Perfect Equilibrium

Seminar Game Theory

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Contents

1 Introduction

We begin by recalling the basic framework of finite strategic games and their mixed extensions, including the notions of best reply and Nash equilibrium. Next, we introduce the concepts of pure best replies and completely mixed strategies. The central focus is on perfect equilibrium: we define it using trembles and prove that every perfect equilibrium is a Nash equilibrium, and that every finite game admits at least one perfect equilibrium. Finally, we show that every perfect equilibrium is undominated, thereby deducing that every finite game has at least one undominated Nash equilibrium.

2 Repetition

We begin by fixing a finite set of players

$$N := \{1, \dots, n\}.$$

Throughout our discussion, each player $i \in N$ chooses a strategy from a nonempty set A_i. We gather all possible strategy profiles in the Cartesian product

$$A := \prod_{i=1}^{n} A_i,$$

so that any element

$$a := (a_1, \dots, a_n) \in A$$

is called a *strategy profile*. Each player i then receives a numerical payoff $u_i(a)$ whenever the profile a is played.

Definition 2.1 (n-Player Strategic Game)**.** *An* n-player strategic game *is a pair*

$$G = (A, u),$$

where

$$A = A_1 \times \cdots \times A_n$$

is the set of all strategy profiles (each A_i is nonempty and is understood to be the strategy set for player i), and

$$u_i : A \longrightarrow \mathbb{R}$$

is the payoff function *of player i for each $i \in N$. Thus we write*

$$u = (u_1, \dots, u_n)$$

for the collective payoff functions. If $a \in A$ is a strategy profile and $a_i' \in A_i$ is some alternative strategy for player i, we denote by (a_{-i}, a_i') the profile obtained from a by replacing player i's strategy a_i with a_i'.

Definition 2.2 (Nash Equilibrium). *Let $G = (A, u)$ be an n-player strategic game. A* Nash equilibrium *of G is a strategy profile*

$$a^* = (a_1^*, \dots, a_n^*) \in A$$

such that for every player $i \in N$ and every alternative strategy $a_i' \in A_i$, we have

$$u_i(a^*) \geq u_i(a_{-i}^*,\, a_i'),$$

where

$$a_{-i}^* := (a_j^*)_{j \neq i}.$$

In other words, no single player can achieve a strictly higher payoff by deviating from a^* and choosing a different strategy on their own.

Definition 2.3 (Best Reply). *Consider again $G = (A, u)$ with $A_i \subset \mathbb{R}^{m_i}$ for each $i \in N$, where each A_i is nonempty and compact, and suppose that each payoff function u_i is continuous on A. For a profile $a_{-i} \in A_{-i}$, we define the* best reply *of player i by*

$$\mathrm{BR}(a_{-i}) := \left\{ a_i \in A_i \;\middle|\; u_i(a_{-i}, a_i) = \max_{\widetilde{a}_i \in A_i} u_i(a_{-i}, \widetilde{a}_i) \right\}.$$

Hence, for any fixed collection of strategies a_{-i} of the other players, the set $\mathrm{BR}(a_{-i})$ *contains precisely those strategies of player i that maximize player i's payoff.*

Theorem 2.1 (Nash's Theorem). *Consider a strategic game $G = (A, u)$ such that for every player $i \in N$ the set $A_i \subset \mathbb{R}^{m_i}$ is nonempty, convex, and compact, each payoff function u_i is continuous on A, and for every profile $a_{-i} \in A_{-i}$ the function $u_i(a_{-i}, \cdot)$ is quasi-concave on A_i. Then G has at least one Nash equilibrium.*

Definition 2.4 (Finite Game). *We say that a strategic game $G = (A, u)$ is* finite *if, for every player $i \in N$, the set A_i of all strategies available to player i is finite (i.e. $|A_i| < \infty$).*

