Biblioteczka Opracowań Matematycznych

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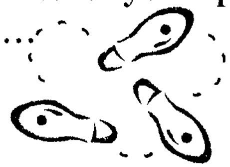
210 całek nieoznaczonych z pełnymi rozwiązaniami krok po kroku...



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Całka nieoznaczona

1. Całkowanie bezpośrednie oraz przez rozkład

Całkowanie bezpośrednie polega na zastosowaniu gotowych wzorów, które można znaleźć w każdych tablicach matematycznych z elementami matematyki wyższej. Poniżej podany jest zestaw podstawowych wzorów do bezpośredniego całkowania.

Całkowanie przez rozkład to najbardziej elementarna metoda całkowania. Polega ona na przekształcaniu wyrażenia podcałkowego tak, aby można było zastosować gotowy wzór, czyli przejść do całkowania bezpośredniego. Oczywiście nie wszystkie wyrażenia podcałkowe można przekształcić tak, aby zastosować gotowy wzór. Najczęściej stosowane przekształcenia polegają na zastosowaniu dobrze znanych wzorów, np.: wzorów skróconego mnożenia, rozdzielenia wyrażeń algebraicznych na sumę prostszych wyrażeń, wzorów trygonometrycznych ujmujących zależności pomiędzy poszczególnymi funkcjami trygonometrycznymi itp. A zatem przed przystąpieniem do całkowania dobrze jest zatrzymać się na chwilę i spróbować dostrzec możliwość zastosowania jakiegoś wzoru.

Tablica Podstawowych Całek:

(1.1) $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ (dla $n \neq -1$)

(1.2) $\int a^x dx = \frac{a^x}{\ln a} + C$ (1.3) $\int \frac{dx}{x} = \ln|x| + C$

(1.4) $\int e^x dx = e^x + C$ (1.5) $\int \cos x dx = \sin x + C$

(1.6) $\int \sin x dx = -\cos x + C$ (1.7.) $\int \frac{dx}{\cos^2 x} = tgx + C$

(1.8) $\int \frac{dx}{\sin^2 x} = -ctgx + C$ (1.9) $\int \frac{dx}{1+x^2} = arctgx + C$

(1.10) $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$ (1.11) $\int \frac{-dx}{1+x^2} = arcctgx + C$

(1.12) $\int \frac{-dx}{\sqrt{1-x^2}} = -\arccos x + C$

Podczas całkowania korzystamy z ważnych własności całek. Poniżej podane są własności całek nieoznaczonych:

(1.13) $k\int fdx = \int kfdx$ (1.14) $\int dx = x + C$

(1.15) $\int (f+g)dx = \int fdx + \int gdx$ - addytywność całki,

(1.16) $d\int fdx = fdx.$

Całkowanie to inaczej wyznaczanie tzw. funkcji pierwotnej. Całkowanie jest działaniem odwrotnym do różniczkowania (wyznaczania pochodnej). Jednak całkowanie nie jest działaniem jednoznacznym co oznacza, że funkcja f(x) może mieć nieskończenie wiele całek (rodzina funkcji). Przy obliczaniu całki nieoznaczonej dopisujemy „C" co oznacza właśnie niejednoznaczność całki.

PRZYKŁADY CAŁKOWANIA

$\int (2x^5 + 6x + \frac{10}{x})dx = \int 2x^5 dx + \int 6x dx + \int \frac{10}{x} dx = 2\int x^5 dx +$

$+ 6\int xdx + 10\int \frac{dx}{x} = \frac{2x^6}{6} + \frac{6x^2}{2} + 10\ln|x| + C = \frac{x^6}{3} + 3x^2 +$

$+10\ln|x| + C.$

2/ $\int \frac{20x^{10}+5}{x^6}dx = \int \frac{20x^{10}}{x^6}dx + \int \frac{5}{x^6}dx = 20\int \frac{x^{10}}{x^6}dx + 5\int \frac{dx}{x^6} =$

$= 20\int x^4 dx + 5\int x^{-6} dx \overset{(1.1)}{=} \frac{20x^5}{5} + \frac{5x^{-5}}{-5} + C = 4x^5 - \frac{1}{x^5} + C$

3/ $\int \frac{2x-5}{x^4}dx = \int \frac{2x}{x^4}dx - \int \frac{5}{x^4}dx = 2\int \frac{1}{x^3}dx - 5\int x^{-4}dx =$

$= 2\int x^{-3}dx - \frac{5x^{-3}}{-3} + C = \frac{2x^{-2}}{-2} + \frac{5}{3x^3} + C = \frac{-1}{x^2} + \frac{5}{3x^3} + C$

4/ $\int \frac{(x^2+1)^2}{x^3}dx = \int \frac{x^4+2x^2+1}{x^3}dx = \int \frac{x^4}{x^3}dx + 2\int \frac{x^2}{x^3}dx + \int \frac{dx}{x^3}$

$= \int xdx + 2\int \frac{dx}{x} + \int x^{-3}dx = \frac{x^2}{2} + 2\ln|x| + \frac{x^{-2}}{-2} + C = \frac{x^2}{2} +$

$+ 2\ln|x| - \frac{1}{2x^2} + C$

5/ $\int (\sqrt[3]{x} + 2\sqrt[5]{x})dx = \int x^{\frac{1}{3}}dx + 2\int x^{\frac{1}{5}}dx = \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + \frac{2x^{\frac{6}{5}}}{\frac{6}{5}} + C =$

$= \frac{3x^{\frac{4}{3}}}{4} + \frac{5x^{\frac{6}{5}}}{3} + C = \frac{3}{4}\sqrt[3]{x^4} + \frac{5}{3}\sqrt[5]{x^6} + C$

6/ $\int \left(\frac{10}{3\sqrt[3]{x}} + \frac{2}{\sqrt[3]{x^2}}\right)dx = \frac{10}{3}\int \frac{dx}{x^{\frac{1}{3}}} + 2\int \frac{dx}{x^{\frac{2}{3}}} = \frac{10}{3}\int x^{\frac{-1}{3}}dx +$

$+ 2\int x^{\frac{-2}{3}}dx = \frac{\frac{10}{3}x^{\frac{2}{3}}}{\frac{2}{3}} + \frac{2x^{\frac{1}{3}}}{\frac{1}{3}} + C = 5x^{\frac{2}{3}} + 6x^{\frac{1}{3}} + C = 5\sqrt[3]{x^2} + 6\sqrt[3]{x} + C$

7/

$$\int 2e^{x}\left(1-\frac{e^{-x}}{x^{2}}\right)dx = 2\int e^{x}dx - \int \frac{2e^{x}e^{-x}}{x^{2}}dx = 2\int e^{x}dx - 2\int x^{-2}dx =$$

$$= 2e^{x} - \frac{2x^{-1}}{-1} + C = 2e^{x} + \frac{2}{x} + C$$

8/

$$\int \frac{4\cos 2x}{3\cos^{2}x\sin^{2}x}dx = \frac{4}{3}\int\frac{\cos^{2}x-\sin^{2}x}{\cos^{2}x\sin^{2}x}dx = \frac{4}{3}\int\frac{\cos^{2}x-1+\cos^{2}x}{\cos^{2}x\sin^{2}x}dx =$$

$$= \frac{4}{3}\int\frac{2\cos^{2}x-1}{\cos^{2}x\sin^{2}x}dx = \frac{4}{3}\int\frac{2\cos^{2}x}{\cos^{2}x\sin^{2}x}dx - \frac{4}{3}\int\frac{dx}{\cos^{2}x\sin^{2}x} = \frac{8}{3}\int\frac{dx}{\sin^{2}x} -$$

$$-\frac{4}{3}\int\frac{\cos^{2}x+\sin^{2}x}{\cos^{2}x\sin^{2}x}dx = -\frac{8}{3}ctgx - \frac{4}{3}\int\frac{\cos^{2}x}{\cos^{2}x\sin^{2}x}dx - \frac{4}{3}\int\frac{\sin^{2}x}{\cos^{2}x\sin^{2}x}dx =$$

$$= -\frac{8}{3}ctgx - \frac{4}{3}\int\frac{dx}{\sin^{2}x} - \frac{4}{3}\int\frac{dx}{\cos^{2}x} = -\frac{8}{3}ctgx + \frac{4}{3}ctgx - \frac{4}{3}tgx + C =$$

$$= -\frac{4}{3}ctgx - \frac{4}{3}tgx + C$$

9/ $\int ctg^{2}x dx = \int\frac{\cos^{2}x}{\sin^{2}x}dx = \int\frac{1-\sin^{2}x}{\sin^{2}x}dx = \int\frac{dx}{\sin^{2}x} - \int dx = -ctgx - x + C$

10/ $\int\frac{10dx}{\sin^{2}x\cos^{2}x} = 10\int\frac{\sin^{2}x+\cos^{2}x}{\sin^{2}x\cos^{2}x}dx = 10\int\frac{\sin^{2}x}{\sin^{2}x\cos^{2}x}dx +$

$$+10\int\frac{\cos^{2}x}{\sin^{2}x\cos^{2}x}dx + C = 10\int\frac{dx}{\cos^{2}x} + 10\int\frac{dx}{\sin^{2}x} = 10tgx - 10ctgx + C$$

W powyższych całkach wykorzystano własności całki, wzory (1.1)-(1.10) oraz wzory:

Wzór skróconego mnożenia: $(a+b)^{2} = a^{2} + 2ab + b^{2}$

Definicje potęgi o wykładniku całkowitym i wymiernym:

(1.17) $x^{-a} = \frac{1}{x^{a}}$, (1.18) $x^{\frac{m}{n}} = \sqrt[n]{x^{m}}$

Wzory trygonometryczne:

(1.19) $\cos 2x = \cos^{2}x - \sin^{2}x$, (1.20) $ctgx = \frac{\cos x}{\sin x}$

(1.21) $\sin^{2}x + \cos^{2}x = 1$

Uwaga:
Najczęściej pojawiającym się błędem przy stosowaniu wzoru (1.1) jest pomniejszanie wykładnika potęgi o wykładniku ujemnym oraz stosowanie wzoru (1.1) dla n = -1.

2.Całkowanie przez podstawianie

Całkowanie przez podstawianie polega na wprowadzeniu nowej zmiennej. Po wprowadzeniu nowej zmiennej wykonuje się także przekształcenie różniczki dx. W praktyce oznacza to, że oznaczając: $G(x) = T(u)$ otrzymujemy $G'(x)dx = T'(u)du$, skąd po dalszych przekształceniach otrzymujemy dx. Sposób ten jasno pokażą przykłady.
W bardziej skomplikowanych przypadkach całkowania metodę tę stosujemy pośrednio przy okazji stosowania innej metody.

PRZYKŁADY CAŁKOWANIA

11/ $\int\frac{1}{2}\sin 4x dx = \frac{1}{2}\int\sin 4x dx = \left|\begin{matrix}4x = t\\ 4dx = dt\end{matrix}\right| = \frac{1}{2}\int\sin t\frac{dt}{4} =$

$$= \frac{1}{8}\int\sin t dt = -\frac{1}{8}\cos t + C = -\frac{1}{8}\cos 4x + C$$

12/ $\int 3\cos\frac{x}{5}dx = 3\int\cos\frac{x}{5}dx = \left|\begin{matrix}\frac{x}{5} = t\\ \frac{dx}{5} = dt\end{matrix}\right| = 3\int 5\cos t dt = 15\int\cos t dt =$

$$= -15\sin t + C = -15\sin\frac{x}{5} + C$$

13/

$$\int (8x-5)^5 dx = \left| \begin{array}{l} 8x-5=t \\ 8dx=dt \\ dx=\frac{dt}{8} \end{array} \right| = \int t^5 \frac{dt}{8} = \frac{1}{8}\int t^5 dt = \frac{t^6}{48} + C = \frac{(8x-5)^6}{48} + C$$

14/

$$\int \frac{4xdx}{3x^2+5} = \left| \begin{array}{l} 3x^2+5=t \\ 6xdx=dt \\ xdx=\frac{dt}{6} \end{array} \right| = 4\int \frac{\frac{dt}{6}}{t} = \frac{2}{3}\int \frac{dt}{t} = \frac{2}{3}\ln|t| + C = \frac{2}{3}\ln\left|3x^2+5\right| + C$$

15/

$$\int \frac{2e^{3x}dx}{2+5e^{3x}} = \left| \begin{array}{l} 2+5e^{3x}=t \\ 15e^{3x}dx=dt \\ e^{3x}dx=\frac{dt}{15} \end{array} \right| = 2\int \frac{\frac{dt}{15}}{t} = \frac{2}{15}\int \frac{dt}{t} = \frac{2}{15}\ln|t| + C = \frac{2}{15}\ln\left|2+5e^{3x}\right| + C$$

16/ .

$$\int ctgx dx = \int \frac{\cos x}{\sin x}dx = \left| \begin{array}{l} \sin x = t \\ \cos x dx = dt \end{array} \right| = \int \frac{dt}{t} = \ln|t| = \ln|\sin x| + C$$

17/

$$\int \sqrt{4x-1}dx = \left| \begin{array}{l} 4x-1=t^2 \\ 4dx=2tdt \\ dx=\frac{tdt}{2} \\ \sqrt{4x-1}=t \end{array} \right| = \int t\frac{tdt}{2} = \frac{1}{2}\int t^2 dt = \frac{1}{6}t^3 + C = \frac{\sqrt{(4x-1)^3}}{6} + C$$

18/

$$\int \sqrt[4]{10-2x}dx = \left| \begin{array}{l} 10-2x=t^4 \\ -2dx=4t^3dt \\ dx=-2t^3dt \\ t=\sqrt[4]{10-2x} \end{array} \right| = \int t\left(-2t^3dt\right) = -2\int t^4 dt = \frac{-2t^5}{5} + C$$

19/

$$\int e^{\frac{-x}{4}} dx = \left| \begin{array}{l} \frac{-x}{4}=t \\ \frac{-dx}{4}=dt \\ dx=-4dt \end{array} \right| = -4\int e^t dt = -4e^t + C = -4e^{\frac{-x}{4}} + C$$

20/

$$5\int e^{-x^3}x^2dx = \left| \begin{array}{l} -x^3=t \\ -3x^2dx=dt \\ x^2dx=\frac{-dt}{3} \end{array} \right| = 5\int e^t\left(\frac{-dt}{3}\right) = \frac{-5}{3}\int e^t dt = \frac{-5}{3}e^t + C = -\frac{5}{3}e^{-x^3} + C$$

21/

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = \left| \begin{array}{l} \sqrt{x}=t \\ \frac{dx}{2\sqrt{x}}=dt \\ \frac{dx}{\sqrt{x}}=2dt \end{array} \right| = 2\int e^{\sqrt{x}}\frac{dx}{\sqrt{x}} = 2\int e^t dt = 2e^t + C = 2e^{\sqrt{x}} + C$$

22/

$$\int \sqrt[4]{2x^5+10}x^4dx = \left| \begin{array}{l} 2x^5+10=t^4 \\ 10x^4dx=4t^3dt \\ x^4dx=\frac{4}{10}t^3dt \\ t=\sqrt[4]{2x^5+10} \end{array} \right| = \frac{2}{5}\int t^4dt = \frac{2}{25}t^5 + C = \frac{2}{25}\sqrt[4]{\left(2x^5+10\right)^5} + C$$

23/

$$\int\frac{2x^2dx}{\sqrt[5]{3+x^3}}=\left|\begin{array}{l}3+x^3=t^5\\3x^2dx=5t^4dt\\x^2dx=\frac{5}{3}t^4dt\\t=\sqrt[5]{3+x^3}\end{array}\right|=2\int\frac{\frac{5}{3}t^4dt}{t}=\frac{10}{3}\int t^3dt=\frac{10}{12}t^4+C=\frac{5}{6}\sqrt[5]{(3+x^3)^4}+C$$

24/

$$\int\frac{12\,xdx}{\sqrt{1-3x^2}}=\left|\begin{array}{l}1-3x^2=t^2\\-6xdx=2tdt\\xdx=-\frac{1}{3}tdt\\t=\sqrt{1-3x^2}\end{array}\right|=12\int\frac{-\frac{tdt}{3}}{t}=-4\int dt=-4t+C=-4\sqrt{1-3x^2}+C$$

25/

$$\int\frac{\cos 2x}{\sin x\cos x}dx=2\int\frac{\cos 2x}{2\sin x\cos x}dx=\int\frac{2\cos 2x}{\sin 2x}dx=\left|\begin{array}{l}\sin 2x=t\\2\cos 2xdx=dt\end{array}\right|=$$

$$=\int\frac{dt}{t}=\ln|t|+C=\ln|\sin 2x|+C$$

26/

$$\int\frac{\sin xdx}{1+3\cos x}=\left|\begin{array}{l}1+3\cos x=t\\-3\sin dx=dt\\\sin xdx=-\frac{dt}{3}\end{array}\right|=\int\frac{-\frac{dt}{3}}{t}=-\frac{1}{3}\ln|t|+C=-\frac{1}{3}\ln|1+3\cos x|+C$$

27/

$$\int\frac{dx}{x(1+\ln x)}=\left|\begin{array}{l}1+\ln x=t\\\frac{dx}{x}=dt\end{array}\right|=\int\frac{dx}{x}\frac{1}{1+\ln x}=\int\frac{dt}{t}=\ln|t|+C=\ln|1+\ln x|+C$$

28/

$$\int\sin^2x\cos xdx=\left|\begin{array}{l}\sin x=t\\\cos xdx=dt\end{array}\right|=\int t^2dt=\frac{t^3}{3}+C=\frac{\sin^3x}{3}+C$$

29/

$$\int\cos^5x\sin xdx=\left|\begin{array}{l}\cos x=t\\-\sin xdx=dt\end{array}\right|=-\int t^5dt=-\frac{t^6}{6}+C=-\frac{1}{6}\cos^6x+C$$

30/

$$\int\frac{\sin xdx}{\cos^5x}=\left|\begin{array}{l}\cos x=t\\-\sin xdx=dt\\\sin xdx=-dt\end{array}\right|=\int\frac{-dt}{t^5}=-\int t^{-5}dt=\frac{t^{-4}}{4}+C=\frac{1}{4\cos^4x}+C$$

31/

$$\int 2e^{\sin x}\cos xdx=\left|\begin{array}{l}\sin x=t\\\cos xdx=dt\end{array}\right|=2\int e^tdt=2e^t+C=2e^{\sin x}+C$$

32/

$$\int 10e^{x^5+1}x^4dx=\left|\begin{array}{l}x^5+1=t\\5x^4dx=dt\\x^4dx=\frac{dt}{5}\end{array}\right|=10\int e^t\frac{dt}{5}=2\int e^tdt=2e^t+C=2e^{x^5+1}+C$$

33/

$$\int\frac{2tgxdx}{\cos^2x}=2\int tgx\frac{dx}{\cos^2x}=\left|\begin{array}{l}tgx=t\\\frac{dx}{\cos^2x}=dt\end{array}\right|=2\int tdt=t^2+C=tg^2x+C$$

34/

$$\int\frac{dx}{e^x+e^{-x}}=\int\frac{dx}{e^x+\frac{1}{e^x}}=\int\frac{dx}{\frac{e^{2x}+1}{e^x}}=\int\frac{e^xdx}{e^{2x}+1}=\left|\begin{array}{l}e^x=t\\e^xdx=dt\end{array}\right|=\int\frac{dt}{t^2+1}$$

$$=arctgt+C=arctg(e^x)+C$$

35/

$$\int\frac{\sqrt{10+\ln|x|}}{2x}dx=\frac{1}{2}\int\frac{\sqrt{10+\ln|x|}}{x}dx=\left|\begin{array}{l}10+\ln|x|=t^2\\\frac{dx}{x}=2tdt\\t=\sqrt{10+\ln x}\end{array}\right|=\frac{1}{2}\int 2t^2dt=\int t^2dt=$$

$$=\frac{t^3}{3}+C=\frac{\sqrt{(10+\ln|x|)^3}}{3}+C$$

36/ $$\int \frac{2\cos x dx}{\sqrt{1-\sin^2 x}} = \left|\begin{array}{l} \sin x = t \\ \cos x dx = dt \end{array}\right| = 2\int \frac{dt}{\sqrt{1-t^2}} = 2\arcsin t + C = 2\arcsin(\sin x) + C$$

37/
$$\int \frac{\sqrt{5+3\ln x}}{2x}dx = \frac{1}{2}\int \frac{\sqrt{5+3\ln x}}{x}dx = \left|\begin{array}{l} 5+3\ln x = t^2 \\ \frac{3dx}{x} = 2tdt \\ \frac{dx}{x} = \frac{2}{3}tdt \\ t = \sqrt{5+3\ln x} \end{array}\right| = \frac{1}{2}\int t\,\frac{2tdt}{3} = \frac{1}{3}\int t^2 dt$$
$$= \frac{t^3}{9} + C = \frac{\sqrt{(5+3\ln x)^3}}{9} + C$$

