.

Consider the symmetry operations that permute the edges at a given vertex *v* of Class A. *v* must be a fixed point of each of those operations. The only possibilities are a rotation about the radial containing *v* and a reflection in a plane through *v*. If there is no such rotation, there can be only one reflection. It will have only one other edge in its orbit. Since all vertices are at least trivalent, a rotation is needed to produce orbits that are sufficiently large.

Therefore, *v* must lie on an axis of rotation of the polyhedron's symmetry group. All other vertices of Class A must be similarly situated and at a common radius from the center of symmetry. The vertices of Class B must also lie on rotation axes and at a common radius. If either of these axes is 2-fold, there must also be a reflection through the axis so that the vertices there have valence greater than 2.

This constraint will drive the remainder of our enumeration. We proceed by first classifying the types of edges (in terms of symmetry group elements) that a vertex-intransitive isotoxal polyhedron can have. Then, for each rotational symmetry group, we identify possible faces that can be bounded by symmetrical sets of such edges and identify polyhedra by combining compatible sets of faces in ways that preserve edge transitivity and that close the surface.

The 3-axes and 2-axes of Tetrahedral symmetry are exactly the same as the 3-axes and 4-axes of Octahedral symmetry. For simplicity we will consider Tetrahedral symmetry in conjunction with Octahedral.

2. Edge Types

We refer to points on k-fold rotation axes as "k-sites" (or simply "sites"). We only consider positive radii, as this simplifies the process of identifying faces. We also generally treat each rotation axis as consisting of two separate rays. There is no operation in the Tetrahedral rotation group that reverses the direction of the 3-fold axes, although there is in T_i, the Tetrahedral group with inversion. As a result, it is often the case that the two rays of each such axis can be distinguished and there are two kinds of 3-sites. Points of the second kind will be referred to as 3'-sites. All k-sites of a given symmetry class that are used as vertices of an isotoxal polyhedron must of course have the same radius.

Pairs of k-sites can be distinguished by the angle between their rays. We do not need to consider the numerical size of the angles, but identify each distinct angle with a letter [2]. This produces symbols of the form "*Gpqr*" where

- G = letter denoting the rotational symmetry group
- p = value of k for Class A
- q = value of k for Class B
- r = sequential letter for the ray angle.

We always assign classes A and B so that $p >= q$. Each symbol denotes an *edge type*. All edges of an isotoxal polyhedron must of course be of the same type. (We will occasionally omit the initial letter when the symmetry group is implied by context.)

The edge types for each type of polyhedral symmetry group are:

Cyclic:	nn
Dihedral:	nn, n2, 22a, 22b, ...
Tetrahedral:	33a, 33'a, 33'b, 32a, 32b, 22a, 22b
Octahedral:	44a, 44b, 43a, 43b, 42a, ... 42c, 33a, ... 33c, 32a, ... 32c, 22a, ... 22d
Icosahedral:	55a, ... 55c, 53a, ... 53d, 52a, ... 52e, 33a, ... 33e, 32a, ... 32g, 22a, ... 22h

For each of these, we note the number of q-fold rays at angle r from a given p-fold ray, and the number of p-fold rays at angle r from a given q-fold ray. For example, for edge type O32b there are six 2-fold rays at the second smallest angle (b) from every 3-fold ray. This is the maximum valence that a vertex of that class can have in a polyhedron of that edge type. We eliminate from consideration any edge type for which either of the counts is less than 3, as that would result in divalent or univalent vertices. Note that this eliminates the Cyclic and Dihedral groups entirely.

The total number of possible edges is the product of the maximum valence and the number of rays of the given class. For most edge types, all of the possible edges must be present in a polyhedron of that type in order to preserve symmetry. There is an exception for types in which both of the maximum valences are even and at least 6. For such types there is a possibility of enantiomorphic pairs using half the total possible edges. This occurs for types I33b and I33c.

For the remaining edge types, these data are shown in Table 2.1. We will be most interested in the total number of face *sides* in the polyhedron, which is twice the total number of edges.

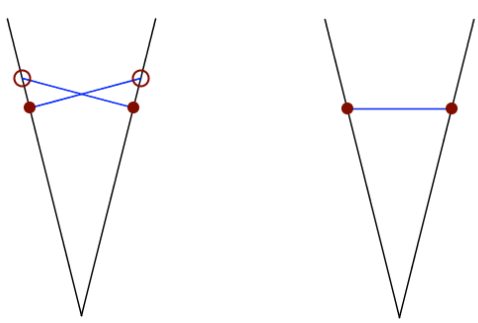
In all cases, each vertex class must contain all possible vertices, that is, one on each k-fold ray for the relevant k.

Because there are odd cycles in all three remaining rotation groups, polyhedra of edge types for which $p = q$ will have two vertices on each ray. Thus, edges cross in pairs as in Figure 2.1a. This is one of the most significant differences from vertex-transitive isotoxal polyhedra, which have but a single edge between any two rays.

[2] See [Br74] for a similar identification.

Table 2.1: Edge Types

Group	Edge Type	Maximum Valence		Number of Rays		Total Edges
		Class A	Class B	Class A	Class B	
T	33a	3	3	4	4	12
T	33'a	3	3	4	4	12
T	22a	4	4	6	6	24
O	44a	4	4	6	6	24
O	43a	4	3	6	8	24
O	43b	4	3	6	8	24
O	33a	3	3	8	8	24
O	33b	3	3	8	8	24
O	32b	6	4	8	12	48
O	22a	4	4	12	12	48
O	22c	4	4	12	12	48
I	55a	5	5	12	12	60
I	55b	5	5	12	12	60
I	53a	5	3	12	20	60
I	53b	5	3	12	20	60
I	53c	5	3	12	20	60
I	53d	5	3	12	20	60
I	52c	10	4	12	30	120
I	33a	3	3	20	20	60
I	33b	6	6	20	20	120 or 60
I	33c	6	6	20	20	120 or 60
I	33d	3	3	20	20	60
I	32b	6	4	20	30	120
I	32d	6	4	20	30	120
I	32f	6	4	20	30	120
I	22a	4	4	30	30	120
I	22b	4	4	30	30	120
I	22c	4	4	30	30	120
I	22d	4	4	30	30	120
I	22e	4	4	30	30	120
I	22f	4	4	30	30	120
I	22g	4	4	30	30	120



a. vertex-intransitive b. vertex-transitive

Figure 2.1: Edge configurations in isotoxal polyhedra when $p = q$

3. Face Types

For a given face, the vertices of either class must lie in a circle that is centered on the point of the face plane nearest the center of symmetry of the given polyhedron.

The edges connect vertices that lie alternately on 2 concentric (possibly coincident) circles. This produces isotoxal faces. Their borders comprise a family of polygons that seems to have received little attention in the literature, especially compared to the interest shown in isogonal ones. Edmund Hess [He74] and Max Brückner [Br00] described their metrical properties in detail but did not touch upon the geometric alignments that characterize the subtypes we define. The most extensive modern treatment appears to be that in which Branko Grünbaum displayed nearly every subtype of isotoxal 14-gon, but did not label or identify them [Gr94]. They were presented primarily as duals of types of isogonal polygons. The notation used by Grünbaum and G. C. Shephard [GS86], oddly, halves the number of vertices.

We use the notation "[m/d]" (or simply "[m]" when $d = 1$) to denote a polygon having m edges and rotation number (or density) d or a face having such a polygon as its boundary. This is similar to the "{m/d}" notation commonly used for regular polygons, with the difference that when the greatest common factor $(m, d) = h$ is greater than 1 this refers to an m-gon rather than to a compound of h (m/h)-gons. A face whose border is such a polygon is a surface that can be understood as made up of triangular patches, each defined by an edge and the center of the face, stitched together along their other sides. Star-polygon faces have density greater than 1, with a branch point at the center of the face.

When $m = 4$, one obtains a rhombus or square. We refer to the latter as a "regular [4]" for consistency. Table 3.1 shows the various types and subtypes of isotoxal faces for $6 <= m <= 10$. An *aligned* face has lines containing 3 or 4 non-consecutive vertices. The *overlapped* [10/3] is a special case in which some edges partially overlap. To clarify its structure, Table 3.1 includes a representation of it with its edges shifted slightly. Each inner vertex is incident with two edges and also lies on two other edges that pass through the same location in space but are as distant from the vertex as are two layers of a Riemann surface. We can think of no good reason to reject such an object as a face for polyhedra that are surfaces, even though the polyhedra that contain them can be difficult to distinguish visually from simpler ones. *Degenerate* faces, on the other hand, have sets of three consecutive collinear vertices. They are essentially $m/2$-gons, require divalent vertices, and will not be considered. Appendix 2 has more details on the issues they raise.

There are isotoxal faces for each combination of even m and $d < m/2$ for which $(d, m/2) = 1$. For other combinations one cannot form isotoxal faces without retracing edges. As m increases, more subtypes arise than are identified here.

Table 3.1: Isotoxal Face Types and Subtypes

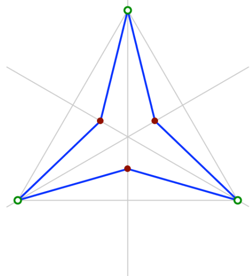
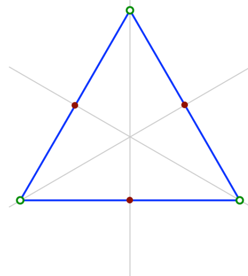
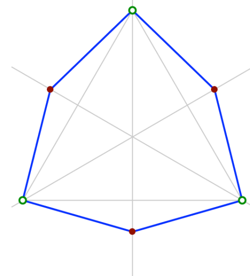
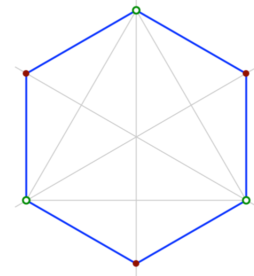
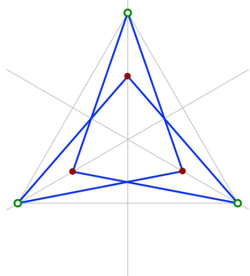
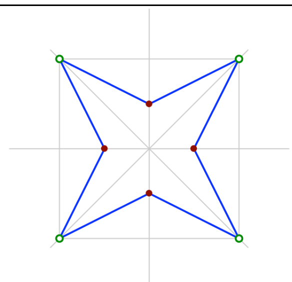
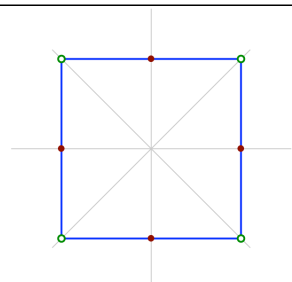
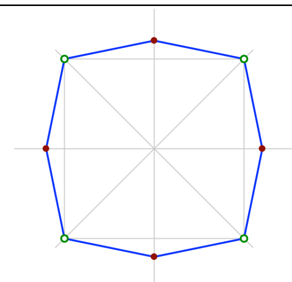
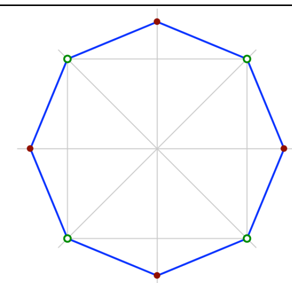
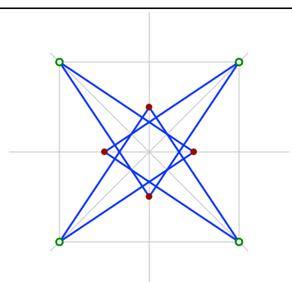
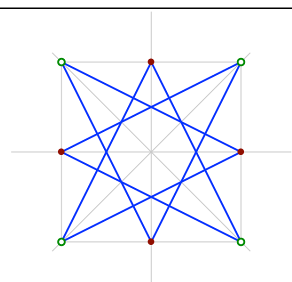
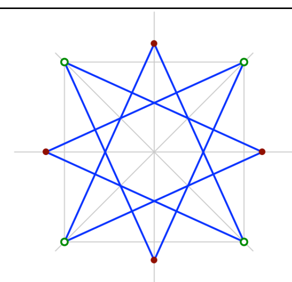
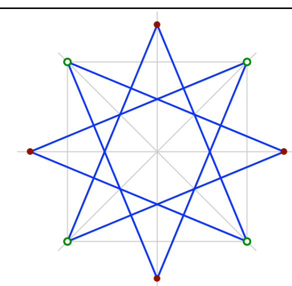
Face Type	Face Subtypes			
[6]	 inner	 degenerate	 outer	 regular
[6/2]				
[8]	 inner	 degenerate	 outer	 regular
[8/3]	 inner	 aligned	 outer	 regular

