Research Technical Note

Interest Rates MC Simulation with the Hull-White Model

June 1, 2007

RiskMetrics Research Department
research.publications@riskmetrics.com

Keywords: Short-rate, Hull-White, Interest rates.

We detail hereafter the implementation process for the one factor Hull-White interest rates model. The process is stated in the Heath-Jarrow-Morton framework. Calibration to caplet and swaption volatilities is proposed in a price-based framework, while the historical calibration is in a volatility-based framework. Relying on HJM, the instantaneous forward rate curve and its derivative are required. We describe how to obtain these curves through smoothed splines.

1 Definitions

We introduce useful quantities and notations. $E_t[.]$ will always denote the expectation conditional on the information available at time t. For homogeneity of notations, times should not be understood as dates per se, but rather as numbers of years from the time-origin (typically, the analysis date).

- Let $B(t, t+\theta)$ be the price of a zero-coupon bond, i.e. the discount factor at date t which pays 1$ at date $t+\theta$, θ being the maturity of the bond.
- The continuous annualized zero-coupon rate or yield-to-maturity $R(t,\theta)$ at date t for a maturity θ is $R(t,\theta) = -\frac{1}{\theta}\ln[B(t,t+\theta)]$.
- The short-rate or the instantaneous risk free rate $r(t)$ at date t is given by, $r(t) = r_t = \lim\limits_{\theta\to 0} R(t,\theta) = -\frac{\partial \log B(t,s)}{\partial s}\lfloor_{s=t}$.
- The instantaneous forward rate at date t which applies at date s is defined by, $f(t,s) = -\frac{\partial \log B(t,s)}{\partial s}$.
- The forward rate at date t starting at the date T with tenor θ is denoted by $F(t,T,T+\theta)$. It is given by $F(t,T,T+\theta) = E_t[R(T,\theta)]$, and we observe that $F(t,T,T+\theta) = \frac{R(t,T+\theta-t)(T+\theta-t)-R(t,T-t)(T-t)}{\theta}$.

- Similarly, the forward bond price over the same period is $B_f(t, T, T+\theta)$. Forwards and forward prices are obviously linked by $B_f(t, T, T+\theta) = exp(-F(t, T, T+\theta)\theta)$. In addition,

$$B_f(t, T, T+\theta) = \frac{B(t, T+\theta)}{B(t, T)}. \tag{1}$$

- The swap rate $S_\theta(t, t)$ and the forward swap rate $S_{n\theta}(t, s)$ at date t for a contract starting at date s and maturing at date $s + n\theta$ is defined by

$$S_{n\theta}(t, s) = \frac{B(t, s) - B(t, s+n\theta)}{\sum\limits_{i=1}^{n} \theta B(t, s+i\theta)}. \tag{2}$$

2 One-Factor Hull-White Model

The Hull-White model can be stated in the HJM framework by the following short-rate dynamics:

$$dr_t = \alpha \left(f(0, t) + \frac{1}{\alpha} \frac{\partial f(0, t)}{\partial t} + \frac{\sigma^2}{2\alpha^2}(1 - e^{-2\alpha t}) - r_t \right) dt + \sigma dW(t) \tag{3}$$

As for other Gaussian-HJM models, the short-rate is allowed to be negative and the long-term mean of the short-rate is time-dependent. This Hull-White model is characterize by a constant volatility of forward bond prices through time for a given tenor. In addition, the volatility of future spot rates is not dependent on their level.

2.1 Simulation

Simulation can easily be done by discretising the SDE (3). Using the simple Euler scheme for a time step Δt, we get,

$$r_{t+\Delta t} = \alpha \left(f(0, t) + \frac{1}{\alpha} \frac{\partial f(0, t)}{\partial t} + \frac{\sigma^2}{2\alpha^2}(1 - e^{-2\alpha t}) \right) \Delta t + (1 - \alpha \Delta t) r_t + \sigma \sqrt{\Delta t} \epsilon_{t+\Delta t} \tag{4}$$

where $\epsilon_{t+\Delta t}$ is a standard normal random variable.

The initial value r_0 has to be read off the spot zero-rate curve. The zero-rate curve, the instantaneous forward rate curve and its derivative are inputs of the model. The parameters α and σ have to be calibrated.