38/ $$\int\left(e^x + e^{-x}\right)^2 dx = \int\left(e^{2x} + 2e^x e^{-x} + e^{-2x}\right)dx = \int e^{2x}dx + 2\int e^x e^{-x}dx + \int e^{-2x}dx$$
$$= \frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + C$$

39/
$$\int \frac{x^3 dx}{\cos^2 x^4} = \left|\begin{array}{l} x^4 = t \\ 4x^3 dx = dt \\ x^3 dx = \frac{dt}{4} \end{array}\right| = \int \frac{\frac{dt}{4}}{\cos^2 t} = \frac{1}{4}\int \frac{dt}{\cos^2 t} = \frac{1}{4}tgt + C = \frac{1}{4}tgx^4 + C$$

40/ $$\int \frac{dx}{5+\sqrt{x}} = \left|\begin{array}{l} x = t^2 \\ dx = 2tdt \end{array}\right| = \int \frac{2tdt}{5+t} = 2\int \frac{tdt}{5+t} = 2\int \frac{t+5-5}{t+5}dt = 2\int dt -$$
$$-10\int \frac{dt}{5+t} = 2t - 10\ln|5+t| + C = 2\sqrt{x} - 10\ln\left|5+\sqrt{x}\right| + C$$

41/
$$\int \frac{x^3 dx}{(x+1)^{10}} = \left|\begin{array}{l} x+1 = t \\ x = t-1 \\ x^3 = (t-1)^3 \\ dx = dt \end{array}\right| = \int \frac{(t-1)^3 dt}{t^{10}} = \int \frac{\left(t^3 - 3t^2 + 3t - 1\right)dt}{t^{10}} =$$
$$= \int \frac{t^3 dt}{t^{10}} - 3\int \frac{t^2 dt}{t^{10}} + 3\int \frac{tdt}{t^{10}} - \int \frac{dt}{t^{10}} = \int t^{-7}dt - 3\int t^{-8}dt + 3\int t^{-9}dt - \int t^{-10}dt =$$
$$= -\frac{1}{6t^6} + \frac{3}{7t^7} - \frac{3}{8t^8} + \frac{1}{9t^9} + C = -\frac{1}{6(x+1)^6} + \frac{3}{7(x+1)^7} - \frac{3}{8(x+1)^8} + \frac{1}{9(x+1)^9} + C$$

42/
$$\int \frac{3x+2}{3x^2+4x+10}dx = \left|\begin{array}{l} 3x^2+4x+10 = t \\ (6x+4)dx = dt \\ (3x+2)dx = \frac{dt}{2} \end{array}\right| = \int \frac{\frac{dt}{2}}{t} = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln|t| + C$$
$$= \frac{1}{2}\ln\left|3x^2+4x+10\right| + C$$

43/
$$\int \frac{dx}{6+x^2} = \frac{1}{6}\int \frac{dx}{1+\left(\frac{x}{\sqrt{6}}\right)^2} = \left|\begin{array}{l} t = \frac{x}{\sqrt{6}} \\ dt = \frac{dx}{\sqrt{6}} \\ dx = \sqrt{6}dt \end{array}\right| = \frac{1}{6}\int \frac{\sqrt{6}dt}{1+t^2} = \frac{\sqrt{6}}{6}\int \frac{dt}{1+t^2} =$$
$$= \frac{1}{\sqrt{6}}arctgt + C = \frac{1}{\sqrt{6}}arctg\frac{x}{\sqrt{6}} + C$$

44/
$$\int \frac{dx}{x^2+16} = \int \frac{dx}{16\left[\left(\frac{x}{4}\right)^2 + 1\right]} = \frac{1}{16}\int \frac{dx}{\left(\frac{x}{4}\right)^2 + 1} = \left|\begin{array}{l} \frac{x}{4} = t \\ \frac{dx}{4} = dt \\ dx = 4dt \end{array}\right| = \frac{1}{16}\int \frac{4dt}{t^2+1}$$
$$= \frac{1}{4}\int \frac{dt}{t^2+1} = \frac{1}{4}arctgt + C = \frac{1}{4}arctg\frac{x}{4} + C$$

45/ $$\int \frac{dx}{\sqrt{5-x^2}} = \int \frac{dx}{\sqrt{5\left(1-\left(\frac{x}{\sqrt{5}}\right)^2\right)}} = \frac{1}{\sqrt{5}}\int \frac{dx}{\sqrt{1-\left(\frac{x}{\sqrt{5}}\right)^2}} = \left|\begin{array}{l} \frac{x}{\sqrt{5}} = t \\ \frac{dx}{\sqrt{5}} = dt \end{array}\right| =$$
$$= \frac{1}{\sqrt{5}}\int \frac{\sqrt{5}dt}{\sqrt{1-t^2}} = \frac{\sqrt{5}}{\sqrt{5}}\int \frac{dt}{\sqrt{1-t^2}} \arcsin t + C = \arcsin\frac{x}{\sqrt{5}} + C$$

46/

$$\int x^2\sqrt{x^3+10}\,dx = \left|\begin{array}{l} x^3+10=t^2 \\ 3x^2dx = 2tdt \\ x^2dx = \frac{2dt}{3} \end{array}\right| = \frac{2}{3}\int t^2 dt = \frac{2}{9}t^3 + C = \frac{2\sqrt{(x^3+10)^3}}{9} + C$$

47/

$$\int \frac{xdx}{x^4+1} = \left|\begin{array}{l} x^2 = t \\ 2xdx = dt \\ xdx = \frac{dt}{2} \end{array}\right| = \frac{1}{2}\int \frac{dt}{t^2+1} = \frac{1}{2}arctgt + C = \frac{1}{2}arctgx^2 + C$$

48/

$$\int \sin^3 xdx = \int \sin^2 x \sin xdx = \int (1-\cos^2 x)\sin xdx = \int \sin xdx - \int \cos^2 x \sin xdx =$$

$$= -\cos x - \left|\begin{array}{l} \cos x = t \\ -\sin xdx = dt \end{array}\right| = -\cos x + \int t^2 dt = -\cos x + \frac{t^3}{3} + C = -\cos x + \frac{\cos^3 x}{3} + C$$

49/

$$\int \frac{5x^2dx}{\sqrt{1-x^6}} = \left|\begin{array}{l} x^3 = t \\ 3x^2dx = dt \\ x^2dx = \frac{dt}{3} \end{array}\right| = 5\int \frac{\frac{dt}{3}}{\sqrt{1-t^2}} = \frac{5}{3}\int \frac{dt}{\sqrt{1-t^2}} = \frac{5}{3}\arcsin t + C = \frac{5}{3}\arcsin x^3 + C$$

Całkowanie przez podstawienie to jedna z najważniejszych metod całkowania. Stosujemy ją wówczas gdy zastosowanie nowej zmiennej dla fragmentu wyrażenia podcałkowego upraszcza całkę. Nie ma niestety ogólnych przepisów kiedy i jak tego dokonać. Umiejętność doboru odpowiedniego podstawienia nabywa się drogą wprawy. Z całą pewnością metodę tą możemy zastosować gdy licznik ułamka podcałkowego jest pochodną mianownika. Korzystamy wówczas ze wzoru:

$$\int \frac{f'(x)dx}{f(x)} = \ln|f(x)| \qquad (1.22)$$

3. Całkowanie przez części

Całkowanie przez części stosuje się wówczas, gdy pod całką występuje iloczyn funkcji algebraicznej lub przestępnej. Całkowanie przez części odbywa się według wzoru:

$$\int udv = uv - \int vdu \qquad (1.23)$$

przy czym jako funkcję u, przyjmuje się funkcję, której rózniczkowanie upraszcza wyrażenie podcałkowe, a za dv tę część wyrażenia podcałkowego, którego całka jest znana lub może być łatwo wyznaczona.

PRZYKŁADY CAŁKOWANIA

50/

$$\int x\sin xdx = \left|\begin{array}{ll} u = x & du = dx \\ dv = \sin xdx & v = -\cos x \end{array}\right| = -x\cos x + \int \cos xdx =$$

$$= -x\cos x + \sin x + C$$

51/

$$\int x^2\cos xdx = \left|\begin{array}{ll} u = x^2 & du = 2xdx \\ dv = \cos xdx & v = \sin x \end{array}\right| = x^2\sin x - 2\int x\sin xdx =$$

$$= x^2\sin x + 2x\cos x - 2\sin x + C$$

(Dla obliczenia całki (51) wykorzystano całkę (50)).

52/

$$\int xe^x dx = \left|\begin{array}{ll} u = x & du = dx \\ dv = e^x dx & v = e^x \end{array}\right| = xe^x - \int e^x dx = xe^x - e^x + C$$

53/

$$\int x^3e^x dx = \left|\begin{array}{ll} u = x^3 & du = 3x^2dx \\ dv = e^x dx & v = e^x \end{array}\right| = x^3e^x - 3\int x^2e^x dx =$$

$$= x^3e^x - 3\left|\begin{array}{ll} u = x^2 & du = 2xdx \\ dv = e^x dx & v = e^x \end{array}\right| = x^3e^x - 3\left(x^2e^x - 2\int xe^x dx\right) =$$

$$= x^3e^x - 3x^2e^x + 6\int xe^x dx = x^3e^x - 3x^2e^x + 6\left|\begin{array}{ll} u = x & du = dx \\ dv = e^x dx & v = e^x \end{array}\right| =$$

$$= x^3e^x - 3x^2e^x + 6xe^x - 6\int e^x dx = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$$

54/ $\int e^x \sin x dx = \left| \begin{array}{ll} u = e^x & du = e^x dx \\ dv = \sin x dx & v = -\cos x \end{array} \right| = -e^x \cos x + \int e^x \cos x dx =$

$= -e^x \cos x + \left| \begin{array}{ll} u = e^x & du = e^x dx \\ dv = \cos dx & v = \sin x \end{array} \right| = -e^x \cos x + e^x \sin x - \int e^x \sin x dx$

A zatem biorąc pod uwagę początek i koniec obliczeń mamy:

$$\int e^x \sin x dx = -e^x \cos x + e^x \sin x - \int e^x \sin x dx$$

$$2 \int e^x \sin x dx = e^x \sin x - e^x \cos x$$

$$\int e^x \sin x dx = \frac{e^x \sin x - e^x \cos x}{2}$$

55/
$\int e^{-2x} \sin 3x dx = \left| \begin{array}{ll} u = e^{-2x} & du = -2e^{-2x} dx \\ dv = \sin 3x dx & v = \frac{-\cos 3x}{3} \end{array} \right| = \frac{-e^{-2x} \cos 3x}{3} - \frac{2}{3} \int e^{-2x} \cos 3x dx$

$= \frac{-e^{-2x} \cos 3x}{3} - \frac{2}{3} \left| \begin{array}{ll} u = e^{-2x} & du = -2e^{-2x} dx \\ dv = \cos 3x dx & v = \frac{\sin 3x}{3} \end{array} \right| = \frac{-e^{-2x} \cos 3x}{3} -$

$-\frac{2}{3}\left(\frac{e^{-2x} \sin 3x}{3} + \frac{2}{3} \int e^{-2x} \sin 3x dx \right) = \frac{-e^{-2x} \cos 3x}{3} - \frac{2e^{-2x} \sin 3x}{9} - \frac{4}{9} \int e^{-2x} \sin 3x dx$

Stąd: $\frac{13}{9} \int e^{-2x} \sin 3x dx = \frac{-e^{-2x} \cos 3x}{3} - \frac{2e^{-2x} \sin 3x}{9}$

$$\int e^{-2x} \sin 3x dx = \frac{-3e^{-2x} \cos 3x}{13} - \frac{2e^{-2x} \sin 3x}{13}$$

56/ $\int \sin^2 \frac{x}{3} dx = \left| \begin{array}{ll} \frac{x}{3} = t & \frac{dx}{3} = dt \\ dx = 3dt & \end{array} \right| = 3 \int \sin^2 t dt = 3 \int \sin t \sin t dt = \left| \begin{array}{ll} u = \sin t & du = \cos t dt \\ dv = \sin t dt & v = -\cos t \end{array} \right| =$

$= -3 \sin t \cos t + 3 \int \cos^2 t dt = -3 \sin t \cos t + 3 \int (1 - \sin^2 t) dt = -3 \sin t \cos t + 3 \int dt - 3 \int \sin^2 t dt$

$3 \int \sin^2 t dt = -3 \sin t \cos t + 3t - 3 \int \sin^2 t dt$

$\int \sin^2 t dt = \frac{-3 \sin t \cos t}{6} + \frac{3t}{6}$ $\qquad \int \sin^2 \frac{x}{3} = \frac{-\sin \frac{x}{3} \cos \frac{x}{3}}{2} + \frac{x}{6}$

57/ $\int \cos^2 \frac{x}{4} dx = \left| \begin{array}{l} \frac{x}{4} = t \\ dx = 4dt \end{array} \right| = 4 \int \cos^2 t dt = \left| \begin{array}{ll} u = \cos t & du = -\sin t dt \\ dv = \cos t dt & v = \sin t \end{array} \right| =$

$4\left(\cos t \sin t + \int \sin^2 t dt\right) = 4 \cos t \sin t + 4 \int (1 - \cos^2 t) dt = 4 \cos t \sin t + 4 \int dt - 4 \int \cos^2 t dt$

$8 \int \cos^2 t dt = 4t + 4 \cos t \sin t$

$\int \cos^2 t dt = \frac{t}{2} + \frac{\cos t \sin t}{2} + C = \frac{x}{8} + \frac{\cos \frac{x}{4} \sin \frac{x}{4}}{2} + C$

58/
$\int \sqrt{x} \ln x dx = \left| \begin{array}{ll} u = \ln x & du = \frac{dx}{x} \\ dv = x^{\frac{1}{2}} dx & v = \frac{2}{3} x^{\frac{3}{2}} \end{array} \right| = \frac{2}{3} x^{\frac{3}{2}} \ln|x| - \frac{2}{3} \int \frac{x^{3/2}}{x} dx = \frac{2}{3} x^{3/2} \ln|x| - \frac{2}{3} \int x^{1/2} dx$

$= \frac{2}{3} x^{3/2} \ln|x| - \frac{4}{9} x^{3/2} + C$

59/ $\int \ln|x| dx = \left| \begin{array}{ll} u = \ln|x| & du = \frac{dx}{x} \\ dv = dx & v = x \end{array} \right| = x \ln|x| - \int dx = x \ln|x| - x + C$

60/ $\int (\ln|x|)^2 dx = \int \ln|x| \ln|x| dx = \left| \begin{array}{ll} u = \ln|x| & du = \frac{dx}{x} \\ dv = \ln|x| dx & v = x \ln|x| - x \end{array} \right|$

$= \ln|x|(x \ln|x| - x) - \int \ln|x| dx + \int dx = x \ln^2|x| - 2x \ln|x| + 2x + C$

61/
$\int \frac{(\ln|x|)^2}{x^5} dx = \left| \begin{array}{ll} u = \ln^2|x| & du = \frac{2 \ln|x| dx}{x} \\ dv = x^{-5} dx & v = \frac{-1}{4x^4} \end{array} \right| = -\frac{(\ln|x|)^2}{4x^4} + \frac{1}{2} \int \frac{\ln|x| dx}{x^5} =$

$= -\frac{(\ln|x|)^2}{4x^4} + \frac{1}{2} \left| \begin{array}{ll} u = \ln|x| & du = \frac{dx}{x} \\ dv = x^{-5} dx & v = \frac{-1}{4x^4} \end{array} \right| = -\frac{(\ln|x|)^2}{4x^4} - \frac{\ln|x|}{8x^4} + \frac{1}{8} \int \frac{dx}{x^5} =$

$$= -\frac{(\ln|x|)^2}{4x^4} - \frac{\ln|x|}{8x^4} - \frac{1}{32x^4} + C$$

62/ $$\int x\ln^2|x|dx = \left|\begin{array}{ll} u = \ln^2|x| & du = \frac{2\ln|x|}{x}dx \\ dv = xdx & v = \frac{x^2}{2}\end{array}\right| = \frac{x^2\ln^2|x|}{2} - \int x\ln x dx = \frac{x^2\ln^2 x}{2} -$$

$$-\left|\begin{array}{ll} u = \ln|x| & du = \frac{dx}{x} \\ dv = x & v = \frac{x^2}{2}\end{array}\right| = \frac{x^2\ln^2|x|}{2} - \frac{x^2\ln|x|}{2} + \frac{1}{2}\int xdx = \frac{x^2\ln^2|x|}{2} - \frac{x^2\ln|x|}{2} + \frac{x^2}{4} + C$$

63/ $$\int\frac{\ln|x|}{x^2}dx = \left|\begin{array}{ll} u = \ln|x| & du = \frac{dx}{x} \\ dv = \frac{dx}{x^2} & v = \frac{-1}{x}\end{array}\right| = -\frac{\ln|x|}{x} + \int\frac{dx}{x^2} = -\frac{\ln|x|}{x} - \frac{1}{x} + C$$

64/ $$\int x\ln|x-1|dx = \left|\begin{array}{ll} u = \ln|x-1| & du = \frac{dx}{x-1} \\ dv = xdx & v = \frac{x^2}{2}\end{array}\right| = \frac{x^2\ln|x-1|}{2} - \frac{1}{2}\int\frac{x^2dx}{x-1} =$$

$$= \frac{x^2\ln|x-1|}{2} - \frac{1}{2}\int x + 1 + \frac{1}{x-1}dx = \frac{x^2\ln|x-1|}{2} - \frac{x^2}{4} - \frac{x}{2} - \frac{\ln|x-1|}{2} + C$$

65/ $$\int\cos(\ln x)dx = \left|\begin{array}{ll} u = \cos(\ln x) & du = -\frac{\sin(\ln x)dx}{x} \\ dv = dx & v = x\end{array}\right| = x\cos(\ln x) + \int\sin(\ln x)dx =$$

$$= x\cos(\ln x) + \left|\begin{array}{ll} u = \sin(\ln x) & du = \frac{\cos(\ln x)}{x} \\ dv = dx & v = x\end{array}\right| = x\cos(\ln x) + x\sin(\ln x) - \int\cos(\ln x)dx$$

$$2\int\cos(\ln x)dx = x\cos(\ln x) + x\sin(\ln x)$$

$$\int\cos(\ln x)dx = \frac{x\cos(\ln x)}{2} + \frac{x\sin(\ln x)}{2} + C$$

66/ $$\int\arcsin x dx = \left|\begin{array}{ll} u = \arcsin x & du = \frac{dx}{\sqrt{1-x^2}} \\ dv = dx & v = x\end{array}\right| = x\arcsin x - \int\frac{xdx}{\sqrt{1-x^2}} =$$

$$\left|\begin{array}{l} 1 - x^2 = t^2 \\ xdx = -tdt\end{array}\right| = x\arcsin x + \int dt = x\arcsin x + \sqrt{1-x^2} + C$$

67/ $$\int arctgx dx = \left|\begin{array}{ll} u = arctgx & du = \frac{dx}{1+x^2} \\ dv = dx & v = x\end{array}\right| = x arctgx - \int\frac{xdx}{1+x^2} =$$

$$= x arctgx - \left|1 + x^2 = t \quad 2xdx = dt \quad xdx = \frac{dt}{2}\right| = x arctgx - \frac{1}{2}\int\frac{dt}{t} =$$

$$= x arctgx - \frac{1}{2}\ln\left|1 + x^2\right| + C$$

68/ $$\int x arctgx dx = \left|\begin{array}{ll} u = arctgx & du = \frac{dx}{1+x^2} \\ dv = xdx & v = \frac{x^2}{2}\end{array}\right| = \frac{x^2 arctgx}{2} - \frac{1}{2}\int\frac{x^2dx}{1+x^2} = \frac{x^2 arctgx}{2} -$$

$$-\frac{1}{2}\int\frac{x^2+1-1}{1+x^2}dx = \frac{x^2 arctgx}{2} - \frac{1}{2}\int dx + \frac{1}{2}\int\frac{dx}{1+x^2} = \frac{x^2 arctgx}{2} - \frac{x}{2} + \frac{arctgx}{2} + C$$

69/ $$\int x^2 arctgx dx = \left|\begin{array}{ll} u = arctgx & du = \frac{dx}{1+x^2} \\ dv = x^2dx & v = \frac{x^3}{3}\end{array}\right| = \frac{x^3 arctgx}{3} - \frac{1}{3}\int\frac{x^3dx}{1+x^2} =$$

$$= \frac{x^3 arctgx}{3} - \frac{1}{3}\int\left(x + \frac{x}{1+x^2}\right)dx = \frac{x^3 arctgx}{3} - \frac{x^2}{6} - \frac{1}{3}\left|\begin{array}{l} 1 + x^2 = t \\ 2xdx = dt\end{array}\right| =$$

$$= \frac{x^3 arctgx}{3} - \frac{x^2}{6} - \frac{1}{6}\int\frac{dt}{t} = \frac{x^3 arctgx}{3} - \frac{x^2}{6} - \frac{\ln\left|1 + x^2\right|}{6} + C$$