Definition 2.5 (Mixed Extension of a Finite Game). *Let $G = (A, u)$ be a finite game. Its* mixed extension *is a new game*

$$E(G) = (S, u),$$

constructed as follows. For each $i \in N$, we let $S_i := \Delta A_i$ be the set of all probability distributions on the finite set A_i. We then write

$$S := S_1 \times \cdots \times S_n$$

for the set of all mixed-strategy profiles $s = (s_1, \ldots, s_n)$*. Given* $s \in S$ *and a pure profile* $a = (a_1, \ldots, a_n) \in A$*, we denote by*

$$s(a) = s_1(a_1) \ \cdots \ s_n(a_n)$$

the product of the probabilities that each player assigns to the respective pure strategies $a_1, \ldots, a_n$*. The* expected payoff *of player* i *under a mixed strategy profile* s *is*

$$u_i(s) = \sum_{a \in A} u_i(a) \, s(a).$$

Hence, $E(G) = (S_1 \times \cdots \times S_n, u)$ *is itself a strategic game, allowing every player* i *to mix over their pure strategies in* A_i*.*

3 PBR & Completely Mixed Strategy

We now continue with a finite game

$$G = (A, u),$$

and its mixed extension

$$E(G) = (S, u).$$

Recall that

$$A = \prod_{i \in N} A_i$$

is the set of all pure strategy profiles, and

$$S = \prod_{i \in N} S_i.$$

is the set of all mixed-strategy profiles, with each $S_i = \Delta A_i$ representing the probability distributions on A_i for player i.

Definition 3.1 (Support, Completely Mixed Strategies, and Pure Best Replies)**.** *Let $s_i \in S_i$ be a mixed strategy for player i, and let $s \in S$ be any mixed-strategy profile.*

1. Support of s_i: *We define*

$$\mathscr{S}(s_i) := \{a_i \in A_i \,|\, s_i(a_i) > 0\},$$

Analogously, the support of the entire profile $s = (s_1, \ldots, s_n)$ is

$$\mathscr{S}(s) := \prod_{i \in N} \mathscr{S}(s_i) = \{a \in A \,|\, s(a) > 0\}.$$

2. Completely Mixed Strategy: *We say that s_i is* completely mixed *if $\mathscr{S}(s_i) = A_i$, meaning that player i assigns strictly positive probability to every pure strategy in A_i. We call the full profile s completely mixed if $\mathscr{S}(s) = A$, or equivalently, if each s_i is completely mixed.*

3. Pure Best Replies: *Consider a mixed-strategy profile*

$$s_{-i} = \big(s_1, \ldots, s_{i-1},\, s_{i+1}, \ldots, s_n\big)$$

in which all players except i choose their mixed strategies. The set of pure best replies *of player i to s_{-i} is*

$$\mathrm{PBR}(s_{-i}) := \Big\{ a_i \in A_i \;\Big|\; u_i\big(s_{-i}, a_i\big) \;\geq\; u_i\big(s_{-i}, \widehat{a}_i\big) \text{ for all } \widehat{a}_i \in A_i \Big\}.$$

Hence, these are precisely the pure strategies of player i that maximize i's expected payoff given s_{-i}. For a full mixed profile $s \in S$, we write

$$\mathrm{PBR}(s) := \prod_{i \in N} \mathrm{PBR}(s_{-i}).$$

Proposition 3.1 (Support of Mixed Strategies implies BR and Nash Equilibrium)**.** *Let $G = (A, u)$ be a finite game, and let $E(G) = (S, u)$ be its mixed extension. Then for every player $i \in N$, each mixed strategy $s_i \in S_i$, and each mixed profile $s \in S$, the following three statements hold:*

1. $s_i \in \mathrm{BR}(s_{-i}) \iff \mathscr{S}(s_i) \subseteq \mathrm{PBR}(s_{-i})$

2. *s is a Nash equilibrium of $E(G)$* $\iff \mathscr{S}(s) \subseteq \mathrm{PBR}(s)$

3. *s is a Nash equilibrium of $E(G)$* $\iff u_i(s) \geq u_i(s_{-i}, \widehat{a}_i)$ *for all $i \in N$, $\widehat{a}_i \in A_i$*

