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Further Studies of Human Locomotion: Postural Stability and Control

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ABSTRACT

In a previous study of human locomotion, an important aspect of our mathematical modeling was the decomposition of the complex body dynamics into two parts: one describes the lower extremity motion; the other pertains to the motion of the upper trunk. The generation of gait patterns via optimal programming was studied in that paper. In the present study the torso motion is considered. This involves an investigation of the unstable equilibrium of the torso about its upright position. A multidimensional "inverted pendulum" model is derived to study this behavior. To stabilize the motion, a linear feedback law coupled with an on-off perturbation is proposed. A Liapunov function is then constructed to show that such a control mechanism provides effective stabilization of the torso for all initial configurations of motion. Significance of the control law is discussed and a possible EMG-photographic experiment is suggested on the basis of this theoretical study.

1. INTRODUCTION

The maintenance of upright posture in walking is an important problem in the overall study of normal and programmed locomotion. Except for the brief, qualitative description presented in an earlier article [1], this question has not been studied quantitatively. The essence of locomotion is to transport the head, arms, and trunk (HAT) section of the body from an initial position to a desired one through the action of the lower extremities. The upper part of the human body is a very heavy mass, being approximately 70% [2] of the body weight, so that inertial and gravitational effects of the torso have considerable influence on the lower extremity behavior. This is clearly reflected by the dominance of the ground reaction forces in determining the actuating moments around the hip and knee joints [3]. Moreover, the bipedal mode of locomotion possesses the property of a

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periodic loss and recovery of equilibrium between successive steps. Clearly, an understanding of the controlling mechanism and stability of the torso is necessary, and the purpose of this article is to contribute thereto by reporting recent modeling, stability, and control studies.

2. BODY DYNAMICS

For simplicity, the inertial and gravitational effects of the head and upper extremities are included with those of the torso. It is only in such cases as running that the upper extremities appear to require separate modeling.

A. *DERIVATION*

Figure 1 describes the situation under study. The torso is considered as a rigid body having three rotational degrees of freedom described by the angles (θ, ψ, ϕ). (This coordinate system is used in studying satellite attitude control [5]). The Ox_0, Oy_0, and Oz_0 axes form the inertial frame

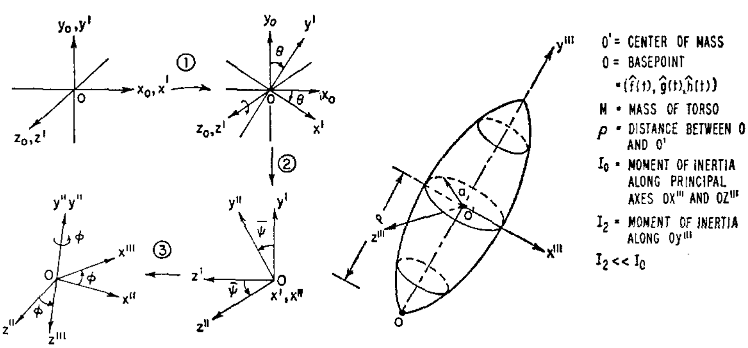

FIG. 1. Coordinate transformation and model of the torso.

of reference while $Ox'y'z'$ represents the three dimensional space spannec by the body axes. The body angles are generated by three successive rotations of the torso about its body axes. Suppose that the torso is firs in the upright position; that is, the two coordinate systems coincide. The angle θ is formed by rotating the torso about Oz' or Oz_0, as shown ir step 1 of the transformation. This rotation describes the pitch motion o the body in the sagittal plane Ox_0y_0. Next, the angle ψ is generated by rotating the body about Ox' as shown in step 2. This corresponds to rollover action of the body. Finally, self-rotation of the torso is describec by the angle ϕ about its axis Oy''. This choice of body angles differs fron the usual Eulerian description in rigid body dynamics in that the Eule angles become ill-conditioned as they become small [6]. In postura regulation, small-angle deviation of the torso from the vertical is of special importance.

Besides rotational motion, the torso also possesses three translational degrees of freedom. Consistent with the device of prescribing the hip motion in our lower extremity study [1], the base point (O) of the torso is assumed to follow a similar trajectory. Let the coordinates of the base point be $[\tilde{f}(t), \tilde{g}(t), \tilde{h}(t)]$ and ρ the distance between the base point (O) and center of mass (O'); then the coordinates of the center of mass (cm) are

$$\begin{aligned} x_c &= \tilde{f}(t) + \rho \sin\theta \cos\psi, \\ y_c &= \tilde{g}(t) + \rho \cos\theta \cos\psi, \\ z_c &= \tilde{h}(t) + \rho \sin\psi. \end{aligned} \tag{1}$$

The translational component of kinetic energy is

$$T_{\text{trans}} = \tfrac{1}{2}M\,.(\dot{x}_c^2 + \dot{y}_c^2 + \dot{z}_c^2). \tag{2}$$

The rotational component can be concisely expressed in terms of the principal axis quantities, that is,

$$T_{\text{rot}} = \tfrac{1}{2}(I_o\omega_1^2 + I_2\omega_2^2 + I_0\omega_3^2) \tag{3}$$

where ω_i, $i = 1, 2, 3$, are the angular velocities along the body axes Ox', Oy', and Oz', respectively. Here, I_0 and I_2 are the principal moments of inertia about these axes. Specifically, the angular velocities are

$$\begin{aligned} \omega_1 &= \dot{\psi}\cos\phi + \dot{\theta}\sin\phi\cos\psi, \\ \omega_2 &= \dot{\phi} - \dot{\theta}\sin\psi, \\ \omega_3 &= \dot{\psi}\sin\phi - \dot{\theta}\cos\phi\cos\psi. \end{aligned} \tag{4}$$

