MATRIX ANALYSIS OF HEAT TRANSFER PROBLEMS

BY

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ABSTRACT

This paper presents a method of analysis of heat conduction problems in solids based on matrix algebra. Use is made of the analogy that exists between the thermal problem and the flow of electricity in an electrical transmission line. It is shown that the use of matrix algebra greatly facilitates the calculation of transient and periodic heat flow in composite solids.

INTRODUCTION

The close analogy that exists between the flow of heat in one dimension and the propagation of electricity in an electric cable has been known for a long time (**1, 2, 3**).[2] During recent years, several electrical engineers and physicists (**4, 5, 6, 7**) have systematized the mathematical analysis of propagation problems along transmission lines and cables by the introduction of the concept of the four-terminal network or quadripole and by the introduction of matrix algebra for the systematic analysis of such networks. It is the purpose of this discussion to call these modern techniques to the attention of engineers who are concerned with heat-conduction problems. The basic principles involved are presented and their use is illustrated by applying them to representative problems.

THE FUNDAMENTAL EQUATIONS

Consider the simple case of a homogeneous rectangular slab of material of uniform thickness shown in Fig. 1. Assume that the heat losses at the edges of the slab are such that they may be neglected and that the temperature of the slab is a function only of the co-ordinate x and the time t. Introduce the following notation:

$e(x,t)$ = temperature of all points of slab situated at a distance x from edge of slab as shown in Fig. 1.

$i(x,t)$ = heat flux or quantity of heat that passes through a plane perpendicular to the x-axis at a distance x from edge of slab per unit area in unit time.

R = thermal or heat resistance per unit length of material of slab in direction of heat flow per unit area. ($1/R$ is thermal conductivity.)

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[2] The boldface numbers in parentheses refer to the references appended to this paper.

C = heat capacitance of slab material. This is the number of heat units required to raise a block of the material of the slab of unit volume 1 deg in temperature. (C = specific heat, x density.)

Then, since Ri is the temperature drop per unit length along the x-axis, it follows that

$$-\frac{\partial e}{\partial x} = Ri. \tag{1}$$

The heat flux at the point x, $i(x,t)$ raises the temperature of a lamina of thickness dx at the rate of $\partial e/\partial t$. Therefore

$$-\frac{\partial i}{\partial x} = C\frac{\partial e}{\partial t}. \tag{2}$$

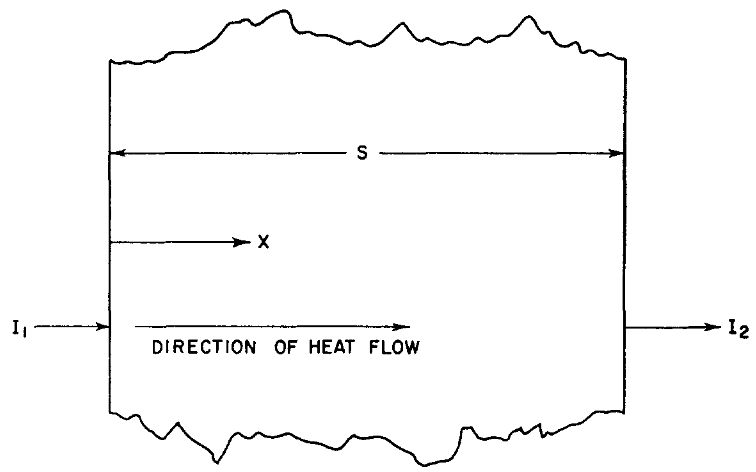

FIG. 1. Heat flow in a homogeneous slab.

If the heat flux, $i(x,t)$ is eliminated between Eqs. 1 and 2, the resulting equation is

$$\frac{\partial^2 e}{\partial x^2} = RC\frac{\partial e}{\partial t}. \tag{3}$$

The quantity

$$K = 1/RC = \frac{\text{(thermal conductivity)}}{\text{(density} \times \text{specific heat)}} \tag{4}$$

is the diffusivity, diffusion coefficient, or thermometric conductivity of the material of the slab.