70/

$$\int \frac{x\,dx}{\sin^2 x} = \left| \begin{array}{ll} u = x & du = dx \\ dv = \dfrac{dx}{\sin^2 x} & v = -ctgx \end{array} \right| = -xctgx + \int ctgx\,dx = -xctgx + \ln|\sin x| + C$$

71/

$$\int \frac{x \cos x\,dx}{\sin^3 x} = \left| \begin{array}{ll} u = x & du = dx \\ dv = \dfrac{\cos x\,dx}{\sin^3 x} & v = \int dv \end{array} \right| = I$$

Pomocniczo obliczamy:

$$\int \frac{\cos x\,dx}{\sin^3 x} = \left| \begin{array}{l} \sin x = t \\ \cos x\,dx = dt \end{array} \right| = \int \frac{dt}{t^3} = \int t^{-3}\,dt = \frac{-1}{2t^2} = -\frac{1}{2\sin^2 x}$$

$$I = -\frac{x}{2\sin^2 x} + \frac{1}{2}\int \frac{dx}{\sin^2 x} = -\frac{x}{2\sin^2 x} - \frac{1}{2}ctgx + C$$

72/

$$\int \arccos x\,dx = \left| \begin{array}{ll} u = \arccos x & du = \dfrac{-dx}{\sqrt{1-x^2}} \\ dv = dx & v = x \end{array} \right| = x\arccos x + \int \frac{x\,dx}{\sqrt{1-x^2}} =$$

$$= x\arccos x + \left| \begin{array}{l} 1-x^2 = t^2 \\ -2x\,dx = 2t\,dt \end{array} \right| = x\arccos x - \int \frac{t\,dt}{t} = x\arccos x - \sqrt{1-x^2} + C$$

73/

$$\int \arccos^2 x\,dx = \left| \begin{array}{ll} u = \arccos x & du = \dfrac{-dx}{\sqrt{1-x^2}} \\ dv = \arccos x\,dx & v = x\arccos x - \sqrt{1-x^2} \end{array} \right| =$$

$$= x\arccos^2 x - \sqrt{1-x^2}\arccos x - \int dx + \int \frac{x\arccos x\,dx}{\sqrt{1-x^2}} = x\arccos^2 x - 2\sqrt{1-x^2}\arccos x - 2x + C$$

74/

$$\int \frac{\arcsin\frac{x}{2}}{\sqrt{2-x}}\,dx = \left| \begin{array}{ll} u = \arcsin\dfrac{x}{2} & du = \dfrac{dx}{2\sqrt{1-\left(\frac{x}{2}\right)^2}} \\ dv = \dfrac{dx}{\sqrt{2-x}} & v = \int dv = -2\sqrt{2-x} \end{array} \right| = -2\sqrt{2-x}\arcsin\frac{x}{2} + \int \frac{\sqrt{2-x}}{\sqrt{1-\frac{x^2}{4}}}\,dx$$

Pomocniczo obliczymy całkę:

$$\int \frac{\sqrt{2-x}}{\sqrt{\frac{4-x^2}{4}}}\,dx = 2\int \sqrt{\frac{2-x}{(2-x)(2+x)}}\,dx = 2\int \frac{dx}{\sqrt{2+x}} = \left| \begin{array}{l} 2+x = z^2 \\ dx = 2z\,dz \end{array} \right| = 4\int dz = 4\sqrt{2+x} + C$$

$$\int \frac{\arcsin\frac{x}{2}}{\sqrt{2-x}}\,dx = 4\sqrt{2+x} - 2\sqrt{2-x}\arcsin\frac{x}{2} + C$$

75/

$$\int x\sin x\cos x\,dx = \frac{1}{2}\int 2x\sin x\cos x\,dx = \frac{1}{2}\int x\sin 2x\,dx = \frac{1}{2}\left| \begin{array}{ll} u = x & du = dx \\ dv = \sin 2x\,dx & v = \dfrac{-\cos 2x}{2} \end{array} \right| = -\frac{x}{4}\cos 2x +$$

$$+ \frac{1}{4}\int \cos 2x\,dx = -\frac{x}{4}\cos 2x + \frac{1}{8}\sin 2x + C$$

76/

$$\int \frac{(x-1)e^x}{x^2}\,dx = \int \frac{xe^x}{x^2}\,dx - \int \frac{e^x}{x^2}\,dx = \int \frac{e^x}{x}\,dx - \int \frac{e^x}{x^2}\,dx = \left| \begin{array}{ll} u = \dfrac{1}{x} & du = \dfrac{-dx}{x^2} \\ dv = e^x\,dx & v = e^x \end{array} \right| - \int \frac{e^x}{x^2}\,dx$$

$$= \frac{e^x}{x} + \int \frac{e^x}{x^2}\,dx - \int \frac{e^x}{x^2}\,dx = \frac{e^x}{x} + C$$

4. Całki funkcji wymiernych

Jeżeli wyrażenie podcałkowe ma postać funkcji wymiernej to mamy do czynienia z całkowaniem funkcji wymiernej. W zależności od postaci tej funkcji możemy stosować różne metody całkowania. Mogą wystąpić zatem przypadki:

a/ ułamek podcałkowy jest właściwy (tzn. mianownik ma niższy stopień niż licznik), wówczas rozkładamy mianownik na czynniki a cały ułamek rozkładamy na sumę ułamków prostych pierwszego lub drugiego rodzaju – *metoda współczynników nieoznaczonych*.

Funkcję wymierną jednej zmiennej postaci:

(1.24) $\frac{A}{(x-a)^n}$ nazywamy ułamkiem prostym pierwszego rodzaju.

Funkcję wymierną jednej zmiennej postaci:

(1.25) $\frac{Ax+B}{(x^2+bx+c)^n}$ nazywamy ułamkiem prostym drugiego rodzaju.

b/ ułamek podcałkowy jest niewłaściwy, wówczas należy wyłączyć wyrażenie całkowite poprzez wykonanie dzielenia wielomianów. Resztę z tego dzielenia należy zapisać w postaci ułamka właściwego i postąpić jw.
c/ licznik ułamka podcałkowego jest pochodną mianownika tego ułamka. Należy wówczas zastosować wzór (1.22) ze str. 14.
d/ licznik ułamka podcałkowego można rozłożyć na składniki, z których jeden jest pochodną mianownika a drugi stanowi nowy przypadek.
e/ funkcję wymierną przez odpowiednie podstawienie da się sprowadzić do postaci funkcji wymiernej, której całka jest postaci *arctgx*.

(1.26) $$\int \frac{dx}{(x-k)^2+b} = \left| \begin{matrix} x-k=\sqrt{b}t \\ dx=\sqrt{b}dt \end{matrix} \right| = \int \frac{\sqrt{b}dt}{bt^2+b} = \frac{\sqrt{b}}{b}\int\frac{dt}{t^2+1} = \frac{1}{\sqrt{b}} arctgt + C = $$
$$= \frac{1}{\sqrt{b}} arctg\left(\frac{x-k}{\sqrt{b}}\right)$$

PRZYKŁADY CAŁKOWANIA

77/ $$\int \frac{dx}{(2x-5)^3} = \left| \begin{matrix} 2x-5=t \\ 2dx=dt \end{matrix} \right| = \frac{1}{2}\int\frac{dt}{t^3} = -\frac{1}{4t^2} + C = -\frac{1}{4(2x-5)^2} + C$$

78/ $$\int \frac{6x+5}{3x^2+5x-3}dx = \ln\left|3x^2+5x-3\right| + C$$

79/ $$\int \frac{4x-20}{2x^2-20x+100}dx = \ln\left|2x^2-20x+100\right| + C$$

80/ $$\int \frac{6+x}{12x+x^2}dx = \frac{1}{2}\int\frac{2(6+x)}{12x+x^2}dx = \frac{1}{2}\int\frac{12+2x}{12x+x^2}dx = \frac{1}{2}\ln\left|12x+x^2\right| + C$$

81/ $$\int \frac{dx}{x^2+4x+4} = \int\frac{dx}{(x+2)^2} = \left| \begin{matrix} x+2=t \\ dx=dt \end{matrix} \right| = \int\frac{dt}{t^2} = -\frac{1}{t} = -\frac{1}{x+2} + C$$

82/ $$I = \int \frac{dx}{-x^2+6x-5} = -\int\frac{dx}{(x-5)(x-1)}$$

Pomocniczo rozkładamy funkcję wymierną na ułamki proste. A zatem:

$$\frac{1}{(x-5)(x-1)} = \frac{A}{x-5} + \frac{B}{x-1} = \frac{A(x-1)+B(x-5)}{(x-5)(x-1)} = \frac{x(A+B)-A-5B}{(x-5)(x-1)}$$

Stąd: $\begin{cases} A+B=0 \\ -A-5B=1 \end{cases}$ $\begin{cases} A=\frac{1}{4} \\ B=-\frac{1}{4} \end{cases}$

$$I = -\frac{1}{4}\int\frac{dx}{x-5} + \frac{1}{4}\int\frac{dx}{x-1} = -\frac{1}{4}\ln|x-5| + \frac{1}{4}\ln|x-1| + C$$

83/ $$\int\frac{dx}{4x-5x^2} = \int\frac{dx}{x(4-5x)} = I$$

$$\frac{1}{x(4-5x)} = \frac{A}{x} + \frac{B}{4-5x} = \frac{x(B-5A)+4A}{x(4-5x)}$$

$\begin{cases} B-5A=0 \\ 4A=1 \end{cases}$ $\begin{cases} A=\frac{1}{4} \\ B=\frac{5}{4} \end{cases}$

$$I = \frac{1}{4}\int\frac{dx}{x} + \frac{5}{4}\int\frac{dx}{4-5x} = \frac{\ln|x|}{4} - \frac{\ln|4-5x|}{4} + C$$

84/ $$\int\frac{3x+4}{x^2-x-2}dx = \int\frac{\frac{3}{2}(2x-1)+\frac{11}{2}}{x^2-x-2}dx = \frac{3}{2}\int\frac{2x-1}{x^2-x-2}dx + \frac{11}{2}\int\frac{dx}{x^2-x-2} =$$
$$= \frac{3}{2}\ln\left|x^2-x-2\right| + \frac{11}{2}\int\frac{dx}{\left(x-\frac{1}{2}\right)^2-\frac{9}{4}} = \frac{3}{2}\ln\left|x^2-x-2\right| + \frac{11}{3} arctg \frac{2\left(x-\frac{1}{2}\right)}{3} + C$$

+ − **85/** $\int \frac{x-4}{(x-2)(x-3)}dx = I$

$$\frac{x-4}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3} = \frac{x(A+B)-3A-2B}{(x-2)(x-3)}$$

$$\begin{cases} A+B=1 \\ -3A-2B=-4 \end{cases} \qquad \begin{cases} A=2 \\ B=-1 \end{cases}$$

?

$$I = 2\int \frac{dx}{x-2} - \int \frac{dx}{x-3} = 2\ln|x-2| - \ln|x-3| + C$$

86/ $\int \frac{2x+7}{x^2+x-2}dx = \int \frac{2x+1+6}{x^2+x-2}dx = \int \frac{2x+1}{x^2+x-2}dx + \int \frac{6dx}{x^2+x-2} =$

$= \ln|x^2+x-2| + I$

$$\frac{6}{x^2+x-2} = \frac{A}{x+2} + \frac{B}{x-1} = \frac{x(A+B)-A+2B}{(x+2)(x-1)}$$

$$\begin{cases} A+B=0 \\ -A+2B=6 \end{cases} \qquad \begin{cases} A=-2 \\ B=2 \end{cases}$$

$$I = \int \frac{-2dx}{x+2} + \int \frac{2dx}{x-1} = -2\ln|x+2| + 2\ln|x-1| + C$$

$$\int \frac{2x+7}{x^2+x-2}dx = \ln|x^2+x-2| - 2\ln|x+2| + 2\ln|x-1| + C$$

? **87/** $\int \frac{3x^2+2x-3}{x^3-x}dx = \int \frac{3x^2-1+2x-2}{x(x-1)}dx = \int \frac{3x^2-1}{x^3-x}dx + \int \frac{2x-2}{x(x^2-1)}dx =$

$= \ln|x^3-x| + 2\int \frac{dx}{x(x+1)} = \ln|x^3-x| + I$

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{x(A+B)+A}{x(x+1)}$$

$$\begin{cases} A+B=0 \\ A=1 \end{cases} \qquad \begin{cases} A=1 \\ B=-1 \end{cases}$$

$$I = 2\int \frac{dx}{x} - 2\int \frac{dx}{x+1} = 2\ln|x| - 2\ln|x+1|$$

$$\int \frac{3x^2+2x-3}{x^3-x}dx = \ln|x^3-x| + 2\ln|x| - 2\ln|x+1| + C$$

? **88/** $\int \frac{(x+1)^3}{x^2-x}dx = \int\left(x+4+\frac{7x+1}{x^2-x}\right)dx = \frac{x^2}{2} + 4x + \int \frac{7x+1}{x^2-x}dx = \frac{x^2}{2} + 4x + I$

$$\frac{7x+1}{x^2-x} = \frac{7x+1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} = \frac{x(A+B)-B}{x(x-1)}$$

$$\begin{cases} A+B=7 \\ -B=1 \end{cases} \qquad \begin{cases} A=8 \\ B=-1 \end{cases}$$

$$I = \int \frac{8dx}{x} - \int \frac{dx}{x-1} = 8\ln|x| - \ln|x-1| + C$$

+1 +8

$$\int \frac{(x+1)^3}{x^2-x}dx = \frac{x^2}{2} + 4x - 8\ln|x| - \ln|x-1| + C$$

? **89/** $\int \frac{dx}{x^4-x^3+x^2} = I$

$$\frac{1}{x^2(x^2-x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2-x+1} = \frac{x^3(A+C)-x^2(A-B-D)+x(A-B)+B}{x^2(x^2-x+1)}$$

$$\begin{cases} A+C=0 \\ -A+B+D=0 \\ A-B=0 \\ B=1 \end{cases} \qquad \begin{cases} A=1 \\ B=1 \\ C=-1 \\ D=0 \end{cases}$$

$$I = \int \frac{dx}{x} + \int \frac{dx}{x^2} - \int \frac{xdx}{x^2-x+1} = \ln|x| - \frac{1}{x} - I_1$$

$$I_1 = \int \frac{xdx}{x^2-x+1} = \int \frac{x-\frac{1}{2}+\frac{1}{2}}{x^2-x+1}dx = \frac{1}{2}\int \frac{2\left(x-\frac{1}{2}\right)}{x^2-x+1} + \frac{1}{2}\int \frac{dx}{x^2-x+1} = \frac{1}{2}\ln|x^2-x+1| + \frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= \frac{1}{2}\ln|x^2-x+1| + \frac{\sqrt{3}}{3}arctg\frac{2x-1}{\sqrt{3}} + C$$

$$I = \ln|x| - \frac{1}{x} - \frac{1}{2}\ln|x^2-x+1| - \frac{\sqrt{3}}{3}arctg\frac{2x-1}{\sqrt{3}} + C$$

90/ $\int \frac{2x+5}{(x-2)^2}dx = \left|\begin{matrix} x-2=t \\ dx=dt \\ x=t+2 \end{matrix}\right| = \int \frac{2(t+2)+5}{t^2}dt = \int \frac{2t+9}{t^2}dt = 2\int \frac{dt}{t} + 9\int \frac{dt}{t^2} = 2\ln|t| - \frac{9}{t} + C =$

$$= 2\ln|x-2| - \frac{9}{x-2} + C$$

91/ $$\int \frac{x-1}{4x^2-4x+1}dx = \int \frac{\frac{1}{8}(8x-4)-\frac{1}{2}}{4x^2-4x+1}dx = \frac{1}{8}\int \frac{8x-4}{4x^2-4x+1}dx - \frac{1}{2}\int \frac{dx}{4x^2-4x+1} =$$

$$= \frac{1}{8}\ln|4x^2-4x+1| - \frac{1}{2}\int \frac{dx}{(2x-1)^2} = \frac{1}{8}\ln|4x^2-4x+1| - \frac{1}{2}\left|\begin{matrix} 2x-1=t \\ 2dx = dt\end{matrix}\right| = \frac{1}{8}\ln|4x^2-4x+1| -$$

$$-\frac{1}{4}\int \frac{dt}{t^2} = \frac{1}{8}\ln|4x^2-4x+1| - \frac{1}{4(2x-1)} + C$$

92/ $$\int \frac{2x^2-5x+1}{x^3-2x^2+x}dx = I$$

$$\frac{2x^2-5x+1}{x^3-2x^2+x} = \frac{2x^2-5x+1}{x(x^2-2x+1)} = \frac{2x^2-5x+1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} = \frac{x^2(A+B)+x(C-B-2A)+A}{x(x-1)^2}$$

$$\begin{cases} A+B=2 \\ C-B-2A=-5 \\ A=1 \end{cases} \qquad \begin{cases} A=1 \\ B=1 \\ C=-2 \end{cases}$$

$$I = \int \frac{dx}{x} + \int \frac{dx}{x-1} - 2\int \frac{dx}{(x-1)^2} = \ln|x| + \ln|x-1| + \frac{2}{x-1} + C$$

93/ $$\int \frac{5x-1}{x^3-3x-2}dx = \int \frac{5x-1}{x^3-2x-x-2}dx = I$$

$$\frac{5x-1}{(x^3-x)-2(x+1)} = \frac{5x-1}{x(x^2-1)-2(x+1)} = \frac{5x-1}{(x+1)^2(x-2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2} =$$

$$= \frac{x^2(A+C)+x(-A+B+2C)+(-2A-2B+C)}{(x+1)^2(x-2)}$$

$$\begin{cases} A+C=0 \\ -A+B+2C=5 \\ -2B-2A+C=-1 \end{cases} \qquad \begin{cases} A=-1 \\ B=2 \\ C=1 \end{cases}$$

$$I = \int \frac{-dx}{x+1} + 2\int \frac{dx}{(x+1)^2} + \int \frac{dx}{x-2} = -\ln|x+1| - \frac{2}{x+1} + \ln|x-2| + C$$

94/ $$\int \frac{5x+2}{x^2+2x+10}dx = \int \frac{\frac{5}{2}(2x+2)-3}{x^2+2x+10}dx = \frac{5}{2}\int \frac{2x+2}{x^2+2x+10}dx - 3\int \frac{dx}{x^2+2x+10} =$$

$$= \frac{5}{2}\ln|x^2+2x+10| - 3\int \frac{dx}{(x+1)^2+9} = \frac{5}{2}\ln|x^2-2x+10| - arctg\frac{x+1}{3} + C$$

(Wykorzystano wzór (1.26)).