2.2 Implied Future Bond Prices

On each path of the short-rate process, we have to be able to compute future zero-rates at each future date, i.e. for each future value r_t of the short-rate. Let us write down the Feynman-Kac equation for bond prices under the risk neutral measure. In absence of arbitrages, the drift term of the bond price process growth at the short-rate speed:

$$\frac{\partial B(t,T)}{\partial t} + \alpha(\eta(t) - r_t)\frac{\partial B(t,T)}{\partial r_t} + \frac{\sigma^2}{2}\frac{\partial^2 B(t,T)}{\partial r_t^2} = r_t B(t,T), \tag{5}$$

where $\eta(t)$ denotes the long term mean of the short-rate process as stated in (3).

Let us further assume that bond prices at time t are an exponential affine function of the state variable r_t, i.e. $B(t,T) = exp\left(\overline{A}(t,T) - \overline{B}(t,T)r_t\right)$. Plugging this expression into (5) and applying the separation of variables principle, solving equation (5) with the boundary condition $B(T,T) = 1$ is equivalent to solve the problem

$$\begin{aligned}
\frac{\partial \overline{B}(t,T)}{\partial t} &= \alpha \overline{B}(t,T) - 1 && (6)\\
\frac{\partial \overline{A}(t,T)}{\partial t} &= -\frac{\sigma^2}{2}\overline{B}^2(t,T) + \alpha\eta(t)\overline{B}(t,T) && (7)
\end{aligned}$$

with $\overline{A}(T,T) = 0$ and $\overline{B}(T,T) = 0$ as boundary conditions.

From the homogenous solution of equation (6) and its particular solution we get,

$$\begin{aligned}
\overline{B}(t,T) &= \frac{1}{\alpha}\left(1 - e^{-\alpha(T-t)}\right)\\
\overline{A}(t,T) &= \frac{\sigma^2}{2\alpha^2}\left[(T-t) - 2\overline{B}(t,T) + \frac{1}{2\alpha}\left(1 - e^{-2\alpha(T-t)}\right)\right] - \int_t^T \left(1 - e^{-\alpha(T-s)}\right)\eta(s)ds
\end{aligned}$$

Future spot rates can then be obtained by

$$R(t, T-t) = \frac{1}{T-t}\left(\overline{B}(t,T)r_t - \overline{A}(t,T)\right). \tag{8}$$

The solutions for $\overline{B}(t,T)$, $\overline{A}(t,T)$, $B(t,T)$ and $R(t,T-t)$ have to be implemented. The computation of the last term of $\overline{A}(t,T)$ can be done analytically. and decomposing $\eta(t)$, we can write,

$$\overline{A}(t,T) = -\frac{\sigma^2}{4\alpha}\overline{B}^2(t,T)\left(1 - e^{-2\alpha t}\right) - F(0,t,T)(T-t) + \overline{B}(t,T)f(0,t). \tag{9}$$

2.3 Calibration

We first underline that Gaussian models of the short-rate are log-normal forward bond price models. As a consequence, calibration can be achieved in a straightforward way as long as a full term structure of forward bond price volatilities can be observed. For that purpose, it is necessary to derive the dynamics of forward bond prices.

The SDE on the short-rate (3) is equivalent to the following SDE on instantaneous forward rates:

$$df(t,T) = \frac{\sigma^2}{\alpha} e^{-\alpha(T-t)}(1 - e^{-\alpha(T-t)})dt + \sigma e^{-\alpha(T-t)} dW(t). \tag{10}$$

Applying Itô's lemma on the variable $exp(-\int_t^T f(t,s)ds)$ provides the dynamics of bond prices:

$$\frac{dB(t,T)}{B(t,T)} = r_t dt - \frac{\sigma}{\alpha}(1 - e^{-\alpha(T-t)})dW(t). \tag{11}$$

Using again Itô's lemma on (1), the SDE on forward bond prices is given by,

$$\frac{dB_f(t,T,T+\theta)}{B_f(t,T,T+\theta)} = \mu_{B_f}(t,T,\theta,B_f)dt + \frac{\sigma}{\alpha} e^{-\alpha(T-t)}(e^{-\alpha\theta} - 1)dW(t). \tag{12}$$