Using relations (1)–(4), we can write the total kinetic energy of the body as

$$\begin{aligned} T &= T_{\text{trans}} + T_{\text{rot}} \\ &= \tfrac{1}{2}I_0(\dot{\psi}^2 + \dot{\theta}^2\cos^2\psi) + \tfrac{1}{2}I_2(\dot{\phi} - \dot{\theta}\sin\psi)^2 + \tfrac{1}{2}M(\dot{x}_c^2 + \dot{y}_c^2 + \dot{z}_c^2) \\ &= \tfrac{1}{2}A\dot{\psi}^2 + \tfrac{1}{2}A\dot{\theta}^2\cos^2\psi + \tfrac{1}{2}I_2(\dot{\phi} - \dot{\theta}\,.\sin\psi)^2 \\ &\quad + \tfrac{1}{2}M\{\dot{\tilde{f}}^2 + \dot{\tilde{g}}^2 + \dot{\tilde{h}}^2 - 2\rho\dot{\tilde{f}}\,.(\dot{\psi}\sin\psi\sin\theta - \dot{\theta}\cos\psi\cos\theta) \\ &\quad - 2\rho\dot{\tilde{g}}\,.(\dot{\psi}\sin\psi\cos\theta + \dot{\theta}\cos\psi\sin\theta) + 2\rho\dot{\tilde{h}}\dot{\psi}\cos\psi\} \end{aligned} \tag{5}$$

where $A \triangleq I_0 + M\rho^2$.

Similarly, the potential energy V can be expressed as

$$V = Mg(\tilde{g}(t) + \rho\cos\psi\cos\theta) \tag{6}$$

where g = gravitational constant (32.2 ft/sec). The equations of motion can be readily obtained by substituting the T and V expressions into

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \left(\frac{\partial T}{\partial q_i}\right) + \left(\frac{\partial V}{\partial q_i}\right) = M_i \tag{7}$$

where q_i represents the body angles and M_i the principal axis moments used to maintain an upright posture. Explicitly, the equations are

$$\begin{aligned}(A\cos^2\psi + I_2\sin^2\psi)\ddot{\theta} - (I_2\sin\psi)\ddot{\phi} &= (A - I_2)\sin(2\psi)\dot{\theta}\dot{\psi} + I_2\cos\psi\dot{\phi}\dot{\psi} \\ &\quad + M\rho(g + \ddot{g}(t))\cos\psi\sin\theta + M_\theta; \\ -(I_2\sin\psi)\ddot{\theta} + I_2\ddot{\phi} &= I_2\cos\psi\dot{\theta}\dot{\psi} + M_\phi; \\ A\ddot{\psi} &= -\tfrac{1}{2}(A - I_2)\sin(2\psi)\dot{\theta}^2 - I_2\cos\psi\dot{\theta}\dot{\phi} + \\ &\quad + M\rho(g + \ddot{g}(t))\cos\theta\sin\psi - M\rho\ddot{h}\cos\psi + M_\psi.\end{aligned} \tag{8}$$

B. *INVERTED PENDULUM MODEL*

Equations (8) describe the dynamic behavior of the torso constrained by the prescribed translation. In a real situation, the self-rotation ϕ is usually small as compared to the θ and ψ angles; thus, a suitable approximation is to study the nonrotation case. This simplification is appealing in that there are now only two equations to consider and the coefficient matrix is decoupled.

$$\begin{aligned}(1 - \mu\sin^2\psi)\ddot{\theta} &= \mu\sin(2\psi)\dot{\theta}\dot{\psi} + c_0(g + \ddot{g})\sin\theta\cos\psi + \frac{1}{A}M_\theta \\ \ddot{\psi} &= -\tfrac{1}{2}\mu\sin(2\psi)\dot{\theta}^2 \\ &\quad + c_0(g + \ddot{g})\sin\psi\cos\theta - c_0\ddot{h}\cos\psi + \frac{1}{A}M_\psi\end{aligned}$$

and

$$\mu = 1 - \left(\frac{I_2}{A}\right), \qquad c_0 = \frac{M\rho}{A}. \tag{9}$$

The similarity of the two equations in (9) suggests that the torso should behave like an inverted pendulum possessing two degrees of freedom and having periodic base motion $\ddot{g}(t)$. Introducing the state variables $x_1 = \theta$, $x_2 = \psi$, $x_3 = \dot{\theta}$, $x_4 = \dot{\psi}$, we can write the system (9) in first-order canonical form as

$$\dot{x}(t) = f(x, u; t) \tag{10}$$

where

$$x = (x_1, x_2, x_3, x_4)^T; \qquad u_1 = \frac{1}{A}M_\theta, \qquad u_2 = \frac{1}{A}M_\psi,$$

$$\begin{aligned}\dot{x}_1 &= x_3, \\ \dot{x}_2 &= x_4, \\ \dot{x}_3 &= (1 - \mu\sin^2(x_2))^{-1}[\mu\sin(2x_2)x_3x_4 \\ &\quad + c_0(g + \ddot{g})\sin(x_1)\cos(x_2) + u_1], \\ \dot{x}_4 &= -\tfrac{1}{2}\mu\sin(2x_2)\,x_3^2 + c_0(g + \ddot{g})\sin(x_2)\cos(x_1) - c_0\ddot{h}\cos(x_2) + u_2.\end{aligned}$$

C. *PRESCRIBED DATA*

Consistent with the prescribed trajectories used in [1], the $\tilde{f}(t)$ and $\tilde{g}(t)$ curves are taken as follows.