95/ $$\int \frac{x+2}{x^3-2x^2}dx = I$$

$$\frac{x+2}{x^3-2x^2} = \frac{x+2}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} = \frac{x^2(A+C)+x(B-2A)-2B}{x^2(x-2)}$$

$$\begin{cases} A+C=0 \\ B-2A=1 \\ -2B=2 \end{cases} \qquad \begin{cases} A=-1 \\ B=-1 \\ C=1 \end{cases}$$

$$I = \int \frac{-dx}{x} + \int \frac{-dx}{x^2} + \int \frac{dx}{2x-2} = \frac{1}{x} + \frac{1}{2}\ln|2x-2| - \ln|x| + C$$

96/ $$\int \frac{x^2dx}{x-2} = \int (x + \frac{4}{x-2})dx = \frac{x^2}{2} + 4\ln(x-2) + C$$

97/ $$\int \frac{x^4dx}{x^2+k^2} = \int (x^2-k^2+\frac{k^4}{x^2+k^2})dx = \frac{x^3}{3} - k^2x - k^4\int \frac{dx}{x^2+k^2} = \frac{x^3}{3} - k^2x - k^4\int \frac{dx}{k^2\left(\left(\frac{x}{k}\right)^2+1\right)} =$$

$$= \frac{x^3}{3} - k^2x - k^3 arctg\left(\frac{x}{k}\right) + C$$

98/ $$\int \frac{2x^2+x+4}{x^3+x^2+4x+4}dx = \int \frac{2x^2+x+4}{x^2(x+1)+4(x+1)}dx = \int \frac{2x^2+x+4}{(x^2+4)(x+1)}dx = I$$

$$\frac{2x^2+x+4}{(x^2+4)(x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4} = \frac{x^2(A+B)+x(B+C)+4A+C}{(x^2+4)(x+1)}$$

$$\begin{cases} A+B=2 \\ B+C=1 \\ 4A+C=4 \end{cases} \qquad \begin{cases} A=1 \\ B=1 \\ C=0 \end{cases} \qquad I = \int \frac{dx}{x+1} + \int \frac{xdx}{x^2+4} = \ln|x+1| + \frac{1}{2}\ln|x^2+4| + C$$

99/ $\int \frac{dx}{x^3+8} = \int \frac{dx}{(x+2)(x^2-2x+4)} = I$

$$\frac{1}{(x+2)(x^2-2x+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2-2x+4} = \frac{x^2(A+B)+x(-2A+2B+C)+4A+2C}{(x+2)(x^2-2x+4)}$$

$$\begin{cases} A+B=0 \\ -2A+2B+C=0 \\ 4A+2C=1 \end{cases} \qquad \begin{cases} A=\frac{1}{12} \\ B=-\frac{1}{12} \\ C=\frac{1}{3} \end{cases}$$

$$I = \frac{1}{12}\int\frac{dx}{x+2} + \int\frac{\frac{-1}{12}x+\frac{1}{3}}{x^2-2x+4}dx = \frac{1}{12}\ln|x+2| - \frac{1}{24}\int\frac{24\left(\frac{1}{12}x-\frac{1}{3}\right)}{x^2-2x+4}dx = \frac{1}{12}\ln|x+2| -$$

$$-\frac{1}{24}\int\frac{2x-8}{x^2-2x+4}dx = \frac{1}{12}\ln|x+2| - \frac{1}{24}\int\frac{2x-2-6}{x^2-2x+4}dx = \frac{1}{12}\ln|x+2| - \frac{1}{24}\ln|x^2-2x+4| + \frac{6}{24}\int\frac{dx}{x^2-2x+4}$$

$$= \frac{1}{12}\ln|x+2| - \frac{1}{24}\ln|x^2-2x+4| + \frac{1}{4}\int\frac{dx}{(x-1)^2+3} = \frac{1}{12}\ln|x+2| - \frac{1}{24}\ln|x^2-2x+4| + \frac{1}{4\sqrt{3}}arctg\frac{x-1}{\sqrt{3}} + C$$

100/ $\int \frac{3x^2+2x+1}{(x+1)^2(x^2+1)}dx = I$

$$\frac{3x^2+2x+1}{(x+1)^2(x^2+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1} = \frac{x^3(A+C)+x^2(A+B+2C+D)+x(A+C+2D)+A+B+D}{(x+1)^2(x^2+1)}$$

$$\begin{cases} A+C=0 \\ A+B+2C+D=3 \\ A+C+2D=2 \\ A+B+D=1 \end{cases} \qquad \begin{cases} A=-1 \\ B=1 \\ C=1 \\ D=1 \end{cases}$$

$$I = \int\frac{-dx}{x+1} + \int\frac{dx}{(x+1)^2} + \int\frac{x+1}{x^2+1}dx = -\ln|x+1| - \frac{1}{x+1} + \frac{1}{2}\ln|x^2+1| + arctgx + C$$

101/ $\int \frac{xdx}{1-x^4} = \int\frac{xdx}{(1-x)(1+x)(1+x^2)} = I$

$$\frac{x}{(1-x)(1+x)(1+x^2)} = \frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{1+x^2} = \frac{x^3(A-B-C)+x^2(A+B-D)+x(A-B+C)+A+B+D}{(1-x)(1+x)(1+x^2)}$$

$$\begin{cases} A-B-C=0 \\ A+B-D=0 \\ A-B+C=1 \\ A+B+D=0 \end{cases} \qquad \begin{cases} A=\frac{1}{4} \\ B=-\frac{1}{4} \\ C=\frac{1}{2} \\ D=0 \end{cases}$$

$$I = \frac{1}{4}\int\frac{dx}{1-x} - \frac{1}{4}\int\frac{dx}{1+x} + \frac{1}{2}\int\frac{xdx}{x^2+1} = -\frac{1}{4}\ln|1-x| - \frac{1}{4}\ln|1+x| + \frac{1}{4}\ln|x^2+1| + C$$

102/ $\int\frac{dx}{x^4+4} = I$

$$\frac{1}{x^4+4} = \frac{1}{(x^2-2x+2)(x^2+2x+2)} = \frac{Ax+B}{x^2-2x+2} + \frac{Cx+D}{x^2+2x+2} =$$

$$= \frac{x^3(A+C)+x^2(2A+B-2C+D)+x(2A+2B+2C-2D)+2B+2D}{x^4+4}$$

$$\begin{cases} A+C=0 \\ 2A+B-2C+D=0 \\ 2A+2B+2C-2D=0 \\ 2B+2D=1 \end{cases} \qquad \begin{cases} A=-\frac{1}{8} \\ B=\frac{1}{4} \\ C=\frac{1}{8} \\ D=\frac{1}{4} \end{cases}$$

$$I = \int\frac{-\frac{1}{8}x+\frac{1}{4}}{x^2-2x+2}dx + \int\frac{\frac{1}{8}x+\frac{1}{4}}{x^2+2x+2}dx = -\frac{1}{8}\int\frac{x-2}{x^2-2x+2}dx + \frac{1}{8}\int\frac{x+2}{x^2+2x+2}dx = -\frac{1}{8}I_1 + \frac{1}{8}I_2$$

$$-\frac{1}{8}I_1 = -\frac{1}{16}\int\frac{2(x-2)}{x^2-2x+2}dx = -\frac{1}{16}\int\frac{2x-2-2}{x^2-2x+2}dx = -\frac{1}{16}\int\frac{2x-2}{x^2-2x+2}dx + \frac{1}{8}\int\frac{dx}{x^2-2x+2} =$$

$$= -\frac{1}{16}\ln|x^2-2x+2| + \frac{1}{8}\int\frac{dx}{(x-1)^2+1} = -\frac{1}{16}\ln|x^2-2x+2| + \frac{1}{8}arctg(x-1)$$

$$= \frac{5}{3}\sqrt{3x^2-2x+1} - \frac{7\sqrt{3}}{9}\ln\left|x-\frac{1}{3}+\sqrt{x^2-\frac{2}{3}x+\frac{1}{3}}\right| + C$$

$$\frac{1}{8}\int\frac{x+2}{x^2+2x+2}dx=\frac{1}{16}\int\frac{2(x+2)dx}{x^2+2x+2}=\frac{1}{16}\int\frac{2x+2+2}{x^2+2x+2}dx=\frac{1}{16}\int\frac{2x+2}{x^2+2x+2}dx+\frac{1}{8}\int\frac{dx}{x^2+2x+2}=$$

$$=\frac{1}{16}\ln\left|x^2+2x+2\right|+\frac{1}{8}arctg(x+1)$$

$$I=\frac{1}{16}\ln\left|\frac{x^2+2x+2}{x^2-2x+2}\right|+\frac{1}{8}arctg(x-1)+\frac{1}{8}arctg(x+1)+C$$

103/ $\displaystyle\int\frac{2x^3dx}{x^6-8}=2\left|\begin{array}{l}x^2=t,\quad xdx=\frac{dt}{2}\end{array}\right|=2\int\frac{t\frac{dt}{2}}{t^3-8}=\int\frac{tdt}{t^3-8}=I$

$$\frac{t}{t^3-8}=\frac{t}{(t-2)(t^2+2t+4)}=\frac{A}{t-2}+\frac{Bt+C}{t^2+2t+4}=\frac{t^2(A+B)+t(2A-2B+C)+4A-2C}{t^3-8}$$

$$\begin{cases}A+B=0\\-2A-2B+C=1\\4A-2C=0\end{cases}\qquad\begin{cases}A=\frac{1}{6}\\B=\frac{-1}{6}\\C=\frac{1}{3}\end{cases}$$

$$I=\frac{1}{6}\int\frac{dt}{t-2}+\int\frac{-\frac{1}{6}t+\frac{1}{3}}{t^2+2t+4}=\frac{1}{6}\ln|t-2|-\frac{1}{12}\int\frac{12\left(\frac{1}{6}t-\frac{1}{3}\right)}{t^2+2t+4}dt=\frac{1}{6}\ln\left|x^2-2\right|-\frac{1}{12}\int\frac{2t-4}{t^2+2t+4}dt$$

$$=\frac{1}{6}\ln\left|x^2-2\right|-\frac{1}{12}\int\frac{2t+2}{t^2+2t+4}dt+\frac{1}{2}\int\frac{dt}{t^2+2t+4}=\frac{1}{6}\ln\left|x^2-2\right|-\frac{1}{12}\ln\left|x^4+2x^2+4\right|+$$

$$+\frac{1}{2}\int\frac{dt}{\left(x^2+1\right)^2+3}=\frac{1}{6}\ln\left|x^2-2\right|-\frac{1}{12}\ln\left|x^4+2x^2+4\right|+\frac{1}{2\sqrt{3}}arctg\frac{x^2+1}{\sqrt{3}}+C$$

Dla obliczenia całek funkcji wymiernych typu:

(1.27) $\displaystyle I_n=\int\frac{dx}{\left(x^2+1\right)^n}$

stosuje się wzór rekurencyjny, którego wyprowadzenie można znaleźć w innych opracowaniach. Wzór ten jest następujący:

(1.28) $\displaystyle I_n=\frac{1}{2n-2}\frac{x}{\left(x^2+1\right)^{n-1}}+\frac{2n-3}{2n-2}I_{n-1}$ gdzie $\displaystyle I_n=\int\frac{dx}{\left(x^2+1\right)^n}$

Całka 104/ jest obliczona z wykorzystaniem wzoru (1.28).

104/ $\displaystyle\int\frac{2x+1}{\left(x^2+2x+5\right)^2}dx=\int\frac{2x+2-1}{\left(x^2+2x+5\right)^2}dx=\int\frac{2x+2}{\left(x^2+2x+5\right)^2}dx-\int\frac{dx}{\left(x^2+2x+5\right)^2}=$

$$=-\frac{1}{x^2+2x+5}-I_2$$

$$I_2=\int\frac{dx}{\left(x^2+2x+5\right)^2}=\int\frac{dx}{\left[(x+1)^2+4\right]^2}=\left|\frac{x+1}{2}=t\right|=\int\frac{2dt}{4^2\left(t^2+1\right)^2}=\frac{1}{8}\int\frac{dt}{\left(t^2+1\right)^2}=$$

$$=\frac{1}{16}\frac{t}{t^2+1}+\frac{1}{16}\int\frac{dt}{t^2+1}=\frac{1}{16}\frac{t}{t^2+1}+\frac{1}{16}arctgt=\frac{1}{8}\frac{x+1}{x^2+2x+5}+\frac{1}{16}arctg\frac{x+1}{2}$$

Ostatecznie więc:

$$I=-\frac{1}{x^2+2x+5}-\frac{1}{8}\frac{x+1}{x^2+2x+5}-\frac{1}{16}arctg\frac{x+1}{2}=\frac{-x-9}{8\left(x^2+2x+5\right)}-\frac{1}{16}arctg\frac{x+1}{2}+C$$

Uwaga:
Najczęściej pojawiające się błędy przy wyznaczaniu całek funkcji wy-miernych to: błędy obliczeniowe, błędne propozycje rozkładu funkcji wymiernej na ułamki proste.
Poniżej podano dodatkowo kilka propozycji rozkładu funkcji wymiernej na ułamki proste.

$$\frac{5x+2}{(x-2)^2x^3(2x+1)^2}=\frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{x}+\frac{D}{x^2}+\frac{E}{x^3}+\frac{F}{2x+1}+\frac{H}{(2x+1)^2}$$

$$\frac{3x+7}{\left(2x^2-7x+3\right)\left(x^2-x-6\right)^2}=\frac{3x+7}{(x-3)(2x-1)(x-3)^2(x+2)^2}=\frac{3x+7}{(x-3)^3(2x-1)(x+2)^2}$$

$$=\frac{A}{x-3}+\frac{B}{(x-3)^2}+\frac{C}{(x-3)^3}+\frac{D}{2x-1}+\frac{E}{x+2}+\frac{F}{(x+2)^2}$$

5. Całki funkcji niewymiernych

Jest wiele postaci funkcji algebraicznych z niewymiernościami.
Stąd też jest wiele metod całkowania funkcji niewymiernych.

Warto wymienić następujące:

a/ jeżeli funkcja podcałkowa jest powyższej postaci $\int R(x,\sqrt[n]{ax+b})dx$ to aby wyznaczyć całkę należy zastosować podstawienie: $ax+b=t^n$.

b/ jeżeli funkcja podcałkowa zawiera pierwiastek kwadratowy z trójmianu kwadratowego typu:

(1.29) $\displaystyle\int\frac{dx}{\sqrt{x^2+k}}$ to dla wyznaczenia całki należy zastosować pierwsze podstawienie Eulera. Pomijając szczegóły wyprowadzenia wzoru otrzymu--jemy:

(1.30) $$\int\frac{dx}{\sqrt{x^2+k}}=\ln\left|x+\sqrt{x^2+k}\right|+C$$

W ten sposób obliczamy każdą całkę postaci $\displaystyle\int\frac{dx}{\sqrt{ax^2+bx+c}}$ gdzie a>0.

c/ każdą całkę postaci $\displaystyle\int\frac{dx}{\sqrt{ax^2+bx+c}}$ gdzie a <0 obliczamy wykorzystując całkę:

(1.31) $$\int\frac{dx}{\sqrt{a^2-x^2}}=\arcsin\frac{x}{a}+C$$

d/ całki postaci $\int\sqrt{ax^2+bx+c}\,dx$ gdzie a < 0 wyznaczamy wykorzystując wzór:

(1.32) $$\int\sqrt{a^2-x^2}\,dx=\frac{a^2}{2}\arcsin\frac{x}{|a|}+\frac{x}{2}\sqrt{a^2-x^2}+C$$

e/ całki postaci $\displaystyle\int\frac{x^2dx}{\sqrt{a^2-x^2}}$ obliczamy korzystając ze wzoru:

(1.33) $$\int\frac{x^2dx}{\sqrt{a^2-x^2}}=\frac{a^2}{2}\arcsin\frac{x}{|a|}-\frac{x}{2}\sqrt{a^2-x^2}+C$$

f/ całki postaci $\int\sqrt{ax^2+bx+c}\,dx$ gdzie a > 0 wyznaczamy ze wzoru:

(1.34) $$\int\sqrt{x^2+k}\,dx=\frac{1}{2}x\sqrt{x^2+k}+\frac{1}{2}k\ln\left|x+\sqrt{x^2+k}\right|+C$$

g/ całki postaci $\displaystyle\int\frac{x^2dx}{\sqrt{ax^2+bx+c}}$ gdzie a > 0, wyznaczamy ze wzoru:

(1.35) $$\int\frac{x^2dx}{\sqrt{x^2+k}}=\frac{1}{2}x\sqrt{x^2+k}-\frac{1}{2}k\ln\left|x+\sqrt{x^2+k}\right|+C$$

h/ przy obliczaniu całek postaci $\displaystyle\int\frac{W_n(x)dx}{\sqrt{ax^2+bx+c}}$ stosujemy metodę współczynników nieoznaczonych. Metoda ta zostanie przedstawiona na przykładach.

PRZYKŁADY CAŁKOWANIA

105/

$$\int\frac{2x+1}{\sqrt{4x+1}}dx=\left|\begin{array}{l}4x+1=t^2\\4dx=2tdt\\2x=\dfrac{t^2-1}{2}\end{array}\right|=\int\frac{\frac{t^2-1}{2}+1}{t}\frac{t}{2}dt=\frac{1}{4}\int t^2+1dt=\frac{t^3}{12}+t+C=$$

$$=\frac{\sqrt{(4x+1)^3}}{12}+\sqrt{4x+1}+C$$

106/ $$\int\frac{dx}{\sqrt[3]{x}+\sqrt{x}}=\left|\begin{array}{l}x=t^6\\dx=6t^5dt\\\sqrt[3]{x}=t^2,\sqrt{x}=t^3\end{array}\right|=\int\frac{6t^5dt}{t^2+t^3}=\int\frac{6t^5dt}{t^2(1+t)}=6\int\frac{t^3dt}{1+t}=$$

$$=6\int\left(t^2-t+1-\frac{1}{t+1}\right)dt=\frac{6t^3}{3}-\frac{6t^2}{2}+\frac{6t}{2}-6\ln|t+1|+C=\frac{6\sqrt{x}}{3}-3\sqrt[3]{x}+3\sqrt[6]{x}-6\ln\left|\sqrt[6]{x}+1\right|+C$$

107/

$$\int \frac{\sqrt[3]{x}dx}{x+\sqrt[6]{x^5}} = \left|\begin{matrix} x = t^6 \\ dx = 6t^5dt \end{matrix}\right| = \int \frac{t^2 6t^5 dt}{t^6+t^5} = 6\int \frac{t^7 dt}{t^5(t+1)} = 6\int \frac{t^2dt}{t+1} = 6\int (t-1+\frac{1}{t+1})dt$$

$$= 3t^2 - 6t + 6\ln|t+1| + C = 3\sqrt[6]{x^2} - 6\sqrt[6]{x} + 6\ln\left|\sqrt[6]{x}+1\right| + C$$

108/

$$\int \sqrt{\frac{x-1}{x-2}}\frac{dx}{(x-1)^2} = \left|\begin{matrix} \frac{x-1}{x-2} = t^2 \\ x = \frac{1-2t^2}{1-t^2} \\ dx = -\frac{2tdt}{(1-t^2)^2} \end{matrix}\right| = \int t\frac{-2tdt}{(1-t^2)^2}\frac{(1-t^2)^2}{t^4} = \int \frac{-2dt}{t^2} =$$

$$= \frac{2}{t} + C = \frac{2}{\sqrt{\frac{x-1}{x-2}}} + C = \frac{2\sqrt{x-2}}{\sqrt{x-1}} + C$$

109/

$$\int \frac{dx}{\sqrt{x^2+3x+2}} = \int \frac{dx}{\sqrt{\left(x+\frac{3}{2}\right)^2 - \frac{1}{4}}} = \left|\begin{matrix} x+\frac{3}{2} = t \\ dx = dt \end{matrix}\right| = \int \frac{dt}{\sqrt{t^2-\frac{1}{4}}} \overset{(1.30)}{=} \ln\left|t+\sqrt{t^2-\frac{1}{4}}\right| + C =$$

$$= \ln\left|x+\frac{3}{2}+\sqrt{x^2+3x+2}\right| + C$$

110/

$$\int \frac{5x-4}{\sqrt{3x^2-2x+1}}dx = \frac{5}{6}\int \frac{\frac{6}{5}(5x-4)}{\sqrt{3x^2-2x+1}}dx = \frac{5}{6}\int \frac{6x-\frac{24}{5}}{\sqrt{3x^2-2x+1}}dx = \frac{5}{6}\int \frac{6x-2}{\sqrt{3x^2-2x+1}}dx -$$

$$-\frac{7}{3}\int \frac{dx}{\sqrt{3\left(x-\frac{1}{3}\right)^2 + \frac{2}{3}}} = \frac{5}{3}\sqrt{3x^2-2x+1} - \frac{7}{3}\int \frac{dx}{\sqrt{3\left[\left(x-\frac{1}{3}\right)^2+\frac{2}{9}\right]}} = \frac{5}{3}\sqrt{3x^2-2x+1} -$$

$$\left|\begin{matrix} x-\frac{1}{3} = t \\ dx = dt \end{matrix}\right| - \frac{7\sqrt{3}}{9}\int \frac{dt}{\sqrt{t^2+\frac{2}{9}}} = \frac{5}{3}\sqrt{3x^2-2x+1} - \frac{7\sqrt{3}}{9}\ln\left|t+\sqrt{t^2+\frac{2}{9}}\right| + C =$$

111/

$$\int \frac{(5x+2)dx}{\sqrt{2x^2+8x-1}} = \frac{5}{4}\int \frac{\frac{4}{5}(5x+2)dx}{\sqrt{2x^2+8x-1}} = \frac{5}{4}\int \frac{4x+\frac{8}{5}}{\sqrt{2x^2+8x-1}}dx = \frac{5}{4}\int \frac{(4x+8)dx}{\sqrt{2x^2+8x-1}} -$$

$$-\frac{5}{4}\int \frac{\frac{32}{5}dx}{\sqrt{2x^2+8x-1}} = \frac{5}{4}2\sqrt{2x^2+8x-1} - 8\int \frac{dx}{\sqrt{2(x+2)^2-9}} = \frac{5}{2}\sqrt{2x^2+8x-1} - \frac{8}{\sqrt{2}}\int \frac{dx}{\sqrt{(x+2)^2-\frac{9}{2}}}\left|\begin{matrix} x+2 = t \\ dx = dt \end{matrix}\right|$$

$$= \frac{5}{2}\sqrt{2x^2+8x-1} - 4\sqrt{2}\int \frac{dt}{\sqrt{t^2-\frac{9}{2}}} = \frac{5}{2}\sqrt{2x^2+8x-1} - 4\sqrt{2}\ln\left|t+\sqrt{t^2-\frac{9}{2}}\right| + C =$$

$$= \frac{5}{2}\sqrt{2x^2+8x-1} - 4\sqrt{2}\ln\left|x+2+\sqrt{x^2+4x-\frac{1}{2}}\right| + C$$

112/

$$\int \frac{x-5}{\sqrt{5+4x-x^2}}dx = -\frac{1}{2}\int \frac{-2(x-5)dx}{\sqrt{9-(x-2)^2}} = -\frac{1}{2}\int \frac{-2x+4}{\sqrt{5+4x-x^2}} - \frac{1}{2}\int \frac{6dx}{\sqrt{9-(x-2)^2}} =$$

$$= -\frac{1}{2}2\sqrt{5+4x-x^2} - 3\int \frac{3dt}{\sqrt{9-9t^2}}\left|\begin{matrix} x-2 = 3t \\ dx = 3dt \end{matrix}\right| = \sqrt{5+4x-4x^2} - 3\int \frac{dt}{\sqrt{1-t^2}} =$$

$$= \sqrt{5+4x-x^2} - 3\arcsin t + C = \sqrt{5+4x-x^2} - 3\arcsin\frac{x-2}{3} + C$$

113/

$$\int \frac{5dx}{\sqrt{2x-x^2}} = \int \frac{5dx}{\sqrt{1-(x-1)^2}}\left|\begin{matrix} x-1 = t \\ dx = dt \end{matrix}\right| = 5\int \frac{dt}{\sqrt{1-t^2}} = 5\arcsin t + C = 5\arcsin(x-1) + C$$

114/

$$\int \frac{3xdx}{\sqrt{1-2x-3x^2}} = -\frac{1}{2}\int \frac{-6xdx}{\sqrt{1-2x-3x^2}} = -\frac{1}{2}\int \frac{-6x-2+2}{\sqrt{1-2x-3x^2}}dx = -\frac{1}{2}\int \frac{-6x-2}{\sqrt{1-2x-3x^2}}dx -$$

$$-\frac{1}{2}\int \frac{2dx}{\sqrt{1-2x-3x^2}} = -\sqrt{1-2x-3x^2} - I + C = -\sqrt{1-2x-3x^2} - \frac{\sqrt{3}}{3}\arcsin\frac{3x+1}{2} + C$$

$$I = -\int \frac{dx}{\sqrt{1-2x-3x^2}} = -\int \frac{dx}{\sqrt{\frac{4}{3}-\left(\sqrt{3}\left(x+\frac{1}{3}\right)\right)^2}} = \left|\sqrt{3}\left(x+\frac{1}{3}\right) = \frac{2}{\sqrt{3}}t\right| = \frac{\sqrt{3}}{3}\arcsin\left(\frac{3x+1}{2}\right)$$

$$(1.36) \qquad \boxed{\int \frac{u'(x)dx}{\sqrt{u(x)}} = 2\sqrt{u(x)} + C}$$

W całkach 110/ - 112/ wykorzystano wzór (1.36).