2.3.1 Using Historical Volatilities

We use a volatility-based calibration. From time series of spot rates, it is straightforward to estimate the parameters of the model. Let us assume that we can observe the set of constant maturity continuous annualized spot rates $R_s(0,\theta_1), .., R_s(0,\theta_K)$ daily over a one year period ending at the analysis date (indexed by s). The θ_1 to θ_K are the nodes of a DMX curve vertice. Then, computing the quantities 13,

$$\begin{aligned} \Delta_s R_s(0,\theta_i) &= R_{s+\delta s}(0,\theta_i - \delta s) - R_s(0,\theta_i) \\ &\approx R_{s+\delta s}(0,\theta_i) - R_s(0,\theta_i), \end{aligned} \tag{13}$$

since future spot rates are linked to future bond prices by $B(t,t+\theta_i) = exp(-R(t,t+\theta_i)\theta_i)$, their instantaneous volatility $\sigma_R(t,t+\theta_i)$ expresses as

$$\begin{aligned} \sigma_R(t,t+\theta_i) &= \frac{\partial R(t,t+\theta_i)}{\partial r_s}\sigma \\ &= \frac{\sigma}{\alpha\theta_i}\left(1 - e^{-\alpha\theta_i}\right), \end{aligned}$$

and the parameters can be calibrated solving,

$$\min_{\alpha,\sigma} \sum_i \left(vol\left(\Delta_s R_s(0,\theta_i)\right)\sqrt{260} - \sigma_R(0,\theta_i) \right)^2 \tag{14}$$

where $vol()$ is the estimated standard deviation of the time series.

2.3.2 Using Caplet Volatilities

We use a price-based calibration. The process reduces in using the standard Black's formula for caplet prices but using a specific Black equivalent volatility.

Since forward bond prices are log-normally distributed, a close-form solution for caplet prices in a price-based settings can easily be derived using standard techniques. Under the forward measure F_T for a notional of N, the caplet price of tenor θ with strike R_X is indeed given by

$$\begin{aligned}
PV(caplet(T,\theta)) &= N\,(1+R_X\theta)\,E\left[\left(\frac{1}{1+R_X\theta} - B(T,T+\theta)\right)_+ \exp\left(-\int_0^T r_s ds\right)\right] \\
&= N\,(1+R_X\theta)\,B(0,T)E^{F_T}\left[\left(\frac{1}{1+R_X\theta} - B(T,T+\theta)\right)_+\right] \\
&= N\,(1+R_X\theta)\,B(0,T)E^{F_T}\left[\left(\frac{1}{1+R_X\theta} - B_f(T,T,T+\theta)\right)_+\right] \\
&= B(0,T)\,(1+R_X\theta)\left(\frac{\Phi(-d_2)}{1+R_X\theta} - B_f(0,T,T+\theta)\Phi(-d_1)\right) \qquad (15)
\end{aligned}$$

with

$$\begin{aligned}
d_1 &= \frac{\ln(B_f(0,T,T+\theta)\,(1+R_X\theta)) + \sigma^2_{Black}(T,\theta)T/2}{\sigma_{Black}(T,\theta)\sqrt{T}} \\
d_2 &= d_1 - \sigma_{Black}(T,\theta)\sqrt{T}
\end{aligned}$$

where $\sigma_{Black}(T,\theta)$ is an equivalent Black's volatility given the Hull-White model, i.e. the average volatility of forward bond prices up to the exercise date T. From the SDE (12), it can be computed as

$$\begin{aligned}
\sigma^2_{Black}(T,\theta) &= \frac{1}{T}\int_0^T \left(\frac{\sigma}{\alpha}e^{-\alpha(T-s)}(e^{-\alpha\theta}-1)\right)^2 ds \\
&= \frac{\sigma^2}{\alpha^2}\left(e^{-\alpha\theta}-1\right)^2 \frac{1-e^{-2\alpha T}}{2\alpha T} \qquad (16)
\end{aligned}$$

The calibration of α and σ is finally achieved by a best fit minimization to caplet prices. We indeed observe a set of cap volatilities $\sigma_{cap}(t_j)$ for different maturities $t_1, .., t_k$ with a given compounding frequency θ. These volatilities are flat volatilities of log-normal forward rates. The following process has thus to be applied[1]:

1. Transform iteratively the cap volatilities $\sigma_{cap}(t_j)$ into θ-months tenor caplet volatilities $\sigma_{caplet}(t_i)$, keeping track of the cap strike rate applied to each of the caplet (this strike should be the corresponding swap rate derived from the spot zero-rate curve).

[1] See *Calculating forward volatility from flat volatility* for more details.

2. Transform each caplet volatility $\sigma_{caplet}(t_i)$ of forward rates into a caplet volatility $\sigma_{obs}(t_i)$ of forward bond prices by matching the prices of the caplet in two approaches (yield-based and price-based).

