

# Differentiation I

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The goal is to reprove important theorems of Analysis I in a consistent and intuitive way.

## 8.1 The Derivative

We know the definition of the derivative from Analysis I. For  $A \subseteq \mathbb{R}$ ,  $f : A \rightarrow \mathbb{R}$ , differentiable at  $x \in A$ :

$$f'(x) \text{ is the derivative of } f \text{ at } x \iff \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = f'(x) \quad (1)$$

We now want to define the derivative using hyperreals.

### Theorem 8.1.1

If  $f$  is differentiable at  $x \in A \subseteq \mathbb{R}$ :

$$L \text{ is the derivative of } f \text{ at } x \iff \forall \varepsilon \simeq 0, \varepsilon \neq 0 : \frac{f(x+\varepsilon) - f(x)}{\varepsilon} \simeq L \quad (2)$$

*Proof.* From Chapter 7.3 we know: If  $c, L \in \mathbb{R}$  and  $f$  is defined on  $A \subseteq \mathbb{R}$ :

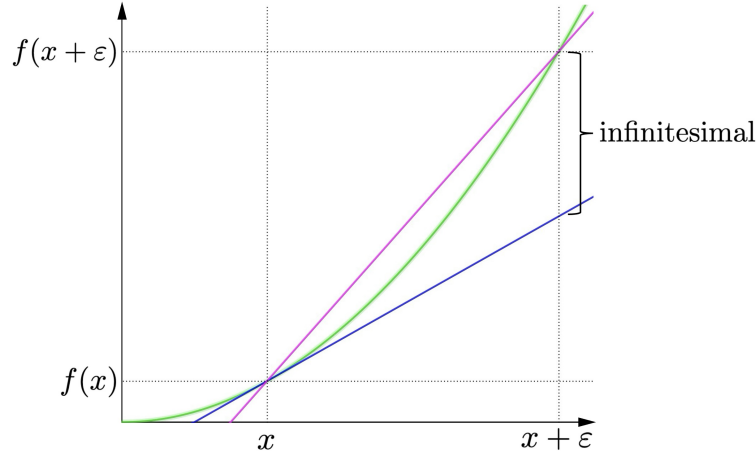
$$\lim_{x \rightarrow c} f(x) = L \iff f(x) \simeq L, \forall x \in {}^*A \text{ with } x \simeq c, x \neq c \quad (3)$$

Define a function  $g(\varepsilon) := \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$  for any fixed  $x \in A \subseteq \mathbb{R}$ .

By Chapter 7.3:

$$g'(\varepsilon) := \lim_{\varepsilon \rightarrow 0} g(\varepsilon) = L \iff g(\varepsilon) \simeq L, \forall \varepsilon \in {}^*A \text{ with } \varepsilon \simeq 0, \varepsilon \neq 0 \quad (4)$$

□



Thus, when  $f$  has a derivative at  $x$ :

$$f'(x) = \text{sh} \left( \frac{f(x + \varepsilon) - f(x)}{\varepsilon} \right), \forall \varepsilon \simeq 0, \varepsilon \neq 0 \quad (5)$$

In standard Analysis, we also have a one-sided limit:

$$\lim_{h \rightarrow 0^+} \frac{f(x + h) - f(x)}{h} = L \iff \frac{f(x + \varepsilon) - f(x)}{\varepsilon} \simeq L, \forall \varepsilon \simeq 0, \varepsilon > 0 \quad (6)$$

$$\lim_{h \rightarrow 0^-} \frac{f(x + h) - f(x)}{h} = L \iff \frac{f(x + \varepsilon) - f(x)}{\varepsilon} \simeq L, \forall \varepsilon \simeq 0, \varepsilon < 0 \quad (7)$$

This is consistent with the property:

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} = L &\iff \lim_{h \rightarrow 0^+} \frac{f(x + h) - f(x)}{h} = L \wedge \lim_{h \rightarrow 0^-} \frac{f(x + h) - f(x)}{h} = L \\ &\iff \frac{f(x + \varepsilon) - f(x)}{\varepsilon} \simeq L, \forall \varepsilon \simeq 0, \varepsilon \neq 0 \end{aligned} \quad (8)$$

### Exercise 8.1.2

Let's show that the derivative of  $\sin(x)$  is  $\cos(x)$  using the definition:

Let  $x^* = x + \varepsilon, x \in \mathbb{R}, \varepsilon \simeq 0, \varepsilon \neq 0$ .

