Standard Continuous Distributions

1 - Uniform Distribution

- PDF:
  - $f(x) = 1 / (b - a)$ for $a < x < b$
  - $f(x) = 0$ otherwise
- CDF:
  - $F(x) = 0$ for $x < a$
  - $F(x) = (x - a) / (b - a)$ for $a < x < b$
  - $F(x) = 1$ for $x > b$

The expected value (mean) and variance of X are:

- $E(x) = (a + b) / 2$
- $Var(X) = (b - a)^2 / 12$

**Inequalities

We will discuss three important inequalities: Markov's inequality, Jensen's inequality, and Chebyshev's inequality. These inequalities are particularly useful when we have limited information about the distribution of random variables but can calculate their expected values and/or variances. They allow us to derive bounds on probabilities.

Markov’s Inequality

- X is a non-negative random variable with $E[x] = \mu$
- for any $k > 0$ --> $P(X \geq k) \leq \mu/k$
  -ex a post office handles, on average, 10 4 letters per day
  (a) what is the probability that, tomorrow, it will handle at least 1.5 10 4 letters ?
  (b) and less than 1.5 10 4 letters ?

Jensen’s Inequality

- the variance of a random variable is always a positive value
  $Var(X) = E[X^2] - (E[X])^2 \geq 0$
- let X be a random variable with finite mean $\mu = E[X]$ and let $g(x) : \mathbb{R} \longrightarrow \mathbb{R}$, a convex function (i.e. $d^2g/dx^2 > 0$)
  - allora $g(E[x]) \leq E[g(x)]$

Chebyshev’s Inequality

- X is a non-negative random variable with $E[x] = \mu$ and $Var(x) = \sigma^2$
- for any $k > 0$, $P(|X - \mu| \geq k) \leq \sigma^2/k^2$

2 - The Erlang distribution

- let's consider again a Poisson process {N(t) : t ≥ 0} where N(t) represents the number of events that happened at or before time t
- j the time between events j − 1 and j
- X be the time of the n-th event
- Exponential is the time to wait for the first event to occur
- Gamma is the time to wait for the n-th event to occur
- an Erlang distribution with parameters (1, λ) is an exponential distribution

3 - Gamma

A random variable X follows a gamma distribution, denoted as X ~ Gamma(α, λ), if its PDF has the form:

$f(x) = \frac{1}{x^{\alpha-1}\lambda^{\alpha}\Gamma(\alpha)}e^{-\lambda x}$

- If X follows a Gamma(α, 1) distribution, then X/λ follows a Gamma(α, λ) distribution.
- The Gamma distribution is a generalization of the exponential density and can exhibit various shapes, from skewed to bell-shaped. It has a mode at a strictly positive value.
- The Gamma distribution naturally arises as the density of the sum of a number of independent exponential random variables.
- The cumulative distribution function (CDF) of the Gamma distribution does not have an explicit form, making it challenging to use the inverse method for variate generation.
- In Bayesian analysis, the Gamma distribution serves as a natural conjugate prior for the standard deviation of a normal distribution.

4 - Beta

A random variable X follows a beta distribution, denoted as X ~ Beta(α, β), if its probability density function (PDF) has the form:

$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$$

where 0 ≤ x ≤ 1 and α, β > 0. The parameters α and β determine the shape of the distribution.

- The beta distribution is commonly used to model random variables that are bounded between 0 and 1, such as probabilities or proportions.
- Beta densities appear in the study of the median of a sample of random points, particularly when generated from the beta distribution.
- The expected value of X is E[X] = α / (α + β), and the variance is Var(X) = αβ / ((α + β)^2(α + β + 1)).
- The central moments of X are given by E[X^n] = (αβ / ((α + β)^2(α + β + 1))) * (Γ(α + n)Γ(α + β) / (Γ(α + β + n)Γ(α))).

5 - Normal