13.20

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a)
$$\int_0^{\pi/4} \frac{1}{1+\sin x\cos x}\,dx = \int_0^{\pi/4} \frac{2}{2+\sin 2x}\,dx = \begin{bmatrix} t=2x & x=t/2 \\ dt=2\,dx & \\ x=0\ t=0 & x=\pi/4\ t=\pi/2 \end{bmatrix} = \int_0^{\pi/2} \frac{1}{2+\sin t}\,dt =$$

$$= \begin{bmatrix} s=\tan t/2 & t=2\arctan s \\ dt=\frac{2}{1+s^2}\,dt & \\ t=0\ s=0 & t=\pi/2\ s=1 \end{bmatrix} = \int_0^1 \frac{1}{\cancel{2}+\frac{\cancel{2}s}{1+s^2}}\cdot\frac{2}{1+s^2}\,ds = \int_0^1 \frac{1}{1+s^2+s}\,ds = \int_0^1 \frac{1}{(s+\frac{1}{2})^2+\frac{3}{4}} =$$

$$= \begin{bmatrix} u=s+1/2 & s=u-1/2 \\ ds=du & \\ s=0\ u=1/2 & s=1\ u=3/2 \end{bmatrix} = \int_{1/2}^{3/2} \frac{1}{u^2+3/4}\,du = \frac{4}{3}\int_{1/2}^{3/2} \frac{1}{\left(\frac{2u}{\sqrt{3}}\right)^2+1}\,du = \left[\frac{\cancel{4}^{\,2}}{\cancel{3}_{\sqrt{3}}}\arctan\frac{2u}{\sqrt{3}}\cdot\frac{\cancel{\sqrt{3}}}{\cancel{2}}\right]_{1/2}^{3/2} =$$

$$= \left[\frac{2}{\sqrt{3}}\arctan\frac{2u}{\sqrt{3}}\right]_{1/2}^{3/2} = \frac{2}{\sqrt{3}}\arctan\sqrt{3} - \frac{2}{\sqrt{3}}\arctan 1/\sqrt{3} = \frac{2}{\sqrt{3}}\left(\frac{\pi}{3}-\frac{\pi}{6}\right) = \frac{\pi}{3^{3/2}}$$

b)

$\arctan t = \frac{x}{2}$ $\quad x = 2\arctan t$

$$\int \frac{\cos x}{1+\cos x}\,dx = \begin{bmatrix} t=\tan x/2 & \cos x=\frac{1-t^2}{1+t^2} \\ dx=\frac{2}{1+t^2}\,dt & \end{bmatrix} = \int \frac{\frac{1-t^2}{1+t^2}}{\frac{1+\cancel{t^2}+1-\cancel{t^2}}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt = \int \frac{1-t^2}{\frac{\cancel{(1+t^2)}\cdot\cancel{2}}{\cancel{1+t^2}}}\cdot\frac{\cancel{2}}{1+t^2}\,dt =$$

$$= \int \frac{1-t^2}{1+t^2}\,dt = \int \frac{1}{1+t^2} - \underbrace{\frac{t^2}{1+t^2}}_{1-\frac{1}{1+t^2}}\,dt = \arctan t - \int 1-\frac{1}{1+t^2}\,dt = \arctan t - t + \arctan t + C = 2\arctan t - t + C =$$

$$= x - \tan\frac{x}{2} + C$$

$$\int_0^{\pi/2} \frac{\cos x}{1-\cos x}\,dx = \left[x-\tan\frac{x}{2}\right]_0^{\pi/2} = \frac{\pi}{2}-1$$