Using addition theorems, we can expand the difference quotient:

$$\frac{\sin(x + \varepsilon) - \sin(x)}{\varepsilon} = \frac{\sin x \cos \varepsilon + \cos x \sin \varepsilon - \sin x}{\varepsilon} \quad (9)$$

We use  $\sin \varepsilon \simeq 0 \iff \sin \varepsilon \simeq \varepsilon$  and  $\cos \varepsilon \simeq 1$ .

Using the transfer property, we can derive these properties from the Taylor expansion:

$$\begin{aligned} \sin(\varepsilon) &= \varepsilon - \frac{\varepsilon^3}{3!} + \frac{\varepsilon^5}{5!} - \dots, \\ \cos(\varepsilon) &= 1 - \frac{\varepsilon^2}{2!} + \frac{\varepsilon^4}{4!} - \dots \end{aligned} \quad (10)$$

Since  $\varepsilon^n \ll \varepsilon$  if  $n > 1$ ,  $\sin \varepsilon \simeq \varepsilon$  and  $\cos \varepsilon \simeq 1$ . Hence:

$$\frac{\sin x \cos \varepsilon + \cos x \sin \varepsilon - \sin x}{\varepsilon} \simeq \frac{\sin x \cdot 1 + \cos x \cdot \varepsilon - \sin x}{\varepsilon} = \frac{\cos x \cdot \varepsilon}{\varepsilon} = \cos(x) \quad (11)$$

$$\Rightarrow \operatorname{sh} \left( \frac{\sin(x + \varepsilon) - \sin(x)}{\varepsilon} \right) = \cos(x) \quad (12)$$

□

## 8.2 Increments and Differentials

Let  $x \in \mathbb{R}$ ,  $\Delta x \simeq 0$ ,  $\Delta x \neq 0$ .

We define  $\Delta f := f(x + \Delta x) - f(x)$ .

$\Delta f$  can also be written as  $\Delta f(x, \Delta x)$ , as it depends both on  $x$  and  $\Delta x$ .

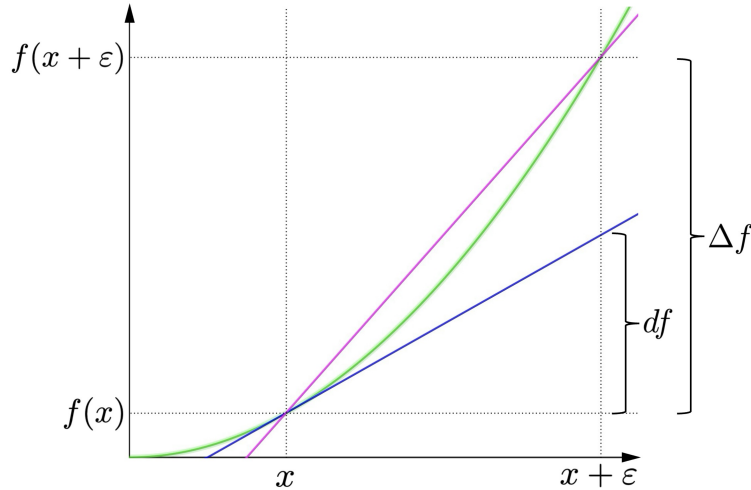
By Theorem 8.1.1, if  $f$  is differentiable at  $x$ :

$$\frac{\Delta f}{\Delta x} \simeq f'(x), \forall \Delta x \quad (13)$$

We define  $df := f'(x)dx$ ,  $\forall dx$  ( $dx = \Delta x$ ).

This can be rewritten into the well-known notation:

$$\frac{df}{dx} = f'(x) \quad (14)$$



**Theorem 8.2.1**

If  $f$  is differentiable at  $x \in \mathbb{R}$ , then  $f$  is continuous at  $x$ .

*Proof.* Let  $\Delta x \simeq 0$ ,  $\Delta x \neq 0$ .

Then  $\Delta f = \frac{\Delta f}{\Delta x} \Delta x$  is infinitesimal.

$$\implies f(x + \Delta x) \simeq f(x) \quad (15)$$

By Theorem 7.1.1:

$$f \text{ is continuous at } c \in \mathbb{R} \iff \forall \bar{x} \in {}^*\mathbb{R} \text{ with } \bar{x} \simeq c : f(\bar{x}) \simeq f(c) \quad (16)$$

Here  $\bar{x} = x + \Delta x, c = x \implies f$  is continuous at  $x$ .  $\square$