Proof. Since each player i's expected payoff from a mixed profile $s = (s_1, \ldots, s_n)$ satisfies

$$u_i(s) = \sum_{a_i \in A_i} s_i(a_i)\, u_i\big(s_{-i},\, a_i\big),$$

we see that s_i can only be a best reply if all pure strategies a_i in its support yield a maximum payoff. Hence part (1) follows by noting that s_i must place zero probability on any strictly suboptimal pure strategy. From this characterization of best replies, part (2) is immediate: s is a Nash equilibrium exactly when each s_i is a best reply, or equivalently, every pure strategy in the support of s_i is a best reply. Finally, part (3) restates the usual "no profitable one-player deviation" criterion, which is clearly equivalent to each s_i being a best reply. $\square$

4 Perfect Equilibrium

The central motivation behind the notion of Nash equilibrium is to identify strategy profiles that are *self-enforcing.* Intuitively, if players are asked (or informally agree) to play according to some profile, then no player should have an incentive to deviate from it. A self-enforcing profile must indeed be a Nash equilibrium, but not every Nash equilibrium is necessarily self-enforcing. We illustrate with an example.

Example 4.1. *Consider the following two-player normal-form game:*

	L_2	R_2
L_1	$(1,1)$	$(10,0)$
R_1	$(0,10)$	$(10,10)$

We first verify that there are indeed two pure Nash equilibria. If Player 1 (the row player) chooses L_1, then Player 2's (the column player's) best response is L_2, and if Player 2 chooses L_2, then Player 1's best response is L_1. Thus (L_1, L_2) is a Nash equilibrium. Similarly, if Player 1 chooses R_1, then Player 2 is indifferent between L_2 and R_2 (both yield the same payoff of 10), and if Player 2 chooses R_2, then Player 1's best response is R_1. Hence (R_1, R_2) is also a Nash equilibrium.

Despite both pairs being Nash equilibria, only (L_1, L_2) is self-enforcing. *If the two players have informally agreed to play (R_1, R_2), each one can profitably deviate. For example, Player 1 can switch to L_1 and guarantee at least as good a payoff. Once one player deviates, the other follows, so they end up playing (L_1, L_2) anyway. In contrast, with (L_1, L_2), no single player can improve their payoff by switching strategies, so it is stable under deviations. This shows that noncooperative analysis focuses on outcomes stable against incentive to deviate, rather than on socially optimal outcomes.*

We continue working with a finite strategic game

$$G = (A, u)$$

and its mixed extension

$$E(G) = (S, u).$$

Definition 4.1 (Tremble)**.** *Let G be a finite strategic game with N players. A* tremble *in G is a vector*

$$\eta = (\eta_1, \ldots, \eta_n),$$

where each $\eta_i : A_i \to \mathbb{R}$ satisfies:

$$\eta_i(a_i) > 0 \quad \textit{for every } a_i \in A_i,$$

and for the sum we have:

$$\sum_{a_i \in A_i} \eta_i(a_i) < 1.$$

We denote by $T(G)$ the set of all such trembles in G. Intuitively, for each player i, the function η_i imposes a strictly positive lower bound on the probability that i must assign to each of their pure strategies.

Definition 4.2 (η-Perturbation)**.** *Let $G = (A, u)$ be a finite game, and let $\eta \in T(G)$. The η-*perturbation *of G is the strategic game*

$$(G, \eta) := \big(S(\eta),\, u\big)$$

constructed as follows:

1. Sets of Strategies: *For each $i \in N$, define*

   $$S_i(\eta_i) := \{s_i \in S_i \,|\, s_i(a_i) \geq \eta_i(a_i) \textit{ for all } a_i \in A_i\},$$

   and set

   $$S(\eta) := \prod_{i \in N} S_i(\eta_i) \;\subseteq\; S.$$

   Hence $S(\eta)$ is the collection of all mixed-strategy profiles in which each player i assigns at least $\eta_i(a_i)$ to each of their pure strategies a_i.