Table 3-1: Isotoxal Face Types and Subtypes, continued

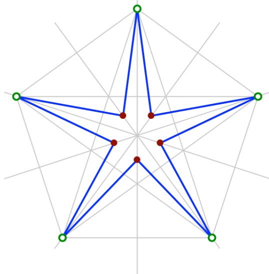
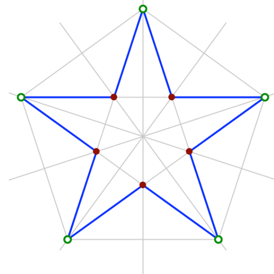
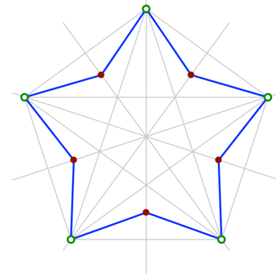
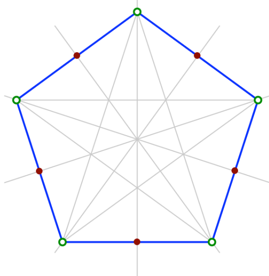
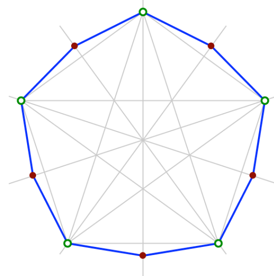
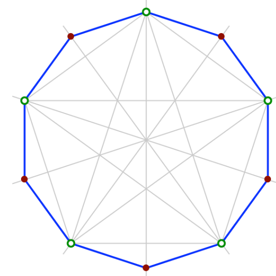
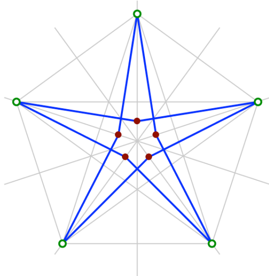
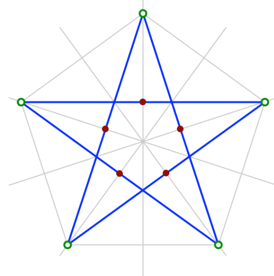
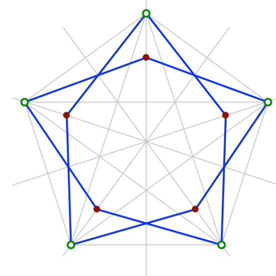
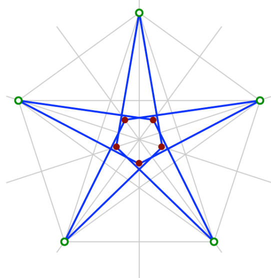
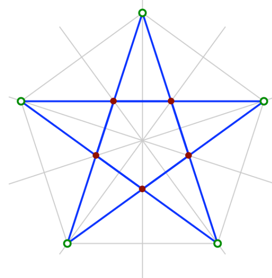
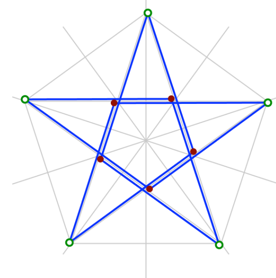
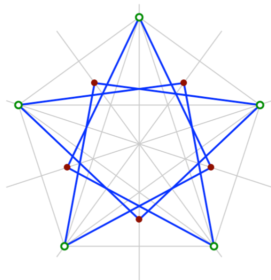
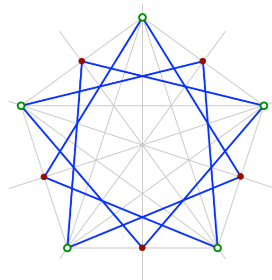
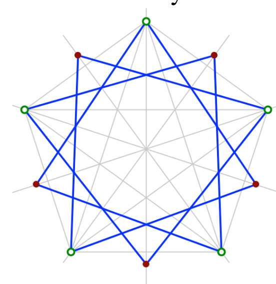
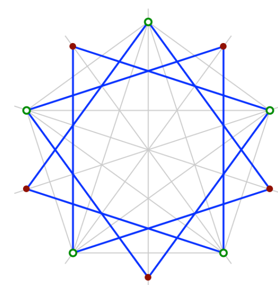
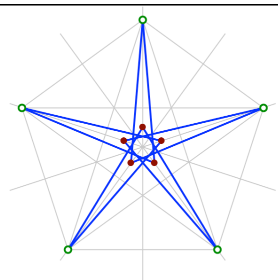
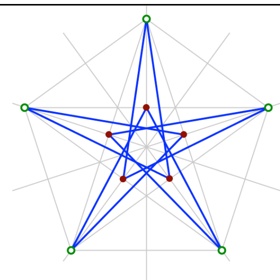
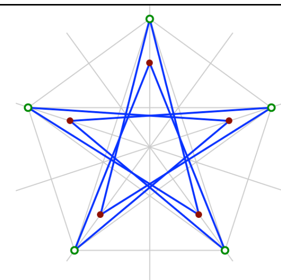
Face Type	Face Subtypes		
[10]	 inner	 aligned	 medial
	 degenerate	 outer	 regular
[10/2]	 inner	 degenerate	 outer

Table 3-1: Isotoxal Face Types and Subtypes, continued

Face Type	Face Subtypes			
[10/3]	 inner	 overlapped	 overlapped with edges shifted to clarify structure	
	 medial	 aligned	 outer	 regular
[10/4]	 inner	 aligned	 outer	

4. Identifying Faces

We accept faces that contain the center of symmetry of the polyhedron, as is common among the uniform polyhedra. These *equatorial* faces lie in planes that contain entire rotation axes. That makes their proportions arbitrary and leads to very different treatment in the enumeration.

We place the center of symmetry at the origin of $\mathbb{R}^3$ and fix the rotation axes.

For a given face, the vertices of each class must define a regular *n*-gon ($n >= 2$), arranged as described at the beginning of the previous section. We will refer to such collections of vertices as "*n-sets*". Each *n*-set is parameterized by a scaling factor that determines the actual radius of its vertices. The two *n*-sets that define a face must be coplanar and be rotationally aligned to produce isotoxal symmetry. When this happens we say that they *match*. Matching *n*-sets will have the same normal vector and contain the same number of *k*-sites. Their values of *k* may differ.

For non-equatorial digons, the normal is defined as a ray from the origin through the center of the digon. Equatorial rhombi are a special case, since equatorial digons have no normal direction. For those, each digon is a pair of sites on opposite rays of a rotation axis and matching *n*-sets will be pairs of sites on any axes perpendicular to the given one. The resulting rhombi will have well-defined normals (up to sign).

The *n*-sets of a given non-equatorial face have radii that are in a fixed ratio that is determined by the geometry of the intersections of the rotation axes and the face plane. The ratio that is relevant is not that of radii from the center of the face, but that from the center of symmetry of the polyhedron. This is part of what determines whether two sets of faces are compatible, that is, can be part of the same polyhedron. Equatorial faces can be adjusted to any ratio to make them compatible with other face sets.

For any face, we define the *radii ratio* as the radius of Class A vertices divided by the radius of the Class B vertices. When $p = q$, we arbitrarily assign Class A to the set of vertices with the greater radius. The radii cannot be equal in such cases or the two classes of vertices would coincide. This makes all ratios for such edge types greater than 1 and simplifies the determination of compatible faces.

Equatorial *n*-sets can only match other equatorial ones. Note than an equatorial *n*-set for odd *n* can match itself, but the radii must be different or vertices will coincide.

For each of the Octahedral and Icosahedral symmetry groups, we proceed as follows:

1) Identify equatorial digons and match perpendicular pairs of them. These matches produce equatorial rhombi.

2) Identify the other *n*-sets.

3) For each equivalence class of *n*-sets, select a representative one with a normal in a canonical position.

4) Identify matching pairs of *n*-sets, producing candidate face sets with the required symmetry.

5) Identify single face sets and compatible pairs of face sets that have the required number of face sides.

6) Identify polyhedra and compounds from such combination that close into surfaces.

5. Octahedral and Tetrahedral Symmetry

We fix the Octahedral rotation axes as follows:

4-fold: $(u, 0, 0)$, $(0, u, 0)$, $(0, 0, u)$
3-fold: (u, u, u), $(u, -u, -u)$, $(-u, u, -u)$, $(-u, -u, u)$
2-fold: $(u, u, 0)$, $(u, -u, 0)$, $(u, 0, u)$, $(-u, 0, u)$, $(0, u, u)$, $(0, u, -u)$

where u is an arbitrary parameter that is constant within any given n-set. Taking signs as shown or all as negated identifies a particular ray of an axis.

The equatorial digons for $k = 4$ have 4-sites and 2-sites on their equators (i.e., the perpendicular bisecting planes), those for $k = 3$ have 2-sites on their equators, and those for $k = 2$ have 4-sites, 3-sites, and 2-sites on their equators. These matches produce rhombi with edge types 44a, 42b, 32b, and 22b. Edge types 42b and 22b have been eliminated. The 44a and 32b rhombi are included below in Table 5.3. The digons and the rhombi they define are omitted from the following discussion of n-sets.

n-sets

n-sets can readily be found by inspection of a model of the Octahedral group on a sphere. For a pair of k-sites at each angular distance, we examine the planes that contain that pair and identify simply-connected regular n-gons having that pair as a side. The vertices of such an n-gon form an n-set. The n-sets are listed in Table 5.1. We list only one n-set of each symmetry orbit, choosing the one with its normal in a given symmetry domain in order to facilitate the identification of matches. We choose those with normals within the solid angle defined by $(1, 0, 0)$, $(1, 1, 0)$, and $(1, 1, 1)$.

Matching n-sets must have the same normal, value of n, and condition of being equatorial or not. Table 5.2 lists the n-sets grouped by these properties.

n-set Pairs

n-sets 14 and 19 have nothing to match. Figure 5.1 shows planes perpendicular to each of the normal directions of non-equatorial potential faces. Pairs of matching n-sets can readily be identified from these figures.

In these and later figures, 2-sites are identified by red dots, 3-sites by green circles, 4-sites by purple squares, and 5-sites by purple pentagons. The lines on these diagrams show collinearity to illustrate the relative locations of the n-sets. They are unrelated to potential edges of faces.

Among equatorial n-sets, 15 and 16 are on rays 45 degrees apart and 17 and 18 are on rays 60 degrees apart. n-sets 15 and 16 match; they define arbitrary [8]s and [8/3]s. When not regular, these exist as coplanar pairs having the same radii ratio, as shown in Figure 5.2. n-sets 17 and 18 also match; they define arbitrary [6]s that also exist as coplanar pairs when not regular. In addition, 17 and 18 each match themselves, having odd n. This results in coplanar pairs of [6/2]s.

The faces determined by n-set pairs $\{3, 4\}$ and $\{5, 6\}$ have radii ratio = 1 and $p = q$, so cannot appear in a vertex-intransitive isotoxal polyhedron with Octahedral symmetry. The $\{3, 4\}$ face set is applicable to Tetrahedral symmetry and will be considered below.

Table 5.3 lists the pairs of matching n-sets, grouped in the same way as the n-sets in Table 5.2. Each entry in the table represents a symmetrical set of faces. These are all the potential faces of vertex-intransitive isotoxal polyhedra having Octahedral or Tetrahedral symmetry. This table includes the equatorial rhombi identified above.

We immediately disregard the pairs that are of eliminated edge types or that define degenerate faces.

Compatible Faces

Face sets that are compatible must have the same radii ratio as well as the same edge type. Table 5.4 lists the remaining potential face sets grouped by edge type. In addition to showing the radii ratio, this table shows the number of faces in each set.

We eliminate from further consideration all equatorial face sets that are not compatible with any non-equatorial ones. Such sets are never sufficient to form a closed surface by themselves.

Face Combinations

It remains only to check all compatible face sets, singly and in pairs, for each Edge Type and radii ratio. As mentioned above, the faces of a set, taken together, must provide the correct number of sides. If two sets of faces are present, both must provide the same number of sides. Edge-transitivity will be preserved as long as there are no more than two face sets. If one set is sufficient for closure, the result is a polyhedron that is face-transitive as well.

The test for closure can be done by direct inspection of virtual models of the faces. By noting as well whether multiple surfaces result or there is only point connectivity at vertices, we identify any compounds as well as single polyhedra that are made up of these face sets.

Table 5.5 shows the results for the remaining possible Edge Types. We find three Octahedral vertex-intransitive isotoxal polyhedra, including two that are face-intransitive and were previously unknown. They are described and briefly discussed below in Section 8.

