# Linear First Order Equations
# The Transport Equation

Introduction to Partial Differential Equations

Raphael Bisang

July 28, 2025

# Contents

# 1 Introduction

In this chapter, we explore first-order linear partial differential equations (PDEs) in two variables. We begin with the homogeneous equation

$$a(x,y)\,u_x + b(x,y)\,u_y \;=\; 0,$$

where $u = u(x,y)$ and

$$u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y}.$$

We will see that $u$ remains constant along certain paths in the $(x,y)$-plane, called characteristic curves, on which this PDE simplifies significantly. We also note that the transport equation, commonly introduced in terms of time $t$ and spatial coordinates, follows the same principle of constancy along characteristics.

More generally, on a domain $V \subset \mathbb{R}^2$, a linear first-order PDE can be written in the form

$$a(x,y)\,u_x + b(x,y)\,u_y \;=\; c(x,y)\,u \;+\; f(x,y),$$

for $(x,y) \in V$.

In our discussions, we always assume that $a(x,y)$ and $b(x,y)$ are not both zero, ensuring a genuine dependence on both variables. This framework prepares us for the methods we will use to solve such equations and highlights the importance of characteristic curves in understanding their behavior.

# 2 Constant coefficient equations

We begin by analyzing a very simple PDE:

$$\frac{\partial u}{\partial x} = 0.$$

Since the derivative of $u$ with respect to $x$ is zero, $u$ does not depend on $x$. Hence, the general solution is a function of $y$ alone:

$$\boxed{u(x,y) = f(y)}.$$

For instance, we can choose $u(x,y) = y^2 - y$ or $u(x,y) = e^y \cos(y)$. Geometrically, these solutions remain constant along horizontal lines (lines of constant $y$) in the $(x,y)$-plane.

We next consider the constant coefficient equation

$$a\,u_x + b\,u_y = 0,$$

where $a$ and $b$ are constants that are not both zero. This generalizes the simpler PDE above by incorporating derivatives in both the $x$- and $y$-directions under a constant coefficient framework. We will now examine the methods for finding its solutions.

## 2.1 Geometric interpretation

**Definition 2.1** (Characteristic Curve)**.** *For the PDE*

$$a\,u_x + b\,u_y = 0,$$

*a curve $\gamma(s) = (x(s), y(s))$ in the $(x,y)$-plane is called a* characteristic curve *if, along this curve, the PDE implies that $u$ remains constant. Equivalently, we can say that the directional derivative of $u$ in the direction of the vector $(a,b)$ is zero along $\gamma$.*

We consider the PDE

$$a\,u_x + b\,u_y = 0,$$

where $a$ and $b$ are constants, not both zero. Here $a\,u_x + b\,u_y$ is precisely the directional derivative of $u$ along the vector $(a,b)$. Since this derivative must vanish, we see that $u$ remains constant when moving in the direction $(a,b)$. Consequently, in order for $u$ to vary, we have to look along the perpendicular direction, given by the vector $(b,-a)$. Lines that are parallel to $(b,-a)$ satisfy

$$b\,x - a\,y = c,$$

for some constant $c$. Along each such line, the value of $u$ does not change, indicating that $u$ depends solely on $b\,x - a\,y$. Therefore, the general solution is given by

$$\boxed{u(x, y) \;=\; f\big(b\,x - a\,y\big)},$$

where $f$ is any single-variable function.

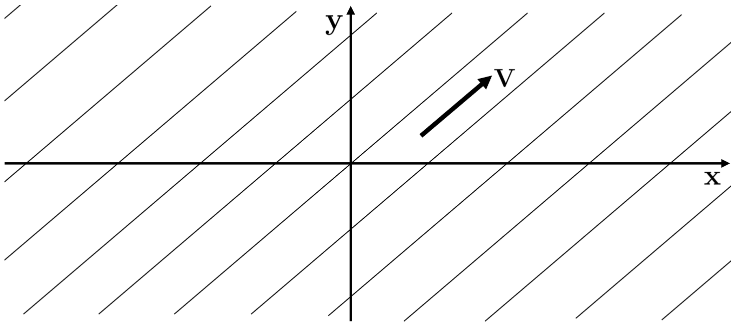


Figure 1: Constant vector field $\mathbf{V}$

## 2.2 Method of characteristics

We revisit the PDE

$$a\,u_x \;+\; b\,u_y \;=\; 0,$$

where $a$ and $b$ are constants, not both zero. To analyze this equation using a change of coordinates, we introduce