**Theorem 8.2.2**

If  $f$  is differentiable at  $x \in \mathbb{R}$  and  $\Delta x \simeq 0$ ,  $\Delta x \neq 0$ , then  $\Delta f$  and  $df$  are infinitesimal and there exists an infinitesimal  $\varepsilon(x, \Delta x)$  such that:

$$\Delta f = f'(x)\Delta x + \varepsilon\Delta x = df + \varepsilon\Delta x \quad (17)$$

$$\iff f(x + \Delta x) - f(x) = df + \varepsilon\Delta x \quad (18)$$

$$\iff f(x + \Delta x) = f(x) + f'(x)\Delta x + \varepsilon\Delta x \quad (19)$$

*Proof.* We know  $\Delta f$  and  $df$  are infinitesimal by definition. By Theorem 8.1.1:

$$df = f'(x)\Delta x \simeq \frac{\Delta f}{\Delta x} \Delta x = \Delta f \implies df \simeq \Delta f \quad (20)$$

Moreover, their difference is infinitesimal compared to  $\Delta x$ :

$$\frac{\Delta f - df}{\Delta x} = \frac{\frac{\Delta f}{\Delta x} \Delta x - f'(x)\Delta x}{\Delta x} = \frac{\Delta x \left( \frac{\Delta f}{\Delta x} - f'(x) \right)}{\Delta x} = \frac{\Delta f}{\Delta x} - f'(x) =: \varepsilon \quad (21)$$

$\varepsilon$  is infinitesimal since  $\frac{\Delta f}{\Delta x} \simeq f'(x)$ . Therefore,  $\Delta f - df$  is infinitesimal compared to  $\Delta x$ . With this  $\varepsilon$ , we can rewrite  $\Delta f$ :

$$\Delta f = \frac{\Delta f}{\Delta x} \Delta x = \frac{\Delta f}{\Delta x} \Delta x + f'(x)\Delta x - f'(x)\Delta x = f'(x)\Delta x + \Delta x \left( \frac{\Delta f}{\Delta x} - f'(x) \right) \quad (22)$$

$$\iff \Delta f = df + \varepsilon\Delta x \quad (23)$$

$$\iff f(x + \Delta x) = f(x) + f'(x)\Delta x + \varepsilon\Delta x \quad (24)$$

Additionally, we can define a linear function:

$$l(\Delta x) := f(x) + f'(x)\Delta x \quad (25)$$

which differs from  $f(x + \Delta x)$  only by  $\varepsilon\Delta x$ , which is infinitesimal compared to  $\Delta x$ .  $\square$

## 8.3 Rules for Derivatives

If  $f, g$  are differentiable at  $x \in \mathbb{R}$ , then so are  $f + g$  and  $f \cdot g$ , and:

1.  $(f + g)'(x) = f'(x) + g'(x)$
2.  $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$

*Proof.* Let  $\Delta x \simeq 0$ ,  $\Delta x \neq 0$ . Since  $f, g$  are differentiable at  $x$ ,  $f(x + \Delta x)$  and  $g(x + \Delta x)$  are defined.

1.  $(f + g)(x + \Delta x) = f(x + \Delta x) + g(x + \Delta x)$ . Then,

$$\frac{\Delta(f + g)}{\Delta x} = \frac{f(x + \Delta x) + g(x + \Delta x) - f(x) - g(x)}{\Delta x} \quad (26)$$

$$= \frac{f(x + \Delta x) - f(x)}{\Delta x} + \frac{g(x + \Delta x) - g(x)}{\Delta x} \simeq f'(x) + g'(x) \quad (27)$$

$$\implies (f + g)'(x) = f'(x) + g'(x) \quad (28)$$

2.  $(fg)(x + \Delta x) = f(x + \Delta x)g(x + \Delta x)$ .

$$\Delta(fg) = f(x + \Delta x)g(x + \Delta x) - f(x)g(x) \quad (29)$$

Making use of the fact

$$\Delta f = f(x + \Delta x) - f(x) \iff f(x + \Delta x) = f(x) + \Delta f \quad (30)$$

$$\Delta(fg) = (f(x) + \Delta f)(g(x) + \Delta g) - f(x)g(x) \quad (31)$$

$$= f(x)g(x) + f(x)\Delta g + \Delta f g(x) + \Delta f \Delta g - f(x)g(x) \quad (32)$$

Dividing by  $\Delta x$  yields:

$$\frac{\Delta(fg)}{\Delta x} = \frac{\Delta g}{\Delta x} f(x) + \frac{\Delta f}{\Delta x} g(x) + \Delta f \frac{\Delta g}{\Delta x} \quad (33)$$