Tetrahedral Symmetry

In reducing from the Octahedral symmetry group, 4-axes and 4-sites become 2-axes and 2-sites. Four of the 3-axis rays and their associated 3-sites remain as such while their inversions become 3'-rays and 3'-sites. The 2-axes and 2-sites are not applicable.

We have seen that the only possible edge types are T33a, T33'a, and T22a. The only faces identified using Octahedral elements that carry over are as follows:

Eq.	Normal	n	Id1	Id2	Face subtype	Edge Type	Radii Ratio
	(1, 0, 0)	2	3	4	regular [4]	O33a → T33'a	1
✓	(1, 0, 0)	2	eq4	eq4	[4]	O44a → T22a	any

The T22a faces are all equatorial and do not close into a surface. The T33'a faces close into a cube. This, denoted T33'a_1, would be the only vertex-intransitive isotoxal polyhedron having only Tetrahedral symmetry if one considered the directionality of the edges to be significant. We do not, for reasons outlined in Appendix 1. Thus, this is an ordinary cube, with transitive vertices and Octahedral symmetry after all.

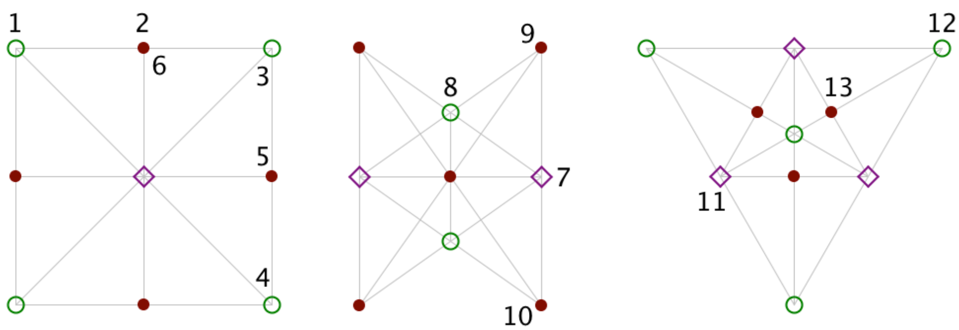


Figure 5.1: Intersections of rotation axes with non-equatorial planes normal to (1, 0, 0), (1, 1, 0), and (1, 1, 1), respectively. One *k*-site in each *n*-set is labeled.

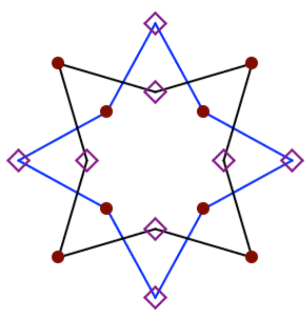


Figure 5.2: Coplanar pairs of irregular equatorial faces.

Table 5.1: Octahedral n-sets

k	Dist.	n	Sites	Eq.	Normal
4	a	2	$(u, 0, 0), (0, u, 0)$		(1, 1, 0)
4	a	3	$(u, 0, 0), (0, u, 0), (0, 0, u)$		(1, 1, 1)
4	a	4	$(0, u, 0), (0, 0, u), (0, -u, 0), (0, 0, -u)$	✓	(1, 0, 0)
3	a	2	$(u, u, u), (u, u, -u)$		(1, 1, 0)
3	a	4	$(u, u, u), (u, u, -u), (u, -u, u), (u, -u, -u)$		(1, 0, 0)
3	b	2	$(u, u, u), (u, -u, -u)$		(1, 0, 0)
3	b	2	$(u, u, -u), (u, -u, u)$		(1, 0, 0)
3	b	3	$(u, u, -u), (u, -u, u), (-u, u, u)$		(1, 1, 1)
2	a	2	$(u, 0, u), (u, u, 0)$		(2, 1, 1)
2	a	3	$(u, u, 0), (u, 0, u), (0, u, u)$		(1, 1, 1)
2	a	4	$(u, 0, u), (u, u, 0), (u, 0, -u), (u, -u, 0)$		(1, 0, 0)
2	a	6	$(u, 0, -u), (0, u, -u), (-u, u, 0), (-u, 0, u), (0, -u, u), (u, -u, 0)$	✓	(1, 1, 1)
2	b	2	$(u, 0, u), (u, 0, -u)$		(1, 0, 0)
2	b	2	$(u, u, 0), (u, -u, 0)$		(1, 0, 0)
2	b	4	$(0, u, u), (0, u, -u), (0, -u, -u), (0, -u, u)$	✓	(1, 0, 0)
2	c	2	$(0, u, u), (u, 0, -u)$		(1, 1, 0)
2	c	2	$(0, u, -u), (u, 0, u)$		(1, 1, 0)
2	c	3	$(0, u, -u), (-u, 0, u), (u, -u, 0)$	✓	(1, 1, 1)
2	c	3	$(0, -u, u), (u, 0, -u), (-u, u, 0)$	✓	(1, 1, 1)

Table 5.2: Octahedral n-sets grouped by equatoriality, normal, and value of n

Id	Eq.	Normal	n	k	Sites
1		(1, 0, 0)	4	3	$(u, u, u), (u, u, -u), (u, -u, u), (u, -u, -u)$
2		(1, 0, 0)	4	2	$(u, 0, u), (u, u, 0), (u, 0, -u), (u, -u, 0)$
3		(1, 0, 0)	2	3	$(u, u, u), (u, -u, -u)$
4		(1, 0, 0)	2	3	$(u, u, -u), (u, -u, u)$
5		(1, 0, 0)	2	2	$(u, 0, u), (u, 0, -u)$
6		(1, 0, 0)	2	2	$(u, u, 0), (u, -u, 0)$
7		(1, 1, 0)	2	4	$(u, 0, 0), (0, u, 0)$
8		(1, 1, 0)	2	3	$(u, u, u), (u, u, -u)$
9		(1, 1, 0)	2	2	$(0, u, +u), (u, 0, -u)$
10		(1, 1, 0)	2	2	$(0, u, -u), (u, 0, +u)$
11		(1, 1, 1)	3	4	$(u, 0, 0), (0, u, 0), (0, 0, u)$
12		(1, 1, 1)	3	3	$(u, u, -u), (u, -u, u), (-u, u, u)$
13		(1, 1, 1)	3	2	$(u, u, 0), (u, 0, u), (0, u, u)$
14		(2, 1, 1)	2	2	$(u, 0, u), (u, u, 0)$
15	✓	(1, 0, 0)	4	4	$(0, u, 0), (0, 0, u), (0, -u, 0), (0, 0, -u)$
16	✓	(1, 0, 0)	4	2	$(0, u, u), (0, u, -u), (0, -u, -u), (0, -u, u)$
17	✓	(1, 1, 1)	3	2	$(0, +u, -u), (-u, 0, +u), (+u, -u, 0)$
18	✓	(1, 1, 1)	3	2	$(0, -u, +u), (+u, 0, -u), (-u, +u, 0)$
19	✓	(1, 1, 1)	6	2	$(u, 0, -u), (0, u, -u), (-u, u, 0), (-u, 0, u), (0, -u, u), (u, -u, 0)$

Table 5.3: Potential faces from matched Octahedral n-sets and equatorial rhombi

Eq.	Normal	n	Id1	Id2	Face Subtype	Edge Type
	(1, 0, 0)	4	1	2	degenerate [8]	32a
	(1, 0, 0)	4	1	2	aligned [8/3]	32b
	(1, 1, 0)	2	7	8	[4]	43a
	(1, 1, 1)	3	11	12	degenerate [6]	43a
	(1, 1, 1)	3	11	13	degenerate [6]	42a
	(1, 1, 1)	3	12	13	[6/2]	32b
✓	(1, 0, 0)	4	15	16	arbitrary [8]	42a
✓	(1, 0, 0)	4	15	16	arbitrary [8/3]	42c
✓	(1, 1, 1)	3	17	17	[6/2]	22c
✓	(1, 1, 1)	3	18	18	[6/2]	22c
✓	(1, 1, 1)	3	17	18	arbitrary [6]	22a
✓	(1, 0, 0)	2	eq4	eq4	[4]	44a
✓	(2, 1, 1)	2	eq3	eq2	[4]	32b

Table 5.4: Octahedral face sets by edge type

Edge Type	Radii Ratio	Equatorial	Face Subtype	Number of Faces
44a	any	✓	[4]	3 or 6
43a	$2/\sqrt{3}$		[4]	12
32b	$\sqrt{3}/\sqrt{2}$		aligned [8/3]	6
32b	$\sqrt{6}$		[6/2]	8
32b	any	✓	[4]	12
22a	any	✓	arbitrary [6]	4 or 8
22c	any	✓	[6/2]	4 or 8

Table 5.5: Octahedral face combinations

Edge Type	Radii Ratio	Face Sets	Contributed Sides	Required Sides	Resulting Polyhedron
43a	$2/\sqrt{3}$	12 [4]s	48	48	O43a_1
32b	$\sqrt{3}/\sqrt{2}$	6 aligned [8/3]s	48	96	
32b	$\sqrt{3}/\sqrt{2}$	6 aligned [8/3]s, 12 equatorial [4]s	96	96	O32b_1
32b	$\sqrt{6}$	8 [6/2]s	48	96	
32b	$\sqrt{6}$	8 [6/2]s, 12 equatorial [4]s	96	96	O32b_2

6. Icosahedral Symmetry

The argument here is exactly analogous to that used for Octahedral symmetry. For the most part, the text in this section will describe only differences from the discussion in the previous section.

Let $\phi = (\sqrt{5} + 1) / 2$, $\tau = (\sqrt{5} - 1) / 2$, and $\lambda = \sqrt{(\phi+2)}$.

We fix the rotation axes as follows:

5-sites: $(u, \tau u, 0)$, $(u, -\tau u, 0)$, $(0, u, \tau u)$, $(0, u, -\tau u)$, $(\tau u, 0, u)$, $(-\tau u, 0, u)$
3- sites: (u, u, u), $(u, -u, -u)$, $(-u, u, -u)$, $(-u, -u, u)$,
$(\phi u, 0, \tau u)$, $(\phi u, 0, -\tau u)$, $(\tau u, \phi u, 0)$, $(-\tau u, \phi u, 0)$, $(0, \tau u, \phi u)$, $(0, -\tau u, \phi u)$
2- sites: $(2u, 0, 0)$, $(0, 2u, 0)$, $(0, 0, 2u)$,
$(\phi u, \tau u, u)$, $(\phi u, -\tau u, u)$, $(\phi u, \tau u, -u)$, $(\phi u, -\tau u, -u)$,
$(u, \phi u, \tau u)$, $(u, \phi u, -\tau u)$, $(-u, \phi u, \tau u)$, $(-u, \phi u, -\tau u)$,
$(\tau u, u, \phi u)$, $(-\tau u, u, \phi u)$, $(\tau u, -u, \phi u)$, $(-\tau u, -u, \phi u)$

The equatorial digons for $k = 5$ have 2-sites on their equators, those for $k = 3$ have 2-sites on their equators, and those for $k = 2$ have 5-sites, 3-sites, and 2-sites on their equators. These matches produce rhombi with edge types 52c, 32d, and 22d, which are included below in Table 6.3. The digons and the rhombi they define are omitted from the following discussion of *n*-sets.

n-sets

The *n*-sets are listed in Table 6.1. For these, we choose representative *n*-sets with normals on the *x-y* and *x-z* planes rather than in a single fundamental domain of the symmetry group. This makes the coordinate lists simpler.

Table 6.2 lists the *n*-sets grouped by matching criteria.

n-set Pairs

n-sets 12, 32, 35, 38, and 39 have nothing to match. *n*-sets 1 and 2 are located in the plane perpendicular to the 5-axis $(-\tau u, 0, u)$. *n*-set 1 consists of two adjacent vertices of an equatorial regular decagon and *n*-set 2 consists of two adjacent vertices of an equatorial regular decagram. When projected onto the same potential face plane, they lie on the same line but would need to be perpendicular in order to match.

Note that *n*-sets 13 and 14 are essentially the same as the matching *n*-sets 6 and 7 of Octahedral symmetry.

Figures 6.1 through 6.3[3] show planes perpendicular to each of the remaining three normal directions of non-equatorial potential faces. Pairs of matching *n*-sets can readily be identified from these figures. As in the Octahedral plane diagrams, the lines on these diagrams show collinearity to illustrate the relative locations of the *n*-sets. Some lines are emphasized to highlight matching.