$$x' \;=\; a\,x \;+\; b\,y, \quad y' \;=\; b\,x \;-\; a\,y.$$

By the chain rule, we express $u_x$ and $u_y$ in terms of $u_{x'}$ and $u_{y'}$:

$$u_x \;=\; u_{x'}\,\frac{\partial x'}{\partial x} \;+\; u_{y'}\,\frac{\partial y'}{\partial x}, \quad u_y \;=\; u_{x'}\,\frac{\partial x'}{\partial y} \;+\; u_{y'}\,\frac{\partial y'}{\partial y}.$$

Since

$$\frac{\partial x'}{\partial x} \;=\; a, \quad \frac{\partial y'}{\partial x} \;=\; b, \quad \frac{\partial x'}{\partial y} \;=\; b, \quad \frac{\partial y'}{\partial y} \;=\; -a,$$

we obtain

$$u_x \;=\; a\,u_{x'} \;+\; b\,u_{y'}, \quad u_y \;=\; b\,u_{x'} \;-\; a\,u_{y'}.$$

Substituting these into $a\,u_x + b\,u_y = 0$ gives

$$a\big(a\,u_{x'} + b\,u_{y'}\big) \;+\; b\big(b\,u_{x'} - a\,u_{y'}\big) \;=\; (a^2 + b^2)\,u_{x'} \;=\; 0.$$

Because $a^2 + b^2$ is nonzero, we must have $u_{x'} = 0$. Hence $u$ does not depend on $x'$ and must therefore be a function of $y'$ only:

$$u(x, y) \;=\; f(y').$$

Recalling $y' = b\,x \;-\; a\,y$, we conclude

$$\boxed{u(x, y) \;=\; f\big(b\,x - a\,y\big)},$$

which is precisely the same solution form we identified using the geometric approach.

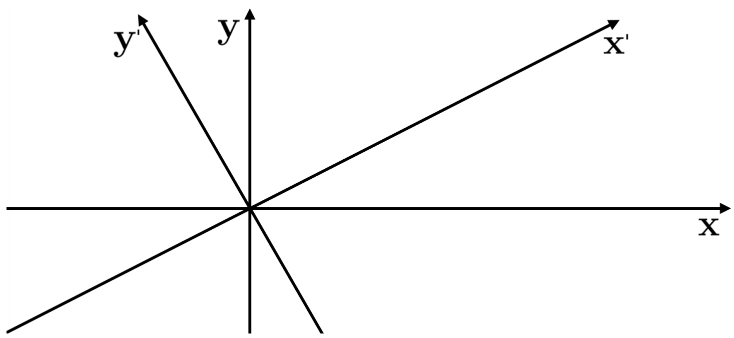


Figure 2: Coordinate change

## 2.3 Example 1

We want to solve the PDE

$$4\,u_x \;-\; 3\,u_y \;=\; 0$$

together with the boundary condition

$$u(0,y) \;=\; y^3.$$

From the constant coefficient approach, we see that $u$ must depend only on the linear combination $-3x - 4y$. Hence, the general solution takes the form

$$u(x,y) \;=\; f(-3x \;-\; 4y).$$

We apply the boundary condition by setting $x = 0$, which gives

$$u(0,y) \;=\; f(-4y) \;=\; y^3.$$

Let us introduce $w = -4y$. Then $y = -w/4$, so

$$f(w) \;=\; \left(-\tfrac{w}{4}\right)^3 \;=\; -\tfrac{w^3}{64}.$$

Translating back to the original variables, we have

$$u(x,y) \;=\; f\big(-3x \;-\; 4y\big) \;=\; -\frac{\big(-3x \;-\; 4y\big)^3}{64} \;=\; \frac{\big(3x \;+\; 4y\big)^3}{64}.$$