$$\begin{aligned}
\tilde{f}(t) &= v_0 t && [\text{ft}], && (11)\\
v_0 &= 2.275 && [\text{ft/sec}], &&\\
\ddot{\tilde{f}}(t) &= 0.0 && [\text{ft/sec}^2], &&\\
\tilde{g}(t) &= e_0 - \frac{0.9}{12}\sin(2\pi t) && [\text{ft}], && (12)\\
e_0 &= 2.763 && [\text{ft}], &&\\
\ddot{\tilde{g}}(t) &= g_0 \sin(2\pi t) && [\text{ft/sec}^2], &&\\
g_0 &= 3.0 && [\text{ft/sec}^2]. &&
\end{aligned}$$

The $\tilde{h}(t)$ curve describes the transfer of body weight between the two lower extremities during rollover. In this study, an approximate expression is used:

$$\begin{aligned}
\tilde{h}(t) &= -\tfrac{1}{12}\cos(\pi t) && [\text{ft}], && (13)\\
\ddot{\tilde{h}}(t) &= \left(\frac{\pi^2}{12}\right)\cos(\pi t) && [\text{ft/sec}^2]. &&
\end{aligned}$$

Note that $(\pi^2/12) < 1$ and $|\ddot{\tilde{h}}(t)|_{\max} < \frac{1}{3}|\ddot{\tilde{g}}(t)|_{\max}$. Since the vertical motion $\ddot{\tilde{g}}(t)$ appears with the gravitational constant g, the effect of $\ddot{\tilde{h}}(t)$ is likely to be small as compared to the $[g + \ddot{\tilde{g}}(t)]$ term.

D. *VALUES OF PHYSICAL PARAMETERS*

Thus far, the values of the various physical parameters have not been specified. In practice, these values have to be determined experimentally [3] because the shape of the torso is analytically very difficult to describe. In previous studies, a common approximation is to assume the torso to be either cylindrical or ellipsoidal in shape. These two shapes give parameter values quite close together. For a body weight of 130 lb and average length (data from Lissner [2]), the parameter values are as shown in Table 1.

The form factor μ is an interesting quantity. It is a measure of " slenderness" of the body. For a fat person, the moment of inertia I_2 is large and this makes μ small For a thin person, μ is close to 1. In all cases, however, μ lies strictly between 0 and 1; that is, $0 < \mu < 1$. From (10) the magnitude of the angular acceleration $\ddot{\theta}$ is magnified as μ approaches unity if the ψ angle is large.

TABLE 1

	Cylindrical	Ellipsoidal
Body weight W	130 lb	130 lb
Mass of torso M	2.826 slugs	2.826 slugs
Base point to center-of-mass distance	1.415 ft [a]	1.415 ft [a]
Body radius, a	0.46 ft [a]	0.46 ft [a]
I_0	$\frac{1}{12}M(3a^2 + 4\rho^2) = 2.03$ slug-ft^2	$\frac{1}{5}M(a^2 + \rho^2) = 1.25$ slug-ft^2
I_2	$\frac{1}{2}Ma^2 = 0.299$ slug-ft^2	$Ma^2 = 0.598$ slug-ft^2
$A = I_0 + M\rho^2$	7.67 slug-ft^2	6.89 slug-ft^2
$\mu = 1 - (I_2/A)$	0.961	0.913
$c_0 = M/A$	0.52 ft^{-1}	0.58 ft^{-1}

[a] Estimated.

3. FEEDBACK STABILIZATION WITH ON–OFF PERTURBATION

From the inverted pendulum model of the torso motion, it is clear that the point $\theta = \psi = \dot{\theta} = \dot{\psi} = 0$ corresponds to unstable equilibrium. Experimental studies indicate, interestingly, that the angular deviations of the torso from the vertical are usually small in normal walking [4, 7]. A typical trajectory for ψ varies between $\pm$ 2.5 deg; the "smallness" of this motion indicates active stabilization on the part of the muscle system.

A common approach to postural regulation is to linearize the torso dynamics around the vertical, and then to derive a suitable *linear* feedback law. Relevant works in this area include those of Frank [8]; McGhee and Kuhner [9]; Kato *et al.* [10]; and Witt [11]. In the two papers by Vukobratovic *et al.* [12, 13], stability of the compass gait was investigated using a nonlinear model. In all these studies, results show that the angular displacements are quite small.

Departing from the previous linearized studies, a "global" stabilization approach is used here. Specifically, it is proposed that a control law of the following form is used to stabilize the torso motion.

$$\begin{aligned} u_1 &= -k\dot{\theta} - [\omega_1^2 + c_0 g + 2c_0 g_0 \operatorname{sat}(\theta\dot{\theta})]\theta, \\ u_2 &= -k\dot{\psi} - [\omega_2^2 + c_0 g + 2c_0 g_0 \operatorname{sat}(\psi\dot{\psi})]\psi \end{aligned} \tag{14}$$

where

$$\operatorname{sat}(\theta\dot{\theta}) \triangleq \begin{cases} 1 & \text{if } \theta\ddot{\theta} \geqslant 0, \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

A similar definition holds for $\operatorname{sat}(\psi\dot{\psi})$. This feedback law is linear in the state variables with the addition of an on–off perturbation term given by $\operatorname{sat}(\theta\dot{\theta})$ and $\operatorname{sat}(\psi\dot{\psi})$. The parameters ω_1^2, ω_2^2, and k are feedback gains to be determined. We have written the constant coefficients in the form of $(\omega_1^2 + c_0 g)$ and $(\omega_2^2 + c_0 g)$ for ease of identification. The on–off function is a perturbation in the feedback gain because $c_0 g_0 \ll (\omega_i^2 + c_0 g)$ for $i = 1, 2$.