Equations 1 and 2 are the equations that govern the propagation of current $i(x,t)$ and potential $e(x,t)$ in a uniform electric cable of capacitance C and resistance R per unit length. This electrical analogy has been used to study heat-conduction problems by several investigators (1, 2, 3). The analogy makes it possible to use all the mathematical techniques of the theory of electric circuits in the study of linear heat-conduction problems. For example, the slab of Fig. 1 is completely analogous to the four-terminal network or quadripole of Fig. 2.

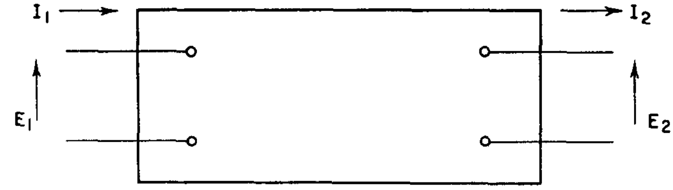

FIG. 2. Electric quadripole.

Let it be supposed that at $t = 0$, the slab of Fig. 1 is at a constant temperature. Let this constant temperature be taken as zero for simplicity. Now at $t = 0 +$ a temperature $e_1(t)$ is impressed on the side $x = 0$ and another temperature $e_2(t)$ is impressed on the side $x = S$; $i_1(t)$ and $i_2(t)$ then represent the heat fluxes entering and leaving the slab. This thermal system is equivalent to the electrical system in which the circuit is inert at $t = 0$ and a potential $e_1(t)$ is impressed on the left terminals and $e_2(t)$ on the right terminals of the quadripole of Fig. 2.

In order to determine the heat fluxes $i_1(t)$ and $i_2(t)$, the simplest procedure is to introduce the following p-multiplied Laplace transforms (4)

$$L\, e(x,t) = p \int_0^\infty e^{-pt} e(x,t)dt = E(x,p) \tag{5}$$

$$L\, i(x,t) = p \int_0^\infty e^{-pt} i(x,t)dt = I(x,p). \tag{6}$$

If it is assumed that the initial temperature $e(x,0)$ and the initial heat flux $i(x,0)$ are both zero at $t = 0$, the transforms of Eqs. 1 and 2 are

$$-\frac{dE}{dx} = RI \tag{7}$$

and

$$-\frac{dI}{dx} = CpE. \tag{8}$$

The transform of Eq. 3 is

$$\frac{d^2E}{dx^2} = RCpE. \tag{9}$$

Let the p-multiplied Laplace transforms of the various quantities involved be represented by capital letters so that $L\,e_k(t) = E_k(p)$ and $L\,i_k(t) = I_k(p)$.

The solutions of the differential Eqs. 7 and 8 subject to the following boundary conditions:

$$\left.\begin{aligned} E(p)_{x=0} &= E_1(p), & I(p)_{x=0} &= I_1(p) \\ E(p)_{x=S} &= E_2(p) & I(p)_{x=S} &= I_2(p) \end{aligned}\right\} \tag{10}$$

are easily computed. The results may be written in the following matrix form:

$$\begin{bmatrix} E_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \cosh(\theta) & Z_0 \sinh(\theta) \\ \dfrac{\sinh(\theta)}{Z_0} & \cosh(\theta) \end{bmatrix} \cdot \begin{bmatrix} E_2 \\ I_2 \end{bmatrix} \tag{11}$$

where

$$\theta = Sa \tag{12}$$

$$a = \sqrt{RCp} = \text{operational propagation constant} \tag{13}$$

$$Z_0 = \sqrt{\frac{R}{Cp}} = \text{operational characteristic impedance.} \tag{14}$$

The quantity a is the operational propagation constant, and Z_0 is the operational characteristic impedance of the quadripole of Fig. 2.

The square matrix

$$[T(\theta)] = \begin{bmatrix} \cosh(\theta) & Z_0 \sinh(\theta) \\ \dfrac{\sinh(\theta)}{Z_0} & \cosh(\theta) \end{bmatrix} = \begin{bmatrix} A & B \\ C & A \end{bmatrix} \tag{15}$$

is the operational transmission matrix of the quadripole of Fig. 2. The operational transmission matrix is very useful in the solution of linear heat-flow problems. If the specific heat of the slab of Fig. 1 is ***negligible*** so that the slab possesses only a *total* thermal resistance of R_T, the operational transmission matrix (15) reduces to

$$[T(\theta)] = \begin{bmatrix} 1 & R_T \\ 0 & 1 \end{bmatrix} \text{(negligible } C\text{).} \tag{16}$$