Among equatorial *n*-sets, 33 and 34 are on rays 60 degrees apart and 36 and 37 are on rays 36 degrees apart. *n*-sets 33 and 34 match; they define arbitrary [6]s that exist as coplanar pairs when not regular. Each also matches itself, which results in coplanar pairs of [6/2]s. *n*-sets 36 and 37 match; they define arbitrary [10]s and [10/3]s that exist as coplanar pairs when not regular. Each also matches itself, which results in coplanar pairs of [10/2]s and [10/4]s.

The faces determined by *n*-set pair {23, 24} have radii ratio = 1 and $p = q$, so cannot appear in a vertex-intransitive isotoxal polyhedron with Icosahedral symmetry. (These faces form the vertex-transitive compound of 5 cubes.)

[3] The first two figures contain the complete stellation diagram of the Dodecahedron and most of that for the Icosahedron. The stellation diagrams result from the intersections of faces planes. Bisecting each of the resulting dihedral angles is a plane through the center that contains multiple rotation axes. The intersections of those planes and one of the face planes are what produce the same lines on these diagrams.

The third figure contains part of the stellation diagram of the rhombic triacontahedron. The positive *x* coordinates of the *k*-sites shown form a geometric series with ratio equal to the Golden Ratio. The same is true for the *y* coordinates.

Table 6.3 lists the pairs of matching *n*-sets, including the equatorial rhombi identified above. These are all the potential faces of vertex-intransitive isotoxal polyhedra having Icosahedral symmetry. The 13-14 rhombus with edge type 32b is the only non-equatorial potential face type not normal to an Icosahedral rotation axis. It is also found in the Octahedral group with a 43a edge.

As before, we immediately disregard the pairs that are of eliminated edge types or that define degenerate faces.

Compatible Faces

Table 6.4 lists the remaining potential face sets grouped by edge type and within that by increasing order of radii ratio. As before, we eliminate from further consideration all equatorial face sets that are not compatible with any non-equatorial ones.

Face Combinations

Table 6.5 shows the results of the compatibility and closure tests for the remaining possible Edge Types.

For edge types 33b and 33c, the face sets are not chiral and do not provide the geometry needed to close with only 120 sides (to form 60 edges).

For edge type 32b and ratio √3/2, when the faces are considered all together one finds that there are multiple surfaces. This is a compound of 5 rhombic dodecahedra.

We find 20 Icosahedral vertex-intransitive isotoxal polyhedra and compounds, most of which were previously unknown including 9 that are face-intransitive. Several have unusual structure, featuring partially-overlapping edges, and have the outward form of more familiar regular or uniform polyhedra. All of these are described and briefly discussed below in Section 8.

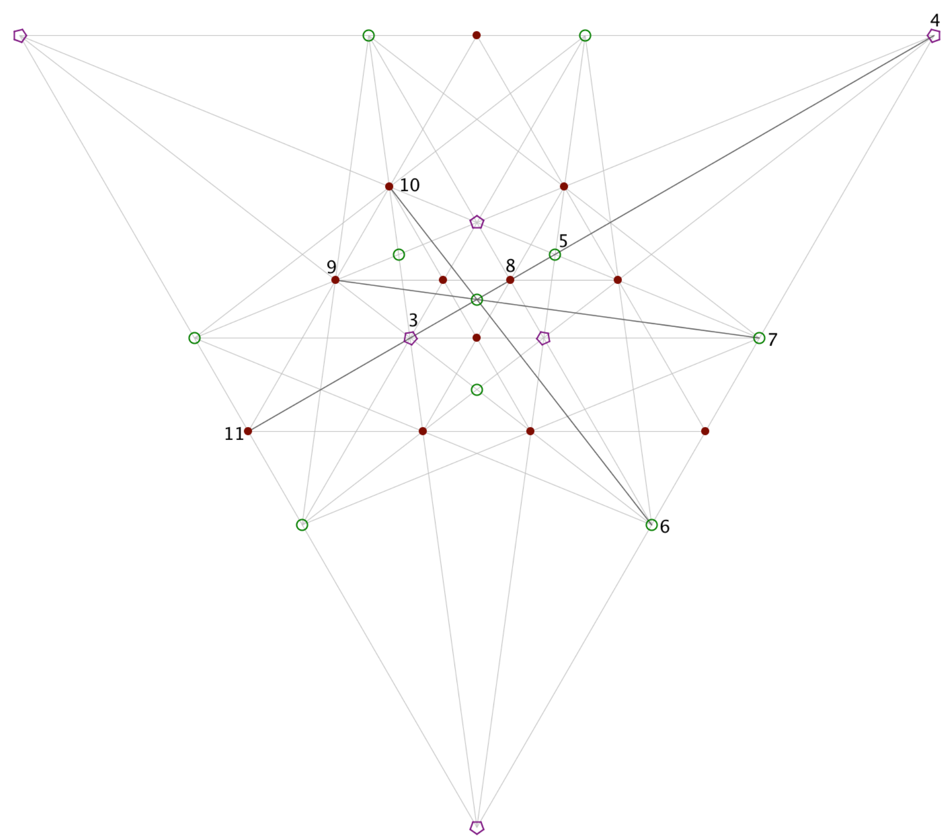


Figure 6.1: Intersections of rotation axes with a non-equatorial plane normal to $(\phi, 0, \tau)$. One *k*-site in each triangular set is labeled. Matching 3-sets are incident with a common line through the center.

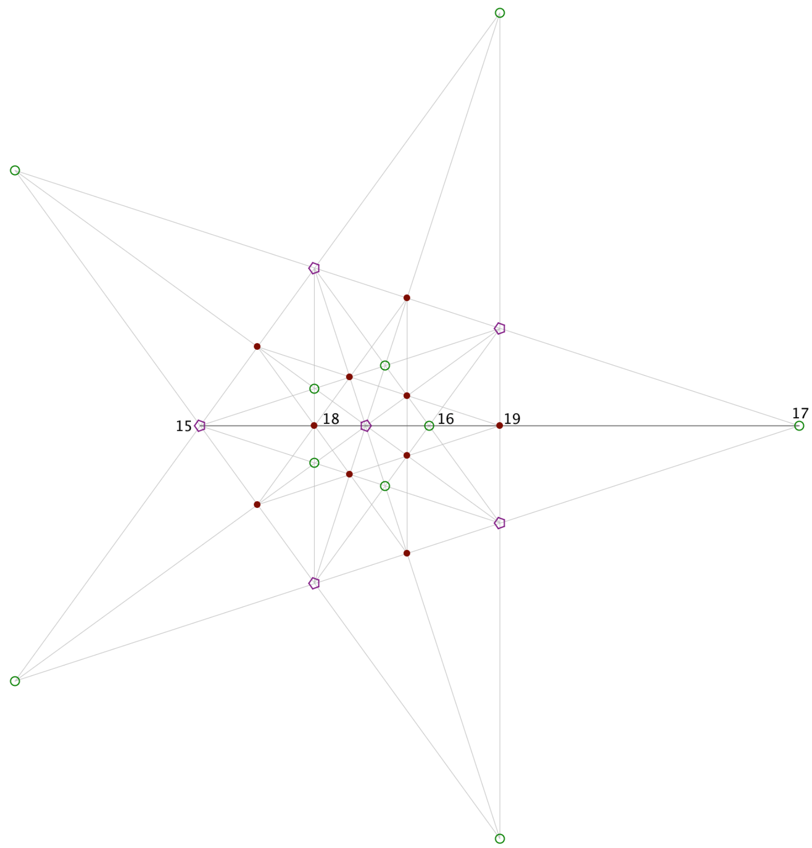


Figure 6.2: Intersections of rotation axes with a non-equatorial plane normal to $(1,\tau,0)$. One k-site in each pentagonal set is labeled. All pairs of 5-sets match as they all are incident with a common line through the center.

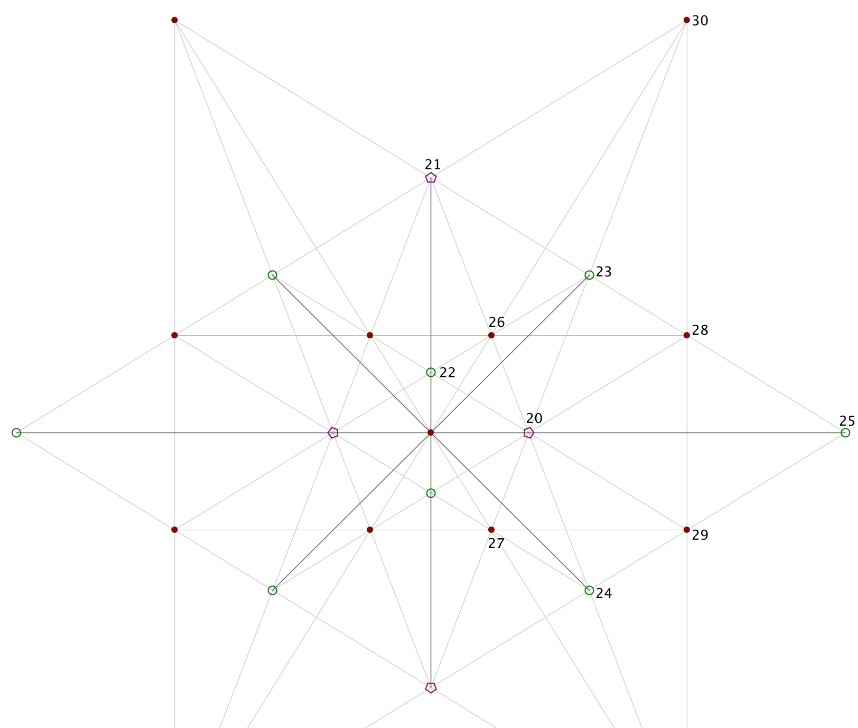


Figure 6.3: Intersections of rotation axes with a non-equatorial plane normal to $(1,0,0)$. One k-site in each opposite pair is labeled. Matching 2-sets lie on perpendicular lines through the center.

Table 6.1: Icosahedral *n*-sets

k	Dist.	n	Sites	Eq.	Normal
5	a	2	$(u, \pm\tau u, 0)$		$(1, 0, 0)$
5	a	3	$(u, \pm\tau u, 0), (\tau u, 0, u)$		$(\phi, 0, \tau)$
5	a	5	$(u, -\tau u, 0), (\tau u, 0, \pm u), (0, u, \pm\tau u)$		$(1, \tau, 0)$
5	b	2	$(\tau u, 0, \pm u)$		$(1, 0, 0)$
5	b	3	$(\tau u, 0, -u), (0, \pm u, \tau u)$		$(\phi, 0, \tau)$
3	a	2	$(\phi u, 0, \pm\tau u)$		$(1, 0, 0)$
3	a	5	$(u, u, \pm u), (\phi u, 0, \pm\tau u), (\tau u, \phi u, 0)$		$(1, \tau, 0)$
3	b	2	$(u, u, u), (u, u, -u)$		$(1, 1, 0)$
3	b	3	$(u, \pm u, u), (\phi u, 0, -\tau u)$		$(\phi, 0, \tau)$
3	b	4	$(u, \pm u, \pm u)$		$(1, 0, 0)$
3	b	5	$(u, -u, \pm u), (0, \tau u, \pm\phi u), (-\tau u, \phi u, 0)$		$(1, \tau, 0)$
3	c	2	$(u, -u, -u), (u, +u, +u)$		$(1, 0, 0)$
3	c	2	$(u, -u, +u), (u, +u, -u)$		$(1, 0, 0)$
3	c	3	$(u, +u, -u), (0, +\tau u, \phi u), (\tau u, -\phi u, 0)$		$(\phi, 0, \tau)$
3	c	3	$(u, -u, -u), (0, -\tau u, \phi u), (\tau u, +\phi u, 0)$		$(\phi, 0, \tau)$
3	d	2	$(\tau u, \pm\phi u, 0)$		$(1, 0, 0)$
2	a	2	$(\phi u, \pm\tau u, u)$		$(\phi, 0, 1)$
2	a	3	$(2u, 0, 0), (\phi u, \pm\tau u, u)$		$(\phi, 0, \tau)$
2	a	5	$(2u, 0, 0), (\phi u, \tau u, \pm u), (u, \phi u, \pm\tau u)$		$(1, \tau, 0)$
2	a	10	$(0, 0, \pm 2u), (-u, \phi u, \pm\tau u), (u, -\phi u, \pm\tau u), (-\tau u, u, \pm\phi u), (\tau u, -u, \pm\phi u)$	✓	$(1, \tau, 0)$
2	b	2	$(\phi u, \tau u, \pm u)$		$(\phi, \tau, 0)$
2	b	5	$(0, 2u, 0), (\phi u, -\tau u, \pm u), (\tau u, u, \pm\phi u)$		$(1, \tau, 0)$
2	b	6	$(0, \pm 2u, 0), (-\tau u, \pm u, \phi u), (\tau u, \pm u, -\phi u)$	✓	$(\phi, 0, \tau)$
2	c	2	$(\phi u, -\tau u, -u), (\phi u, +\tau u, +u)$		$(1, 0, 0)$
2	c	2	$(\phi u, -\tau u, +u), (\phi u, +\tau u, -u)$		$(1, 0, 0)$
2	c	5	$(0, 0, +2u), (-u, \phi u, +\tau u), (u, -\phi u, +\tau u), (-\tau u, u, -\phi u), (\tau u, -u, -\phi u)$	✓	$(1, \tau, 0)$
2	c	5	$(0, 0, -2u), (-u, \phi u, -\tau u), (u, -\phi u, -\tau u), (-\tau u, u, +\phi u), (\tau u, -u, +\phi u)$	✓	$(1, \tau, 0)$
2	d	2	$(2u, 0, 0), (0, 2u, 0)$		$(1, 1, 0)$
2	d	3	$(\tau u, +u, \phi u), (u, -\phi u, \tau u), (\phi u, +\tau u, -u)$		$(\phi, 0, \tau)$
2	d	3	$(\tau u, -u, \phi u), (u, +\phi u, \tau u), (\phi u, -\tau u, -u)$		$(\phi, 0, \tau)$
2	d	4	$(0, \pm 2u, 0), (0, 0, \pm 2u)$	✓	$(1, 0, 0)$
2	e	2	$(u, \pm\phi u, \tau u)$		$(\phi, 0, 1)$
2	e	3	$(0, 0, 2u), (u, \pm\phi u, -\tau u)$		$(\phi, 0, \tau)$
2	f	2	$(u, -\phi u, -\tau u), (u, +\phi u, +\tau u)$		$(1, 0, 0)$
2	f	2	$(u, -\phi u, +\tau u), (u, +\phi u, -\tau u)$		$(1, 0, 0)$
2	f	3	$(0, +2u, 0), (-\tau u, -u, \phi u), (\tau u, -u, -\phi u)$	✓	$(\phi, 0, \tau)$
2	f	3	$(0, -2u, 0), (-\tau u, +u, \phi u), (\tau u, +u, -\phi u)$	✓	$(\phi, 0, \tau)$
2	g	2	$(\tau u, -u, -\phi u), (\tau u, +u, +\phi u)$		$(1, 0, 0)$
2	g	2	$(\tau u, -u, +\phi u), (\tau u, +u, -\phi u)$		$(1, 0, 0)$