Last step consists in minimizing the squared differences between caplet prices using the observed and computed volatilities:

$$\min_{\alpha,\sigma} \sum_i \left(PV(caplet(t_i,\theta), \sigma_{obs}(t_i)) - PV(caplet(t_i,\theta), \sigma_{Black}(t_i,\theta))\right)^2 \tag{17}$$

2.3.3 Using Swaption Volatilities

An exact calibration to swaption prices can be performed using again a price-based approach as swaptions are put options on fixed coupon bonds. We still rely on equation (16) of equivalent Black volatility for the Hull-White model. Using the forward measure F_T, it can be shown that the present value of an European swaption contract with exercise date T, strike S_X and which runs over n tenors θ verifies:

$$\begin{aligned} PV(swaption(T,n\theta)) &= B(0,T)\Bigg(\Phi(d) - \sum_{i=1}^{n} \theta S_X B_f(0,T,T+i\theta)\Phi(d-v_i) \\ &\qquad - B_f(0,T,T+n\theta)\Phi(d-v_n)\Bigg) \end{aligned} \tag{18}$$

where

$$\begin{aligned} v_i^2 &= \frac{\sigma^2}{\alpha^2}\left(e^{-\alpha i\theta} - 1\right)^2 \frac{1-e^{-2\alpha T}}{2\alpha} \\ &= \sigma^2_{Black}(T,i\theta)T \end{aligned}$$

and where d is defined such that there is no arbitrage on the forward coupon bond price under the forward measure, i.e. such that the following equality holds:

$$1 = \sum_{j=1}^{n} \theta S_X B_f(0,T,T+j\theta)\exp\left(v_j d - \frac{v_j^2}{2}\right) + B_f(0,T,T+n\theta)\exp\left(v_n d - \frac{v_n^2}{2}\right). \tag{19}$$

The observed swaption volatilities consist in a set of $\sigma_{swaption}(t_i, n_j\theta)$ for various i and j. Observed European swaption prices $ES(t_i, n_j\theta)$ can then be computed in a yield-based setting using the At-The-Money forward swap rate as strike rate.[2] The calibration procedure is performed by minimizing the squared differences in prices:

$$\min_{\alpha,\sigma} \sum_i \sum_j \left(PV(swaption(t_i, n_j\theta)) - ES(t_i, n_j\theta)\right)^2, \tag{20}$$

[2]See *European Swaption Pricing* [**?**].

with,

$$ES(t_i, n_j\theta) = \sum_{k=1}^{n_j} \left(B(0, t_i + k\theta)\,\theta \right) S_{n_j\theta}(0, t_i)(\Phi(d_1) - \Phi(d_2)), \tag{21}$$

where $d_1 = \frac{1}{2}\sigma_{obs}(t_i)\sqrt{t_i}$ and $d_2 = -d_1$.

It is worthwhile to notice that it is necessary to solve for d through (19) for each swaption within the optimization procedure.

3 Construction and Interpolation of Spot Curves

The HJM setting requires as input the instantaneous forward curve $t \mapsto f(0,t)$. It has to be derived as a continuous function from the market spot curve $\theta \mapsto R(0,\theta)$. In addition, in order to describe the short-rate dynamics, it is necessary to obtain the derivative of the instantaneous forwards $t \mapsto \frac{\partial f(0,t)}{\partial t}$.

Since these curves cannot be derived from the flat forward assumption currently in use in the application, we detail another interpolation technique which can be substituted: smoothed splines.

3.1 Definition

Assume that we can observe the quantities $z_0, .., z_N$ at times $t_0, .., t_N$. Smoothed splines are the unique functions which minimize a double criteria of error and smoothing,

$$\min \left[\int_{t_0}^{t_N} \gamma(s) \left(\frac{\partial^2 v(s)}{\partial s^2} \right)^2 ds + \sum_{i=0}^{N} (v(t_i) - z_i)^2 \right] \tag{22}$$

under the following conditions:

- the restriction of v to $[t_i, t_{i+1}]$ is a polynomial of order 3 for all $i \in [1, N-2]$,
- restrictions of v to $[t_0, t_1]$ and $[t_{N-1}, t_N]$ are linear functions,
- $v \in C^2([t_0, t_N])$,
- $\gamma(s)$ is a predetermined cost function balancing the dominant criterion over the curve.