115/ $\int\sqrt{1-4x^2}\,dx=\int\sqrt{4\left(\frac{1}{4}-x^2\right)}dx=2\int\sqrt{\frac{1}{4}-x^2}\,dx=\frac{1}{4}\arcsin 2x+x\sqrt{\frac{1}{4}-x^2}+C$

116/

$$\int\sqrt{10+3x-x^2}\,dx=\int\sqrt{\frac{49}{4}-\left(x-\frac{3}{2}\right)^2}dx=\left|\begin{matrix}x-\frac{3}{2}=t\\ dx=dt\end{matrix}\right|\int\sqrt{\frac{49}{4}-t^2}\,dt=\frac{49}{8}\arcsin\frac{2t}{7}+\frac{t}{2}\sqrt{\frac{49}{4}-t^2}+C=$$

$$=\frac{49}{8}\arcsin\frac{2x-3}{7}+\frac{x-\frac{3}{2}}{2}\sqrt{10+3x-x^2}+C$$

117/ $\int\sqrt{3x^2+10x+9}\,dx=\int\sqrt{3\left(x+\frac{5}{3}\right)^2+\frac{2}{3}}dx=\int\sqrt{3\left[\left(x+\frac{5}{3}\right)^2+\frac{2}{9}\right]}dx=\sqrt{3}\int\sqrt{\left(x+\frac{5}{3}\right)^2+\frac{2}{9}}dx=$

$$\left|\begin{matrix}x+\frac{5}{3}=t\\ dx=dt\end{matrix}\right|=\sqrt{3}\int\sqrt{t^2+\frac{2}{9}}dt=\sqrt{3}\frac{t}{2}\sqrt{t^2+\frac{2}{9}}+\frac{\sqrt{3}}{2}\frac{2}{9}\ln\left|t+\sqrt{t^2+\frac{2}{9}}\right|+C=$$

$$\frac{\sqrt{3}}{2}\left(x+\frac{5}{3}\right)\sqrt{\frac{3x^2+10x+9}{3}}+\frac{\sqrt{3}}{9}\ln\left|x+\frac{5}{3}+\sqrt{\frac{3x^2+10x+9}{3}}\right|+C=\frac{3x+5}{6}\sqrt{3x^2+10x+9}+$$

$$+\frac{\sqrt{3}}{9}\ln\left|3x+5+\sqrt{3\left(3x^2+10x+9\right)}\right|+C$$

118/ $\int\frac{5x^2-2x+10}{\sqrt{3x^2-5x+8}}dx=(ax+b)\sqrt{3x^2-5x+8}+\int\frac{A\,dx}{\sqrt{3x^2-5x+8}}$

Różniczkujemy obustronnie:

$$\frac{5x^2-2x+10}{\sqrt{3x^2-5x+8}}=a\sqrt{3x^2-5x+8}+(ax+b)\frac{6x-5}{2\sqrt{3x^2-5x+8}}+\frac{A}{\sqrt{3x^2-5x+8}}$$

Obustronnie mnożymy przez $\sqrt{3x^2-5x+8}$

$$5x^2-2x+10=a\left(3x^2-5x+8\right)+(ax+b)\frac{6x-5}{2}+A$$

Stąd układ równań:

$$\begin{cases}10=12a\\-4=-10a-5a+6b\\20=16a-5b+2A\end{cases}\qquad\begin{cases}a=\frac{5}{6}\\ b=\frac{17}{12}\\ A=\frac{165}{24}\end{cases}$$

Wstawiamy do wyjściowego wzoru:

$$I=\left(\frac{5}{6}x+\frac{17}{12}\right)\sqrt{3x^2-5x+8}+\frac{55}{8}\int\frac{dx}{\sqrt{3x^2-5x+8}}$$

$$I_1=\frac{55\sqrt{3}}{24}\int\frac{dt}{\sqrt{t^2+\frac{71}{36}}}=\frac{55\sqrt{3}}{24}\ln\left|t+\sqrt{t^2+\frac{71}{36}}\right|+C=\frac{55\sqrt{3}}{24}\ln\left|\frac{6x-5}{6}+\sqrt{\frac{3x^2-5x+8}{3}}\right|$$

$$I=\left(\frac{5}{6}x+\frac{17}{12}\right)\sqrt{3x^2-5x+8}+\frac{55\sqrt{3}}{24}\ln\left|3x-\frac{5}{2}+\sqrt{3}\sqrt{3x^2-5x+8}\right|+C$$

119/ $\int\frac{x^3-x+1}{\sqrt{x^2+2x+2}}dx=\left(ax^2+bx+c\right)\sqrt{x^2+2x+2}+A\int\frac{dx}{\sqrt{x^2+2x+2}}$

$$\frac{x^3-x+1}{\sqrt{x^2+2x+2}}=(2ax+b)\sqrt{x^2+2x+2}+\left(ax^2+bx+c\right)\frac{2(x+1)}{2\sqrt{x^2+2x+2}}+\frac{A}{\sqrt{x^2+2x+2}}$$

$$x^3-x+1=(2ax+b)\left(x^2+2x+2\right)+\left(ax^2+bx+c\right)(x+1)+A$$

$$x^3-x+1=x^3(3a)+x^2(5a+2b)+x(4a+3b+c)+(2b+c+A)$$

$$\begin{cases}3a=1\\5a+2b=0\\4a+3b+c=-1\\2b+c+A=1\end{cases}\qquad\begin{cases}a=\frac{1}{3}\\ b=\frac{-5}{6}\\ c=\frac{1}{6}\\ A=\frac{15}{6}\end{cases}$$

$$I=\left(\frac{1}{3}x^2-\frac{5}{6}x+\frac{1}{6}\right)\sqrt{x^2+2x+2}+\frac{15}{6}\int\frac{dx}{\sqrt{x^2+2x+2}}$$

$$I_1=\frac{5}{2}\int\frac{dx}{\sqrt{x^2+2x+2}}=\frac{5}{2}\int\frac{dx}{\sqrt{(x+1)^2+1}}\left|\begin{matrix}x+1=t\\ dx=dt\end{matrix}\right|=\frac{5}{2}\int\frac{dt}{\sqrt{t^2+1}}=\frac{5}{2}\ln\left|x+1\right|+\sqrt{x^2+2x+2}+C$$

$$I=\left(\frac{1}{3}x^2-\frac{5}{6}x+\frac{1}{6}\right)\sqrt{x^2+2x+2}+\frac{5}{2}\ln\left|x+1+\sqrt{x^2+2x+2}\right|+C$$

Do obliczenia całek 118/ i 119/ zastosowano metodę współczynników nieoznaczonych.

120/ $\int (2x-5)\sqrt{2+3x-x^2}\,dx = \int \frac{(2x-5)(2+3x-x^2)}{\sqrt{2+3x-x^2}}dx = \int \frac{-2x^3+11x^2-11x-10}{\sqrt{2+3x-x^2}}dx$

Dla wyznaczenia calki 120/ należy w dalszej kolejności zastosować metodę współczynników nieoznaczonych.

121/

$$\int \frac{dx}{x\sqrt{10x-x^2}} = \left| \begin{array}{l} \frac{1}{x}=t \\ \frac{1}{t}=x \\ dx=\frac{-dt}{t^2} \end{array} \right| = \int \frac{\frac{-dt}{t^2}}{\frac{1}{t}\sqrt{\frac{10}{t}-\frac{1}{t^2}}} = -\int \frac{dt}{t\sqrt{\frac{10t-1}{t^2}}} = -\int \frac{t\,dt}{t\sqrt{10t-1}} = -\int \frac{dt}{\sqrt{10t-1}} =$$

$$\left| \begin{array}{l} 10t-1=z^2 \\ 10dt=2zdz \\ dt=\frac{1}{5}zdz \end{array} \right| = -\frac{1}{5}\int dz = -\frac{1}{5}z + C = -\frac{1}{5}\sqrt{10t-1}+C = -\frac{1}{5}\sqrt{\frac{10}{x}-1}+C = -\frac{\sqrt{10x-x^2}}{5x}+C$$

122/

$$\int \frac{dx}{(x+1)\sqrt{x^2+2x+2}} = \left| \frac{1}{x+1}=t, \quad x=\frac{1}{t}-1, \quad dx=-\frac{dt}{t^2} \right| = -\int \frac{dt}{t\sqrt{\frac{1+t^2}{t^2}}} = -\int \frac{dt}{\sqrt{t^2+1}} =$$

$$= -\ln\left|t+\sqrt{t^2+1}\right| + C = -\ln\left|\frac{1}{x+1}+\sqrt{\left(\frac{1}{x+1}\right)^2+1}\right| + C = \ln\left|\frac{x+1}{1+\sqrt{x^2+2x+2}}\right| + C$$

123/

$$\int \frac{dx}{x^3\sqrt{1+x^2}} = \left| \frac{1}{x}=t \quad x=\frac{1}{t} \quad dx=-\frac{dt}{t^2} \right| = \int \frac{-t^3dt}{t^2\sqrt{1+\frac{1}{t^2}}} = -\int \frac{t^2dt}{\sqrt{t^2+1}} =$$

$$= -\frac{1}{2}t\sqrt{t^2+1}+\frac{1}{2}\ln\left|t+\sqrt{t^2+1}\right| + C = -\frac{\sqrt{1+x^2}}{2x^2}+\frac{1}{2}\ln\left|\frac{1+\sqrt{1+x^2}}{x}\right| + C$$

Dla wyznaczenia całek 121/, 122/, 123/ zastosowano poniższe podstawienie:
Dla całek $\int \frac{dx}{(x-k)^n\sqrt{ax^2+bx+c}}$

należy podstawić (1.37) $\frac{1}{x-k}=t$

6. Całki funkcji trygonometrycznych

Jedną z ważniejszych metod całkowania funkcji trygonometrycznych jest zastosowanie jednego z podstawień tzw. uniwersalnego. Wzory (1.38) oraz (1.39) dotyczą właśnie tych dwóch podstawień. Podstawienia te pozwalają sprowadzić całkę zawierającą funkcje trygonometryczne do całki funkcji wymiernej.

(1.38) $$\boxed{\begin{array}{llll} tg\frac{x}{2}=u & x=2arctgu & dx=\frac{2du}{1+u^2} & \sin x=\frac{2u}{1+u^2} \\ \cos x=\frac{1-u^2}{1+u^2} & tgx=\frac{2u}{1-u^2} & & \end{array}}$$

(1.39) $$\boxed{\begin{array}{llll} tgx=u & x=arctgu & dx=\frac{du}{1+u^2} & \sin^2 x=\frac{u^2}{1+u^2} \\ \cos^2 x=\frac{1}{1+u^2} & & & \end{array}}$$

124/

$$\int \frac{dx}{3\sin x+4\cos x} = \int \frac{\frac{2du}{1+u^2}}{3\frac{2u}{1+u^2}+4\frac{1-u^2}{1+u^2}} = \int \frac{du}{-2u^2+3u+2} = \int \frac{du}{-2(u-2)\left(u+\frac{1}{2}\right)} =$$

$$= -\int \frac{du}{(u-2)(2u+1)} = I$$

$$\frac{1}{(u-2)(2u+1)} = \frac{A}{u-2}+\frac{B}{2u+1} = \frac{u(2A+B)+A-2B}{(u-2)(2u+1)}$$

$$\begin{cases} 2A+B=0 \\ A-2B=-1 \end{cases} \qquad \begin{cases} A=-\frac{1}{5} \\ B=\frac{2}{5} \end{cases}$$

$$I = -\frac{1}{5}\int \frac{du}{u-2}+\frac{2}{5}\int \frac{du}{2u+1} = -\frac{1}{5}\ln|u-2| + \left| \begin{array}{l} 2u+1=t \\ du=\frac{dt}{2} \end{array} \right| \frac{2}{5}\int \frac{dt}{2t} = -\frac{1}{5}\ln|u-2|+\frac{1}{5}\ln|2u+1| =$$

$$= \frac{1}{5}\ln\left|\frac{2u+1}{u-2}\right| = \frac{1}{5}\ln\left|\frac{2tg\frac{x}{2}+1}{tg\frac{x}{2}-2}\right| + C$$

125/
$$\int \frac{dx}{1+3\cos^2 x} = \int \frac{\frac{du}{1+u^2}}{1+3\frac{1}{1+u^2}} = \int \frac{du}{u^2+4} = \int \frac{du}{4\left[\left(\frac{u}{2}\right)^2+1\right]} = \frac{1}{4}\int \frac{du}{\left(\frac{u}{2}\right)^2+1} = \left|\begin{matrix}\frac{u}{2}=z\\ du=2dz\end{matrix}\right| =$$
$$= \frac{1}{2}\int \frac{dz}{z^2+1} = \frac{1}{2}arctgz + C = \frac{1}{2}arctg\frac{u}{2} + C = \frac{1}{2}arctg\frac{tgx}{2} + C$$

Dla całki **124/** zastosowano wzory (1.38), a dla całki **125/** wzory (1.39). Aby wyznaczyć niektóre całki zawierające funkcje trygonometryczne należy przekształcać wyrażenia podcałkowe wykorzystując wzory trygonometryczne. Wyznaczenie poniższych całek odbywa się z wykorzystaniem wzorów trygonometrycznych.

126/ $\int \cos 2x \cos 3x dx = \frac{1}{2}\int 2\cos 2x \cos 3x dx = \frac{1}{2}\int (\cos ax + \cos bx)dx = I$

Wykorzystano wzór:

(1.40) $\cos ax + \cos bx = 2\cos\frac{ax+bx}{2}\cos\frac{ax-bx}{2}$

$$\begin{cases}\frac{a+b}{2}=2\\ \frac{a-b}{2}=3\end{cases} \qquad \begin{cases}a=5\\ b=-1\end{cases}$$

$$I = \frac{1}{2}\int (\cos 5x + \cos(-x))dx = \frac{1}{2}\int \cos 5x dx + \frac{1}{2}\int \cos x dx = \frac{1}{10}\sin 5x + \frac{1}{2}\sin x + C$$

127/ $\int \sin 2x \sin 5x dx = -\frac{1}{2}\int (\cos ax - \cos bx)dx = I$

Dla wyznaczenia całki **127/** wykorzystamy wzór:

(1.41) $\cos ax - \cos bx = -2\sin\frac{ax+bx}{2}\sin\frac{ax-bx}{2}$

$$\begin{cases}\frac{a+b}{2}=2\\ \frac{a-b}{2}=5\end{cases} \qquad \begin{cases}a=7\\ b=-3\end{cases}$$

$$I = -\frac{1}{2}\int (\cos 7x - \cos(-3x))dx = -\frac{1}{2}\int \cos 7x dx + \frac{1}{2}\int \cos 3x dx = -\frac{1}{14}\sin 7x + \frac{1}{6}\sin 3x + C$$

128/
$$\int \frac{dx}{\sin x} = \int \frac{\sin x}{\sin^2 x}dx = \int \frac{\sin x dx}{1-\cos^2 x} = \left|\begin{matrix}\cos x = z\\ -\sin x dx = dz\end{matrix}\right| = -\int \frac{dz}{1-z^2} = \frac{1}{2}\int \frac{dz}{z-1} - \frac{1}{2}\int \frac{dz}{z+1} =$$
$$\frac{1}{2}\ln|z-1| - \frac{1}{2}\ln|z+1| + C = \frac{1}{2}\ln\left|\frac{\cos x - 1}{\cos x + 1}\right| + C$$

129/
$$\int \frac{\sin^5 x}{\cos^3 x}dx = \int \frac{\sin^4 x \sin x dx}{\cos^3 x} = \int \frac{(1-\cos^2 x)^2 \sin x dx}{\cos^3 x} = \left|\begin{matrix}\cos x = t\\ -\sin x dx = dt\end{matrix}\right| = -\int \frac{(1-t^2)^2 dt}{t^3} =$$
$$= -\int \frac{1-2t^2+t^4}{t^3}dt = -\int \frac{dt}{t^3} + 2\int \frac{dt}{t} - \int t dt = \frac{1}{2t^2} + 2\ln|t| - \frac{t^2}{2} + C = \frac{1}{2\cos^2 x} + 2\ln|\cos x| - \frac{\cos^2 x}{2} + C$$

130/
$$\int \frac{dx}{\cos x} = \int \frac{dx}{\sin\left(\frac{\pi}{2}+x\right)} = \left|\begin{matrix}\frac{\pi}{2}+x=t\\ dx=dt\end{matrix}\right| = \int \frac{dt}{\sin t} = \int \frac{dt}{2\sin\frac{t}{2}\cos\frac{t}{2}} = \int \frac{dt}{2\frac{\sin\frac{t}{2}\cos^2\frac{t}{2}}{\cos\frac{t}{2}}} = \frac{1}{2}\int \frac{dt}{tg\frac{t}{2}\cos^2\frac{t}{2}} =$$
$$= \frac{1}{2}\ln\left|tg\frac{t}{2}\right| + C = \frac{1}{2}\ln\left|tg\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + C$$

131/
$$\int \frac{dx}{\sin x\cos^2 x} = \int \frac{\cos^2 x + \sin^2 x}{\sin x \cos^2 x}dx = \int \frac{\cos^2 x dx}{\sin x \cos^2 x} + \int \frac{\sin^2 x dx}{\sin x \cos^2 x} = \int \frac{dx}{\sin x} + \int \frac{\sin x}{\cos^2 x}dx =$$
$$= \frac{1}{2}\ln|tgx| - \frac{1}{t} + C = \frac{1}{2}\ln|tgx| - \frac{1}{\cos x} + C$$

132/ $\int tg^2 x dx = \int \frac{dx}{\cos^2 x} - \int dx = tgx - x + C$

Wykorzystano wzór:

(1.42) $tg^2 x = \frac{1}{\cos^2 x} - 1$

133/
$$\int \frac{x dx}{\cos^2 x} = \left|\begin{matrix}u = x & du = dx\\ dv = \frac{dx}{\cos^2 x} & v = tgx\end{matrix}\right| = xtgx - \int tgx dx = xtgx - \int \frac{\sin x}{\cos x}dx = xtgx + \ln|\cos x| + C$$

