

$$a) \det(\lambda I - A) = \begin{vmatrix} \lambda - 5 & 0 & -4 \\ 0 & \lambda - 4 & 0 \\ -4 & 0 & \lambda - 5 \end{vmatrix} =$$

$$= (\lambda - 5)^2 (\lambda - 4) - (-4)^2 (\lambda - 4) =$$

$$= (\lambda - 4) (\lambda^2 - 10\lambda + 9) = 0$$

$$\lambda_1 = 4 \quad \lambda_2 = 9 \quad \lambda_3 = 1$$

$$\underline{(4I - A)x = 0:}$$

$$-4 \begin{cases} -x_1 & -4x_3 = 0 \\ & 0 = 0 \\ -4x_1 & -x_3 = 0 \end{cases} \Leftrightarrow \begin{cases} -x_1 & -4x_3 = 0 \\ & \\ & 15x_3 = 0 \end{cases}$$

$$x_3 = 0, \quad x_1 = 0, \quad x_2 = t$$

$$x = t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad t \neq 0$$

$$\underline{(9I - A)x = 0:}$$

$$\begin{cases} 4x_1 & -4x_3 = 0 \\ & 5x_2 = 0 \\ -4x_1 & 4x_3 = 0 \end{cases} \Leftrightarrow \begin{cases} 4x_1 & -4x_3 = 0 \\ & x_2 = 0 \\ & 0 = 0 \end{cases}$$

$$x_1 = t, \quad x_2 = 0, \quad x_3 = t$$

$$x = t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad t \neq 0$$

$$\underline{(I - A)x = 0:}$$

$$\begin{cases} -4x_1 & -4x_3 = 0 \\ & -3x_2 = 0 \\ -4x_1 & -4x_3 = 0 \end{cases} \Leftrightarrow \begin{cases} -4x_1 & -4x_3 = 0 \\ & x_2 = 0 \\ & 0 = 0 \end{cases}$$

$$x_1 = t, \quad x_2 = 0, \quad x_3 = -t$$

$$x = t \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad t \neq 0$$

$$S = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

b)

$$A^{1/2} = S \cdot D^{1/2} \cdot S^{-1} =$$

$$= \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 4^{1/2} & 0 & 0 \\ 0 & 9^{1/2} & 0 \\ 0 & 0 & 1^{1/2} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 3 & 1 \\ 2 & 0 & 0 \\ 0 & 3 & -1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} =$$

$$= \frac{1}{2} \begin{pmatrix} 4 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 4 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}}}$$