Thus, the unique solution that satisfies both the PDE and the boundary condition is

$$\boxed{u(x,y) \;=\; \frac{\big(3x \;+\; 4y\big)^3}{64}}.$$

# 3 Variable coefficient equations

We now consider the PDE

$$u_x + y\,u_y = 0,$$

which remains linear and homogeneous but has the variable coefficient $y$. Geometrically, we observe that this implies the directional derivative of $u$ along the vector $(1, y)$ is zero, so $u$ must be constant on curves in the $(x, y)$-plane whose slope is $y$.

These *characteristic curves* satisfy the ordinary differential equation

$$\frac{dy}{dx} = y,$$

whose general solution is

$$y = C\,e^x,$$

where $C$ is an arbitrary constant. As $C$ varies, these curves fill the plane, and on each one, the value of $u$ remains constant. Therefore, $u$ can only depend on combinations of $x$ and $y$ that do not change along these characteristic curves. We note that $y\,e^{-x}$ remains invariant whenever $\frac{dy}{dx} = y$, so the general solution is

$$\boxed{u(x, y) = f\big(y\,e^{-x}\big)},$$

where $f$ is an arbitrary single-variable function.

**Remark 3.1.** *The solution derived here applies to the specific case in which the variable coefficient is $y$. More generally, if we have $b(x, y)$ as a variable coefficient, there may be no simple closed-form formula for the solution. However, the characteristic method still provides a systematic way to investigate solutions by solving the associated ODE for the characteristic curves, whenever it is feasible to do so.*

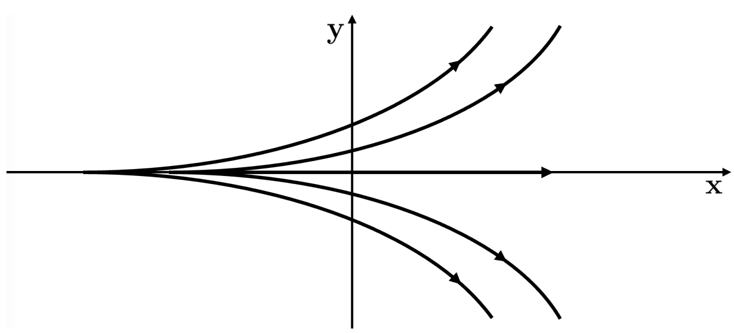


Figure 3: Characteristic curves

## 3.1 Example 2

We revisit the PDE

$$u_x + y\,u_y = 0$$

together with the boundary condition

$$u(0,y) = y^3.$$

Since we know from our general solution form that

$$u(x,y) = f\big(y\,e^{-x}\big),$$

we apply the boundary condition by setting $x = 0$, yielding

$$u(0,y) = f(y) = y^3.$$

Therefore, $f(\xi) = \xi^3$. Substituting back, we obtain

$$u(x,y) = \big(y\,e^{-x}\big)^3 = e^{-3x}\,y^3.$$

Hence, the solution that satisfies both the PDE and the given boundary condition is

$$\boxed{u(x,y) = e^{-3x}\,y^3}.$$

The auxiliary condition is essential here, as it uniquely determines the form of $f$ among infinitely many solutions to the homogeneous equation.

## 3.2 Example 3

We consider the PDE

$$x\,u_x + 2\,y\,u_y = 0,$$

which has coefficients depending on both $x$ and $y$. To solve it, we look for *characteristic curves* along which $u$ remains constant.

**Case 1:** $x \neq 0$. In this case, the characteristic curves satisfy

$$\frac{dy}{dx} = \frac{2\,y}{x}.$$

Separating the variables,

$$\frac{dy}{y} = 2\,\frac{dx}{x},$$

and integrating gives

$$\ln(y) = 2\,\ln(x) + \ln(C) \quad\Longrightarrow\quad y = C\,x^2,$$

where $C$ is an arbitrary constant. Hence the ratio $y/x^2$ is invariant along each characteristic curve. Since $u$ is constant on these curves,

$$u(x,y) = f\Big(\tfrac{y}{x^2}\Big),$$

for some function $f$. One can verify by direct differentiation that any such $f\big(y/x^2\big)$ satisfies the PDE on $\{x \neq 0\}$.