COMPOSITE SLABS

Slabs in Series

The transmission matrix is particularly well-adapted for the study of heat-conduction problems in composite solids. For example, consider a composite wall composed of n slabs of thicknesses $S_1, S_2, \cdots S_n$, thermal resistances per unit length of $R_1, \cdots R_n$, and heat capacitances $C_1 \cdots C_n$. It is evident that since the temperatures at the interfaces of adjacent slabs are equal and the heat flux that emerges from one slab enters the adjacent slab, therefore the over-all operational transmission matrix of the entire wall is given by the products of the individual transmission matrices of the separate slabs in the form

$$[T(\theta)] = [T(\theta_1)]\,[T(\theta_2)] \cdots [T(\theta_n)] \tag{17}$$

where $[T(\theta)]$ is the over-all transmission matrix of the entire wall, and $[T(\theta_k)]$ is the transmission matrix of the kth slab given by

$$[T(\theta_k)] = \begin{bmatrix} \cosh(\theta_k) & Z_{0K}\sinh(\theta_k) \\ \dfrac{\sinh(\theta_k)}{Z_{0K}} & \cosh(\theta_k) \end{bmatrix} \tag{18}$$

and

$$\theta_k = S_k\sqrt{R_kC_kp}, \qquad Z_{0K} = \sqrt{R_k/C_kp}. \tag{19}$$

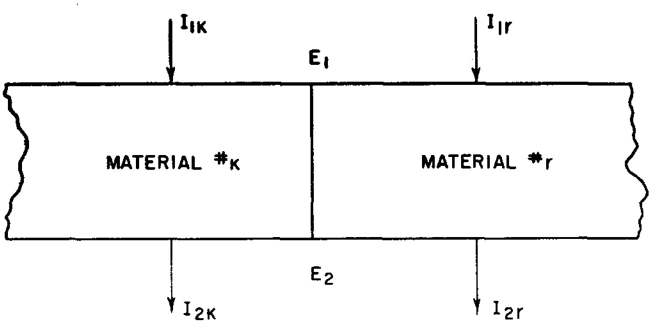

FIG. 3. Slabs in parallel.

The over-all transmission matrix is thus seen to be obtained by a process of matrix multiplication. It enables one to write the relations between the temperatures and heat fluxes on both sides of the composite wall in a most direct manner. The process of matrix multiplication replaces the laborious evaluation of arbitrary constants at the various interfaces as is done when the classical method is employed to solve this type of problem (8).

Slabs in Parallel

Consider the composite wall of Fig. 3. This wall consists of n-slabs of different materials in parallel. Matrix algebra facilitates the solution of problems involving composite walls of this type provided a certain simplifying approximation can be made.

Let the transforms of the temperature on the two sides of the wall be E_1 and E_2. Let I_{1k} be the transform of the heat flux entering the kth slab and I_{2k} be the transform of the heat flux emerging from the kth slab. If it can be assumed that the heat flux across boundaries between any two slabs of different materials can be *neglected* in comparison with the heat fluxes in these slabs that flows normal to the wall, the following relations will exist for each slab

$$\begin{bmatrix} E_1 \\ I_{1k} \end{bmatrix} = \begin{bmatrix} \cosh(\theta_k) & Z_{0K}\sinh(\theta_k) \\ \dfrac{\sinh(\theta_k)}{Z_{0K}} & \cosh(\theta_k) \end{bmatrix} \cdot \begin{bmatrix} E_2 \\ I_{2k} \end{bmatrix} = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} \begin{bmatrix} E_2 \\ I_{2k} \end{bmatrix}. \tag{20}$$

The above relation may also be written in the following form:

$$\begin{bmatrix} I_{1k} \\ I_{2k} \end{bmatrix} \begin{bmatrix} A_k/B_k & -1/B_k \\ 1/B_k & -A_k/B_k \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}. \tag{21}$$