Table 6.2: Icosahedral n-sets grouped by equatoriality, normal, and value of n

Id	Eq.	Normal	n	k	Sites
1		$(\phi, 0, 1)$	2	2	$(\phi u, \pm\tau u, u)$
2		$(\phi, 0, 1)$	2	2	$(u, \pm\phi u, \tau u)$
3		$(\phi, 0, \tau)$	3	5	$(u, \pm\tau u, 0), (\tau u, 0, u)$
4		$(\phi, 0, \tau)$	3	5	$(\tau u, 0, -u), (0, \pm u, \tau u)$
5		$(\phi, 0, \tau)$	3	3	$(u, \pm u, u), (\phi u, 0, -\tau u)$
6		$(\phi, 0, \tau)$	3	3	$(u, +u, -u), (0, +\tau u, \phi u), (\tau u, -\phi u, 0)$
7		$(\phi, 0, \tau)$	3	3	$(u, -u, -u), (0, -\tau u, \phi u), (\tau u, +\phi u, 0)$
8		$(\phi, 0, \tau)$	3	2	$(2u, 0, 0), (\phi u, \pm\tau u, u)$
9		$(\phi, 0, \tau)$	3	2	$(\tau u, +u, \phi u), (u, -\phi u, \tau u), (\phi u, +\tau u, -u)$
10		$(\phi, 0, \tau)$	3	2	$(\tau u, -u, \phi u), (u, +\phi u, \tau u), (\phi u, -\tau u, -u)$
11		$(\phi, 0, \tau)$	3	2	$(0, 0, 2u), (u, \pm\phi u, -\tau u)$
12		$(\phi, \tau, 0)$	2	2	$(\phi u, \tau u, \pm u)$
13		$(1, 1, 0)$	2	3	$(u, u, u), (u, u, -u)$
14		$(1, 1, 0)$	2	2	$(2u, 0, 0), (0, 2u, 0)$
15		$(1, \tau, 0)$	5	5	$(u, -\tau u, 0), (\tau u, 0, \pm u), (0, u, \pm\tau u)$
16		$(1, \tau, 0)$	5	3	$(u, u, \pm u), (\phi u, 0, \pm\tau u), (\tau u, \phi u, 0)$
17		$(1, \tau, 0)$	5	3	$(u, -u, \pm u), (0, \tau u, \pm\phi u), (-\tau u, \phi u, 0)$
18		$(1, \tau, 0)$	5	2	$(2u, 0, 0), (\phi u, \tau u, \pm u), (u, \phi u, \pm\tau u)$
19		$(1, \tau, 0)$	5	2	$(0, 2u, 0), (\phi u, -\tau u, \pm u), (\tau u, u, \pm\phi u)$
20		$(1, 0, 0)$	2	5	$(u, \pm\tau u, 0)$
21		$(1, 0, 0)$	2	5	$(\tau u, 0, \pm u)$
22		$(1, 0, 0)$	2	3	$(\phi u, 0, \pm\tau u)$
23		$(1, 0, 0)$	2	3	$(u, -u, -u), (u, +u, +u)$
24		$(1, 0, 0)$	2	3	$(u, -u, +u), (u, +u, -u)$
25		$(1, 0, 0)$	2	3	$(\tau u, \pm\phi u, 0)$
26		$(1, 0, 0)$	2	2	$(\phi u, -\tau u, -u), (\phi u, +\tau u, +u)$
27		$(1, 0, 0)$	2	2	$(\phi u, -\tau u, +u), (\phi u, +\tau u, -u)$
28		$(1, 0, 0)$	2	2	$(u, -\phi u, -\tau u), (u, +\phi u, +\tau u)$
29		$(1, 0, 0)$	2	2	$(u, -\phi u, +\tau u), (u, +\phi u, -\tau u)$
30		$(1, 0, 0)$	2	2	$(\tau u, -u, -\phi u), (\tau u, +u, +\phi u)$
31		$(1, 0, 0)$	2	2	$(\tau u, -u, +\phi u), (\tau u, +u, -\phi u)$
32		$(1, 0, 0)$	4	3	$(u, \pm u, \pm u)$
33	✓	$(\phi, 0, \tau)$	3	2	$(0, +2u, 0), (-\tau u, -u, \phi u), (\tau u, -u, -\phi u)$
34	✓	$(\phi, 0, \tau)$	3	2	$(0, -2u, 0), (-\tau u, +u, \phi u), (\tau u, +u, -\phi u)$
35	✓	$(\phi, 0, \tau)$	6	2	$(0, \pm 2u, 0), (-\tau u, \pm u, \phi u), (\tau u, \pm u, -\phi u)$
36	✓	$(1, \tau, 0)$	5	2	$(0, 0, +2u), (-u, \phi u, +\tau u), (u, -\phi u, +\tau u), (-\tau u, u, -\phi u), (\tau u, -u, -\phi u)$
37	✓	$(1, \tau, 0)$	5	2	$(0, 0, -2u), (-u, \phi u, -\tau u), (u, -\phi u, -\tau u), (-\tau u, u, +\phi u), (\tau u, -u, +\phi u)$
38	✓	$(1, \tau, 0)$	10	2	$(0, 0, \pm 2\text{v}), (-\text{v}, \phi\text{v}, \pm\tau\text{v}), (\text{v}, -\phi\text{v}, \pm\tau\text{v}), (-\tau\text{v}, \text{v}, \pm\phi\text{v}), (\tau\text{v}, -\text{v}, \pm\phi\text{v})$
39	✓	$(1, 0, 0)$	4	2	$(0, \pm 2u, 0), (0, 0, \pm 2u)$

Table 6.3: Potential faces from matched Icosahedral *n*-sets and equatorial rhombi

Eq.	Normal	n	Id1	Id2	Face Subtype	Edge Type
	$(\phi, 0, \tau)$	3	3	4	inner [6]	55a
			3	5	outer [6]	53a
			3	8	degenerate [6]	52a
			3	11	[6/2]	52c
			4	5	[6/2]	53c
			4	8	[6/2]	52c
			4	11	degenerate [6]	52b
			5	8	[6/2]	32b
			5	11	inner [6]	32b
			6	10	degenerate [6]	32b
			7	9	degenerate [6]	32b
			8	11	inner [6]	22b
	$(1, 1, 0)$	2	13	14	[4]	32b
	$(1, \tau, 0)$	5	15	16	aligned [10]	53a
			15	16	overlapped [10/3]	53b
			15	17	aligned [10]	53a
			15	17	overlapped [10/3]	53c
			15	18	degenerate [10/2]	52b
			15	18	aligned [10/4]	52c
			15	19	degenerate [10]	52a
			15	19	aligned [10/3]	52c
			16	17	inner [10/2]	33b
			16	17	inner [10/4]	33c
			16	18	degenerate [10]	32a
			16	18	aligned [10/3]	32b
			16	19	outer [10/2]	32b
			16	19	outer [10/4]	32d
			17	18	inner [10]	32b
			17	18	inner [10/3]	32d
			17	19	degenerate [10/2]	32c
			17	19	aligned [10/4]	32f
			18	19	aligned [10]	22a
			18	19	overlapped [10/3]	22c
	$(1, 0, 0)$	2	20	21	[4]	55a
			20	22	[4]	53a
			21	25	[4]	53b
			22	25	[4]	33b
✓	$(\phi, 0, \tau)$	3	33	33	[6/2]	22f
			34	34	[6/2]	22f
			33	34	arbitrary [6]	22b
✓	$(1, \tau, 0)$	5	36	36	arbitrary [10/2]	22c
			36	36	arbitrary [10/4]	22g
			37	37	arbitrary [10/2]	22c
			37	37	arbitrary [10/4]	22g
			36	37	arbitrary [10]	22a
			36	37	arbitrary [10/3]	22e
✓	$(\phi, 0, 1)$	2	eq5	eq2	arbitrary [4]	52c
✓	$(\phi, \tau, 0)$	2	eq3	eq2	arbitrary [4]	32d
✓	$(1, 0, 0)$	2	eq2	eq2	arbitrary [4]	22d

Table 6.4: Icosahedral face sets by edge type

Edge Type	Radii Ratio	Decimal Ratio	Equatorial	Face Subtype	Number of Faces
55a	ϕ	1.61803		[4]	30
55a	ϕ^3	4.23607		inner [6]	20
53a	$\tau^2\lambda/\sqrt{3}$	0.41947		aligned [10]	12
53a	$\tau^3\lambda^3/\sqrt{3}$	0.93796		outer [6]	20
53a	$\lambda/\sqrt{3}$	1.09819		[4]	30
53a	$\phi\lambda/\sqrt{3}$	1.77690		aligned [10]	12
53b	$\tau\lambda/\sqrt{3}$	0.67872		[4]	30
53b	$\phi\lambda/\sqrt{3}$	1.77690		overlapped [10/3]	12
53c	$\tau^2\lambda/\sqrt{3}$	0.41947		overlapped [10/3]	12
53c	$\lambda^3/\sqrt{3}$	3.97327		[6/2]	20
52c	$\tau^3\lambda$	0.44903		[6/2]	20
52c	$\tau\lambda$	1.17557		aligned [10/3]	12
52c	λ	1.90211		aligned [10/4]	12
52c	$\phi^2\lambda$	4.97980		[6/2]	20
52c	any	---	✓	[4]	30
33b	ϕ^2	2.61803		[4]	30
33b	ϕ^3	4.23607		inner [10/2]	12
33c	ϕ^3	4.23607		inner [10/4]	12
32b	$\sqrt{3}/\lambda^2$	0.47873		inner [6]	20
32b	$\sqrt{3}\tau^2$	0.66158		outer [10/2]	12
32b	$\sqrt{3}/2$	0.86603		[4]	60
32b	$\sqrt{3}\tau$	1.07047		aligned [10/3]	12
32b	$\sqrt{3}\phi^2/\lambda^2$	1.25332		[6/2]	20
32b	$\sqrt{3}\phi^2$	4.53457		inner [10]	12
32d	$\sqrt{3}\tau^2$	0.66158		outer [10/4]	12
32d	$\sqrt{3}\phi^2$	4.53457		inner [10/3]	12
32d	any	---	✓	[4]	30
32f	$\sqrt{3}\phi$	2.80252		aligned [10/4]	12
22a	ϕ	1.61803		aligned [10]	12
22a	any	---	✓	arbitrary [10]	6+6
22b	ϕ^2	2.61803		inner [6]	20
22b	any	---	✓	arbitrary [6]	10+10
22c	ϕ	1.61803		overlapped [10/3]	12
22c	any	---	✓	arbitrary [10/2]	6+6