134/
$$\int \frac{5\sin x + \cos x}{2\sin^2 x\cos x + 2\cos^3 x}dx = \int \frac{5\sin x + \cos x}{2\sin^2 x\cos x + 2\cos^3 x}\cdot\frac{\frac{1}{\cos^3 x}}{\frac{1}{\cos^3 x}}dx = \int \frac{\frac{5\sin x}{\cos x} + \frac{\cos x}{\cos x}}{\frac{2\sin^2 x}{\cos^2 x} + 2}\frac{dx}{\cos^2 x} = \int \frac{5tgx+1}{2tg^2 x + 2}\frac{dx}{\cos^2 x}$$
$$= \left|\begin{matrix}tgx = t\\ \frac{dx}{\cos^2 x} = dt\end{matrix}\right| = \frac{1}{2}\int \frac{5t+3}{t^2+1}dt = \frac{5}{4}\int \frac{\frac{2}{5}(5t+3)}{t^2+1}dt = \frac{5}{4}\int \frac{2t + \frac{6}{5}}{t^2+1}dt = \frac{5}{4}\int \frac{2t dt}{t^2+1} + \frac{6}{4}\int \frac{dt}{t^2+1} =$$
$$= \frac{5}{4}\ln|t^2+1| + \frac{3}{2}arctgt + C = \frac{5}{4}\ln|tg^2 x + 1| + \frac{3}{2}arctg(tgx) + C$$

135/

$$\int\frac{dx}{\sin^2 x\cos x}=\left|\begin{array}{l}\sin x=t\\ \cos x=\sqrt{1-t^2}\\ dx=\frac{dt}{\sqrt{1-t^2}}\end{array}\right|=\int\frac{\frac{dt}{\sqrt{1-t^2}}}{t^2\sqrt{1-t^2}}=\int\frac{dt}{t^2\left(1-t^2\right)}=\int\frac{-dt}{t^2(t-1)(t+1)}=\int\frac{dt}{t^2}+\frac{1}{2}\int\frac{dt}{t+1}-\frac{1}{2}\int\frac{dt}{t-1}$$

$$=-\frac{1}{t}+\frac{1}{2}\ln|t+1|-\frac{1}{2}\ln|t-1|+C=-\frac{1}{\sin x}+\frac{1}{2}\ln\left|\frac{\sin x+1}{\sin x-1}\right|+C$$

136/

$$\int\frac{\cos^3 x dx}{\sin^2 x+1}=\int\frac{\cos^2 x\cos x}{\sin^2 x+1}dx=\left|\sin x=t\quad \cos xdx=dt\right|=\int\frac{\left(1-t^2\right)dt}{t^2+1}=\int(-1+\frac{2}{t^2+1})dt=$$

$$=-\int dt+2\int\frac{dt}{t^2+1}=-t+2arctgt+C=-\sin x+2arctg(\sin x)+C$$

137/

$$\int\frac{tgxdx}{tgx+2}=\left|tgx=\frac{\sin x}{\cos x}\right|=\int\frac{\sin xdx}{\sin x+2\cos x}=\left|\begin{array}{ll}tgx=t & dx=\frac{dt}{1+t^2}\\ \sin x=\frac{t}{\sqrt{1+t^2}} & \cos x=\frac{1}{\sqrt{1+t^2}}\end{array}\right|=$$

$$=\int\frac{\frac{t}{\sqrt{1+t^2}}\frac{dt}{1+t^2}}{\frac{t}{\sqrt{1+t^2}}+\frac{2}{\sqrt{1+t^2}}}=\int\frac{tdt}{(t+2)\left(1+t^2\right)}=-\frac{2}{5}\int\frac{dt}{t+2}+\frac{1}{5}\int\frac{(2t+1)dt}{t^2+1}=-\frac{2}{5}\ln|t+2|+\frac{1}{5}\ln\left|t^2+1\right|+\frac{1}{5}arctgt+C=$$

$$=-\frac{2}{5}\ln|tgx+2|+\frac{1}{5}\ln\left|tg^2x+1\right|+\frac{1}{5}arctg(tgx)+C$$

138/ $$\int\frac{dx}{\sin^2 x\cos^3 x}=\int\frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos^3 x}dx=\int\frac{dx}{\cos^3 x}+\int\frac{dx}{\sin^2 x\cos x}=I_1+I_2$$

$$I_1=\int\frac{dx}{\cos^3 x}=\int\frac{\sin^2 x+\cos^2 x}{\cos^3 x}dx=\int\frac{\sin^2 xdx}{\cos^3 x}+\int\frac{dx}{\cos x}=\int\sin x\frac{\sin x}{\cos^3 x}dx+\int\frac{dx}{\cos x}=$$

$$=\left|\begin{array}{ll}u=\sin x & du=\cos xdx\\ dv=\frac{\sin xdx}{\cos^3 x} & v=\int\frac{\sin xdx}{\cos^3 x}\end{array}\right|+\frac{1}{2}\ln\left|tg\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|$$

Pomocniczo wyznaczamy:

$$\int\frac{\sin xdx}{\cos^3 x}=\left|\cos x=t\quad -\sin xdx=dt\right|=-\int\frac{dt}{t^3}=\frac{1}{2t^2}+C=\frac{1}{2\cos^2 x}+C$$

$$I_1=\frac{\sin x}{2\cos^2 x}+\frac{1}{2}\ln\left|tg\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|+C$$

$$I_2=\int\frac{dx}{\sin^2 x\cos x}=\int\frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos x}dx=\int\frac{dx}{\cos x}+\int\frac{\cos xdx}{\sin^2 x}=\ln\left|tg\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|+$$

$$+\left|\begin{array}{l}\sin x=t\\ \cos xdx=dt\end{array}\right|=\ln\left|tg\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|+\int\frac{dt}{t^2}=\ln\left|tg\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|-\frac{1}{\sin x}+C$$

Ostatecznie otrzymujemy:

$$I=\frac{\sin x}{2\cos^2 x}-\frac{1}{\sin x}+\frac{3}{2}\ln\left|tg\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|+C$$

139/

$$\int\sin^4 x\cos^5 xdx=\int\sin^4 x\cos^4 x\cos xdx=\int\sin^4 x\left(1-\sin^2 x\right)^2\cos xdx=\left|\sin x=t\quad \cos xdx=dt\right|=$$

$$=\int t^4\left(1-t^2\right)dt=\int\left(t^4-2t^6+t^8\right)dt=\frac{t^5}{5}-\frac{2t^7}{7}+\frac{t^9}{9}+C=\frac{\sin^5 x}{5}-\frac{2\sin^7 x}{7}+\frac{\sin^9 x}{9}+C$$

140/

$$\int\frac{\sqrt{tgx}dx}{\sin 2x}=\int\frac{\sqrt{tgx}dx}{2\sin x\cos x}=\frac{1}{2}\int\frac{\sqrt{tgx}dx}{\sin x\cos x}\frac{\cos x}{\cos x}=\frac{1}{2}\int\sqrt{tgx}\frac{\cos x}{\sin x}\frac{dx}{\cos^2 x}=\left|tgx=t\quad dt=\frac{dx}{\cos^2 x}\right|=$$

$$=\frac{1}{2}\int t^{\frac{1}{2}}t^{-1}dt=t^{\frac{1}{2}}+C=\sqrt{tgx}+C$$

7.Całki funkcji wykładniczych i logarytmicznych

Całki postaci $\int R\left(e^x\right)dx$ wyznacza się przez podstawienie $e^x=t$.

P R Z Y K Ł A D Y   C A Ł K O W A N I A

141/

$$\int\left(e^{4x}+\sqrt{e^x}\right)dx=\left|e^x=t\quad e^xdx=dt\quad dx=\frac{dt}{t}\right|=\int\left(t^4+t^{\frac{1}{2}}\right)\frac{dt}{t}=\frac{t^3}{3}-2t^{\frac{1}{2}}+C=\frac{1}{3}e^{3x}-2e^{\frac{1}{2}}+C$$

142/

$$\int\frac{e^x+1}{e^x-1}dx=\left|e^x=t\quad dx=\frac{dt}{t}\right|=\int\frac{t+1}{t-1}\frac{dt}{t}=\int\frac{t+1}{t(t-1)}dt=I$$

Pomocniczo rozkładamy funkcję wymierną na ułamki proste:

$$\frac{t+1}{t(t-1)} = -\frac{1}{t} + \frac{2}{t-1}$$

$$I = \int \frac{-dt}{t} + 2\int \frac{dt}{t-1} = -\ln|t| + 2\ln|t-1| + C = -\ln e^x + \ln\left(e^x - 1\right)^2 + C$$

142/

$$\int \frac{e^{3x}dx}{e^x+2} = \left| e^x = t \quad dx = \frac{dt}{t} \right| = \int \frac{t^3dt}{t(t+2)} = \int \left(t - 2 + \frac{4t}{t(t+2)}\right)dt = \frac{t^2}{2} - 2t + 4\ln|t+2| + C =$$

$$= \frac{e^{2x}}{2} - 2e^x + 4\ln\left|e^x + 2\right| + C$$

143/

$$\int \frac{dx}{e^{3x} - e^x} = \left| e^x = t \quad dx = \frac{dt}{t} \right| = \int \frac{dt}{t^2\left(t^2-1\right)} = \int \frac{dt}{t^2\left(t^2-1\right)} = -\int \frac{dt}{t^2} + \frac{1}{2}\int \frac{dt}{t-1} - \frac{1}{2}\int \frac{dt}{t+1} =$$

$$= \frac{1}{t} + \frac{1}{2}\ln|t-1| - \frac{1}{2}\ln|t+1| = \frac{1}{e^x} + \frac{1}{2}\ln\left|e^x - 1\right| - \frac{1}{2}\ln\left|e^x + 1\right| + C$$

Pomocniczo rozłożono funkcję wymierną na ułamki proste:

$$\frac{1}{t^2\left(t^2-1\right)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t-1} + \frac{D}{t+1} = -\frac{1}{t^2} + \frac{1}{2(t-1)} - \frac{1}{2(t+1)}$$

144/

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx = \left| e^x = t \quad dx = \frac{dt}{t} \right| = \int \frac{t - \frac{1}{t}}{t + \frac{1}{t}}\frac{dt}{t} = \int \frac{t^2-1}{t^2+1}\frac{dt}{t} = \int\left(-\frac{1}{t} + \frac{2t}{t^2+1}\right)dt = -\int\frac{dt}{t} + 2\int\frac{tdt}{t^2+1} =$$

$$= -\ln\left|e^x\right| + \ln\left|e^{2x}+1\right| + C = \ln\left|e^{2x}+1\right| - x + C$$

145/

$$\int x^3e^{-x}dx = \left| \begin{matrix} u = x^3 & du = 3x^2dx \\ dv = e^{-x}dx & v = -e^{-x} \end{matrix} \right| = -xe^{-x} + 3\int x^2e^{-x}dx \left| \begin{matrix} u = x^2 & du = 2xdx \\ dv = e^{-x}dx & v = -e^{-x} \end{matrix} \right| =$$

$$= -xe^{-x} - 3x^2e^{-x} + 6\int xe^{-x}dx = -x^3e^{-x} - 3x^2e^{-x} + 6\left| \begin{matrix} u = x & du = dx \\ dv = e^{-x} & v = -e^{-x} \end{matrix} \right| =$$

$$= -x^3e^{-x} - 3x^2e^{-x} - 6xe^{-x} + 6\int e^{-x}dx = -xe^{-x} - 3x^2e^{-x} - 6xe^{-x} - 6e^{-x} + C$$

146/

$$\int \frac{a^xdx}{a^{2x}+1} = \left| a^x = t \quad \ln a^x = \ln t \quad dx = \frac{dt}{t\ln a} \right| = \int \frac{dt}{\ln a\left(t^2+1\right)} = \frac{1}{\ln a}\int\frac{dt}{t^2+1} =$$

$$= \frac{1}{\ln a}arctgt + C = \frac{1}{\ln a}arctg\left(a^x\right) + C$$

147/

$$\int \log_3 xdx = \int\frac{\ln xdx}{\ln 3} = \frac{1}{\ln 3}\int \ln xdx = \frac{1}{\ln 3}\left| \begin{matrix} u = \ln x & du = \frac{dx}{x} \\ dv = dx & v = x \end{matrix} \right| = \frac{1}{\ln 3}(x\ln x - x) + C$$

Do wyznaczenia całki **147/** wykorzystano wzór:

$$3^{\log_3 x} = x \quad \ln 3\log_3 x = \ln x \quad \log_3 x = \frac{\ln x}{\ln 3} \qquad (1.43)$$

148/

$$\int \frac{e^{-3x}dx}{\sqrt{1+e^{-3x}}} = \left| \sqrt{1+e^{-3x}} = t \quad -\frac{3e^{-3x}dx}{2\sqrt{1+e^{-3x}}} = dt \right| = -\frac{2}{3}\int dt = -\frac{2t}{3} + C = -\frac{2\sqrt{1+e^{-3x}}}{3} + C$$

8. Całki funkcji hiperbolicznych

Całki funkcji hiperbolicznych wyznacza się tymi samymi sposobami co inne całki. Należy wykorzystywać wzory dotyczące związków pomiędzy tymi funkcjami oraz wzory dotyczące całkowania funkcji hiperbolicznych. Poniżej podane są najważniejsze z nich.

(1.44) $\sinh x = shx = \frac{e^x - e^{-x}}{2}$

(1.45) $\cosh x = chx = \frac{e^x + e^{-x}}{2}$

(1.46) $tghx = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x - e^{-x}}$

(1.47) $ctghx = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

(1.48) $\cosh^2 x = 1 + \sinh^2 x$

(1.49) $\cosh 2x = \cosh^2 x + \sinh^2 x$

(1.50) $ch^2x = \frac{ch2x+1}{2}$

(1.51) $sh^2x = \frac{ch2x-1}{2}$

(1.52) $shxchx = \frac{sh2x}{2}$

(1.53) $\int shxdx = chx + C$

(1.54) $\int chxdx = shx + C$

(1.55) $\int \frac{dx}{ch^2x} = thx + C$

(1.56) $\int \frac{dx}{sh^2 x} = -cthx + C$

PRZYKLADY CAŁKOWANIA

149/

$$\int sh^2 4x dx = \int \frac{ch8x-1}{2}dx = \frac{1}{2}\int ch8x dx - \frac{1}{2}dx = \frac{1}{16}\int chtdt \left|\begin{array}{l} 8x = t \\ dx = \frac{dt}{8}\end{array}\right| - \frac{1}{2}x = \frac{1}{16}sh8x - \frac{1}{2}x + C$$

150/

$$\int ch^3 x dx = \int chxch^2 x dx = \int chx(1+sh^2 x)dx = \left|shx = t \quad chxdx = dt\right| = \int (1+t^2)dt =$$

$$= t + \frac{t^3}{3} + C = shx + \frac{sh^3 x}{3} + C$$

151/

$$\int \frac{dx}{x^2\sqrt{x^2+1}} = \left|x = sht \quad \sqrt{x^2+1} = cht \quad dx = chtdt\right| = \int \frac{chtdt}{sh^2 tcht} = \int \frac{dt}{sh^2 t} = -ctht + C$$

$$= -\frac{cht}{sht} + C = -\frac{\sqrt{x^2+1}}{x} + C$$

152/

$$\int \frac{\sqrt{x^2-1}}{5x}dx = \left|x = cht \quad \sqrt{x^2-1} = sht \quad dx = shtdt\right| = \frac{1}{5}\int \frac{shtsht}{cht}dt = \frac{1}{5}\int \frac{sh^2 t}{ch^2 t}chtdt =$$

$$= \frac{1}{5}\int \frac{sh^2 t}{sh^2 t+1}chtdt = \left|\begin{array}{l} sht = k \\ chtdt = dk\end{array}\right| = \frac{1}{5}\int \frac{k^2 dk}{k^2+1} = \frac{1}{5}k - \frac{1}{5}arctgk + C = \frac{1}{5}sht - \frac{1}{5}arctgsht + C =$$

$$= \frac{1}{5}\sqrt{x^2-1} - \frac{1}{5}arctg\sqrt{x^2-1} + C$$

153/

$$\int \frac{dx}{\sqrt{(x^2+4)^3}} = \left|x = 2sht \quad \sqrt{x^2+4} = 2cht \quad dx = 2chtdt\right| = \int \frac{2chtdt}{(2cht)^3} = \frac{1}{4}\int \frac{dt}{ch^2 t} =$$

$$= \frac{1}{4}tht + C = \frac{1}{4}\frac{sht}{cht} + C = \frac{1}{4}\frac{\frac{x}{2}}{\frac{\sqrt{x^2+4}}{2}} + C = \frac{1}{4}\frac{x}{\sqrt{x^2+4}} + C$$

154/

$$\int \sqrt{x^2+9}dx = \left|x = 3sht \quad \sqrt{x^2+9} = 3cht \quad dx = 3chtdt\right| = 9\int ch^2 tdt = 9\int \frac{ch2t+1}{2} =$$

$$= \frac{9}{2}\int ch2tdt + \frac{9}{2}\int dt = \frac{9}{2}sh2t + \frac{9}{2}t + C = \frac{9}{2}shtcht + \frac{9}{2}t + C = \frac{x\sqrt{x^2+9}}{2} + \frac{9}{2}\ln\left|\frac{x+\sqrt{x^2+9}}{3}\right| + C$$

9. Całki różne

155/ $\int |x^2 - x|dx = I_1 + I_2 + I_3$

$I_1 = \int (x^2 - x)dx = \frac{x^3}{3} - \frac{x^2}{2} + C_1$ dla $x \in (-\infty, 0]$

$I_2 = \int (x - x^2)dx = -\frac{x^3}{3} + \frac{x^2}{2} + C_2$ dla $x \in [0, 1]$

$I_3 = \int (x^2 - x)dx = \frac{x^3}{3} - \frac{x^2}{2} + C_3$ dla $x \in [1, \infty)$

Ostatecznie pozostalo dobrać tak stale C_1, C_2, C_3 aby funkcja podcałkowa pozostala ciągla w punktach x = 0 oraz x = 1.
Jeżeli $C_1 = C = C_2$ oraz $C_3 = 1/3 + C_1$ to warunek ten będzie spełniony.

156/

$$\int \frac{x+1}{x+\sqrt{x+2}}dx = \left|\begin{array}{l} x+2 = t^2 \\ dx = 2tdt \\ x = t^2 - 2\end{array}\right| = \int \frac{t^2-1}{t^2-2+t}2tdt = 2\int \frac{t^3-t}{t^2+t-2}dt = 2\int (t - 1 + \frac{2t-2}{t^2+t-2})dt =$$

$$= t^2 - 2t + 4\int \frac{t-1}{t^2+t-2}dt = x + 2 - 2\sqrt{x+2} + 4\int \frac{t-1}{(t-1)(t+2)}dt = x + 2 - 2\sqrt{x+2} + 4\int \frac{dt}{t+2} =$$

$$= x + 2 - 2\sqrt{x+2} + 4\ln|t+2| + C = x + 2 - 2\sqrt{x+2} + 4\ln\left|\sqrt{x+2} + 2\right| + C$$

157/

$$\int \frac{x+\sqrt{2x-3}}{x-1}dx = \left|\begin{array}{l} 2x-3 = t^2 \\ 2dx = 2tdt \\ x = \frac{t^2+3}{2}\end{array}\right| = \int \frac{\frac{t^2+3}{2}+t}{\frac{t^2+3}{2}-1}tdt = \int \frac{t^3+2t^2+3t}{t^2+1}dt = \int (t + 2 + \frac{2t-2}{t^2+1})dt$$

$$= \frac{t^2}{2} + 2t + \int \frac{2tdt}{t^2+1} - 2\int \frac{dt}{t^2+1} = \frac{2x-3}{2} + 2\sqrt{2x-3} + \ln|t^2+1| - 2arctgt + C = \frac{2x-3}{2} + 2\sqrt{2x-3} +$$

$$+ \ln|2x-2| - 2arctg\sqrt{2x-3} + C$$

158/

$$\int \frac{dx}{x - \sqrt{x^2-x+1}} = \left|\sqrt{x^2-x+1} = x - t \quad x = \frac{t^2-1}{2t-1} \quad dx = \frac{2(t^2-t+1)}{(2t-1)^2}dt\right| =$$

$$= \int \frac{2(t^2 - t + 1)}{(2t-1)^2 t} dt = I$$

$$\frac{t^2 - t + 1}{t(2t-1)^2} = \frac{A}{t} + \frac{B}{2t-1} + \frac{C}{(2t-1)^2}$$

$$\begin{cases} 4A + 2B = 1 \\ C - 4A - B = -1 \\ A = 1 \end{cases} \quad \begin{cases} A = 1 \\ B = -\frac{3}{2} \\ C = \frac{3}{2} \end{cases}$$

$$I = 2\int \frac{dt}{t} - 3\int \frac{dt}{2t-1} + 3\int \frac{dt}{(2t-2)^2} = 2\ln|t| - \frac{3}{2}\ln|2t-1| - \frac{3}{2t} = 2\ln\left|x - \sqrt{x^2 - x + 1}\right| -$$

$$- \frac{3}{2}\ln\left|2x - 2\sqrt{x^2 - x + 1} - 1\right| - \frac{3}{2\left(x - \sqrt{x^2 - x + 1} - 1\right)} + C$$

159/

$$\int \frac{\sin^2 x\,dx}{1 + \cos^2 x} = \left| \begin{array}{l} tgx = t \\ x = arctgt \\ dx = \frac{dt}{1+t^2} \end{array} \right| = \int \frac{\frac{t^2}{1+t^2}}{1 + \frac{1}{1+t^2}} \frac{dt}{1+t^2} = \int \frac{t^2 dt}{(2+t^2)(1+t^2)} = \int \frac{2dt}{t^2+2} - \int \frac{dt}{t^2+1} =$$

$$= 2\int \frac{dt}{2\left(\left(\frac{t}{\sqrt{2}}\right)^2 + 1\right)} - arctgt = \sqrt{2} arctg\left(\frac{tgx}{\sqrt{2}}\right) - arctg(tgx) + C$$

160/

$$\int |x| dx = F(x)$$

$$F'(x) = |x| = \begin{cases} x, & x > 0 \\ 0 & x = 0 \\ -x, & x < 0 \end{cases} \qquad F(x) = \begin{cases} \frac{x^2}{2} + C_1 & x > 0 \\ C & x = 0 \\ -\frac{x^2}{2} + C_2 & x < 0 \end{cases}$$

Funkcja F(x) jest różniczkowalna w R, więc jest także różniczkowalna dla x = 0. Stąd:

$$\lim_{x \to 0^+} F(x) = \lim_{x \to 0^-} F(x) = F(0)$$

$$\lim_{x \to 0^+}\left(\frac{x^2}{2} + C_1\right) = \lim_{x \to 0^-}\left(-\frac{x^2}{2} + C_2\right) = C$$

$$C_1 = C_2 = C$$

gdzie C jest dowolną stałą.