**Case 2:** $x = 0$. Along the line $x = 0$, the PDE becomes

$$x\,u_x + 2\,y\,u_y \;=\; 0 \quad\Longrightarrow\quad 2\,y\,u_y = 0.$$

For $y \neq 0$, this forces $u_y = 0$, hence $u$ is constant with respect to $y$. At the single point $(0, 0)$, the equation is $0 = 0$ and thus does not impose any further condition on $u$ unless an auxiliary condition is given.

# 4 Transport equation

We now introduce one of the simplest yet most fundamental partial differential equations, known as the *transport equation*. In its standard form, we let $u = u(t, \mathbf{x})$ be a function of time $t \geq 0$ and of the spatial position $\mathbf{x} \in \mathbb{R}^n$. A constant vector $\mathbf{b} \in \mathbb{R}^n$ represents the velocity or direction of transport. The equation is given by

$$u_t \ + \ \mathbf{b} \cdot \nabla u \ = \ 0,$$

where $u_t$ denotes the partial derivative of $u$ with respect to $t$, and $\nabla u$ is the gradient of $u$ with respect to $\mathbf{x}$.

We interpret this PDE to mean that $u$ remains unchanged as we move in the direction of $\mathbf{b}$. Concretely, if we define the path

$$\mathbf{z}(s) \ = \ \mathbf{x} \ + \ s\,\mathbf{b},$$

then the directional derivative of $u$ along $\mathbf{b}$ vanishes. Hence, $u$ remains constant along each line in $\mathbb{R}^n$ whose direction is parallel to $\mathbf{b}$. We can view these lines as *characteristic curves*, reflecting that the value of $u$ is simply *transported* along them without changing.

This equation arises naturally in many physical settings. For example, if $u$ represents the concentration of some substance and $\mathbf{b}$ is a constant flow velocity, the transport equation models how the substance is carried along by this flow. More sophisticated equations in fluid dynamics and wave propagation often build on this same transport principle.

## 4.1 Example from physics

Consider a fluid flowing at a constant speed $c$ along a horizontal pipe of fixed cross-section, in the positive $x$-direction. Suppose a pollutant is suspended in the fluid, and let $u(x, t)$ denote its concentration (in grams per centimeter, for instance) at position $x$ and time $t$. Neglecting diffusion, we assume the pollutant is simply carried by the fluid without spreading, so the transport equation becomes

$$u_t \ + \ c\,u_x \ = \ 0.$$

To understand this equation, we track the total pollutant mass in a segment of fluid. For instance, consider the segment initially between $x = 0$ and $x = b$. After time $t$, all fluid that was originally in $[0, b]$ has moved to $[c\,t,\ b + c\,t]$. If no pollutant is created or destroyed and no lateral diffusion occurs, the mass of pollutant in that moving segment remains constant. Differentiating this statement with respect to time yields precisely

$$u_t \ + \ c\,u_x \ = \ 0.$$

Geometrically, the PDE asserts that $u$ is constant along straight lines in the $(x, t)$-plane for which $\frac{dx}{dt} = c$. Each fluid particle (and any pollutant it carries) travels to the right at speed $c$. Solving the PDE, for example via the characteristic method, shows that $u$ depends solely on $x - ct$. Physically, this indicates that the pollutant's concentration profile translates to the right at speed $c$ without any change in shape.

