Differentiation I

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The goal is to reprove important theorems of Analysis I in a consistent and intuitive way.

8.1 The Derivative

We know the definition of the derivative from Analysis I. For $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, differentiable at $x \in A$:

$$f'(x) \text{ is the derivative of } f \text{ at } x \iff \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x) \tag{1}$$

We now want to define the derivative using hyperreals.

Theorem 8.1.1

If f is differentiable at $x \in A \subseteq \mathbb{R}$:

$$L \text{ is the derivative of } f \text{ at } x \iff \forall \varepsilon \simeq 0,\, \varepsilon \neq 0 : \frac{f(x+\varepsilon) - f(x)}{\varepsilon} \simeq L \tag{2}$$

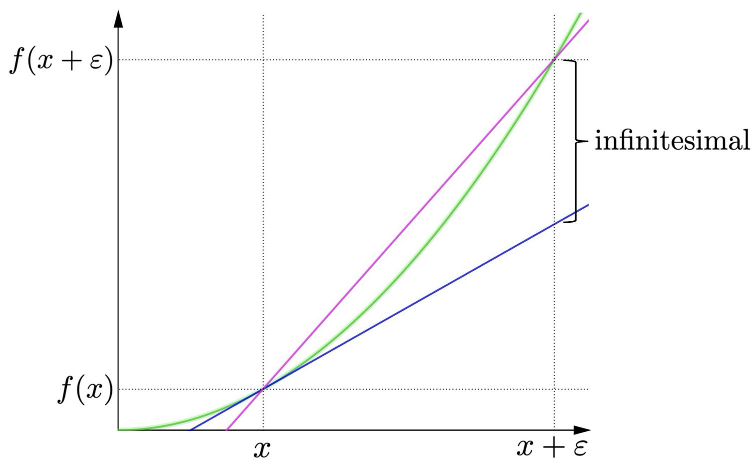
Proof. From Chapter 7.3 we know: If $c, L \in \mathbb{R}$ and f is defined on $A \subseteq \mathbb{R}$:

$$\lim_{x \to c} f(x) = L \iff f(x) \simeq L,\, \forall x \in {}^*A \text{ with } x \simeq c,\, x \neq c \tag{3}$$

Define a function $g(\varepsilon) := \frac{f(x+\varepsilon)-f(x)}{\varepsilon}$ for any fixed $x \in A \subseteq \mathbb{R}$.
By Chapter 7.3:

$$g'(\varepsilon) := \lim_{\varepsilon \to 0} g(\varepsilon) = L \iff g(\varepsilon) \simeq L,\, \forall \varepsilon \in {}^*A \text{ with } \varepsilon \simeq 0,\, \varepsilon \neq 0 \tag{4}$$

$\square$



Thus, when f has a derivative at x:

$$f'(x) = \mathrm{sh}\left(\frac{f(x+\varepsilon)-f(x)}{\varepsilon}\right),\ \forall \varepsilon \simeq 0,\ \varepsilon \neq 0 \tag{5}$$

In standard Analysis, we also have a one-sided limit:

$$\lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} = L \iff \frac{f(x+\varepsilon)-f(x)}{\varepsilon} \simeq L,\ \forall \varepsilon \simeq 0,\ \varepsilon > 0 \tag{6}$$

$$\lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} = L \iff \frac{f(x+\varepsilon)-f(x)}{\varepsilon} \simeq L,\ \forall \varepsilon \simeq 0,\ \varepsilon < 0 \tag{7}$$

This is consistent with the property:

$$\begin{aligned}\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = L &\iff \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} = L \wedge \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} = L \\ &\iff \frac{f(x+\varepsilon)-f(x)}{\varepsilon} \simeq L,\ \forall \varepsilon \simeq 0,\ \varepsilon \neq 0\end{aligned} \tag{8}$$

Exercise 8.1.2

Let's show that the derivative of $\sin(x)$ is $\cos(x)$ using the definition:

Let $x^* = x+\varepsilon, x \in \mathbb{R}$, $\varepsilon \simeq 0$, $\varepsilon \neq 0$.
Using addition theorems, we can expand the difference quotient:

$$\frac{\sin(x+\varepsilon)-\sin(x)}{\varepsilon} = \frac{\sin x \cos\varepsilon + \cos x \sin\varepsilon - \sin x}{\varepsilon} \tag{9}$$