Table 6.5: Icosahedral face combinations

Edge Type	Radii Ratio	Face Sets	Contributed Sides	Required Sides	Resulting Polyhedron
55a	1.61803	30 [4]s	120	120	I55a_1
55a	4.23607	20 inner [6]s	120	120	I55a_2
53a	0.41947	12 aligned [10]s	120	120	I53a_1
53a	0.93796	20 outer [6]s	120	120	I53a_2
53a	1.09819	30 [4]s	120	120	I53a_3
53a	1.77690	12 aligned [10]s	120	120	I53a_4
53b	0.67872	30 [4]s	120	120	I53b_1
53b	1.77690	12 overlapped [10/3]s	120	120	I53b_2
53c	0.41947	12 overlapped [10/3]s	120	120	I53c_1
53c	3.97327	20 [6/2]s	120	120	I53c_2
52c	0.44903	20 [6/2]s	120	240	
52c	0.44903	20 [6/2]s, 30 equatorial [4]s	240	240	I52c_1
52c	1.17557	12 aligned [10/3]s	120	240	
52c	1.17557	12 aligned [10/3]s, 30 equatorial [4]s	240	240	I52c_2
52c	1.90211	12 aligned [10/4]s	120	240	
52c	1.90211	12 aligned [10/4]s, 30 equatorial [4]s	240	240	I52c_3
52c	4.97980	20 [6/2]s	120	240	
52c	4.97980	20 [6/2]s, 30 equatorial [4]s	240	240	I52c_4
33b	2.61803	30 [4]s	120	240 or 120	
33b	4.23607	12 inner [10/2]s	120	240 or 120	
33c	4.23607	12 inner [10/4]s	120	240 or 120	
32b	0.47873	20 inner [6]s	120	240	
32b	0.66158	12 outer [10/2]s	120	240	
32b	0.86603	60 [4]s	240	240	I32b_1
32b	1.07047	12 aligned [10/3]s	120	240	
32b	1.25332	20 [6/2]s	120	240	
32b	4.53457	12 inner [10]s	120	240	
32d	0.66158	12 outer [10/4]s	120	240	
32d	0.66158	12 outer [10/4]s, 30 equatorial [4]s	240	240	I32d_1
32d	4.53457	12 inner [10/3]s	120	240	
32d	4.53457	12 inner [10/3]s, 30 equatorial [4]s	240	240	I32d_2
32f	2.80252	12 aligned [10/4]s	120	240	
22a	1.61803	12 aligned [10]s	120	240	
22a	1.61803	12 aligned [10]s, 6 medial equatorial [10]s	180	240	
22a	1.61803	12 aligned [10]s, 12 medial equatorial [10]s	240	240	I22a_1
22b	2.61803	20 inner [6]s	120	240	
22b	2.61803	20 inner [6]s, 10 inner equatorial [6]s	180	240	
22b	2.61803	20 inner [6]s, 20 inner equatorial [6]s	240	240	I22b_1
22c	1.61803	12 overlapped [10/3]s	120	240	
22c	1.61803	12 overlapped [10/3]s, 6 outer equatorial [10/2]s	180	240	
22c	1.61803	12 overlapped [10/3]s, 12 outer equatorial [10/2]s	240	240	I22c_1

7. Compounds

A component polyhedron of an isotoxal compound must itself be isotoxal.

Components can share vertices. Compounds for which this occurs are *bound*. Those that are made rigid thereby are *fully bound*. Others are *partially bound*. Compounds in which the components do not share vertices are *free*. The components simply occupy the same region of space.

Edges cannot be shared among component polyhedra of an isotoxal compound, as that would result in all the edges being incident with 4 faces. Such duplication of all edges under the symmetry constraint leads only to mere superposition of components.

The procedure described in the preceding sections finds all free vertex-intransitive isotoxal compounds. It remains to find compounds that are bound or vertex-transitive (isogonal).

We consider four categories of isotoxal compounds, depending on the vertex-transitivity of the compound and of its components.

Isogonal Compounds of Isogonal Polyhedra

Such components would be among those listed below in Table 8.3. J. Skilling published a complete list of uniform compounds of uniform polyhedra [Sk76]. To find the isotoxal ones among them, we eliminate the ones with only prismatic symmetry as well as any for which the total number of edges among the components fails to divide the order of the symmetry group of the compound.

This leaves the following:

Skilling No.	Count	Components	Symmetry
4	2	tetrahedra	O_h
5	5	tetrahedra	I
6	10	tetrahedra	I_h
9	5	cubes	I_h
12	4	octahedra	O_h
15	10	octahedra	I_h
16	10	octahedra	I_h
17	5	octahedra	I_h
18	5	tetrahemihexahedra	I
59	5	cuboctahedra	I_h
60	5	cubohemioctahedra	I_h
61	5	octahemioctahedra	I_h

None has rotational freedom, which could affect the equivalence of edges.

For those not clearly regular, it is straightforward to check virtual models for edge-transitivity. All but the compounds of 4 and 10 octahedra are isotoxal.

Isogonal Compounds of Non-isogonal Polyhedra

In order to form such a compound, the radii ratio of the component's faces must equal 1. There are no such non-isogonal isotoxal polyhedra and therefore no compounds in this category.

Non-isogonal Compounds of Isogonal Polyhedra

The edge type of such a compound must have equal values of p and q. For such types, edges cross in pairs as in Figure 2.1 and vertices are at two different radii. But an isogonal component has a single edge between such rays and cospherical radii. Thus isogonal components cannot form non-isogonal isotoxal compounds.

Non-isogonal Compounds of Non-isogonal Polyhedra

For any compound in this category, the number of edges in a component must be less than and divide the total number of edges for the edge type of the compound. All single polyhedra already have all possible edges for their type, as there are no enantiomorphic pairs. Thus a compound cannot be of the same type but must have a larger symmetry group that contains a compatible edge type.

Based on edge counts, the only possibility for components is O43a_1. As an Icosahedral compound this would have edge type I32b. It is the compound of 5 rhombic dodecahedra that we found in the course of enumerating Icosahedral polyhedra. It is the only vertex-intransitive isotoxal compound.

While the rhombic dodecahedron has full Octahedral symmetry, it only has 24 edges. For each pair of edges, there are two symmetry operations that map one to the other. It is always the case that one of those operations is in the T_h group, which is a subgroup of the compound's I_h group. Thus, that polyhedron is still edge-transitive when 5 are combined with their 4-axes aligned with Icosahedral 2-axes and their 3-axes aligned with Icosahedral 3-axes.

Table 8.4 below lists all the isotoxal compounds.

8. Discussion

The vertex-transitive isotoxal polyhedra and the isotoxal compounds are all well known, but their isotoxal nature is rarely mentioned. The same is true of most of the vertex-intransitive, face-transitive ones as well.

The following tables list various properties of the isotoxal polyhedra and compounds. Columns "F", "E", and "V" contain the total numbers of faces, edges, and vertices, respectively. Columns "F1" and "F2" contain the numbers of faces of types 1 and 2.

Table 8.1: Isotoxal Polyhedra that are neither Vertex-transitive nor Face-transitive

Polyhedron	Face Type 1	Face Type 2	F1	F2	F	E	V	Sides	Genus
O32b_1	aligned [8/3]	[4]	6	12	18	48	20	1	6
O32b_2	[6/2]	[4]	8	12	20	48	20	1	5
I52c_1	[6/2]	[4]	20	30	50	120	42	1	15
I52c_2	aligned [10/3]	[4]	12	30	42	120	42	1	19
I52c_3	aligned [10/4]	[4]	12	30	42	120	42	1	19
I52c_4	[6/2]	[4]	20	30	50	120	42	1	15
I32d_1	outer [10/4]	[4]	12	30	42	120	50	1	15
I32d_2	inner [10/3]	[4]	12	30	42	120	50	1	15
I22a_1	aligned [10]	medial [10]	12	12	24	120	60	1	19
I22b_1	inner [6]	inner [6]	20	20	40	120	60	1	11
I22c_1	overlapped [10/3]	outer [10/2]	12	12	24	120	60	1	19

Table 8.2: Vertex-intransitive Isotoxal Polyhedra that are Face-transitive

Polyhedron	Face Type	F	E	V	Sides	Genus	Density	Novel
O43a_1	[4]	12	24	14	2	0	1	
I55a_1	[4]	30	60	24	2	4	3	
I55a_2	inner [6]	20	60	24	2	9	4	
I53a_1	aligned [10]	12	60	32	2	9	4	✓
I53a_2	outer [6]	20	60	32	2	5	2	
I53a_3	[4]	30	60	32	2	0	1	
I53a_4	aligned [10]	12	60	32	2	9	2	✓
I53b_1	[4]	30	60	32	2	0	7	
I53b_2	overlapped [10/3]	12	60	32	2	9	4	✓
I53c_1	overlapped [10/3]	12	60	32	2	9	10	✓
I53c_2	[6/2]	20	60	32	2	5	6	

Table 8.3: Vertex-Transitive Isotoxal Polyhedra. The irregular ones are listed in groups that have the same edges.

Vertex Configuration	Face Type 1	Face Type 2	F1	F2	F	E	V	Sides	Genus	Density
(3, 3, 3)	{3}		4		4	6	4	2	0	1
(4, 4, 4)	{4}		6		6	12	8	2	0	1
(3, 3, 3, 3)	{3}		8		8	12	6	2	0	1
(5, 5, 5)	{5}		12		12	30	20	2	0	1
(3, 3, 3, 3, 3)	{3}		20		20	30	12	2	0	1
(5/2, 5/2, 5/2, 5/2, 5/2)	{5/2}		12		12	30	12	2	4	3
(5, 5, 5, 5, 5)/2	{5}		12		12	30	12	2	4	3
(5/2, 5/2, 5/2)	{5/2}		12		12	30	20	2	0	7
(3, 3, 3, 3, 3)/2	{3}		20		20	30	12	2	0	7
(3, ±4, -3, ±4)	{3}	{4}	4	3	7	12	6	1	1	--
(3, 4, 3, 4)	{3}	{4}	8	6	14	24	12	2	0	1
(3, ±6, -3, ±6)	{3}	{6}	8	4	12	24	12	2	1	2
(4, ±6, -4, ±6)	{4}	{6}	6	4	10	24	12	1	4	--
(3, 5, 3, 5)	{3}	{5}	20	12	32	60	30	2	0	1
(3, ±10, -3, ±10)	{3}	{10}	20	6	26	60	30	1	6	--
(5, ±10, -5, ±10)	{5}	{10}	12	6	18	60	30	1	14	--
(5/2, 5, 5/2, 5)	{5/2}	{5}	12	12	24	60	30	2	4	3
(5/2, ±6, -5/2, ±6)	{5/2}	{6}	12	10	22	60	30	1	10	--
(5, ±6, -5, ±6)	{5}	{6}	12	10	22	60	30	1	10	--
(5/2, 3, 5/2, 3)	{5/2}	{3}	12	20	32	60	30	2	0	7
(5/2, ±10/3, -5/2, ±10/3)	{5/2}	{10/3}	12	6	18	60	30	1	14	--
(3, ±10/3, -3, ±10/3)	{3}	{10/3}	20	6	26	60	30	1	6	--
(5/2, 3, 5/2, 3, 5/2, 3)	{5/2}	{3}	12	20	32	60	20	2	5	2
(-5/2, 5, -5/2, 5, -5/2, 5)	{5/2}	{5}	12	12	24	60	20	2	9	4
(3, 5, 3, 5, 3, 5)/2	{3}	{5}	20	12	32	60	20	2	5	6

Table 8.4: Isotoxal Compounds

ID	Compound	Face Type 1	Face Type 2	F1	F2	F	E	V	Binding
I32b_1	5 rhombic dodecahedra	[4]		60		60	120	50	full
Sk04	2 tetrahedra	{3}		8		8	12	8	free
Sk05	5 tetrahedra	{3}		20		20	30	20	free
Sk06	10 tetrahedra	{3}		40		40	60	20	full
Sk09	5 cubes	{4}		30		30	60	20	full
Sk17	5 octahedra	{3}		40		40	60	30	free
Sk18	5 tetrahemihexahedra	{3}	{4}	20	15	35	60	30	free
Sk59	5 cuboctahedra	{3}	{4}	40	30	70	120	60	free
Sk60	5 cubohemioctahedra	{4}	{6}	30	20	50	120	60	free
Sk61	5 octahemioctahedra	{3}	{6}	40	20	60	120	60	free

The remainder of this section describes the vertex-intransitive polyhedra listed above.