3.2 Deriving Coefficients

Actually, the unique solution to the objective function 22 can be easily obtained by inverting a matrix. Let us denote by $Y = [y_0, y_0', .., y_N, y_N']$ the vector of the values and first derivatives of the function v at the observation points $z_0, .., z_N$. Let us now define $h_i = t_{i+1} - t_i$ and $\gamma_i = \gamma(t_i)$. Writing the gradient of 22 and rearranging the terms, we can define the matrix M as

$$M = \begin{bmatrix} B_0 & C_0 & & & & & \\ A & B & C & & & 0 & \\ & \ddots & \ddots & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & 0 & & & A & B & C \\ & & & & & A_N & B_N \end{bmatrix}$$

where,

$$A = \begin{bmatrix} -\frac{12}{h_{i-1}^2}\gamma_i & -\frac{6}{h_{i-1}}\gamma_i \\ \frac{3}{h_{i-1}^2} & \frac{1}{h_{i-1}} \end{bmatrix} B = \begin{bmatrix} 12\gamma_i\left(\frac{1}{h_i^2} + \frac{1}{h_{i-1}^2}\right) + 1 & 6\gamma_i\left(\frac{1}{h_i} - \frac{1}{h_{i-1}}\right) \\ 3\left(\frac{1}{h_i^2} - \frac{1}{h_{i-1}^2}\right) & 2\left(\frac{1}{h_i} + \frac{1}{h_{i-1}}\right) \end{bmatrix}$$

$$C = \begin{bmatrix} -\frac{12}{h_i^2}\gamma_i & \frac{6}{h_i}\gamma_i \\ -\frac{3}{h_i^2} & \frac{1}{h_i} \end{bmatrix} B_0 = \begin{bmatrix} \frac{12\gamma_0}{h_0^2} + 1 & \frac{6\gamma_0}{h_0} \\ \frac{3}{h_0} & 2 \end{bmatrix} C_0 = \begin{bmatrix} -\frac{12\gamma_0}{h_0^2} & \frac{6\gamma_0}{h_0} \\ -\frac{3}{h_0} & 1 \end{bmatrix}$$

$$A_N = \begin{bmatrix} -\frac{12\gamma_N}{h_N^2} & -\frac{6\gamma_N}{h_N} \\ \frac{3}{h_N} & 1 \end{bmatrix} B_N = \begin{bmatrix} \frac{12\gamma_N}{h_N^2} + 1 & -\frac{6\gamma_N}{h_N} \\ -\frac{3}{h_N} & 2 \end{bmatrix}.$$

The solution Y we are looking for is then given by $Y = M^{-1}Z$ with $Z = [z_0, 0, .., z_N, 0]$.

3.3 Interpolated Curves

Let us apply the technique to the spot zero-rates curve, substituting z_i for $R(0, \theta_i)$ where θ_i is one node of RMG curves. Set $m_i = \frac{y_{i+1} - y_i}{h_i}$. For $s \in [\theta_i, \theta_{i+1}[$, we get the

- spot zero-rate $R(0, s)$ by,

$$R(0, s) = v(s) = y_i + y_i'(s - \theta_i) + \frac{m_i - y_i'}{h_i}(s - \theta_i)^2 + \frac{y_{i+1}' - 2m_i + y_i'}{h_i^2}(s - \theta_i)^2(s - \theta_{i+1}),$$

- the instantaneous forward rate $f(0, s)$ by,

$$\begin{aligned} f(0, s) &= R(0, s) + sv'(s) \\ v'(s) &= y_i' + 2\frac{m_i - y_i'}{h_i}(s - \theta_i) + 2\frac{y_{i+1}' - 2m_i + y_i'}{h_i^2}(s - \theta_i)(s - \theta_{i+1}) + \frac{y_{i+1}' - 2m_i + y_i'}{h_i^2}(s - \theta_i)^2, \end{aligned}$$

- the derivative of the instantaneous forward rate by,

$$\frac{\partial f(0,s)}{\partial s} = 2v'(s) + s\left[2\frac{m_i - y'_i}{h_i} + 4\frac{y'_{i+1} - 2m_i + y'_i}{h_i^2}(s-\theta_i) + 2\frac{y'_{i+1} - 2m_i + y'_i}{h_i^2}(s-\theta_{i+1})\right].$$

References

[1] RiskMetrics Research Department, *European swaption pricing*, Research Technical Note.