If D_k is the normal area of the kth slab to the direction of heat flow and if ϕ_{1k} and ϕ_{2k} are the transforms of the total rates of heat per second entering and leaving the kth slab, we have

$$\begin{bmatrix} \phi_{1k} \\ \phi_{2k} \end{bmatrix} = D_k \begin{bmatrix} A_k/B_k & -1/B_k \\ 1/B_k & -A_k/B_k \end{bmatrix} \cdot \begin{bmatrix} E_1 \\ E_1 \end{bmatrix} = \begin{bmatrix} U_k & -V_k \\ V_k & -U_k \end{bmatrix} \cdot \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}. \tag{22}$$

If ϕ_1 and ϕ_2 are the transforms of the total rates of heat per second entering and leaving the composite wall, we have

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \sum_{k=1}^{n} \begin{bmatrix} \phi_{1k} \\ \phi_{2k} \end{bmatrix} = \sum_{k=1}^{n} \begin{bmatrix} U_k & -V_k \\ V_k & -U_k \end{bmatrix} \cdot \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}. \tag{23}$$

If the underlying approximations under which Eq. 23 is valid can be made, it may be used to advantage in studying the heat flow in buildings when they are undergoing periodic temperature variations. The periodic heating may be assumed to be produced by periodic internal heating or by a periodically varying external temperature. The transmission matrices of the various parts of the building such as walls, doors, windows, air spaces, etc., may be obtained and the over-all matrix

$$[W] = \sum_{k=1}^{n} \begin{bmatrix} U_k & -V_k \\ V_k & -U_k \end{bmatrix} \tag{24}$$

calculated. The transforms of the rates of heat entering and leaving the entire structure are then given by

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = [W] \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}. \tag{25}$$

TRANSIENT HEAT-CONDUCTION PROBLEMS

When the operational transmission matrix (15) is known, problems involving transient heat conduction may be solved by the use of the Laplace transformation (9) with an economy of algebraic effort. The computation of the inverse transforms involved may present formidable difficulties, but the matrix formulation of the problem reduces the algebraic labor involved to a minimum. As a simple example of the use of the transmission matrix in heat-conduction problems, consider the following problems.

Heat Extraction from a Refrigerator

A household refrigerator consists of a rectangular box, the walls being made of thermal insulating material of thickness S. Originally the inner and outer temperatures and the walls of the refrigerator are at a constant temperature which can be taken as a zero reference temperature. At $t = 0$, the temperature of the inner surface of the refrigerator is lowered to $-e_0$, and it is desired to maintain it at this value. It is required to determine the rate of extraction of heat by the refrigerator mechanism on the assumption that the problem can be treated as if the refrigerator can be considered as an infinite slab one side being at temperature zero and the other side at temperature $-e_0$.

In order to solve this problem, write Eq. 11 in the form

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} A/B & -1/B \\ 1/B & -A/B \end{bmatrix} \cdot \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \tag{26}$$

where

$$A = \cosh(\theta), \qquad B = Z_0 \sinh(\theta). \tag{27}$$

Since the wall $x = 0$ is at zero temperature, $E_1 = 0$, and since $e_2 = -e_0$, $E_2 = -e_0$. If these values of E_1 and E_2 are inserted in Eq. 26 the result is

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} A/B & -1/B \\ 1/B & -A/B \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -e_0 \end{bmatrix}. \tag{28}$$

Therefore by matrix multiplication the following results are obtained:

$$I_1 = e_0/B = e_0/Z_0 \sinh(\theta) \tag{29}$$

$$I_2 = e_0 A/B = e_0 \cosh(\theta)/Z_0 \sinh(\theta). \tag{30}$$

The inverse transforms of Eqs. 29 and 30 give the heat fluxes passing the interior and exterior walls of the refrigerator. The heat flux entering the refrigerator through its inner wall is given by

$$i_2 = L^{-1}I_2(p) = L^{-1}\frac{e_0 \cosh(S\sqrt{RCp})}{\sqrt{\frac{R}{Cp}}\cdot \sinh(S\sqrt{RCp})}. \tag{31}$$

This inverse transform may be calculated by the theory of residues in the usual manner (4). The result is

$$i_2 = \frac{e_0}{RS}\left[1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi^2 t/S^2CR}\right]. \tag{32}$$

This is the rate of extraction of heat per unit area at the inner surface of the refrigerator. The series (32) is uniformly convergent for $t > 0$.