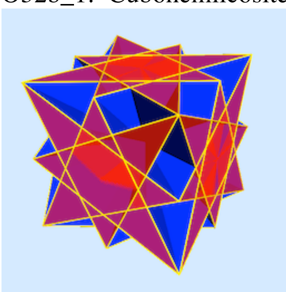
The static images supplied here cannot do justice to the interior structure of most of the polyhedra listed here. For full effect, one needs virtual models of them with transparent faces and full navigation all the way through. This is a consequence of the fact that we are stuck in 3-dimensional space. Just as A Square of Flatland cannot perceive star polygons without seeing transparent ones and having a good imagination, we have difficulty seeing the true beauty of these objects lacking a view from a fourth dimension.

Face-Intransitive Individual Polyhedra

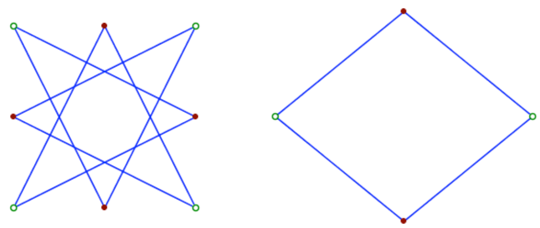
These are the ones that many authors have claimed are provably non-existent. All have equatorial faces and all are non-orientable.

The first two have equatorial faces that are not quite parallel to those of the dual of a (3, 4, 4, 4), but are normal to the radii of midpoints of equatorial [6]s that the symmetry group defines.

O32b_1: Cubohemiicositetrahedron

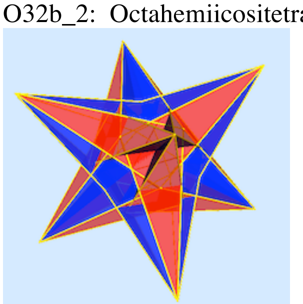


Faces:



This fits snugly inside a cube. There are several Uniform polyhedra in which pairs of non-adjacent faces share 2 vertices. In this one, some such pairs share three vertices. This is only possible with aligned faces, not regular ones.

O32b_2: Octahemiicositetrahedron



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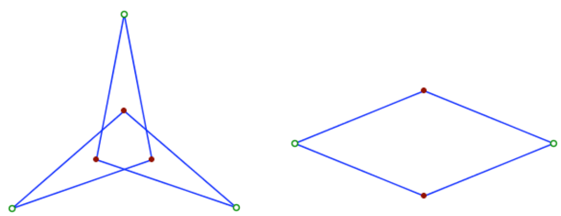
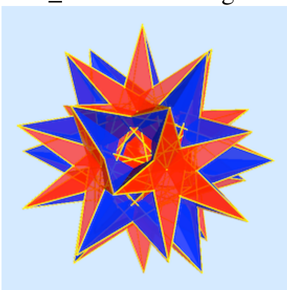


This fits snugly inside a Stella Octangula. Many pairs of non-adjacent faces share two vertices. The Class B (2-fold) vertices are in the valleys formed by interpenetrating [6/2]s.

The next four have equatorial faces that are not quite parallel to those of the dual of a (3, 4, 5, 4), but are normal to the radii of midpoints of equatorial [10]s that the symmetry group defines.

I52c_1: Small Hexagrammic Hemihexecontahedron



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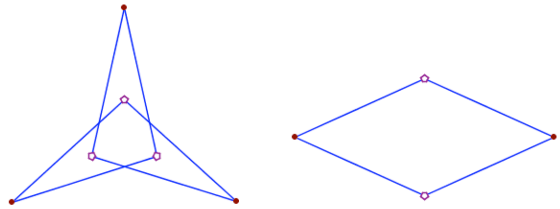
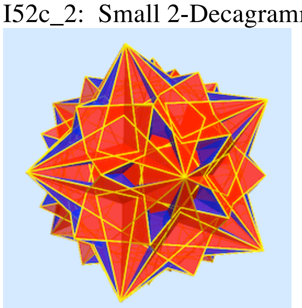


The Class A (5-fold) vertices are at the bottom of star-shaped valleys. Pairs of non-adjacent faces share two of these vertices.

I52c_2: Small 2-Decagrammic Hemihexecontahedron



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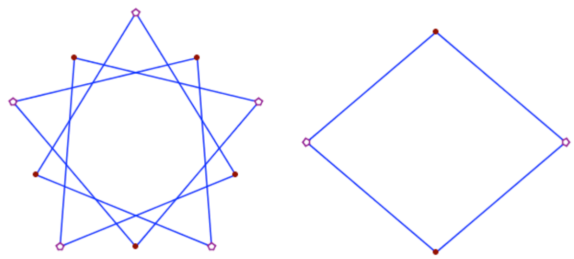
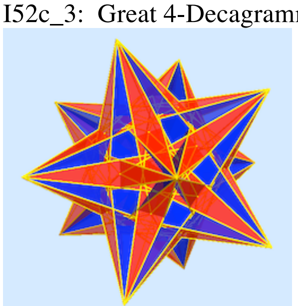
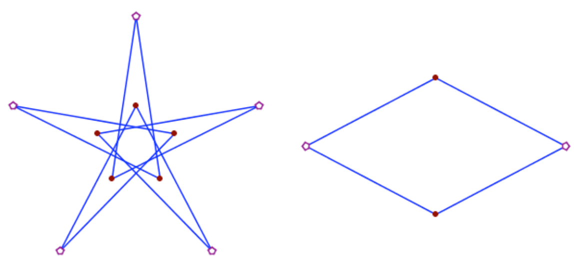


This fits snugly inside a Great Dodecahedron. Some pairs of non-adjacent faces share three vertices.

I52c_3: Great 4-Decagrammic Hemihexecontahedron

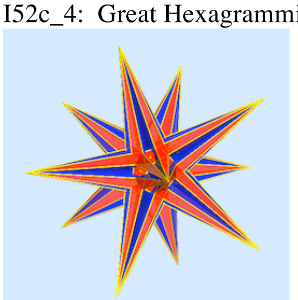


Faces:

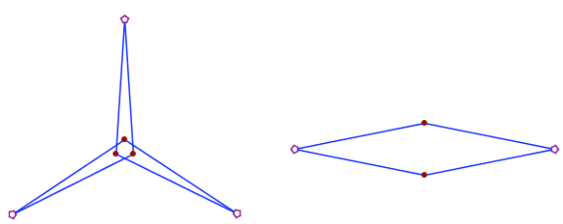


This fits snugly inside a Small Stellated Dodecahedron. The Class B (2-fold) vertices are in the valleys formed by interpenetrating [10/4]s. Some pairs of non-adjacent faces share three vertices.

I52c_4: Great Hexagrammic Hemihexecontahedron



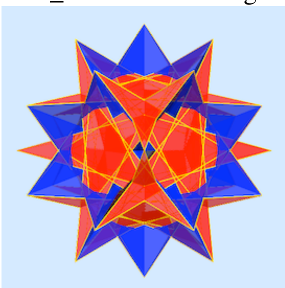
Faces:



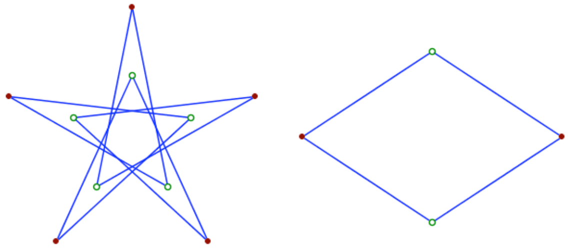
This fits snugly inside a Great Triambic Icosahedron. The Class B (2-fold) vertices are in the valleys formed by interpenetrating [6/2]s, in the center of canyons. Some pairs of non-adjacent faces share two vertices.

The next two have equatorial faces that are normal to the radii of midpoints of equatorial [6]s. Thus the hexecontahedron of which these are "hemi" forms is a different one than for the I52c polyhedra.

I32d_1: Small 4-Decagrammic Hemihexecontahedron

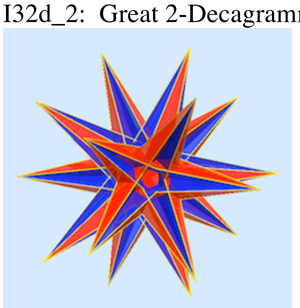


Faces:

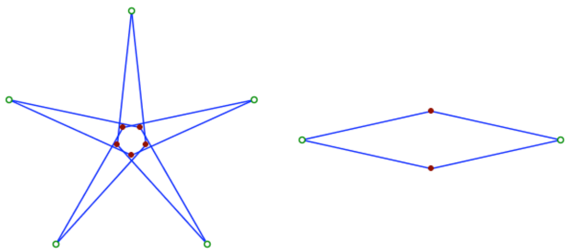


The Class A (3-fold) vertices are at the bottom of star-shaped valleys. Pairs of non-adjacent faces share two of these vertices.

I32d_2: Great 2-Decagrammic Hemihexecontahedron



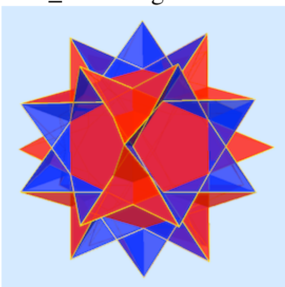
Faces:



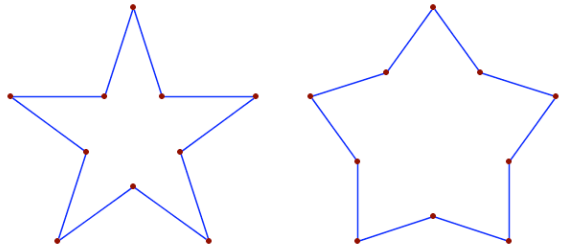
This fits snugly inside a Great Stellated Dodecahedron. The Class B (2-fold) vertices are well within the outer surface layer. Pairs of non-adjacent faces share two Class A vertices.

The following three have equatorial faces that are in the same planes as those of the uniform "...hemidodecahedra" and "...hemiicosahedra" but exist as coplanar pairs.

I22a_1: Decagonal Dihemidodecahedron

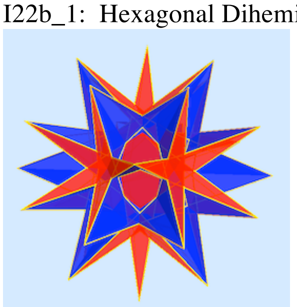


Faces:

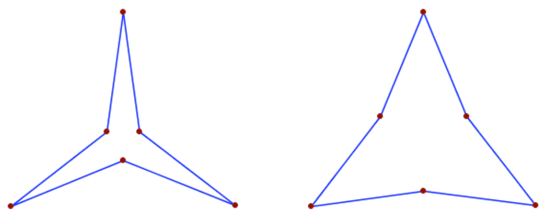


This surface coincides with that of a (5/2, ±10/3, -5/2, ±10/3). Inner vertices sit at the bases of the tall spires.

I22b_1: Hexagonal Dihemiicosahedron

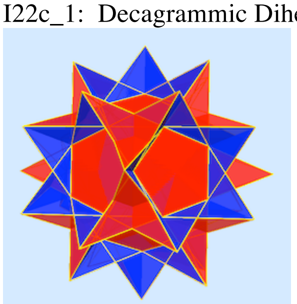


Faces:

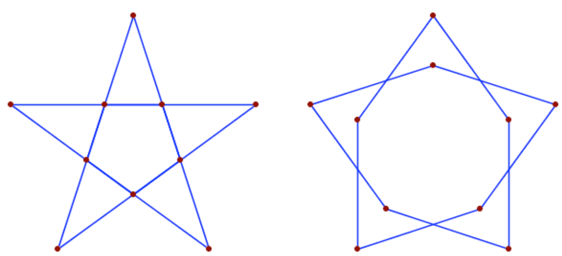


Inner vertices sit at the bases of the tall spires.

I22c_1: Decagrammic Dihemidodecahedron



Faces:

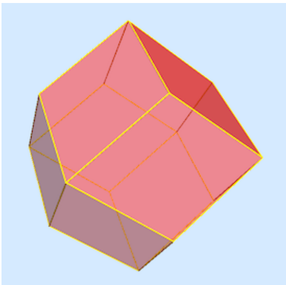


This surface also coincides with that of a (5/2, ±10/3, -5/2, ±10/3). Inner vertices coincide with non-vertex edge crossings and sit at the bases of the tall spires.