The foregoing form of controller has been chosen for four reasons: First, previous studies, using linear models, yield linear controllers. Second, experimental evidence indicates that angular deviations away from the vertical are small in normal walking; thus, one could expect that a linear controller, or slight modification thereof, would be adequate. Third, the unstable inverted pendulum effect is removed by including in the controller the linear terms $c_0 g\theta$ and $c_0 g\psi$. Fourth, our intuition suggests that the gains should increase if $(\theta\dot{\theta})$ and/or $(\psi\dot{\psi})$ are positive because this means that the angle and rate of change of angle are of the same sign, indicating that the torso is falling away from the vertical (and hence a larger restoring control is required). The periodic base motion $\tilde{g}(t)$ is a significant factor in causing the torso motion to deviate from the vertical. The magnitude of the perturbation controller, being directly proportional to the amplitude of the base motion, counteracts this deviation tendency.

4. STABILITY VIA LIAPUNOV'S SECOND METHOD

The advantages of the foregoing control scheme are its simplicity of form and ease of implementation. It is essentially linear except for the small on–off action. However, this perturbation of the feedback gain is sufficient to achieve asymptotic stability of the torso motion. In the previous linearized studies, validity of the feedback law for the nonlinear system and the size of the stability region were not considered at all.

A. *HARD SPRING ANALOGY*

Substituting the control expressions (14) into (9), we have

$$\begin{aligned}
\ddot{\theta} &= (1 - \mu \sin^2 \psi)^{-1}(\mu \sin(2\psi)\dot{\theta}\dot{\psi} - c_0 g(\theta - \sin\theta\cos\psi) \\
&\qquad -2c_0 g_0 \operatorname{sat}(\theta\dot{\theta})\theta + c_0 g_0 \sin(2\pi t)\sin\theta - k\dot{\theta} - \omega_1^2\theta), \\
\ddot{\psi} &= -\tfrac{1}{2}\mu \sin(2\psi)\dot{\theta}^2 - c_0 g(\psi - \sin\psi\cos\theta) - 2c_0 g_0 \operatorname{sat}(\psi\dot{\psi})\psi \\
&\qquad + c_0 g_0 \sin(2\pi t)\sin\psi - k\dot{\psi} - \omega_2^2\psi.
\end{aligned} \tag{16}$$

An immediate observation is that the inverted pendulum effect is now stabilized. Define

$$f_\theta(\theta, \psi) = c_0 g(\theta - \sin\theta\cos\psi); \tag{17}$$

then $f_\theta(-\theta, \psi) = -f_\theta(\theta, \psi)$, that is, odd function in θ

$$(\theta - \sin\theta\cos\psi) > 0, \forall\, \theta > 0 \qquad \text{and} \qquad -\pi < \psi < \pi.$$

This also holds for

$$f_\psi(\theta, \psi) = c_0 g(\psi - \sin\psi\cos\theta).$$

Thus, the term $f_\theta(\theta, \psi)$ always opposes the angular acceleration $\ddot{\theta}$, whereas previously it aided the acceleration. The effect of this is one of hard, cubic spring action over a fairly wide range of θ and ψ; that is,

$$\begin{aligned}\theta - \sin\theta\cos\psi &= \theta - \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)\left(1 - \frac{\psi^2}{2!} + \frac{\psi^4}{4!} + \cdots\right) \\ &\cong \tfrac{1}{2}(\psi^2 + \tfrac{1}{3}\theta^2)\theta.\end{aligned} \tag{18}$$

Now, consider a cubic spring model in dimensionless coordinates [14].

$$\ddot{x}(t) + x(t) + \varepsilon x^3(t) = 0. \tag{19}$$

Define $y = \dot{x}$; then $\ddot{x} = dy/dt = y\,dy/dx$. This gives

$$y\,dy + (x + \varepsilon x^3)\,dx = 0. \tag{20}$$

The "energy" integral of (20), giving trajectories in the state plane (x, y), demonstrates the existence of periodic solutions. We have

$$\tfrac{1}{2}y^2 + \tfrac{1}{2}x^2 + \tfrac{1}{4}\varepsilon x^4 = \text{energy}, \qquad E(x, y) = \text{constant}, \tag{21}$$

$$\frac{dE(x, y)}{dt} = 0.$$

For $\varepsilon > 0$, all paths for different energies $E(> 0)$ are closed curves, corresponding to periodic solutions. For $\varepsilon < 0$, there are still closed paths in the neighborhood of the origin. Thus, for small negative ε or $\varepsilon > 0$ there are always periodic solutions and these are *stable*. Note that the energy integral satisfies

$$\begin{gathered} E(x = 0, y = 0; t) = 0, \\ E(x \to \infty \text{ or } y \to \infty; t) \to \infty \quad \forall\, t, \\ \frac{dE}{dt} = 0. \end{gathered} \tag{22}$$

B. *CONSTRUCTING A LIAPUNOV FUNCTION*

To generalize the hard spring analogy, first suppose that the spring force is the dominant contribution to the motion. Consider the simplified, symmetrical model

$$\ddot{\theta} = -c_0 g(\theta - \sin\theta\cos\psi) - \omega^2\theta, \tag{23}$$

$$\ddot{\psi} = -c_0 g(\psi - \sin\psi\cos\theta) - \omega^2\psi. \tag{24}$$

On adding and subtracting (23)–(24), we have

$$(\ddot{\theta} + \ddot{\psi}) + \omega^2(\theta + \psi) = -c_0 g(\theta + \psi - \sin(\psi + \theta)), \tag{25}$$

$$(\ddot{\theta} - \ddot{\psi}) + \omega^2(\theta - \psi) = -c_0 g(\theta - \psi - \sin(\theta - \psi)). \tag{26}$$