Temperature Distribution of an Insulated Slab

In order to give another illustration of the use of the operational transmission matrix, consider a slab of conducting material that is perfectly insulated at the face $x = S$ so that no heat can flow through this face. Let it be assumed that initially the entire slab is at zero temperature and that at $t = 0$ the face $x = 0$ is suddenly elevated to a temperature e_0. Let it be required to determine the subsequent temperature of the face $x = S$.

In this case, Eq. 11 reduces to

$$\begin{bmatrix} e_0 \\ I_1 \end{bmatrix} = \begin{bmatrix} \cosh(\theta) & Z_0 \sinh(\theta) \\ \dfrac{\sinh(\theta)}{Z_0} & \cosh(\theta) \end{bmatrix} \cdot \begin{bmatrix} E_2 \\ 0 \end{bmatrix}. \tag{33}$$

By matrix multiplication, we obtain

$$e_0 = E_2 \cosh(\theta), \qquad I_1 = E_2 \sinh(\theta)/Z_0. \tag{34}$$

These equations may be solved for E_2 and I_1 and the results are

$$E_2 = e_0/\cosh(\theta) \tag{35}$$

and

$$I_1 = e_0 \tanh(\theta)/Z_0. \tag{36}$$

In order to determine the temperature of the face $x = S$, it is necessary to compute the inverse transform of E_2. To do this, E_2 may be written in the following form

$$E_2 = 2E_0/(e^{\theta} + e^{-\theta}) = 2E_0(e^{-\theta} - e^{-3\theta} + e^{-5\theta} - e^{-7\theta} + \cdots). \quad (37)$$

Let

$$a = S\sqrt{RC}, \qquad \theta = a\sqrt{p}. \quad (38)$$

With this notation, Eq. 37 may be written in the following form

$$E_2 = 2e_0(e^{-a\sqrt{p}} - e^{-3a\sqrt{p}} + e^{-5a\sqrt{p}} - e^{-7a\sqrt{p}} + \cdots). \quad (39)$$

The series (39) is uniformly convergent because $Re p > 0$ as is shown in works on the Laplace transform theory (9).

The following result may be obtained from a table of Laplace transforms (4)

$$L^{-1}e^{-k\sqrt{p}} = erfc(k/2\sqrt{t}), \qquad k > 0 \quad (40)$$

where the function $erfc(x)$ is the complementary error function defined by the expression

$$erfc(x) = 1 - \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du. \quad (41)$$

This function is well known and extensive tables of it exist (9). The inverse transform of (39) is, therefore,

$$e_2 = 2e_0[erfc(a/2\sqrt{t}) - erfc(3a/2\sqrt{t}) + erfc(5a/2\sqrt{t}) - erfc(7a/2\sqrt{t}) + \cdots]. \quad (42)$$

This is the temperature of the insulated face of the slab. The heat flux entering the slab may be obtained by computing the inverse transform of I_1 as given by Eq. 36 in a similar manner.

PROBLEMS INVOLVING OSCILLATORY TEMPERATURE VARIATIONS

Problems involving temperature variations which are periodic functions of the time are of great practical importance in the study of the fluctuations of the temperature of the earth's crust due to the periodic heating of the sun, in various experimental arrangements for the determination of the diffusivity, in the calculation of the periodic temperature variation of the cylinder walls of steam and internal-combustion engines and in the theory of automatic temperature-control systems.

The operational transmission matrix can be adapted to facilitate the solution of heat-conduction problems involving harmonic temperature variations. In problems of this type the temperature and heat flux oscillate with respect to time and contain a factor of sin $(wt + \phi)$

where ϕ is a phase angle and $w = 2\pi \times$ the frequency of the temperature oscillation. To solve problems of this type it is necessary to modify the operational propagation constant a and the operational characteristic impedance Z_0 as given by Eqs. 13 and 14 to introduce the complex propagation constant $\bar{a}$ and the complex characteristic impedance $\bar{Z}_0$ defined by the equations

$$\bar{a} = (a)_{p=jw} = \sqrt{jwRC}, \qquad j = \sqrt{-1} \tag{43}$$

and

$$\bar{Z}_0 = (Z_0)_{p=jw} = \sqrt{R/jwc}. \tag{44}$$

If we let

$$\bar{\theta} = s\bar{a} = s\sqrt{jwRC} \tag{45}$$

then, instead of Eq. 11 we may write

$$\begin{bmatrix} \bar{E}_1 \\ \bar{I}_1 \end{bmatrix} = \begin{bmatrix} \cosh(\bar{\theta}) & \bar{Z}_0 \sinh(\bar{\theta}) \\ \dfrac{\sinh(\bar{\theta})}{\bar{Z}_0} & \cosh(\bar{\theta}) \end{bmatrix} \cdot \begin{bmatrix} \bar{E}_2 \\ \bar{I}_2 \end{bmatrix}. \tag{46}$$