2. Payoff Functions: *We use the same payoff functions u_i from the mixed extension $E(G)$, now restricted to the smaller domain $S(\eta)$. That is,*

   $$(G, \eta) = \big(S(\eta),\, u_1, \ldots, u_n\big).$$

   Since every $s_i \in S_i(\eta_i)$ must assign strictly positive probability (never below $\eta_i(a_i)$) to each pure strategy a_i, the game (G, η) remains finite, convex, compact, and continuous, thereby satisfying the conditions of Nash's Theorem. In particular, (G, η) admits at least one Nash equilibrium.

Definition 4.3 (Perfect Equilibrium)**.** *Let G be a finite game with mixed extension $E(G) = (S, u)$. A mixed strategy profile $s \in S$ is called a* perfect equilibrium *of $E(G)$ if there exist*

$$\text{(i)} \quad \{\eta^k\}_{k=1}^{\infty} \subset T(G) \qquad \textit{with } \eta^k \to 0,$$

$$\text{(ii)} \quad \{s^k\}_{k=1}^{\infty} \subset S \qquad \textit{with } s^k \to s,$$

such that, for each $k \in \mathbb{N}$, the profile s^k is a Nash equilibrium of the perturbed game $\big(G, \eta^k\big)$. Concretely, this definition means we can approximate s by Nash equilibria of the η-perturbations (G, η^k) as η^k approaches 0. One can verify that every perfect equilibrium of $E(G)$ is indeed a Nash equilibrium of $E(G)$, and that every finite game admits at least one perfect equilibrium in its mixed extension.

We again place ourselves in the setting of Definition 4.3, where G is a finite game with mixed extension $E(G) = (S, u)$, and a perfect equilibrium $s \in S$ is characterized as the limit of Nash equilibria of η-perturbations $\big(G, \eta^k\big)$ as $\eta^k \to 0$.

Theorem 4.1 (Perfect Equilibrium implies Nash Equilibrium)**.** *If $s \in S$ is a perfect equilibrium of $E(G)$, then s is a Nash equilibrium of $E(G)$.*

Proof. By Definition 4.3 of a perfect equilibrium, there exist sequences

$$\{\eta^k\}_{k=1}^{\infty} \subset T(G) \qquad \text{with } \eta^k \to 0,$$

$$\{s^k\}_{k=1}^{\infty} \subset S \qquad \text{with } s^k \to s,$$

and each s^k is a Nash equilibrium of the perturbed game $\big(G, \eta^k\big)$.

Recall that in $\big(G, \eta^k\big)$, each player i must assign at least $\eta_i^k(a_i)$ to every pure strategy a_i. Hence, if some pure strategy a_i is *not* a best reply under s_{-i}^k, the equilibrium condition forces $s_i^k(a_i)$ to be exactly that minimal probability: $s_i^k(a_i) = \eta_i^k(a_i)$.

Suppose for contradiction that s is *not* a Nash equilibrium of $E(G)$. By Proposition 3.1 (2) and (3), there must be a player i and a pure strategy $a_i \in A_i$ such that

$$a_i \notin \mathrm{PBR}(s_{-i}) \quad \text{and} \quad s_i(a_i) > 0.$$

Since $s^k \to s$ and $s_i(a_i) > 0$, eventually $s_i^k(a_i) > \varepsilon$ for some $\varepsilon > 0$ and all large k. Meanwhile, $\eta_i^k(a_i) \to 0$ implies that $\eta_i^k(a_i) < \varepsilon$ for all sufficiently large k. Thus, for large k,

$$s_i^k(a_i) \;>\; \eta_i^k(a_i).$$

However, once s^k is close enough to s, continuity of payoffs implies $a_i \notin \mathrm{PBR}(s_{-i}^k)$. But in $\big(G, \eta^k\big)$, a non–best-reply pure strategy is forced to receive exactly $\eta_i^k(a_i)$. Hence for large k, Nash equilibrium in $\big(G, \eta^k\big)$ would require $s_i^k(a_i) = \eta_i^k(a_i)$, contradicting $s_i^k(a_i) > \eta_i^k(a_i)$. Therefore our assumption that s is not a Nash equilibrium must be false.