The energy integrals of (25) and (26) are, respectively,

$$\tfrac{1}{2}(\dot\theta + \dot\psi)^2 + \tfrac{1}{2}(\theta + \psi)^2 + c_0 g(\tfrac{1}{2}(\theta + \psi)^2 + \cos(\psi + \theta)) = C_1, \tag{27}$$

$$\tfrac{1}{2}(\dot\theta + \dot\psi)^2 + \tfrac{1}{2}(\theta - \psi)^2 + c_0 g(\tfrac{1}{2}(\theta - \psi)^2 + \cos(\theta - \psi)) = C_2. \tag{28}$$

Combining (27) and (28), we have

$$\dot\theta^2 + \dot\psi^2 + \omega^2\theta^2 + \omega^2\psi^2 + c_0 g(\theta^2 + \psi^2 + 2\cos\theta\cos\psi) = C_3. \tag{29}$$

Expression (29) is the basis for the construction of a suitable Liapunov function for stability verification. Specifically, we have in mind the following.

$$\begin{aligned} x &\triangleq (x_1, x_2, x_3, x_4)^T = (\theta, \psi, \dot\theta, \dot\psi)^T, \\ V(x) &= \tfrac{1}{2}(\omega_1^2 + c_0 g_0)\theta^2 + \tfrac{1}{2}(\omega_2^2 + c_0 g_0)\psi^2 + \tfrac{1}{2}(1 - \mu \sin^2\psi)\dot\theta^2 \\ &\qquad + \tfrac{1}{2}\dot\psi^2 + c_0 g f(\theta, \psi), \\ f(\theta, \psi) &= \tfrac{1}{2}\theta^2 + \tfrac{1}{2}\psi^2 + \cos\theta\cos\psi - 1. \end{aligned} \tag{30}$$

Note that

$$f(\theta, \psi) \geqq 0, \quad \forall\, \theta, \psi; \qquad \cos\theta\cos\psi \leqq 1, \quad \forall\, \theta, \psi.$$

Therefore,

$$W_1(x) \leqq V(x) \leqq W_2(x) \tag{31}$$

where

$$W_1(x) = \tfrac{1}{2}(\omega_1^2 + c_0 g_0)\theta^2 + \tfrac{1}{2}(\omega_2^2 + c_0 g_0)\psi^2 + \tfrac{1}{2}(1 - \mu\sin^2\psi)\dot\theta^2 + \tfrac{1}{2}\dot\psi^2,$$

$$W_2(x) = \tfrac{1}{2}(\omega_1^2 + c_0 g_0 + c_0 g)\theta^2 + \tfrac{1}{2}(\omega_2^2 + c_0 g_0 + c_0 g)\psi^2 + \tfrac{1}{2}\dot\theta^2 + \tfrac{1}{2}\dot\psi^2.$$

The functions $W_1(x)$ and $W_2(x)$ are positive definite for $\omega_1^2 \geqq 0$, $\omega_2^2 \geqq 0$ because $c_0 g$ and $c_0 g_0$ are positive quantities. Hence, $V(x)$ is positive definite. It would also be a Liapunov function of the controlled system if its derivative along the state trajectory satisfies further properties.

C. *DERIVATIVE ALONG A TRAJECTORY*

$$\frac{dV}{dt} = \frac{\partial V}{\partial \theta}\dot\theta + \frac{\partial V}{\partial \psi}\dot\psi + \frac{\partial V}{\partial \dot\theta}\ddot\theta + \frac{\partial V}{\partial \dot\psi}\ddot\psi. \tag{32}$$

From (30), we have

$$\begin{aligned} \frac{\partial V}{\partial \theta} &= (\omega_1^2 + c_0 g_0)\theta + c_0 g(\theta - \sin\theta\cos\psi), \\ \frac{\partial V}{\partial \dot\theta} &= (1 - \mu\sin^2\psi)\dot\theta, \\ \frac{\partial V}{\partial \psi} &= (\omega_2^2 + c_0 g_0)\psi + c_0 g(\psi - \sin\psi\cos\theta) - \tfrac{1}{2}\mu\sin(2\psi)\dot\theta^2, \\ \frac{\partial V}{\partial \dot\psi} &= \dot\psi. \end{aligned} \tag{33}$$

Therefore,

$$\frac{dV}{dt} = (\omega_1^2 + c_0 g_0)\theta\dot{\theta} + c_0 g(\theta - \sin\theta\cos\psi)\dot{\theta} + \dot{\theta}[\mu\sin(2\psi)\dot{\theta}\dot{\psi}$$
$$-c_0 g(\theta - \sin\theta\cos\psi) + c_0 g_0\sin(2\pi t)\sin\theta\cos\psi - k\dot{\theta} - \omega_1^2\theta$$
$$-2c_0 g_0\,\mathrm{sat}(\theta\dot{\theta})\theta] + (\omega_2^2 + c_0 g_0)\psi\dot{\psi} + c_0 g(\psi - \sin\psi\cos\theta)\dot{\psi}$$
$$-\tfrac{1}{2}\mu\sin(2\psi)\dot{\theta}^2\dot{\psi} + \dot{\psi}[-\tfrac{1}{2}\mu\sin(2\psi)\dot{\theta}^2 - c_0 g(\psi - \sin\psi\cos\theta)$$
$$+c_0 g_0\sin(2\pi t)\sin\psi\cos\theta - k\dot{\psi} - \omega_2^2\psi - 2c_0 g_0\,\mathrm{sat}(\psi\dot{\psi})\psi]. \quad (34)$$