In Eq. 46, $\bar{E}_1$ and $\bar{E}_2$ are the complex temperatures and $\bar{I}_1$ and $\bar{I}_2$ are the complex heat fluxes. These quantities correspond exactly to the complex potentials and currents of an electric cable in the steady-state periodic alternating-current case. The square matrix of Eq. 46 in the complex transmission matrix of the slab.

To illustrate the use of the complex transmission matrix in the solution of a period heat-conduction problem, consider the following representative problem:

Let the temperature variation of the face $x = 0$ of the slab of Fig. 1 be maintained at a temperature of $e_1 = E_m \sin(wt)$ and let the temperature of the face $x = S$ be maintained at zero. When a steady periodic state has been reached, let it be required to determine the heat flux that passes through the face $x = S$.

In order to solve this problem, we follow the usual procedure of alternating current theory (10) and write

$$e_1(t) = E_m \sin(wt) = \text{Im}\,[E_m e^{jwt}] \tag{47}$$

where the symbol "Im" means "the imaginary part of." Equation 46 in this case reduces to

$$\begin{bmatrix} E_m \\ \bar{I}_1 \end{bmatrix} = \begin{bmatrix} \cosh(\bar{\theta}) & Z_0 \sinh(\bar{\theta}) \\ \dfrac{\sinh(\bar{\theta})}{\bar{Z}_0} & \cosh(\theta) \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \bar{I}_2 \end{bmatrix}. \tag{48}$$

In Eq. 48 E_m is used as a reference temperature and all the phases are determined with respect to the temperature $e_1 = E_m \sin(wt)$.

By matrix multiplication the following equations are obtained

$$E_m = \bar{Z}_0 \sinh(\theta)\bar{I}_2 \tag{49}$$

and

$$\bar{I}_1 = \bar{I}_2 \cosh(\bar{\theta}). \tag{50}$$

Hence

$$\bar{I}_2 = E_m/\bar{Z}_0 \sinh(\bar{\theta}) \tag{51}$$

$$I_1 = E_m/\bar{Z}_0 \tanh(\bar{\theta}). \tag{52}$$

The instantaneous heat flux that passes through the wall $x = S$ is given by

$$i_2 = \operatorname{Im}[\bar{I}_2 e^{iwt}]. \tag{53}$$

Since $\bar{Z}_0$ and $\sinh(\bar{\theta})$ are complex numbers, they may be written in the following polar form

$$\bar{Z}_0 = |Z_0|e^{i\phi_1}, \qquad \sinh(\bar{\theta}) = |\sinh(\bar{\theta})|e^{i\phi_2}. \tag{54}$$

Hence

$$\left.\begin{aligned} i_2 &= \operatorname{Im}[E_m e^{i(wt-\phi_1-\phi_2)}/|\bar{Z}_0|\cdot|\sinh(\bar{\theta})|] \\ &= E_m \sin(wt - \phi_1 - \phi_2)/|Z_0|\cdot|\sinh(\bar{\theta})| \end{aligned}\right\}. \tag{55}$$

This expression gives the amplitude and phase of the heat flux entering the wall $x = S$ with respect to the temperature variation $E_m \sin(wt)$ of the wall $x = 0$. The flux is thus seen to have a phase lag of $(\phi_1 + \phi_2)$. It can be seen from Eq. 53 that the *modulus* of $\bar{I}_2$ gives the amplitude of the heat flux and the *argument* of $\bar{I}_2$ determines the phase of the heat flux.

By the use of complex transmission matrices, problems involving periodic heat conduction in composite slabs may be solved readily by the theory given in the third section of the paper. The solution of problems of this type involves manipulation of complex numbers and the matrix notation greatly facilitates the algebraic manipulations involved.

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