We conclude that s is indeed a Nash equilibrium of $E(G)$. □

Theorem 4.2 (Existence of Perfect Equilibrium)**.** *The mixed extension $E(G)$ of any finite game G admits at least one perfect equilibrium.*

Proof. We begin by choosing a sequence

$$\{\eta^k\} \subset T(G) \quad \text{with } \eta^k \to 0.$$

For each k, we let s^k be a Nash equilibrium of the perturbed game $\big(G, \eta^k\big)$; by Nash's Theorem, each s^k exists and assigns strictly positive probability to every pure strategy. Since $S = \Delta A_1 \times \cdots \times \Delta A_n$ is a product of simplices, it is compact, so the sequence $\{s^k\}$ has a convergent subsequence $\{s^{k_\ell}\}$ converging to some $s \in S$. Because $\eta^{k_\ell} \to 0$ and each s^{k_ℓ} is a Nash equilibrium of $\big(G, \eta^{k_\ell}\big)$, the limit pair $\big(\eta^{k_\ell}, s^{k_\ell}\big)$ meets the conditions of Definition 4.3, making s a perfect equilibrium of $E(G)$. □

5 Undominated Nash Equilibrium

Definition 5.1 (ε-Perfect Equilibrium)**.** *Let $G = (A, u)$ be a finite game with mixed extension $E(G) = (S, u)$, and fix $\varepsilon > 0$. A mixed strategy profile $s \in S$ is called an* ε-perfect equilibrium *of $E(G)$ if:*

1. *s is* completely mixed, *that is, for every player $i \in N$ and each strategy $a_i \in A_i$, we have*
$$s_i(a_i) \;>\; 0.$$
2. *Whenever $u_i\big(s_{-i}, a_i\big) \;<\; u_i\big(s_{-i},\, \widehat{a}_i\big)$ for some $a_i, \widehat{a}_i \in A_i$, it follows that*
$$s_i(a_i) \;\leq\; \varepsilon.$$

In other words, any pure strategy a_i that is strictly suboptimal for player i (when facing s_{-i}) is assigned probability at most ε in s_i. Thus ε-perfect equilibria limit the probability placed on any pure strategy that is not a best reply.

Proposition 5.1 (Perfect Equilibrium Is Equivalent to the Limit of ε-Perfect Equilibria)**.** *Let $E(G)$ be the mixed extension of a finite game G. For any $s \in S$, the following three statements are equivalent:*

1. *s is a perfect equilibrium of $E(G)$.*
2. *There exist sequences $\{\varepsilon^k\}_{k=1}^{\infty} \subset (0, \infty)$ with $\varepsilon^k \to 0$, and $\{s^k\}_{k=1}^{\infty} \subset S$ with $s^k \to s$, such that each s^k is an ε^k-perfect equilibrium of $E(G)$.*
3. *There is a sequence $\{s^k\}_{k=1}^{\infty} \subset S$ of completely mixed profiles such that $s^k \to s$, and for every $k \in \mathbb{N}$ and each $i \in N$ we have*
$$s_i \;\in\; \mathrm{BR}\big(s^k_{-i}\big).$$

Proof. We show
$$\text{i)} \implies \text{ii)} \implies \text{iii)} \implies \text{i)}$$
in turn.

i) $\implies$ ii) Suppose s is a perfect equilibrium of $E(G)$. By Definition 4.3, there are sequences

$$\{\eta^k\}_{k=1}^{\infty} \;\subset\; T(G),$$
$$\{s^k\}_{k=1}^{\infty} \;\subset\; S.$$

such that $\eta^k \to 0$, $s^k \to s$, and each s^k is a Nash equilibrium of the η^k-perturbed game $\big(G, \eta^k\big)$. In (G, η^k), if a pure strategy a_i is not a best reply for player i, then $s_i^k(a_i) = \eta_i^k(a_i)$. We define