We use $\sin\varepsilon \simeq 0 \iff \sin\varepsilon \simeq \varepsilon$ and $\cos\varepsilon \simeq 1$.
Using the transfer property, we can derive these properties from the Taylor expansion:

$$\begin{aligned}\sin(\varepsilon) &= \varepsilon - \frac{\varepsilon^3}{3!} + \frac{\varepsilon^5}{5!} - \cdots, \\ \cos(\varepsilon) &= 1 - \frac{\varepsilon^2}{2!} + \frac{\varepsilon^4}{4!} - \cdots\end{aligned} \tag{10}$$

Since $\varepsilon^n \ll \varepsilon$ if $n > 1$, $\sin\varepsilon \simeq \varepsilon$ and $\cos\varepsilon \simeq 1$. Hence:

$$\frac{\sin x\cos\varepsilon + \cos x\sin\varepsilon - \sin x}{\varepsilon} \simeq \frac{\sin x \cdot 1 + \cos x \cdot \varepsilon - \sin x}{\varepsilon} = \frac{\cos x \cdot \varepsilon}{\varepsilon} = \cos(x) \tag{11}$$

$$\implies \operatorname{sh}\left(\frac{\sin(x+\varepsilon) - \sin(x)}{\varepsilon}\right) = \cos(x) \tag{12}$$

□

8.2 Increments and Differentials

Let $x \in \mathbb{R}$, $\Delta x \simeq 0$, $\Delta x \neq 0$.

We define $\Delta f := f(x + \Delta x) - f(x)$.
Δf can also be written as $\Delta f(x, \Delta x)$, as it depends both on x and Δx.

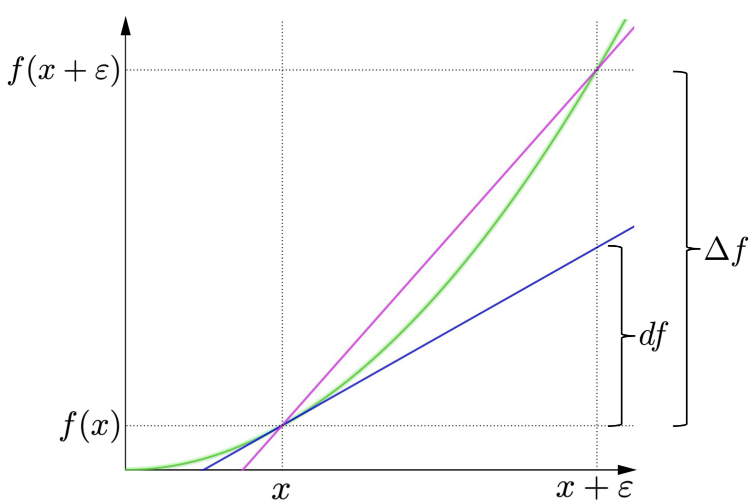
By Theorem 8.1.1, if f is differentiable at x:

$$\frac{\Delta f}{\Delta x} \simeq f'(x),\ \forall \Delta x \tag{13}$$

We define $df := f'(x)dx$, $\forall dx \quad (dx = \Delta x)$.

This can be rewritten into the well-known notation:

$$\frac{df}{dx} = f'(x) \tag{14}$$



Theorem 8.2.1

If f is differentiable at $x \in \mathbb{R}$, then f is continuous at x.

Proof. Let $\Delta x \simeq 0$, $\Delta x \neq 0$.

Then $\Delta f = \frac{\Delta f}{\Delta x}\Delta x$ is infinitesimal.

$$\implies f(x+\Delta x) \simeq f(x) \tag{15}$$

By Theorem 7.1.1:

$$f \text{ is continuous at } c \in \mathbb{R} \iff \forall \bar{x} \in {}^*\mathbb{R} \text{ with } \bar{x} \simeq c : f(\bar{x}) \simeq f(c) \tag{16}$$