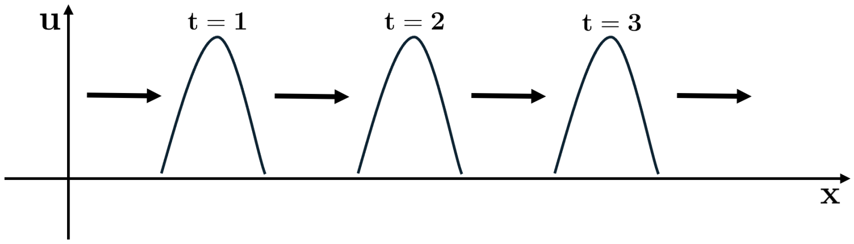


Figure 4: Transport of substance

## 4.2 Initial value problem

We now consider the transport equation

$$u_t + \mathbf{b} \cdot \nabla u = 0 \quad \text{in} \quad \mathbb{R}^n \times (0, \infty),$$

together with the initial condition

$$u(\mathbf{x}, 0) = g(\mathbf{x}) \quad \text{on} \quad \Gamma = \{ (\mathbf{x}, 0) \mid \mathbf{x} \in \mathbb{R}^n \}.$$

Here, $\mathbf{b} \in \mathbb{R}^n$ is a constant velocity vector, and $g$ is the given initial data.

Since the PDE indicates that the directional derivative of $u$ along $\mathbf{b}$ is zero, $u$ must remain constant along each straight line in $\mathbb{R}^n$ whose direction is parallel to $\mathbf{b}$. We can interpret these lines as characteristic curves. In particular, the characteristic curve originating at $(\mathbf{x}_0, 0)$ moves through the points $\big(\mathbf{x}_0 + \mathbf{b}\,t,\ t\big)$. If we assume $u$ is sufficiently smooth (for instance, $C^1$), the value of $u$ along this curve is given by $g(\mathbf{x}_0)$. This connection provides a unique solution for all $(\mathbf{x}, t)$.

**Remark 4.1.** *If the initial data $g$ is not $C^1$, we cannot generally expect a classical ($C^1$) solution. Nevertheless, even with nonsmooth or discontinuous data, the transport equation can admit solutions in a weaker sense. For example, one may seek a* weak solution *in a suitable function space (such as an $L^p$ space, or in the sense of distributions) that retains the essential physics of transport. This issue is especially relevant in more complicated situations, such as nonlinear transport equations that develop discontinuities or* shock waves. *In these cases, solutions are often treated with measure-theoretic or distributional frameworks to ensure the PDE is satisfied in an integral or weak sense.*

## 4.3 Nonhomogeneous transport equation

We now turn to the nonhomogeneous transport equation

$$u_t + \mathbf{b} \cdot \nabla u = f(\mathbf{x}, t) \quad \text{in } \mathbb{R}^n \times (0, \infty),$$

together with the initial condition

$$u(\mathbf{x},0) \;=\; g(\mathbf{x}) \quad \text{for all } \mathbf{x}\in\mathbb{R}^n.$$

To find $u$, we fix a point $(\mathbf{x},t)$ and trace the characteristic line

$$\mathbf{z}(s) \;=\; \mathbf{x} \;+\; \mathbf{b}\,\bigl(s-t\bigr).$$

At $s=0$, this becomes $\mathbf{z}(0)=\mathbf{x}-t\,\mathbf{b}$, which lies on the initial plane $t=0$. Along this line, the transport equation implies that the directional derivative of $u$ in the direction $\mathbf{b}$ equals $f$. Integrating from $s=0$ to $s=t$ shows

$$u(\mathbf{x},t) \;-\; g\bigl(\mathbf{x}-t\,\mathbf{b}\bigr) \;=\; \int_0^t f\bigl(\mathbf{z}(s),\, s\bigr)\,ds.$$

Thus, the unique solution is

$$\boxed{u(\mathbf{x},t) \;=\; g\bigl(\mathbf{x}-t\,\mathbf{b}\bigr) \;+\; \int_0^t f\Bigl(\mathbf{x}-\mathbf{b}\,\bigl(t-s\bigr),\; s\Bigr)\,ds}.$$