Hence,

$$\frac{dV}{dt} = I_1 + I_2 + I_3 \quad (35)$$

where

$$I_1 = \dot{\theta}[-2c_0 g_0\,\mathrm{sat}(\theta\dot{\theta})\theta + c_0 g_0\theta + c_0 g_0\sin(2\pi t)\cos\psi\sin\theta],$$
$$I_2 = \dot{\psi}[-2c_0 g_0\,\mathrm{sat}(\psi\dot{\psi})\psi + c_0 g_0\psi + c_0 g_0\sin(2\pi t)\cos\theta\sin\psi],$$
$$I_0 = -k(\dot{\theta}^2 + \dot{\psi}^2).$$

Consider the term I_1:

$$I_1 = c_0 g_0\dot{\theta}[-2\,\mathrm{sat}(\theta\dot{\theta})\theta + \theta + (\sin(2\pi t)\cos\psi)\sin\theta].$$

This is nonpositive for all values of θ, $\dot{\theta}$, ψ, as can easily be seen by examining the signs of θ and $\dot{\theta}$. Table 2 shows the four possible combinations.

TABLE 2

$$\mathrm{Sat}(\theta\dot{\theta}) \triangleq \begin{cases} 1 & \text{if } \theta\dot{\theta} \geqq 0, \\ 0 & \text{if } \theta\dot{\theta} < 0. \end{cases}$$

θ	$\dot{\theta}$	$\dot{\theta}[-2\mathrm{sat}(\theta\dot{\theta})\theta + \theta + (\sin(2\pi t)\cos\psi)\sin\theta]$	I_1
$\geqq 0$	$\geqq 0$	$\geqq 0[(-2)(\geqq 0) + (\geqq 0) + ('')(\geqq 0)]$	$\leqq 0$
$\leqq 0$	$\geqq 0$	$\geqq 0[\quad 0 \quad + (\leqq 0) + ('')(\leqq 0)]$	$\leqq 0$
> 0	< 0	$< 0[\quad 0 \quad + (> 0) + ('')(> 0)]$	< 0
< 0	< 0	$< 0[(-2)(< 0) + (< 0) + ('')(< 0)]$	< 0

Note 1.

$$|\theta| \geqq |\sin(2\pi t)\cos\psi|\cdot|\sin\theta|; \qquad |\sin(2\pi t)\cos\psi| \leqq 1.$$

Note 2. The on–off function $\mathrm{sat}(\theta\dot{\theta})\theta\dot{\theta}$ is continuous. This means that I_1 is continuous, and so is dV/dt.

Since I_2 is of the same form as I_1, with θ replaced by ψ, it is true that $I_2 \leqq 0$. Obviously,

$$I_3 = -k(\dot{\theta}^2 + \dot{\psi}^2) \leqq 0 \quad \forall\, k \geqq 0. \quad (36)$$

Therefore,

$$\frac{dV}{dt} = I_1 + I_2 + I_3 \leqq 0. \tag{37}$$

Thus, the function V is a Liapunov function for the system under study, showing that it is stable under all initial conditions of angular displacements and velocities.

D. *ASYMPTOTIC STABILITY*

From expression I_3 (36), it can be further concluded that the torso dynamics is asymptotically stable under all initial conditions for $k > 0$. From the system dynamics, we have an equation of the "isocline"

$$\begin{aligned}\frac{d\dot{\theta}}{d\theta} = \frac{d\dot{\theta}}{dt}\frac{dt}{d\theta} = \frac{\ddot{\theta}}{\dot{\theta}} = \frac{1}{\dot{\theta}}(1 - \mu \sin^2 \psi)^{-1}[&\mu \sin(2\psi)\dot{\theta}\dot{\psi} \\ &+ c_0 g(\theta - \sin\theta\cos\psi) - 2c_0 g_0 \operatorname{sat}(\theta\dot{\theta})\theta \\ &+ c_0 g_0 \sin(2\pi t)\theta - k\dot{\theta} - \omega_1^2\theta].\end{aligned}$$

On the θ axis where $\dot{\theta} = 0$, the slope of the isocline becomes unbounded. This means that the state trajectory cannot be sustained along $\dot{\theta} = 0$ except by crossing over at isolated time instants. A similar behavior holds along the $\dot{\psi} = 0$ axis. Since I_3 vanishes only if both $\dot{\theta} = 0$ and $\dot{\psi} = 0$, it is thus negative ($I_3 < 0$) except for isolated instants of time. From this, the strengthened condition that

$$\frac{dV}{dt} \leqq -k(\dot{\theta}^2 + \dot{\psi}^2) < 0, \quad \forall k > 0$$

except for isolated time instants shows that the controlled system is asymptotically stable under all initial conditions.

E. *SENSITIVITY OF THE CONTROL LAW*

For the proposed control law, we have written the angular feedback gains in the form $(\omega_1^2 + c_0 g)$ and $(\omega_2^2 + c_0 g)$ for ease of identification and have shown that the system is stable for $\omega_1^2 > 0$, $\omega_2^2 > 0$. The $c_0 g$ term essentially cancels the inverted pendulum effects of the torso. However, cancellation does not always work, due to parameter variations. To investigate the sensitivity of the feedback control, let the expression (14) be modified to

$$\begin{aligned} u_1 &= -k\dot{\theta} - \gamma c_0 g\theta - [\omega_1^2 + 2c_0 g_0 \operatorname{sat}(\theta\dot{\theta})]\theta, \\ u_2 &= -k\dot{\psi} - \gamma c_0 g\psi - [\omega_2^2 + 2c_0 g_0 \operatorname{sat}(\psi\dot{\psi})]\psi \end{aligned} \tag{38}$$

for $\gamma \neq 1$. Consider a modified scalar function of the form

$$\begin{aligned} V' = \tfrac{1}{2}(\omega_1^2 + c_0 g_0)\theta^2 + \tfrac{1}{2}(\omega_2^2 + c_0 g_0)\psi^2 + \tfrac{1}{2}(1 - \mu \sin^2 \psi)\dot{\theta}^2 \\ + \tfrac{1}{2}\dot{\psi}^2 + c_0 g[\tfrac{1}{2}\gamma\theta^2 + \tfrac{1}{2}\gamma\psi^2 + \cos\theta\cos\psi - 1] \end{aligned} \tag{39}$$

