1 Cosine-sine unitary decomposition

The cosine-sine unitary decomposition (CSD) is based on the mathematical technique of the same name *cosine-sine decomposition* [1, 2]. Assuming a unitary $U \in \mathbb{C}^{N\times N}$, where N is even, the CSD theorem states that U can always be decomposed in the following form

$$U = RD_{cs}L = \begin{bmatrix} R_0 & 0 \\ 0 & R_1 \end{bmatrix} \begin{bmatrix} C & -S \\ S & C \end{bmatrix} \begin{bmatrix} L_0 & 0 \\ 0 & L_1 \end{bmatrix}, \tag{1}$$

composed by unitaries $L_i, R_i \in \mathbb{C}^{N/2\times N/2}$ and real diagonal matrices C e S such that $C^2+S^2 = \mathbb{I}_{N/2}$, where $C = \mathrm{diag}(\cos(\theta_0), \ldots, \cos(\theta_{N/2}))$ e $S = \mathrm{diag}(\sin(\theta_0), \ldots, \sin(\theta_{N/2}))$.

The CSD algorithm [3, 4] applies the cosine-sine decomposition recursively. It starts with the operator U of dimension $N \times N$, where $N = 2^n$ and $n \geq 1$ is the number of qubits. U is decomposed into the matrix D_{cs}, two unitaries L_i and two unitaries R_i (Equation 1 and Figure 1). Next, the four unitaries produced in the previous step are decomposed into four central matrices and sixteen unitaries, as in Equation (2). The unitaries form four sets $R_{R_0} + R_{R_1}$, $L_{R_0} + L_{R_1}$, $R_{L_0} + R_{L_1}$ e $L_{L_0} + L_{L_1}$ (Figure 2), each set with four operators.

$$\begin{aligned} U = &\begin{bmatrix} \begin{bmatrix} R_{R_0,0} & 0 \\ 0 & R_{R_0,1} \end{bmatrix} \begin{bmatrix} C_{R_0} & -S_{R_0} \\ S_{R_0} & C_{R_0} \end{bmatrix} \begin{bmatrix} L_{R_0,0} & 0 \\ 0 & L_{R_0,1} \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} R_{R_1,0} & 0 \\ 0 & R_{R_1,1} \end{bmatrix} \begin{bmatrix} C_{R_1} & -S_{R_1} \\ S_{R_1} & C_{R_1} \end{bmatrix} \begin{bmatrix} L_{R_1,0} & 0 \\ 0 & L_{R_1,1} \end{bmatrix} \end{bmatrix} \\ &\begin{bmatrix} C & -S \\ S & C \end{bmatrix} \\ &\begin{bmatrix} \begin{bmatrix} R_{L_0,0} & 0 \\ 0 & R_{L_0,1} \end{bmatrix} \begin{bmatrix} C_{L_0} & -S_{L_0} \\ S_{L_0} & C_{L_0} \end{bmatrix} \begin{bmatrix} L_{L_0,0} & 0 \\ 0 & L_{L_0,1} \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} R_{L_1,0} & 0 \\ 0 & R_{L_1,1} \end{bmatrix} \begin{bmatrix} C_{L_1} & -S_{L_1} \\ S_{L_1} & C_{L_1} \end{bmatrix} \begin{bmatrix} L_{L_1,0} & 0 \\ 0 & L_{L_1,1} \end{bmatrix} \end{bmatrix}. \end{aligned} \tag{2}$$

The recursion continues until the unitaries have dimension 2×2 (one-qubit operators). In the end, after $n-1$ iterations, there will be 2^{n-1} sets of unitaries, each with 2^{n-1} one-qubit operators. Each set corresponds to an operator uniformly controlled by the most significant qubits and applied to the least significant qubit (Figure 1) [4]. These sets are separated by the central operators

Figure 1: Identities used for the cosine-sine decomposition (CSD) of quantum operators.

(Figure 2), which have the same structure as the uniformly controlled rotation R_y applied to the qubit corresponding to the step of the recursion [4]. This decomposition requires a total of $4^n - 2^{n+1}$ CNOTs.

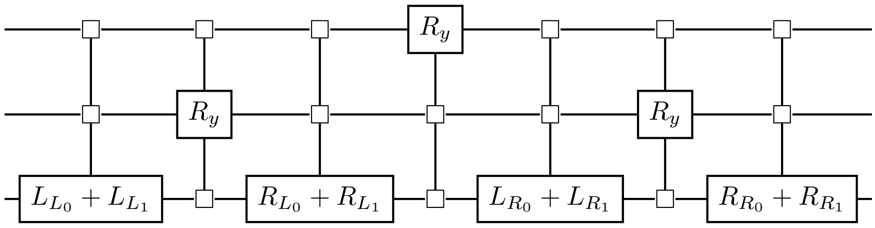

Figure 2: Circuit for the cosine-sine decomposition (CSD) of a three-qubit quantum operator.

Listing 1: Cosine-Sine decomposition of operators using qclib.

```
from qiskit import transpile
from scipy.stats import unitary_group
from qclib.unitary import unitary as decompose
table = '|␣n␣|␣cnots␣␣|␣depth␣␣|\n'
table += '|:-:|:------:|:------:|\n'
for n_qubits in range(3, 9):
    U = unitary_group.rvs(2 ** n_qubits)
    circuit = decompose(U, decomposition='csd')
    t_circuit = transpile(circuit, basis_gates=['u', 'cx'],
                          optimization_level=0)
    n_depth = t_circuit.depth()
    n_cx = t_circuit.count_ops().get('cx', 0)
    table += f'|␣{n_qubits}␣|␣{n_cx:^6d}␣|␣{n_depth:^6d}␣|\n'
print(table)
# | n | cnots  | depth  |
# |:-:|:------:|:------:|
# | 3 |   47   |  125   |
# | 4 |  223   |  546   |
# | 5 |  959   |  2307  |
# | 6 |  3967  |  9546  |
# | 7 | 16127  | 39073  |
# | 8 | 65023  | 158771 |
```

References

[1] G. W. Stewart and Ji-guang Sun. Matrix perturbation theory. Computer science and scientific computing. Academic Press, Boston, 1990.

[2] C.C. Paige and M. Wei. History and generality of the cs decomposition. Linear Algebra and its Applications, 208-209:303–326, 1994.

[3] Robert R. Tucci. A rudimentary quantum compiler(2cnd ed.). 1999.

[4] V.V. Shende, S.S. Bullock, and I.L. Markov. Synthesis of quantum-logic circuits. IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems, 25(6):1000–1010, 2006.