Here $\bar{x} = x + \Delta x, c = x \implies f$ is continuous at x. □

Theorem 8.2.2

If f is differentiable at $x \in \mathbb{R}$ and $\Delta x \simeq 0$, $\Delta x \neq 0$, then Δf and df are infinitesimal and there exists an infinitesimal $\varepsilon(x, \Delta x)$ such that:

$$\Delta f = f'(x)\Delta x + \varepsilon\Delta x = df + \varepsilon\Delta x \tag{17}$$

$$\iff f(x+\Delta x) - f(x) = df + \varepsilon\Delta x \tag{18}$$

$$\iff f(x+\Delta x) = f(x) + f'(x)\Delta x + \varepsilon\Delta x \tag{19}$$

Proof. We know Δf and df are infinitesimal by definition. By Theorem 8.1.1:

$$df = f'(x)\Delta x \simeq \frac{\Delta f}{\Delta x}\Delta x = \Delta f \implies df \simeq \Delta f \tag{20}$$

Moreover, their difference is infinitesimal compared to Δx:

$$\frac{\Delta f - df}{\Delta x} = \frac{\frac{\Delta f}{\Delta x}\Delta x - f'(x)\Delta x}{\Delta x} = \frac{\Delta x\left(\frac{\Delta f}{\Delta x} - f'(x)\right)}{\Delta x} = \frac{\Delta f}{\Delta x} - f'(x) =: \varepsilon \tag{21}$$

ε is infinitesimal since $\frac{\Delta f}{\Delta x} \simeq f'(x)$. Therefore, $\Delta f - df$ is infinitesimal compared to Δx. With this ε, we can rewrite Δf:

$$\Delta f = \frac{\Delta f}{\Delta x}\Delta x = \frac{\Delta f}{\Delta x}\Delta x + f'(x)\Delta x - f'(x)\Delta x = f'(x)\Delta x + \Delta x\left(\frac{\Delta f}{\Delta x} - f'(x)\right) \tag{22}$$

$$\iff \Delta f = df + \varepsilon\Delta x \tag{23}$$

$$\iff f(x+\Delta x) = f(x) + f'(x)\Delta x + \varepsilon\Delta x \tag{24}$$

Additionally, we can define a linear function:

$$l(\Delta x) := f(x) + f'(x)\Delta x \tag{25}$$

which differs from $f(x+\Delta x)$ only by $\varepsilon\Delta x$, which is infinitesimal compared to Δx. □

8.3 Rules for Derivatives

If f, g are differentiable at $x \in \mathbb{R}$, then so are $f+g$ and $f \cdot g$, and:

1. $(f+g)'(x) = f'(x) + g'(x)$
2. $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$

Proof. Let $\Delta x \simeq 0$, $\Delta x \neq 0$. Since f, g are differentiable at x, $f(x+\Delta x)$ and $g(x+\Delta x)$ are defined.

1. $(f+g)(x+\Delta x) = f(x+\Delta x) + g(x+\Delta x)$. Then,

$$\frac{\Delta(f+g)}{\Delta x} = \frac{f(x+\Delta x) + g(x+\Delta x) - f(x) - g(x)}{\Delta x} \tag{26}$$

$$= \frac{f(x+\Delta x) - f(x)}{\Delta x} + \frac{g(x+\Delta x) - g(x)}{\Delta x} \simeq f'(x) + g'(x) \tag{27}$$

$$\implies (f+g)'(x) = f'(x) + g'(x) \tag{28}$$

2. $(fg)(x+\Delta x) = f(x+\Delta x)g(x+\Delta x)$.

$$\Delta(fg) = f(x+\Delta x)g(x+\Delta x) - f(x)g(x) \tag{29}$$

Making use of the fact

$$\Delta f = f(x+\Delta x) - f(x) \iff f(x+\Delta x) = f(x) + \Delta f \tag{30}$$

$$\Delta(fg) = (f(x) + \Delta f)(g(x) + \Delta g) - f(x)g(x) \tag{31}$$

$$= f(x)g(x) + f(x)\Delta g + \Delta f g(x) + \Delta f \Delta g - f(x)g(x) \tag{32}$$

Dividing by Δx yields:

$$\frac{\Delta(fg)}{\Delta x} = \frac{\Delta g}{\Delta x} f(x) + \frac{\Delta f}{\Delta x} g(x) + \Delta f \frac{\Delta g}{\Delta x} \tag{33}$$