Adding and subtracting $\frac{1}{2}c_0g(\theta^2 + \psi^2)$ to V', we have

$$V' = V + \tfrac{1}{2}(\gamma - 1)(\theta^2 + \psi^2)c_0g \tag{40}$$

where V is the original Liapunov function constructed earlier. Carrying out the same manipulations as in Section 4.C, it is obvious that the total derivative of V' along a trajectory, dV'/dt, is nonpositive for all instants of time. The function V' is thus a Liapunov function for the controls (38) if we demand that

$$\omega_i^2 + c_0g\alpha > 0 \qquad \text{for} \quad i = 1, 2,$$
$$\alpha \triangleq \gamma - 1 \tag{41}$$

or the feedback gains

$$k_i \triangleq (\omega_i^2 + \gamma c_0 g) > c_0 g \qquad \text{for} \quad i = 1, 2. \tag{42}$$

Hence, the feedback law is largely insensitive to incomplete cancellation as long as requirement (42) is satisfied.

5. NUMERICAL EXAMPLE

To substantiate our Liapunov approach to torso stability, a numerical example is presented. The physical and geometrical parameters of the torso are taken from Table 1 for a cylindrically shaped model. Initial conditions of the problem are set at

$\theta(0) = 0.3$ rad($\sim$ 17 deg), $\psi(0) = -0.25$ rad(~ -14.5 deg),
$\dot\theta(0) = 0.0$ rad/sec, $\dot\psi(0) = 0.0$ rad/sec.

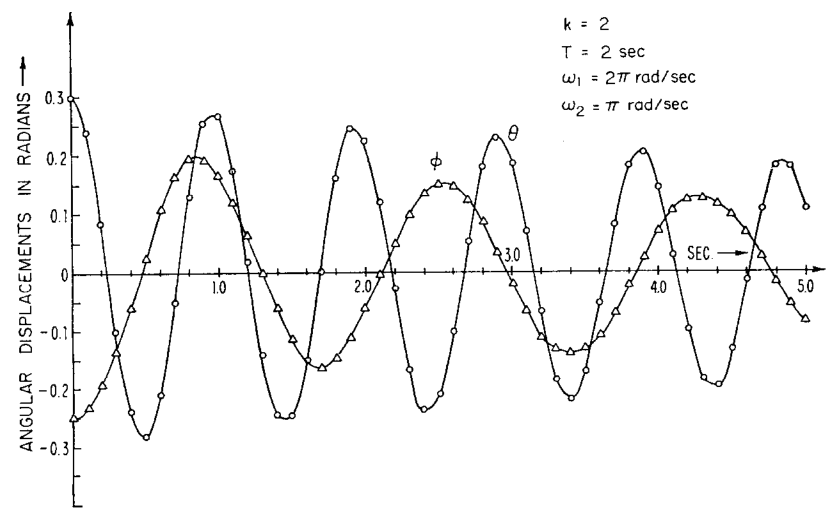

FIG. 2. Angular displacements of torso motion.

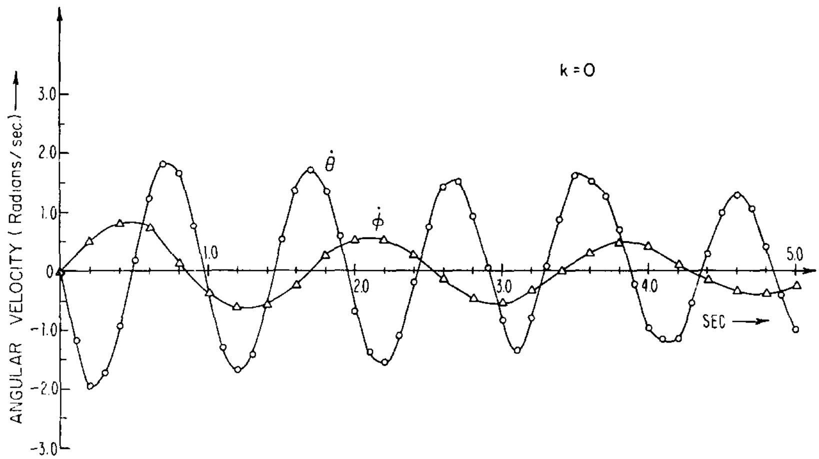

FIG. 3. Angular velocities of torso motion.

The feedback constants of the control law are chosen as

$$\omega_1 = \frac{2\pi}{T_1} = 2\pi \text{ rad/sec} \quad (T_1 = 1 \text{ sec}),$$

$$\omega_2 = \frac{2\pi}{T_2} = \pi \text{ rad/sec} \quad (T_2 = 2 \text{ sec}),$$

$$k = 0 \text{ and } 2.$$

Results of the computation are presented in Fig. 2–6.