161/ $\int |x - 1| dx = I$

$$F'(x) = |x - 1| = \begin{cases} x - 1 & x > 1 \\ 0 & x = 1 \\ -x + 1 & x < 1 \end{cases} \qquad F(x) = \begin{cases} \frac{x^2}{2} - x + C_1 & x > 1 \\ C & x = 1 \\ -\frac{x^2}{2} + x + C_2 & x < 1 \end{cases}$$

Ponieważ F(x) jest różniczkowalna w R, więc jest różniczkowalna w punkcie x = 1 a stąd ciągła dla x = 1. Stąd:

$$\lim_{x \to 1^+} F(x) = \lim_{x \to 1^-} F(x) = F(1)$$

$$\lim_{x \to 1^+}\left(\frac{x^2}{2} - x + C_1\right) = \lim_{x \to 1^-}\left(-\frac{x^2}{2} + x + C_2\right) = C$$

$$C_1 - \frac{1}{2} = C_2 + \frac{1}{2} = C$$

$$C_2 = C_1 - 1$$

gdzie C_1 jest dowolną stałą.

162/

$$\int \frac{\arcsin x\,dx}{\sqrt{(1-x^2)^3}} = \int \frac{\arcsin x\,dx}{\sqrt{1-x^2}(1-x^2)} = \left| \begin{array}{ll} \arcsin x = t & x = \sin t \\ dt = \frac{dx}{\sqrt{1-x^2}} & x^2 = \sin^2 t \quad 1 - x^2 = \cos^2 t \end{array} \right| = \int \frac{tdt}{\cos^2 t} =$$

$$= \left| \begin{array}{ll} u = t & du = dt \\ dv = \frac{dt}{\cos^2 t} & v = tgt \end{array} \right| = ttgt - \int tgtdt = ttgt + \ln|\cos t| + C = \arcsin x \frac{\sin(\arcsin x)}{\cos(\arcsin x)} + \ln\left|\sqrt{1-x^2}\right| + C =$$

$$= \frac{x \arcsin x}{\sqrt{1 - \sin^2(\arcsin x)}} + \ln\left|(1-x^2)^{\frac{1}{2}}\right| + C = \frac{x \arcsin x}{\sqrt{1-x^2}} + \frac{1}{2}\ln\left|1 - x^2\right| + C$$

163/

$$\int \frac{x^2 arctgx dx}{1+x^2} = \left| \begin{array}{ll} arctgx = t & x = tgt \\ \frac{dx}{1+x^2} = dt & x^2 = tg^2 t \end{array} \right| = \int t tg^2 t dt = \left| \begin{array}{ll} u = t & du = dt \\ dv = tg^2 t dt & v = \int tg^2 t dt = \int \left(\frac{1}{\cos^2 t} - 1\right) dt = tgt - t \end{array} \right| =$$

$$= ttgt - t^2 - \int tgtdt + \int tdt = arctgx(tg(arctgx)) + \ln|\cos t| - \frac{1}{2}(arctgx)^2 =$$

$$= xarctgx + \ln\left|\cos(arctgx)\right| - \frac{1}{2}(arctgx)^2 + C$$

164/

$$\int\frac{xarctgx\,dx}{\left(1+x^2\right)^2}=\left|\begin{matrix}u=arctgx & du=\dfrac{dx}{1+x^2}\\ dv=\dfrac{xdx}{\left(1+x^2\right)^2} & v=\int dv=-\dfrac{1}{2\left(1+x^2\right)}\end{matrix}\right|=-\frac{arctgx}{2\left(1+x^2\right)^2}+\frac{1}{2}\int\frac{dx}{\left(1+x^2\right)^2}=$$

$$=-\frac{arctgx}{2\left(1+x^2\right)^2}+\frac{1}{2}\left(\frac{x}{2\left(x^2+1\right)}+\frac{1}{2}arctgx\right)=\frac{\left(x^2-1\right)arctgx+x}{4\left(x^2+1\right)}+C$$

Do obliczenia całki 164/ wykorzystano wzór rekurencyjny (1.28).

165/

$$\int\frac{\arcsin x}{x^2}dx=\left|\begin{matrix}u=\arcsin x & du=\dfrac{dx}{\sqrt{1-x^2}}\\ dv=\dfrac{dx}{x^2} & v=-\dfrac{1}{x}\end{matrix}\right|=-\frac{\arcsin x}{x}+\int\frac{dx}{x\sqrt{1-x^2}}=-\frac{\arcsin x}{x}+I$$

$$I=\int\frac{dx}{x\sqrt{1-x^2}}\left|\begin{matrix}\dfrac{1}{x}=t & x=\dfrac{1}{t}\\ -\dfrac{dx}{x^2}=dt & dx=-\dfrac{dt}{t^2}\end{matrix}\right|=\int\frac{-\frac{dt}{t^2}}{\frac{1}{t}\sqrt{1-\frac{1}{t^2}}}=\int\frac{-\frac{dt}{t^2}}{\frac{1}{t^2}\sqrt{t^2-1}}=\int\frac{-dt}{\sqrt{t^2-1}}=-\ln\left|t+\sqrt{t^2-1}\right|=$$

$$=-\ln\left|\frac{1}{x}+\sqrt{\frac{1}{x^2}-1}\right|+C$$

$$\int\frac{\arcsin x}{x^2}dx=-\frac{\arcsin x}{x}-\ln\left|\frac{1}{x}+\sqrt{\frac{1}{x^2}-1}\right|+C=-\frac{\arcsin x}{x}-\ln\left|\frac{1+\sqrt{1-x^2}}{x}\right|+C$$

166/

$$\int\frac{\arcsin e^x}{e^x}dx=\left|\begin{matrix}e^x=t\\ e^xdx=dt\\ dx=\dfrac{dt}{t}\end{matrix}\right|=\int\frac{\arcsin t\,dt}{t^2}=\left|\begin{matrix}u=\arcsin t & du=\dfrac{dt}{\sqrt{1-t^2}}\\ dv=\dfrac{dt}{t^2} & v=-\dfrac{1}{t}\end{matrix}\right|=-\frac{\arcsin t}{t}+\int\frac{dt}{t\sqrt{1-t^2}}=$$

$$=-\frac{\arcsin e^x}{e^x}+I$$

$$I=\int\frac{dt}{t\sqrt{1-t^2}}=\left|\begin{matrix}\dfrac{1}{t}=z\\ t=\dfrac{1}{z}\\ -\dfrac{dt}{t^2}=dz\end{matrix}\right|=\int\frac{-\frac{dz}{z^2}}{\frac{1}{z}\sqrt{1-\frac{1}{z^2}}}=\int\frac{-dz}{\sqrt{z^2-1}}=-\ln\left|z+\sqrt{z^2-1}\right|=-\ln\left|\frac{1}{t}+\sqrt{\frac{1}{t^2}-1}\right|=$$

$$I=-\ln\left|\frac{1+\sqrt{1-e^{2x}}}{e^x}\right|=-\ln\left|1+\sqrt{1-e^{2x}}\right|+\ln e^x=x-\ln\left|1+\sqrt{1-e^{2x}}\right|$$

Ostatecznie otrzymujemy:

$$\int\frac{\arcsin e^x}{e^x}dx=-e^{-x}\arcsin e^x+x-\ln\left|1+\sqrt{1-e^{2x}}\right|+C$$

167/

$$\int x^3arctgx\,dx=\left|\begin{matrix}u=arctgx & du=\dfrac{dx}{1+x^2}\\ dv=x^3dx & v=\dfrac{x^4}{4}\end{matrix}\right|=\frac{x^4arctgx}{4}-\frac{1}{4}\int\frac{x^4dx}{1+x^2}=\frac{x^4arctgx}{4}-\frac{1}{4}\int\left(x^2-1+\frac{1}{1+x^2}\right)dx=$$

$$\frac{x^4arctgx}{4}-\frac{x^3}{4}+\frac{x}{4}-\frac{arctgx}{4}=\frac{\left(x^4-1\right)arctgx-x^3+x}{4}+C$$

168/

$$\int\frac{dx}{\left(1+4x^2\right)\left(arctg2x\right)^2}=\left|u=arctg2x\quad du=\frac{2dx}{1+4x^2}\right|=\frac{1}{2}\int\frac{du}{u^2}=-\frac{1}{2u}=-\frac{1}{2arctg2x}+C$$

169/

$$\int\frac{dx}{\sqrt{1-x^2}\arccos^2x}=\left|\begin{matrix}u=\arccos x\\ du=-\dfrac{dx}{\sqrt{1-x^2}}\end{matrix}\right|=\int-\frac{du}{u^2}=\frac{1}{u}=\frac{1}{\arccos x}+C$$

170/

$$\int\ln\left(x+\sqrt{x^2+1}\right)dx=\left|\begin{matrix}u=\ln\left(x+\sqrt{x^2+1}\right) & du=\dfrac{dx}{\sqrt{x^2+1}}\\ dv=dx & v=x\end{matrix}\right|=x\ln\left|x+\sqrt{x^2+1}\right|-\int\frac{xdx}{\sqrt{x^2+1}}=$$

$$=x\ln\left|x+\sqrt{x^2+1}\right|-\frac{1}{2}\int\frac{2xdx}{\sqrt{x^2+1}}=x\ln\left|x+\sqrt{x^2+1}\right|-\sqrt{x^2+1}+C$$

171/

$$\int\ln|3+4x|dx=\left|\begin{matrix}3+4x=t\\ 4dx=dt\end{matrix}\right|=\frac{1}{4}\int dt\ln t=\frac{1}{4}\left|\begin{matrix}u=\ln t & du=\dfrac{dt}{t}\\ dv=dt & v=t\end{matrix}\right|=\frac{1}{4}t\ln t-\frac{1}{4}\int dt=$$

$$=\frac{1}{4}(3+4x)\ln|3+4x|-\frac{1}{4}(3+4x)+C$$

172/

$$\int\frac{dx}{x\left(1+\ln^2|x|\right)}=\left|\begin{matrix}\ln|x|=t\\ \dfrac{dx}{x}=dt\end{matrix}\right|=\int\frac{dt}{1+t^2}=arctgt+C=arctg\left(\ln|x|\right)+C$$

174/

$$\int x^{-2}\ln|x|dx=\left|\begin{array}{ll} u=\ln|x| & du=\frac{dx}{x} \\ dv=\frac{dx}{x^2} & v=-\frac{1}{x}\end{array}\right|=-\frac{1}{x}\ln|x|+\int\frac{dx}{x^2}=-\frac{1}{x}\ln|x|-\frac{1}{x}+C$$

175/

$$\int(4+3x)^2\ln|x|dx=\left|\begin{array}{ll} u=\ln|x| & du=\frac{dx}{x} \\ dv=(4+3x)^2 & v=\int\left(16+24x+9x^2\right)dx=16x+12x^2+3x^3\end{array}\right|=$$

$$=\ln|x|\left(16x+12x^2+3x^3\right)-\int\left(16+12x+3x^2\right)dx=\left(3x^3+12x^2+16x\right)\ln|x|-16x-6x^2-x^3+C$$

176/

$$\int x^3\ln\left(x^2+3\right)dx=\left|\begin{array}{ll} u=\ln\left(x^2+3\right) & du=\frac{2xdx}{x^2+3} \\ dv=x^3dx & v=\frac{x^4}{4}\end{array}\right|=\frac{x^4}{4}\ln\left|x^2+3\right|-\frac{1}{2}\int\frac{x^5dx}{x^2+3}=\frac{x^4}{4}\ln\left|x^2+3\right|-$$

$$-\frac{1}{2}\int\left(x^3-3x+\frac{9x}{x^2+3}\right)dx=\frac{x^4}{4}\ln\left(x^2+3\right)-\frac{x^4}{8}+\frac{3x^2}{4}-\frac{9}{2}\int\frac{xdx}{x^2+3}=\frac{x^4}{4}\ln\left(x^2+3\right)-\frac{x^4}{8}+\frac{3x^2}{4}-\frac{9}{4}\ln\left(x^2+3\right)+C$$

177/

$$\int x3^xdx=\left|\begin{array}{ll} u=x & du=dx \\ dv=3^xdx & v=\frac{3^x}{\ln 3}\end{array}\right|=x\frac{3^x}{\ln 3}-\frac{1}{\ln 3}\int 3^xdx=x\frac{3^x}{\ln 3}-\frac{1}{\ln 3}\frac{3^x}{\ln 3}=\frac{3^x}{\ln 3}\left(x-\frac{1}{\ln 3}\right)+C$$

178/

$$\int\frac{x^2dx}{\sqrt{x^2-3}}=\left|\begin{array}{c} x^2-3=t^2 \\ x=\sqrt{t^2+3} \\ dx=\frac{tdt}{x}=\frac{tdt}{\sqrt{t^2+3}}\end{array}\right|=\int\frac{\left(t^2+3\right)tdt}{t\sqrt{t^2+3}}=\int\sqrt{t^2+3}dt=\frac{3}{2}\ln\left|t+\sqrt{t^2+3}\right|+\frac{1}{2}t\sqrt{t^2+3}+C=$$

$$=\frac{3}{2}\ln\left|x+\sqrt{x^2-3}\right|+\frac{1}{2}x\sqrt{x^2-3}+C$$

179/ $$\int\frac{dx}{x^4+3x^2}=\int\frac{dx}{x^2\left(x^2+3\right)}=I$$

$$\frac{1}{x^2\left(x^2+3\right)}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+3}=\frac{x^3(A+C)+x^2(B+D)+3Ax+3B}{x^2\left(x^2+3\right)}$$

$$\begin{cases} A+C=0 \\ B+D=0 \\ 3A=0 \\ 3B=1\end{cases}\qquad\begin{cases} A=0 \\ B=\frac{1}{3} \\ C=0 \\ D=-\frac{1}{3}\end{cases}$$

Wracając do całki:

$$I=\frac{1}{3}\int\frac{dx}{x^2}-\frac{1}{3}\int\frac{dx}{x^2+3}=-\frac{1}{3x}-\frac{\sqrt{3}}{9}arctg\frac{x}{\sqrt{3}}+C$$

180/

$$\int\frac{dx}{x^4-x^2-2}\int\frac{dx}{\left(x^2+1\right)\left(x-\sqrt{2}\right)\left(x+\sqrt{2}\right)}=\frac{\sqrt{2}}{2}\int\frac{dx}{x-\sqrt{2}}-\frac{\sqrt{2}}{2}\int\frac{dx}{x+\sqrt{2}}-\frac{1}{2}\int\frac{dx}{x^2+1}$$

Powyższe współczynniki otrzymano z rozkładu podcałkowej funkcji wymiernej. A zatem:

$$\int\frac{dx}{x^4-x^2-2}=\frac{\sqrt{2}}{2}\ln\left|x-\sqrt{2}\right|-\frac{\sqrt{2}}{2}\ln\left|x+\sqrt{2}\right|-\frac{1}{2}arctgx+C$$

181/

$$\int\frac{dx}{(2x+1)\left(1+\sqrt{2x+1}\right)}=\left|\begin{array}{l} 2x+1=t^2 \\ 2dx=2tdt\end{array}\right|=\int\frac{tdt}{t^2(1+t)}=\int\frac{dt}{t(1+t)}=\int\frac{dt}{t}-\int\frac{dt}{t+1}=\ln|t|-\ln|t+1|+C=$$

$$=\ln\left|\sqrt{2x+1}\right|-\ln\left|1+\sqrt{2x+1}\right|+C$$

182/

$$\int\frac{\cos x}{\cos 3x}dx=\int\frac{\cos xdx}{\cos x\left(\cos^2x-3\sin^2x\right)}=\int\frac{dx}{1-\sin^2x-3\sin^2x}=\int\frac{dx}{1-4\sin^2x}=$$

$$=\left|\begin{array}{c} tgx=t \\ dx=\frac{dt}{1+t^2} \\ \sin^2x=\frac{t^2}{1+t^2}\end{array}\right|=\int\frac{\frac{dt}{1+t^2}}{1-\frac{4t^2}{1+t^2}}=\int\frac{dt}{1-3t^2}=I$$

$$\frac{1}{1-3t^2}=\frac{A}{1-\sqrt{3}t}+\frac{B}{1+\sqrt{3}t}=\frac{t\left(A\sqrt{3}-B\sqrt{3}\right)+A+B}{1-3t^2}$$

$$\begin{cases} A\sqrt{3}-B\sqrt{3}=0 \\ A+B=1\end{cases}\qquad\begin{cases} A=\frac{1}{2} \\ B=\frac{1}{2}\end{cases}$$

$$I=\frac{1}{2}\int\frac{dt}{1-\sqrt{3}t}+\frac{1}{2}\int\frac{dt}{1+\sqrt{3}t}=-\frac{1}{2\sqrt{3}}\ln\left|1-\sqrt{3}t\right|+\frac{1}{2\sqrt{3}}\ln\left|1+\sqrt{3}t\right|+C=$$

$$=\frac{1}{2\sqrt{3}}\ln\left|\frac{1+\sqrt{3}tgx}{1-\sqrt{3}tgx}\right|+C$$

183/

$$\int\frac{dx}{x\sqrt{x^3-1}}=\left|x^3-1=t^2\quad 3x^2dx=2tdt\quad x^3=t^2+1\right|=\frac{2}{3}\int\frac{tdt}{t(t^2+1)}=\frac{2}{3}arctgt=\frac{2}{3}arctg\sqrt{x^3-1}+C$$

184/

$$\int\frac{dx}{1+tgx}=\left|\begin{matrix}tgx=t\\ \frac{dt}{1+t^2}=dx\end{matrix}\right|=\int\frac{\frac{dt}{1+t^2}}{1+t}=\int\frac{dt}{(1+t)(1+t^2)}=I$$