Since  $f'(x) \simeq \frac{\Delta f}{\Delta x}$ ,  $g'(x) \simeq \frac{\Delta g}{\Delta x}$ , and  $\Delta f \frac{\Delta g}{\Delta x} \simeq 0$ ,

$$\iff \frac{\Delta(fg)}{\Delta x} \simeq g'(x)f(x) + f'(x)g(x) + 0 \quad (34)$$

$$\implies (f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x) \quad (35)$$

□

## 8.4 Chain Rule

If  $f$  is differentiable at  $x \in \mathbb{R}$  and  $g$  is differentiable at  $f(x)$ , then  $g \circ f$  is differentiable at  $x$  with

$$(g \circ f)'(x) = g'(f(x))f'(x). \quad (36)$$

*Proof.* Let  $\Delta x \simeq 0, \Delta x \neq 0$ .

Since  $f$  is differentiable at  $x$  and  $g$  is differentiable at  $f(x)$ ,  $f(x + \Delta x)$  and  $g(f(x) + \Delta f)$  is defined.

By definition of the increment:

$$\Delta f = f(x + \Delta x) - f(x) \quad (37)$$

$$\Delta(g \circ f) = g(f(x + \Delta x)) - g(f(x)) \quad (38)$$

We use  $\Delta f = f(x + \Delta x) - f(x) \iff f(x + \Delta x) = f(x) + \Delta f$

$$\iff \Delta(g \circ f) = g(f(x) + \Delta f) - g(f(x)) \quad (39)$$

Thus  $\Delta(g \circ f)$  is an increment of  $g$  at  $f(x)$

$$\iff \Delta(g \circ f) = \Delta g(f(x), \Delta f) \quad (40)$$

Now we apply the equation (23):

$$\Delta g = dg + \varepsilon \Delta x, \text{ here } \Delta x = \Delta f \quad (41)$$

Thus  $\Delta g(f(x), \Delta f) = g'(f(x))\Delta f + \varepsilon \Delta f$

Dividing by  $\Delta x$  yields:

$$\frac{\Delta(g \circ f)}{\Delta x} = g'(f(x))\frac{\Delta f}{\Delta x} + \varepsilon \frac{\Delta f}{\Delta x} \quad (42)$$

We use  $\frac{\Delta f}{\Delta x} \simeq f'(x), \frac{\Delta f}{\Delta x}$  limited

$$\iff \frac{\Delta(g \circ f)}{\Delta x} \simeq g'(f(x))f'(x) \quad (43)$$

$$\implies (g \circ f)'(x) = g'(f(x))f'(x) \quad (44)$$

□

## 8.5 Critical Point Theorem

If  $f(x)$  is a maximum or minimum for some  $x \in (a, b) \subseteq \mathbb{R}$  and if  $f$  is differentiable at  $x$ , then  $f'(x) = 0$ .

*Proof.* Let  $x$  be a maximum.  $\Delta x \simeq 0$ ,  $\Delta x \neq 0$ . By the transfer principle:

$$f(x + \Delta x) \leq f(x), \forall \Delta x \quad (45)$$

Let  $\varepsilon > 0$ ,  $\delta < 0$  be infinitesimals. Then:

$$f(x + \varepsilon) - f(x) \leq 0 \quad (46)$$

$$f(x + \delta) - f(x) \leq 0 \quad (47)$$

Dividing by  $\varepsilon$  and  $\delta$  respectively yields:

$$f'(x) \simeq \frac{f(x + \varepsilon) - f(x)}{\varepsilon} \leq \frac{0}{\varepsilon} = 0 \quad (48)$$

$$0 = \frac{0}{\delta} \leq \frac{f(x + \delta) - f(x)}{\delta} \simeq f'(x) \quad (49)$$

Combining these, we get:

$$f'(x) = \text{sh} \left( \frac{f(x + \varepsilon) - f(x)}{\varepsilon} \right) \leq 0 \leq \text{sh} \left( \frac{f(x + \delta) - f(x)}{\delta} \right) = f'(x) \quad (50)$$

Therefore,

$$f'(x) = 0. \quad (51)$$

A similar argument holds for a minimum at  $x$ . □

Using the critical point and extreme value theorems (Section 7.6), the following results can be derived about a function  $f$  that is continuous on  $[a, b] \subseteq \mathbb{R}$  and differentiable on  $(a, b)$ . The proofs do not require further reasoning about infinitesimals or limits.

- Rolle's Theorem
- Mean Value Theorem
- If  $f'$  is zero/positive/negative on  $(a, b)$ , then  $f$  is constant/increasing/decreasing on  $[a, b]$