13.34

torsdag 22 december 2022 17:06

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}(k+1)} \leq \frac{\pi+1}{2}$$

$$\int_1^{\infty} \frac{1}{\sqrt{x}(x+1)}\,dx = \left[\begin{matrix} t=\sqrt{x} & x=t^2 \\ dx = 2t\,dt & \\ x=1\ t=1 & x=\infty\ t=\infty \end{matrix}\right] = \int_1^{\infty} \frac{1}{\cancel{t}(t^2+1)} \cdot 2\cancel{t}\,dt = \Big[2\arctan t\Big]_1^{\infty} = 2(\underbrace{\arctan\infty}_{\pi/2} - \underbrace{\arctan 1}_{\pi/4}) = \frac{\pi}{2}$$

$$\sum_{k=1}^{n} f(k) + f(n) \leq \sum_{k=1}^{n} f(k) \leq \sum_{k=1}^{n} f(k) + f(1) \iff \sum_{k=1} f(k) = \frac{\pi}{2} \leq \frac{\pi+1}{2} = \sum_{k=1}^{n} f(k) + f(1)$$

$$\boxed{\int_1^n f(x) + f(1) = \frac{1}{2} + \int_1^n \frac{1}{\sqrt{x}(x+1)}\,dx = \frac{1}{2} + \Big[2\arctan\sqrt{x}\Big]_1^n = 1/2 + \underbrace{2\arctan\sqrt{n}}_{\pi\text{, då } n\to\infty} - \frac{\pi}{2} = \frac{1+\pi}{2}}$$

$$\frac{1}{\sqrt{1}(1+1)}$$