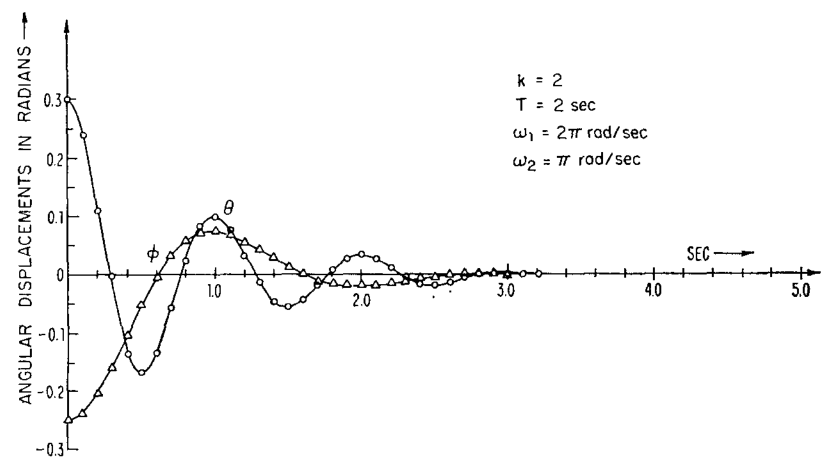

FIG. 4. Angular displacements of torso motion.

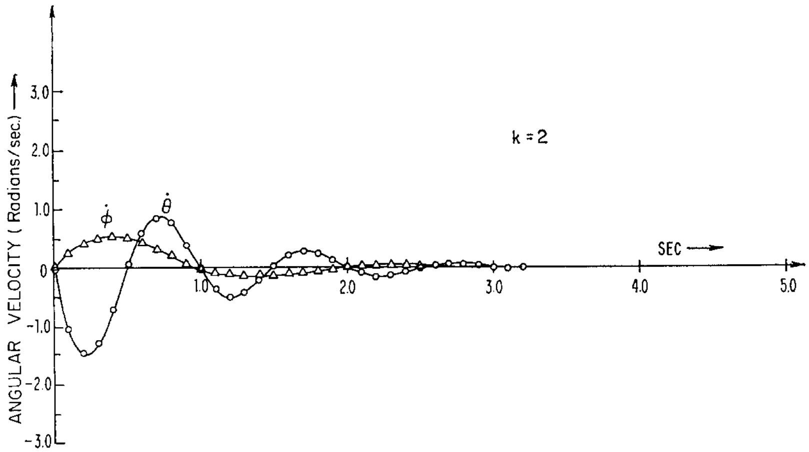

FIG. 5. Angular velocities of torso motion.

When there is no velocity feedback, $k = 0$, the angular trajectories are quite oscillatory with the maximum amplitude slowly diminishing (Fig. 2 and 3). When $k = 2$, however, the motion dies down quickly (Fig. 4 and 5). This illustrates that velocity feedback is very effective in achieving stability of the torso motion. Indeed, from Eq. 35, for $k > 0$, the term I_3 is probably the significant negative contribution to the total derivative as compared to I_1 or I_2.

Another interesting observation is that with the constants ω_1 and ω_2 set at 2π and π, the θ and ψ trajectories have periods close to 1 and 2 sec,

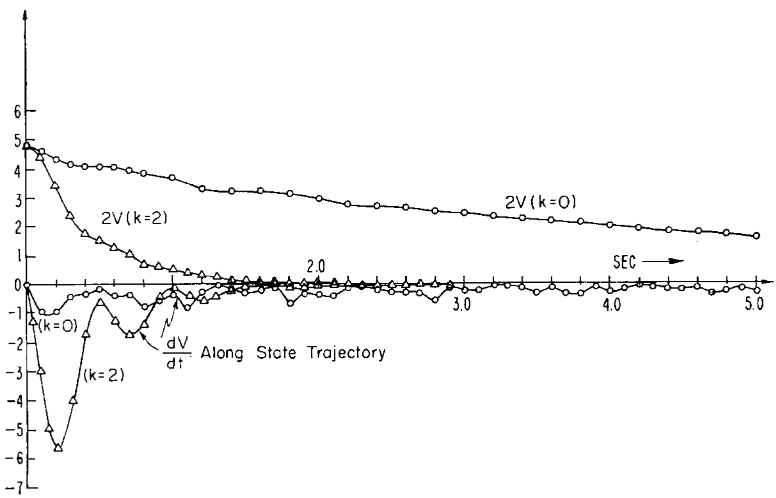

FIG. 6. Time histories of Liapunov function and its derivatives.

respectively. That the system behavior is predominantly linear indicates effective cancellation of the unstable term in Eq. (38). Also, the other nonlinear terms are small as compared to the linear ones.

The motion is clearly stable, as shown in Fig. 6. The value of the Liapunov function is positive and monotonically decreasing at all times. Likewise, the total derivative along the motion is negative throughout the time interval.

6. CONCLUSION: A POSSIBLE EXPERIMENT

The perturbation terms in the control law have an interesting interpretation. Since the function $\mathrm{sat}(\theta\dot{\theta}) = 1$ for $\theta\dot{\theta} \geqq 0$, it is nonzero in the first and third quadrants of the θ–$\dot{\theta}$ plane. In real motion, this means that the perturbation becomes effective as the torso moves away from the vertical position. When it moves toward the vertical, the perturbation vanishes.

To strengthen the validity of the perturbation hypothesis in feedback realization a combined photographic and EMG experiment should be undertaken. The EMG activity of the abductor, adductor, and erector spinae muscle groups would need to be measured. Simultaneously, synchronized motion pictures would be taken to deduce the torso motion. Superposition of the two sets of data could reveal any differential muscle activity as the torso moves away from or toward the vertical position. Since the perturbation coefficient is small as compared to the feedback constants, $c_0 g_0 \ll (\omega_i^2 + c_0 g)$, $i = 1, 2$, extra care must be taken in experimental procedure.

The foregoing experiment is directly suggested by our stability analysis, and indeed this is one of the major contributions of this type of theoretical study; namely, to suggest new experiments that will confirm or refute a certain theory (in this case, a theory of the stability mechanism).

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