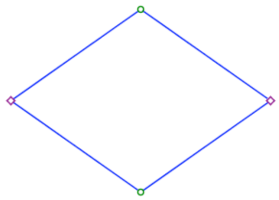
Face-Transitive Individual Polyhedra

Four of these bear superficial resemblance to regular star polyhedra, but they have significant topological differences. The others are quite well known.

O43a_1: Rhombic Dodecahedron

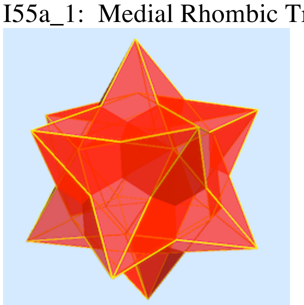


Face:

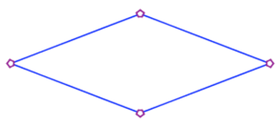


Well known. This is often claimed to be one of only two isotoxal polyhedra that are not vertex-transitive.

I55a_1: Medial Rhombic Triacontahedron

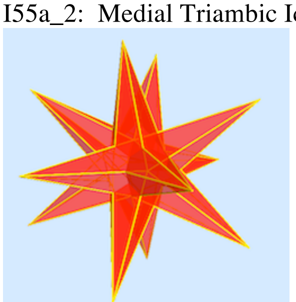


Face:

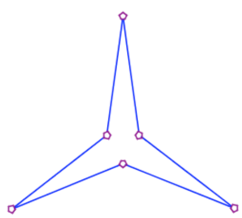


A well-known stellation of the rhombic triacontahedron. Figure 6-3 shows these faces (defined by *n*-sets 20 and 21) in relation to those of the rhombic triacontahedron (*n*-sets 20 and 22).

I55a_2: Medial Triambic Icosahedron

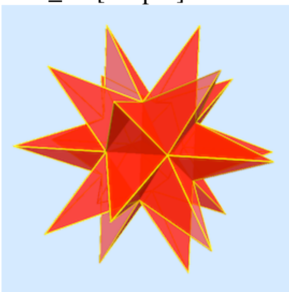


Face:

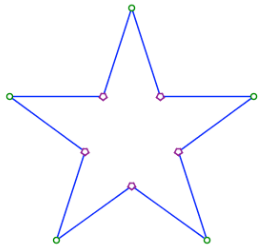


A stellation of the icosahedron. This has the outward appearance of Coxeter and DuVal's $\mathbf{De_2f_2}$ [Co82], but is a surface with internal structure.

I53a_1: [Proper] Great Stellated Dodecahedron

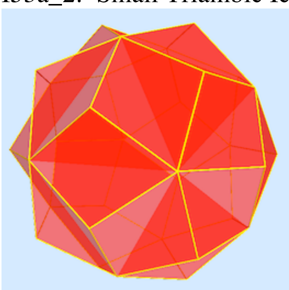


Face:

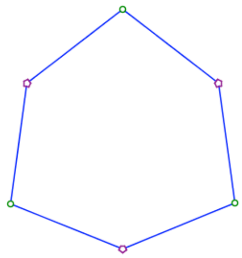


This is a form of the final stellation of the dodecahedron, one that properly corresponds to the notion of *extending* a face of the base polyhedron. The more familiar Kepler-Poinsot form has a branch point in the center of each face. Adjacent faces share 2 collinear (but non-adjacent) edges.

I53a_2: Small Triambic Icosahedron

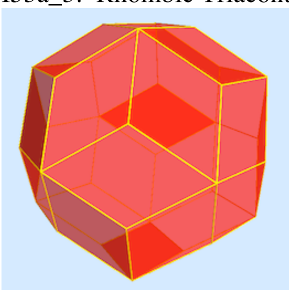


Face:

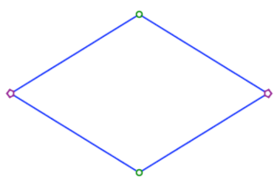


A stellation of the icosahedron. This has the outward appearance of Coxeter and DuVal's **B** [Co82], but is a surface with internal structure.

I53a_3: Rhombic Triacontahedron

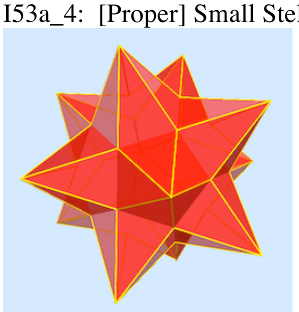


Face:

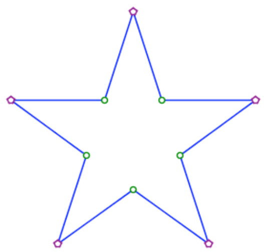


Well known. This is often claimed to be one of only two isotoxal polyhedra that are not vertex-transitive.

I53a_4: [Proper] Small Stellated Dodecahedron



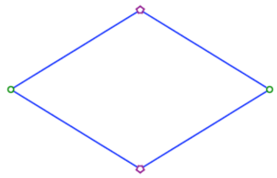
Face:



This is a form of the first stellation of the dodecahedron, one that properly corresponds to the notion of *extending* a face of the base polyhedron. The more familiar Kepler-Poinsot form has a branch point in the center of each face. Adjacent faces share 2 collinear (but non-adjacent) edges.

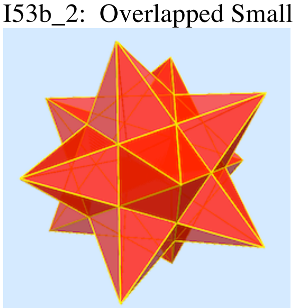
I53b_1: Great Rhombic Triacontahedron

Face:

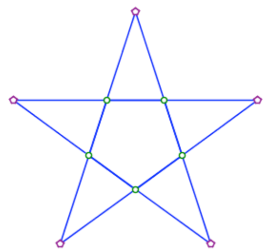


A well-known stellation of the Rhombic Triacontahedron. Figure 6-3 shows these faces (defined by *n*-sets 21 and 25) in relation to those of the rhombic triacontahedron (*n*-sets 20 and 22).

I53b_2: Overlapped Small Stellated Dodecahedron

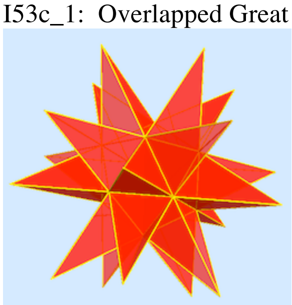


Face:

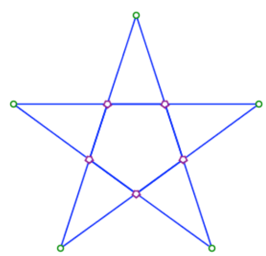


A new form of the first stellation of the dodecahedron, having faces of density 3. Perhaps the term "tristellated" could be applied? Adjacent faces share 2 collinear (partially overlapping but non-adjacent) edges.

I53c_1: Overlapped Great Stellated Dodecahedron

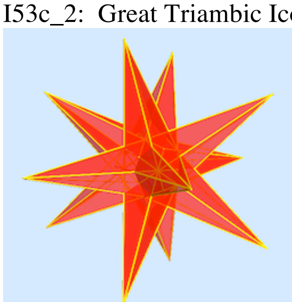


Face:

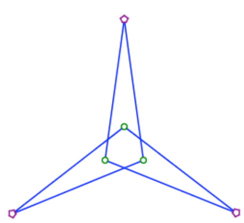


A new form of the final stellation of the dodecahedron, having faces of density 3. Perhaps the term "tristellated" could be applied? Adjacent faces share 2 collinear (partially overlapping but non-adjacent) edges.

I53c_2: Great Triambic Icosahedron



Face:



A stellation of the icosahedron. This also has the outward appearance of Coxeter and DuVal's $\mathbf{De_2f_2}$ [Co82], but is a surface with internal structure and faces of density 2.

Appendix 1: On Directed Edges

In Section 5 we considered and dismissed T33'a_1, a directed cube of Tetrahedral symmetry. Allowing such decorations and non-geometrical, non-topological aspects admits only one other possibility. Due to its even valence, the octahedron can also be viewed as a "tetratetrahedron", with directed edges or alternating colors of faces. That is both edge- and vertex-transitive.

The directed-edge polyhedra can be obtained by removing the edge-reversing elements from the symmetry group of another polyhedron. The direction assigned to the edges is arbitrary, however, and can be reversed as part of a new symmetry operation. Thus, each of these actually has the full Octahedral symmetry of the original non-directed polyhedron; the directionality ends up adding nothing new.

In the case of T33'a_1, this is so because the choice of which set of rays to designate as 3 and which as 3' is arbitrary to begin with. When the radii are equal, the geometric distinction vanishes. A cube with a pattern would still have some features to distinguish the two sets as the vertex neighborhoods would be mirror images, but the underlying polyhedron ends up with eight indistinguishable vertices and insufficient justification for its own identity.

Both of the above can be assembled into compounds of 5 components each. As Skilling points out [Sk76], those compounds only use tetrahedral symmetry of the component polyhedra. This eliminates the symmetry operations that reverse the orientation of the edges, so those compounds already essentially have directed edges. These add nothing new either.

Finally, face-intransitive polyhedra can all be considered as having a direction to their edges without changing anything about them.

Thus, we conclude that the directed edges do not add anything of significance.

Appendix 2: On Degenerate Faces

In the enumeration we rejected degenerate faces from consideration. It is worth noting what would result if we allowed them and their necessarily divalent vertices. Their inclusion would essentially produce the 9 regular polyhedra, the compounds of 2, 5, and 10 tetrahedra, and the tetrahemihexahedron, as shown below. All of these would have their edges bisected by an additional divalent vertex. No additional surface structure or geometry would be obtained; the divalent vertices add nothing of interest.

In general, such vertices are not well defined as they could be placed anywhere on an edge without contributing anything to the geometry or topology. Any number of them could be added, leading to the entire edge being made up of "vertices".

For these reasons we do not accept adjacent collinear edges.

Table A.1: Polyhedra and compounds resulting from the acceptance of degenerate faces.

Edge Type	Degenerate n-gon(s)	Defining n-set Pair(s)	Resulting Polyhedron or Compound
O43a	8 [6]s	{11,12}	Stella Octangula
O43a	4 [6]s	{11,12}	tetrahedron
O42a	8 [6]s	{11,13}	octahedron
O42a	4 [6]s and 3 equatorial [8]s	{11,13} and {15,16}	tetrahemihexahedron
O32a	6 [8]s	{1,2}	cube
I52a	20 [6]s	{3,8}	icosahedron
I52a	12 [10]s	{15,19}	great dodecahedron
I52b	20 [6]s	{4,11}	great icosahedron
I52b	12 [10/2]s	{15,18}	small stellated dodecahedron
I32a	12 [10]s	{16,18}	dodecahedron
I32b	20 [6]s	{6,10} or {7,9}	5 tetrahedra
I32b	40 [6]s	both {6,10} and {7,9}	10 tetrahedra
I32c	12 [10/2]s	{17,19}	great stellated dodecahedron

References

[Br74] N. J. Bridge, "Facetting the Dodecahedron", *Acta Crystallographica* A30 (1974), pp. 548-552.

[Br00] Max Brückner, *Vielecke und Vielflache: Theorie und Geschichte*, Teubner, Leipzig, 1900.

[Co82] H. S. M. Coxeter, P. DuVal, H. T. Flather, J. F. Petrie, *The fifty-nine icosahedra*, 2nd Edition, Springer-Verlag, NY, 1982.

[Gr94] Branko Grünbaum, "Metamorphoses of Polygons", *The Lighter Side of Mathematics*, pp 35-48, MAA, 1994.

[GS86] Branko Grünbaum and G. C. Shephard, *Tilings and Patterns*, W. H. Freeman, NY, 1986.

[GS90] Branko Grünbaum and G. C. Shephard, "Rotation and Winding Numbers for Planar Polygons and Curves", Trans. Amer. Math. Soc., Vol. 322, No. I, Nov. 1990, pp 169-187

[He74] Edmund Hess, "Ueber gleicheckige und gleichkantige Polygone", *Schriften der Gesellschaft zur Beförderung der gesammten Naturwissenschaften zu Marburg*, Band 10, Zwölfte Abhandlung, pp 611-743, 29 figures, Th. Kay, Cassel, 1874

[Sk76] J. Skilling, "Uniform compounds of uniform polyhedra", *Math. Proc. Camb. Phil. Soc.* (1976), 79, pp. 447