$$\varepsilon^k := \max_{i \in N} \max_{a_i \in A_i} \eta_i^k(a_i)$$

Since $\eta^k \to 0$, we have $\varepsilon^k \to 0$ as well. We claim each s^k is then an ε^k*-perfect equilibrium* (see Definition 5.1). Indeed, if a_i is strictly suboptimal for player i, we have

$$s_i^k(a_i) = \eta_i^k(a_i) \ \le \ \varepsilon^k.$$

so condition (2) of an ε^k-perfect equilibrium is satisfied. Moreover, each s^k is completely mixed by construction, since $s_i^k(a_i) \ge \eta_i^k(a_i) > 0$. Hence s^k is indeed ε^k-perfect, and $\varepsilon^k \to 0$ together with $s^k \to s$ shows **(ii)**.

ii) $\Longrightarrow$ iii) Now assume there are sequences $\{\varepsilon^k\} \to 0$ and $\{s^k\} \subset S$ with $s^k \to s$, where each s^k is an ε^k-perfect equilibrium. By definition, if

$$u_i\big(s_{-i}^k, a_i\big) < u_i\big(s_{-i}^k, \widehat{a}_i\big),$$

it follows that

$$s_i^k(a_i) \le \varepsilon^k.$$

Hence any pure strategy a_i that appears in the support $\mathscr{S}(s_i^k)$ of s_i^k (i.e. $s_i^k(a_i) > 0$) cannot be strictly suboptimal. Thus a_i must be a best reply; in other words, $a_i \in \mathrm{PBR}\big(s_{-i}^k\big)$ and so $s_i^k \in \mathrm{BR}\big(s_{-i}^k\big)$ for each k. The convergence $s^k \to s$ implies that for each $i \in N$ and for all sufficiently large k,

$$s_i \in \mathrm{BR}\big(s_{-i}^k\big).$$

Thus, we obtain a sequence of completely mixed strategies $\{s^k\} \to s$ such that for each i and all sufficiently large k, $s_i \in \mathrm{BR}(s_{-i}^k)$, which establishes **(iii)**.

iii) $\Longrightarrow$ i) Finally, suppose $\{s^k\} \subset S$ is a sequence of completely mixed profiles converging to s, and for each k and each $i \in N$ we have $s_i^k \in \mathrm{BR}\big(s_{-i}^k\big)$. We construct trembles $\eta^k \in T(G)$ so that s^k becomes a Nash equilibrium of $\big(G, \eta^k\big)$. Set, for each $i \in N$ and $a_i \in A_i$,

$$\eta_i^k(a_i) := \begin{cases} s_i^k(a_i), & \text{if } a_i \notin \mathscr{S}\big(s_i\big), \\ 1/k, & \text{otherwise.} \end{cases}$$

We observe that $\{\eta^k\} \to 0$. Moreover, we choose $\bar{k} \in \mathbb{N}$ such that, for each $k \ge \bar{k}$, $\eta^k \in T(G)$, (G, η^k) is well defined, and $s^k \in S(\eta^k)$. Now, for each $k \ge \bar{k}$, each $i \in N$, and each $a_i \in A_i$, since $s_i \in \mathrm{BR}_i(s_{-i}^k)$, we obtain that

$$a_i \notin \mathrm{PBR}_i\big(s_{-i}^k\big) \ \Longrightarrow \ s_i(a_i) = 0 \ \Longrightarrow \ s_i^k(a_i) \ = \ \eta_i^k(a_i).$$

Hence, for each $k \ge \bar{k}$, we conclude that s^k is a Nash equilibrium of (G, η^k).