Since $f'(x) \simeq \frac{\Delta f}{\Delta x}$, $g'(x) \simeq \frac{\Delta g}{\Delta x}$, and $\Delta f \frac{\Delta g}{\Delta x} \simeq 0$,

$$\iff \frac{\Delta(fg)}{\Delta x} \simeq g'(x)f(x) + f'(x)g(x) + 0 \tag{34}$$

$$\implies (f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x) \tag{35}$$

□

8.4 Chain Rule

If f is differentiable at $x \in \mathbb{R}$ and g is differentiable at $f(x)$, then $g \circ f$ is differentiable at x with

$$(g \circ f)'(x) = g'(f(x))f'(x). \tag{36}$$

Proof. Let $\Delta x \simeq 0, \Delta x \neq 0$.

Since f is differentiable at x and g is differentiable at $f(x)$, $f(x+\Delta x)$ and $g(f(x)+\Delta f)$ is defined.

By definition of the increment:

$$\Delta f = f(x+\Delta x) - f(x) \tag{37}$$

$$\Delta(g \circ f) = g(f(x+\Delta x)) - g(f(x)) \tag{38}$$

We use $\Delta f = f(x+\Delta x) - f(x) \iff f(x+\Delta x) = f(x) + \Delta f$

$$\iff \Delta(g \circ f) = g(f(x)+\Delta f) - g(f(x)) \tag{39}$$

Thus $\Delta(g \circ f)$ is an increment of g at $f(x)$

$$\iff \Delta(g \circ f) = \Delta g(f(x), \Delta f) \tag{40}$$

Now we apply the equation (23):

$$\Delta g = dg + \varepsilon \Delta x, \text{ here } \Delta x = \Delta f \tag{41}$$

Thus $\Delta g(f(x), \Delta f) = g'(f(x))\Delta f + \varepsilon \Delta f$

Dividing by Δx yields:

$$\frac{\Delta(g \circ f)}{\Delta x} = g'(f(x))\frac{\Delta f}{\Delta x} + \varepsilon \frac{\Delta f}{\Delta x} \tag{42}$$

We use $\frac{\Delta f}{\Delta x} \simeq f'(x), \frac{\Delta f}{\Delta x}$ limited

$$\iff \frac{\Delta(g \circ f)}{\Delta x} \simeq g'(f(x))f'(x) \tag{43}$$

$$\implies (g \circ f)'(x) = g'(f(x))f'(x) \tag{44}$$

□

8.5 Critical Point Theorem

If $f(x)$ is a maximum or minimum for some $x \in (a,b) \subseteq \mathbb{R}$ and if f is differentiable at x, then $f'(x) = 0$.

Proof. Let x be a maximum. $\Delta x \simeq 0$, $\Delta x \neq 0$. By the transfer principle:

$$f(x + \Delta x) \leq f(x), \forall \Delta x \tag{45}$$

Let $\varepsilon > 0, \delta < 0$ be infinitesimals. Then:

$$f(x + \varepsilon) - f(x) \leq 0 \tag{46}$$

$$f(x + \delta) - f(x) \leq 0 \tag{47}$$

Dividing by ε and δ respectively yields:

$$f'(x) \simeq \frac{f(x+\varepsilon) - f(x)}{\varepsilon} \leq \frac{0}{\varepsilon} = 0 \tag{48}$$

$$0 = \frac{0}{\delta} \leq \frac{f(x+\delta) - f(x)}{\delta} \simeq f'(x) \tag{49}$$

Combining these, we get:

$$f'(x) = \mathrm{sh}\left(\frac{f(x+\varepsilon) - f(x)}{\varepsilon}\right) \leq 0 \leq \mathrm{sh}\left(\frac{f(x+\delta) - f(x)}{\delta}\right) = f'(x) \tag{50}$$

Therefore,

$$f'(x) = 0. \tag{51}$$

A similar argument holds for a minimum at x. □

Using the critical point and extreme value theorems (Section 7.6), the following results can be derived about a function f that is continuous on $[a,b] \subseteq \mathbb{R}$ and differentiable on (a,b). The proofs do not require further reasoning about infinitesimals or limits.

- Rolle's Theorem
- Mean Value Theorem
- If f' is zero/positive/negative on (a,b), then f is constant/increasing/decreasing on $[a,b]$