Pomocniczo należy rozłożyć funkcję wymierną na ułamki proste:

$$\frac{1}{(1+t)(1+t^2)}=\frac{A}{1+t}+\frac{Bt+C}{1+t^2}=\frac{t^2(A+B)+t(B+C)+A+C}{(1+t)(1+t^2)}$$

$$\begin{cases}A+B=0\\B+C=0\\A+C=1\end{cases}\qquad\begin{cases}A=\frac{1}{2}\\B=-\frac{1}{2}\\C=\frac{1}{2}\end{cases}$$

$$I=\frac{1}{2}\int\frac{dt}{1+t}-\frac{1}{2}\int\frac{t-1}{1+t^2}=\frac{1}{2}\ln|1+t|-\frac{1}{2}\int\frac{tdt}{1+t^2}+\frac{1}{2}\int\frac{dt}{1+t^2}=\frac{1}{2}\ln|1+t|-\frac{1}{4}\ln|1+t^2|+\frac{1}{2}arctgt=$$

$$=\frac{1}{2}\ln|1+tgx|-\frac{1}{4}\ln|1+tg^2x|+\frac{x}{2}+C=\frac{1}{2}\left(x+\ln\left|\frac{1+tgx}{\sqrt{1+tg^2x}}\right|\right)=\frac{1}{2}\ln\left|\frac{\frac{\cos x+\sin x}{\cos x}}{\sqrt{\frac{\cos^2x+\sin^2x}{\cos^2x}}}\right|+\frac{x}{2}+C$$

$$=\frac{x}{2}+\frac{1}{2}\ln|\cos x+\sin x|+C$$

185/

$$\int\frac{\sin 2x}{\cos^4x}dx=\int\frac{2\sin x\cos xdx}{\cos^4x}=2\int\frac{\sin x}{\cos^3x}dx=\left|\begin{matrix}\cos x=t\\-\sin xdx=dt\end{matrix}\right|=-2\int\frac{dt}{t^3}=\frac{1}{t^2}+C=\frac{1}{\cos^2x}+C$$

186/

$$\int\frac{\ln(\cos x)dx}{\sin^2x}=\left|\begin{matrix}u=\ln(\cos x) & du=-\frac{\sin x}{\cos x}dx\\ dv=\frac{dx}{\sin^2x} & v=-ctgx\end{matrix}\right|=-ctgx\ln|\cos x|-\int dx=-ctgx\ln|\cos x|-x+C$$

187/

$$\int\sqrt{1-\sin x}dx=\int\frac{\sqrt{(1-\sin x)(1+\sin x)}}{\sqrt{1+\sin x}}dx=\int\frac{\sqrt{1-\sin^2x}}{\sqrt{1+\sin x}}dx=\int\frac{\cos xdx}{\sqrt{1+\sin x}}=\left|\begin{matrix}\sin x=t\\\cos xdx=dt\end{matrix}\right|$$

$$=\int\frac{dt}{\sqrt{1+t}}=\left|\begin{matrix}1+t=z\\dt=dz\end{matrix}\right|=\int z^{-1/2}dz=2z^{1/2}+C=2\sqrt{z}+C=2\sqrt{1+t}+C=2\sqrt{1+\sin t}+C$$

188/

$$\int\frac{\ln(x^2+1)}{x^3}dx=\left|\begin{matrix}u=\ln(x^2+1) & du=\frac{2xdx}{x^2+1}\\ dv=\frac{dx}{x^3} & v=-\frac{1}{2x^2}\end{matrix}\right|=-\frac{\ln(x^2+1)}{2x^2}+\int\frac{2xdx}{2x^2(x^2+1)}=$$

$$=-\frac{\ln(x^2+1)}{2x^2}+\int\frac{dx}{x(x^2+1)}=-\frac{\ln(x^2+1)}{2x^2}+\ln|x|-\frac{1}{2}\ln(x^2+1)+C$$

Do obliczenia całki wykorzystano rozkład funkcji wymiernej:

$$\frac{1}{x(x^2+1)}=\frac{A}{x}+\frac{Bx+C}{x^2+1}=\frac{x^2(A+B)+Cx+A}{x(x^2+1)}=\frac{1}{x}-\frac{x}{x^2+1}$$

189/

$$\int\frac{a^xdx}{a^{2x}+1}=\left|a^x=t\quad a^{2x}=t^2\quad a^x\ln adx=dt\quad a^xdx=\frac{dt}{\ln a}\right|=\frac{1}{\ln a}\int\frac{dt}{t^2+1}=\frac{1}{\ln a}arctg(a^x)+C$$

190/

$$\int\frac{1-\sin\sqrt{x}}{\sqrt{x}}dx=\left|\sqrt{x}=t^2\quad\frac{dx}{2\sqrt{x}}=2tdt\right|=2\int(1-\sin t^2)2tdt=4\int tdt-4\int t\sin t^2dt=2t^2-\left|\begin{matrix}t^2=z\\2tdt=dz\end{matrix}\right|$$

$$=2t^2-\int\sin zdz=2(t^2+\cos z)=2(\sqrt{x}+\cos t^2)+C=2\sqrt{x}+2\cos\sqrt{x}+C$$

191/

$$\int\sqrt{\frac{(x+1)^3}{(x-1)^2}}dx=\left|x+1=t^2\quad x-1=t^2-2\quad dx=2tdt\right|=\int\frac{\sqrt{(x+1)^2}\sqrt{x+1}}{\sqrt{(t^2-2)^2}}2tdt=\int\frac{2t^4dt}{t^2-2}$$

$$\int(2t^2+4+\frac{8}{t^2-2})dt=\frac{2t^3}{3}+4t+8\int\frac{dt}{t^2-2}=I$$

$$8\int\frac{dt}{t^2-2}=2\sqrt{2}\int\frac{dt}{t-\sqrt{2}}-2\sqrt{2}\int\frac{dt}{t+\sqrt{2}}=2\sqrt{2}\ln|t-\sqrt{2}|-2\sqrt{2}\ln|t+\sqrt{2}|=2\sqrt{2}\ln\left|\frac{\sqrt{x+1}-\sqrt{2}}{\sqrt{x+1}+\sqrt{2}}\right|+$$

$$I=\frac{2(x+1)\sqrt{x+1}+12\sqrt{x+1}}{3}+2\sqrt{2}\ln\left|\frac{\sqrt{x+1}-\sqrt{2}}{\sqrt{x+1}+\sqrt{2}}\right|+C$$

192/

$$\int\frac{dx}{x^2\sqrt{x^2-1}}=\left|\begin{array}{l}\frac{1}{x}=t\quad -\frac{dx}{x^2}=dt\end{array}\right|=-\int\frac{dt}{\sqrt{\frac{1}{t^2}-1}}=-\int\frac{dt}{\sqrt{\frac{1-t^2}{t^2}}}=-\int\frac{tdt}{\sqrt{1-t^2}}=\left|\begin{array}{l}1-t^2=z\\ -tdt=\frac{dz}{2}\end{array}\right|$$

$$=\frac{1}{2}\int\frac{dz}{\sqrt{z}}=\frac{1}{2}2\sqrt{z}+C=\sqrt{z}+C=\sqrt{1-t^2}+C=\frac{\sqrt{x^2-1}}{x}+C$$

193/

$$\int\frac{4x+1}{2x^3+x^2-x}dx=\int\frac{4x+1}{x(x+1)(2x-1)}dx=\int\frac{-dx}{x}+\frac{1}{3}\int\frac{dx}{x+1}+\frac{4}{3}\int\frac{dx}{2x-1}=-\ln|x|+\frac{1}{3}\ln|x+1|+\frac{2}{3}\ln|2x-1|+C$$

Do obliczenia całki **193/** wykorzystano rozkład funkcji wymiernej:

$$\frac{4x+1}{x(x+1)(2x-1)}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{2x-1}=\frac{x^2(2A+2B+C)+x(A-B+C)-A}{x(x+1)(2x-1)}$$

Stąd $A=-1,\quad B=\frac{1}{3},\quad C=\frac{4}{3}.$

194/

$$\int|2^x-2|dx=F(x)+C$$

$$F'(x)=\begin{cases}-2^x+2 & dla\quad x\in(-\infty,1>\\ 2^x-2 & dla\quad x\in(1,\infty).\end{cases}\qquad F(x)=\begin{cases}-\frac{2^x}{\ln 2}+2x+C_1 & dla\quad x\in(-\infty,1>\\ \frac{2^x}{\ln 2}-2x+C_2 & dla\quad x\in(1,\infty)\end{cases}.$$

Aby funkcja F(x) była ciągła dla x = 1 musi być spełniony warunek:

$$\lim_{x\to1^-}-\frac{2^x}{\ln 2}+2x+C_1=\lim_{x\to1^+}\frac{2^x}{\ln 2}-2x+C_2=F(1)$$

$$-\frac{2}{\ln 2}+2+C_1=\frac{2}{\ln 2}-2+C_2$$

$$C_1=\frac{4}{\ln 2}-4+C_2$$

195/

$$\int\frac{x^3dx}{\sqrt{1-x^4}}=\left|x^4=t,\quad 4x^3dx=dt\right|=\frac{1}{4}\int\frac{dt}{\sqrt{1-t^2}}=\frac{1}{4}\arcsin t+C=\frac{1}{4}\arcsin x^4+C$$

196/

$$\int\frac{2^x-5^x}{10^x}dx=\int\frac{2^x}{10^x}dx-\int\frac{5^x}{10^x}dx=\int\left(\frac{1}{5}\right)^x dx-\int\left(\frac{1}{2}\right)^x dx=\int5^{-x}dx-\int2^{-x}dx=-\frac{5^{-x}}{\ln 5}+\frac{2^{-x}}{\ln 2}+C$$

197/

$$\int\frac{x^3+\sqrt[3]{x^2}-1}{\sqrt{x}}dx=\int\frac{x^3}{x^{1/2}}dx+\int\frac{x^{2/3}}{x^{1/2}}dx-\int x^{-\frac{1}{2}}dx=\int x^{5/2}dx+\int x^{1/6}dx-\int x^{-1/2}dx=$$

$$=\frac{2}{7}x^3\sqrt{x}+\frac{6}{7}x\sqrt[6]{x}-2\sqrt{x}+C$$

198/

$$\int\frac{1-x}{1-\sqrt[3]{x}}dx=\left|t=\sqrt[3]{x},\quad x=t^3,\quad dx=3t^2dt\right|=\int\frac{1-t^3}{1-t}dt=3\int(1+t+t^2)t^2dt=3\int(t^2+t^3+t^4)dt=$$

$$=t^3+\frac{3t^4}{4}+\frac{3t^5}{5}+C=x+\frac{3\sqrt[3]{x^4}}{4}+\frac{3\sqrt[3]{x^5}}{5}+C$$

199/

$$\int x^22^xdx=\left|\begin{array}{ll}u=x^2 & du=2xdx\\ dv=2^xdx & v=\frac{2^x}{\ln 2}\end{array}\right|=\frac{1}{\ln 2}x^22^x-\frac{2}{\ln 2}\int x2^xdx=\frac{1}{\ln 2}x^22^x-\frac{2}{\ln 2}\left|\begin{array}{ll}u=x & du=dx\\ dv=2^xdx & v=\frac{2^x}{\ln 2}\end{array}\right|=$$

$$=\frac{1}{\ln 2}x^22^x-\frac{2}{\ln 2}\left(x\frac{2^x}{\ln 2}-\frac{1}{\ln 2}\int2^xdx\right)=\frac{x^22^x}{\ln 2}-\frac{2x2^x}{(\ln 2)^2}-2\frac{2^x}{(\ln 2)^3}+C=\frac{2^x}{\ln 2}\left(x^2-\frac{2x}{\ln 2}-\frac{2}{(\ln 2)^2}\right)+C$$

200/

$$\int\sqrt{x}\,arctg\sqrt{x}dx=\left|x=t^2,\quad \sqrt{x}=t,\quad dx=2tdt\right|=\int2t^2arctgt\,dt=\left|\begin{array}{ll}u=arctgt & du=\frac{dt}{1+t^2}\\ dv=2t^2dt & v=\frac{2t^3}{3}\end{array}\right|=$$

$$=\frac{2t^3}{3}arctgt-\frac{2}{3}\int\frac{t^3dt}{1+t^2}=\frac{2t^3}{3}arctgt-\frac{2}{3}\int\left(t-\frac{t}{1+t^2}\right)dt=\frac{2}{3}t^3arctgt-\frac{t^2}{3}-\frac{1}{3}\ln|1+t^2|$$

$$=\frac{2}{3}x\sqrt{x}arctg\sqrt{x}-\frac{1}{3}x-\frac{1}{3}\ln|1+t^2|+C$$

201/

$$\int\frac{(x-1)e^xdx}{x^2}=\int\frac{e^xdx}{x}-\int\frac{e^xdx}{x^2}=\left|\begin{array}{ll}u=\frac{1}{x} & du=-\frac{dx}{x^2}\\ dv=e^xdx & v=e^x\end{array}\right|-\int\frac{e^xdx}{x^2}=\frac{e^x}{x}+\int\frac{e^xdx}{x^2}-\int\frac{e^xdx}{x^2}=$$

$$=\frac{e^{x}}{x}+C$$

202/

$$\int(6-2x)^{12}dx=\left|\begin{matrix}6-2x=t\\ dx=-\frac{dt}{2}\end{matrix}\right|=-\frac{1}{2}\int t^{12}dt=-\frac{t^{13}}{26}+C=-\frac{1}{26}(6-2x)^{13}+C$$

203/

$$\int x^3e^{x^2}dx=\left|\begin{matrix}e^{x^2}=t,\quad 2xe^{x^2}dx=dt,\quad \ln e^{x^2}=\ln t\\ x^2\ln e=\ln t,\quad x^2=\ln t\end{matrix}\right|=\int x^2xe^{x^2}dx=\frac{1}{2}\int\ln t\,dt=$$

$$=\frac{1}{2}\left|\begin{matrix}u=\ln t & du=\frac{1}{t}\\ dv=dt & v=t\end{matrix}\right|=\frac{1}{2}t\ln|t|-\frac{1}{2}t+C=\frac{1}{2}e^{x^2}\ln e^{x^2}-\frac{1}{2}e^{x^2}+C=\frac{1}{2}e^{x^2}\left(x^2-1\right)+C$$

204/

$$\int x^2\sqrt[6]{7x^3+6}\,dx=\left|7x^3+6=t^6,\quad x^2dx=\frac{6t^5dt}{21}\right|=\frac{2}{7}\int t^6dt=\frac{2t^7}{49}+C=\frac{2}{49}\sqrt[6]{\left(7x^3+6\right)^7}+C$$

205/

$$\int\left(x^2+1\right)\cos\left(x^3+3x+5\right)dx=\left|\begin{matrix}x^3+3x+5=t\\ \left(3x^2+3\right)dx=dt\end{matrix}\right|=\int3\left(x^2+1\right)\cos\left(x^3+3x+5\right)dx=\int\cos t\,dt=$$

$$\sin t+C=\sin\left(x^3+3x+5\right)+C$$

206

$$\int\frac{x^3dx}{x+1}=\int\left(x^2-x+1-\frac{1}{x+1}\right)dx=\frac{x^3}{3}-\frac{x^2}{2}+x-\ln|x+1|+C$$

207/

$$\int\frac{3dx}{3\sin x+4\cos x+5}=\left|tg\frac{x}{2}=t,\quad dx=\frac{2dt}{1+t^2},\quad \sin x=\frac{2t}{1+t^2}\quad \cos x=\frac{1-t^2}{1+t^2}\right|=$$

$$=3\int\frac{\frac{2dt}{1+t^2}}{\frac{6t+4\left(1-t^2\right)+5\left(1+t^2\right)}{1+t^2}}=6\int\frac{dt}{t^2+6t+9}=6\int\frac{dt}{(t+3)^2}=-\frac{6}{t+3}+C=-\frac{6}{tg\frac{x}{2}+3}+C$$

208/

$$\int\frac{dx}{1-\sin^4x}=\left|tgx=t,\quad dx=\frac{dt}{1+t^2},\quad \sin^2x=\frac{t^2}{1+t^2}\right|=\int\frac{\frac{dt}{1+t^2}}{1-\left(\frac{t^2}{1+t^2}\right)^2}=$$

$$=\int\frac{\left(1+t^2\right)dt}{2t^2+1}=\int\frac{1+t^2}{2t^2+1}dt=\int\left(\frac{1}{2}+\frac{\frac{1}{2}}{2t^2+1}\right)dt=\frac{1}{2}\int dt+\frac{1}{2}\int\frac{dt}{2t^2+1}=\frac{1}{2}t+\frac{1}{2}\int\frac{dt}{2\left(t^2+\frac{1}{2}\right)}=$$

$$=\frac{1}{2}tgx+\frac{1}{4}\int\frac{dt}{t^2+\frac{1}{2}}=\frac{1}{2}tgx+\frac{\sqrt{2}}{4}arctg\frac{2t}{\sqrt{2}}+C=\frac{1}{2}tgx+\frac{\sqrt{2}}{4}arctg\sqrt{2}tgx+C$$

209/

$$\int\frac{\sin x\cos x\,dx}{1+\sin^4x}=\left|tgx=t,\quad dx=\frac{dt}{1+t^2},\quad \sin^2x=\frac{t^2}{1+t^2},\quad \sin x\cos x=\frac{t}{1+t^2}\right|=\int\frac{\frac{t}{1+t^2}\frac{dt}{1+t^2}}{\frac{\left(1+t^2\right)^2+t^4}{\left(1+t^2\right)^2}}=$$

$$=\int\frac{tdt}{2t^4+2t^2+1}=\left|\begin{matrix}t^2=z\\ 2tdt=dz\end{matrix}\right|=\frac{1}{2}\int\frac{dz}{2z^2+2z+1}=\frac{1}{2}\int\frac{dz}{2\left(z+\frac{1}{2}\right)^2+\frac{1}{2}}=\frac{1}{4}\int\frac{dz}{\left(z+\frac{1}{2}\right)^2+\frac{1}{4}}=$$

$$=\frac{1}{2}arctg(2z+1)+C=\frac{1}{2}arctg\left(2tg^2x+1\right)+C$$

210/

$$\int\frac{\sin^3x+\cos^3x}{\sin^2x-\sin x\cos x+\cos^2x}dx=\left|\frac{a^3+b^3}{a^2-ab+b^2}=a+b\right|=\int(\sin x+\cos x)dx=$$

$$=\int\sin x\,dx+\int\cos x\,dx=-\cos x+\sin x+C$$

10. Częściej używane wzory całek:

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ dla $n \neq -1$, $x > 0$.

$\int dx = x + C$

$\int \frac{dx}{x^2} = -\frac{1}{x} + C$

$\int \frac{dx}{\sqrt{x}} = 2\sqrt{x} + C$

$\int \frac{dx}{x} = \ln|x| + C$ dla $x \neq 0$.

$\int e^x dx = e^x + C$

$\int a^x dx = \frac{a^x}{\ln a} + C$ dla $a > 0$, $a \neq 1$.

$\int \sin x dx = -\cos x + C$

$\int shx dx = chx + C$

$\int \frac{dx}{\sin^2 x} = -ctgx + C$

$\int \cos x dx = \sin x + C$

$\int chx dx = shx + C$

$\int \frac{dx}{\cos^2 x} = tgx + C$

$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$ dla $-1 < x < 1$.

$\int -\frac{dx}{\sqrt{1-x^2}} = -\arccos x + C$

$\int \frac{dx}{1+x^2} = arctgx + C = -arcctgx + C$

$\int \frac{dx}{\sqrt{1+x^2}} = \ln\left|x + \sqrt{1+x^2}\right| + C$

$\int \frac{dx}{\sqrt{x^2-1}} = \ln\left|x + \sqrt{x^2-1}\right| + C$

$\int \frac{f'(x)dx}{f(x)} = \ln|f(x)| + C$

$\int \frac{dx}{(x-k)^2 + b} = \frac{1}{\sqrt{b}} arctg \frac{x-k}{\sqrt{b}} + C$ gdzie $b > 0$.

$\int \frac{dx}{(x^2+1)^n} = \frac{1}{2n-2} \frac{x}{(x^2+1)^{n-1}} + \frac{2n-3}{2n-2} \int \frac{dx}{(x^2+1)^{n-1}}$ dla $n \in N$.

$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{|a|} + C$

$\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + C$ gdzie $f(x) > 0$.

$\int \sqrt{k^2-x^2}\, dx = \frac{k^2}{2} \arcsin \frac{x}{|k|} + \frac{x}{2}\sqrt{k^2-x^2} + C$

$\int \frac{x^2 dx}{\sqrt{k^2-x^2}} = \frac{k^2}{2} \arcsin \frac{x}{|k|} - \frac{x}{2}\sqrt{k^2-x^2} + C$

$\int \sqrt{x^2+k}\, dx = \frac{x}{2}\sqrt{x^2+k} + \frac{k}{2}\ln\left|x + \sqrt{x^2+k}\right| + C$

$\int \frac{x^2 dx}{\sqrt{x^2+k}} = \frac{x}{2}\sqrt{x^2+k} - \frac{k}{2}\ln\left|x + \sqrt{x^2+k}\right| + C$

$\int \cos^2 x dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$

$\int \sin^2 x dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$

$\int \sin^n x dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x dx$

$\int \cos^n x dx = \int \sin^n\left(\frac{\pi}{2} + x\right) dx$

$\int tg^n x dx = \frac{1}{n-1} tg^{n-1} x - \int tg^{n-2} x dx$

$\int \log_p x dx = \frac{1}{\ln p} \int \ln x dx$

Spis treści

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