□

We continue working with a finite strategic game

$$G = (A, u),$$

and its mixed extension

$$E(G) = (S, u).$$

Definition 5.2 (Dominance)**.** *Let G be a finite game with mixed extension $E(G)$, and suppose $s_i, \widetilde{s_i} \in S_i$ are two strategies of player i. We say that $\widetilde{s_i}$* dominates *s_i if:*

1. *For* every *$s_{-i} \in S_{-i}$,*
$$u_i\big(s_{-i}, \widetilde{s_i}\big) \;\geq\; u_i\big(s_{-i}, s_i\big),$$
2. *There is* at least one *$s_{-i} \in S_{-i}$ for which*
$$u_i\big(s_{-i}, \widetilde{s_i}\big) \;>\; u_i\big(s_{-i}, s_i\big).$$

Equivalently, using pure strategies in A_{-i} instead of mixed strategies in S_{-i}, we have:

$$u_i\big(\,\widehat{a}_{-i},\, \widetilde{s_i}\big) \;\geq\; u_i\big(\,\widehat{a}_{-i},\, s_i\big) \quad \textit{for all} \quad \widehat{a}_{-i} \in A_{-i},$$

with strict inequality for at least one $\widehat{a}_{-i} \in A_{-i}$.

Definition 5.3 (Undominated Strategy Profile)**.** *A mixed strategy $s_i \in S_i$ for player i is called* undominated *if there is no $\widetilde{s_i} \in S_i$ that dominates s_i. A strategy profile $s = (s_1, \ldots, s_n) \in S$ is an* undominated strategy profile *if each s_i is undominated. Finally, an* undominated Nash equilibrium *of $E(G)$ is a Nash equilibrium $s \in S$ in which each s_i is undominated.*

Theorem 5.1 (Perfect Equilibrium Implies Undominated Nash Equilibrium)**.** *Let $E(G)$ be the mixed extension of a finite game G. Then every perfect equilibrium $s^* \in S$ of $E(G)$ is an undominated Nash equilibrium of $E(G)$.*

Proof. Let $s^* = (s_1^*, \ldots, s_n^*)$ be a perfect equilibrium of $E(G)$. Then s^* is a Nash equilibrium (by the usual property that every perfect equilibrium is also a Nash equilibrium). It remains to show that no s_i^* can be dominated by some other strategy $\widetilde{s_i} \in S_i$. Suppose, toward a contradiction, that $\widetilde{s_i}$ *does* dominate s_i^*. By the equivalent definition of dominance using pure profiles, we have

$$u_i\big(a_{-i},\, \widetilde{s_i}\big) \;\geq\; u_i\big(a_{-i},\, s_i^*\big) \quad \text{for all } a_{-i} \in A_{-i}.$$

with strict inequality for at least one $a_{-i} \in A_{-i}$.

Since s^* is perfect, there are sequences $\{\eta^k\} \to 0$ in $T(G)$ and $\{s^k\} \to s^*$ in S such that each s^k is a Nash equilibrium of $\big(G, \eta^k\big)$. In each perturbed game (G, η^k), any pure strategy that fails to be a best reply is assigned probability exactly $\eta_i^k(\cdot)$ by the corresponding player. But if $\widetilde{s}_i$ truly yields strictly higher payoffs against some nearby pure profile a_{-i}^*, then for large k the same strict improvement would occur in (G, η^k) for s_i^k close to s_i^*. This would contradict the fact that s_i^k is a best reply in $\big(G, \eta^k\big)$ if a strictly better alternative $\widetilde{s}_i$ were available. Hence no such dominating $\widetilde{s}_i$ can exist, so each s_i^* is undominated. Combining these facts, s^* is both a Nash equilibrium and consists of undominated strategies, and is therefore an *undominated Nash equilibrium.* □

Corollary 5.1 (Existence of an Undominated Nash Equilibrium)**.** *The mixed extension $E(G)$ of any finite game G admits at least one undominated Nash equilibrium.*

Proof. By Theorem 4.2, there is at least one perfect equilibrium s^* in $E(G)$. From Theorem 5.1, every perfect equilibrium is an undominated Nash equilibrium. Consequently, s^* is an undominated Nash equilibrium of $E(G)$. Hence at least one such undominated equilibrium must exist. □

6 References

[1] Julio Gonzalez-Diaz, Ignacio Garcia-Jurado, and M. Gloria Fiestras-Janeiro. *An Introductory Course on Mathematical Game Theory*. Graduate Studies in Mathematics, Volume 115, American Mathematical Society, 2010.