**Remark 4.2.** *This integral formula shows how the solution at $(\mathbf{x},t)$ is obtained by tracing back to the initial data $g$ and collecting contributions from the source term $f$ along the characteristic path. This process is a direct application of the method of characteristics, effectively reducing the PDE to an ODE on each characteristic curve. The value of $u$ propagates forward from the initial plane $t=0$ (as specified by $g$) and is modified by integrating the source term $f$ over the characteristic.*

## 4.4 Example of a nonhomogeneous transport equation

We consider the PDE

$$u_t \;+\; c\,u_x \;=\; x \;+\; t \quad \text{for } x\in\mathbb{R},\; t\geq 0,$$

along with the initial condition

$$u(x,0) \;=\; \sin(x).$$

We use the standard formula for the one-dimensional transport equation

$$u_t \;+\; c\,u_x \;=\; f(x,t), \quad u(x,0) \;=\; g(x),$$

whose general solution is

$$u(x,t) \;=\; g\bigl(x-c\,t\bigr) \;+\; \int_0^t f\bigl(x-c\,(t-s),\, s\bigr)\,ds.$$

In our specific case, $g(\xi)=\sin(\xi)$ and $f(\xi,s)=\xi+s$. Substituting these into the general solution gives

$$u(x,t) \;=\; \sin\bigl(x-c\,t\bigr) \;+\; \int_0^t \Bigl[\bigl(x-c\,(t-s)\bigr) \;+\; s\Bigr]\,ds.$$

We now simplify the integral:

$$\int_0^t \Big(x - c\,(t - s) + s\Big)\,ds = \int_0^t \Big(x - c\,t + c\,s + s\Big)\,ds = \int_0^t \Big(x - c\,t + (c+1)\,s\Big)\,ds.$$

Breaking this into separate integrals yields

$$\int_0^t x\,ds = x\,t, \quad \int_0^t (-c\,t)\,ds = -c\,t^2, \quad \int_0^t (c+1)\,s\,ds = (c+1)\,\frac{t^2}{2}.$$

Summing these results, we find

$$\int_0^t \Big(x - c\,t + (c+1)\,s\Big)\,ds = x\,t - c\,t^2 + (c+1)\,\frac{t^2}{2} = x\,t + \frac{(1-c)\,t^2}{2}.$$

Hence, the solution becomes

$$u(x,t) = \sin\big(x - c\,t\big) + x\,t + \frac{(1-c)\,t^2}{2}.$$

We verify that this satisfies the PDE and the initial condition. Clearly, $u(x,0) = \sin(x)$. A direct computation of $u_t + c\,u_x$ shows that it equals $x + t$. Therefore, the solution to

$$u_t + c\,u_x = x + t, \quad u(x,0) = \sin(x),$$

is

$$\boxed{u(x,t) = \sin\big(x - c\,t\big) + x\,t + \frac{(1-c)\,t^2}{2}}.$$

# 5 Exercises

## 5.1 Exercise 1

### Problem

Find the function $u(x,t)$ that satisfies the first-order PDE

$$2\,u_t(x,t) \;+\; 3\,u_x(x,t) \;=\; 0 \quad \text{for } t \geq 0,\; x \in \mathbb{R},$$

subject to the initial condition

$$u(x,0) \;=\; \sin(x).$$

### Solution

We first divide both sides of the PDE by 2 to rewrite it in the standard transport form:

$$u_t \;+\; \tfrac{3}{2}\,u_x \;=\; 0.$$

Since this is a transport equation of the type $u_t + c\,u_x = 0$, its solutions depend on the combination $x - c\,t$. Therefore, we write

$$u(x,t) \;=\; F\Big(x - \tfrac{3}{2}\,t\Big),$$

where $F$ is some function of one variable to be determined. To find $F$, we use the initial condition $u(x,0) = \sin(x)$. Substituting $t = 0$ into the general solution, we obtain

$$F(x) \;=\; \sin(x).$$

Hence, the unique solution that satisfies both the PDE and the given initial condition is

$$\boxed{u(x,t) \;=\; \sin\Big(x - \tfrac{3}{2}\,t\Big)}.$$

## 5.2 Exercise 2

### Problem

Find the function $u(x,y)$ that satisfies the first-order PDE

$$u_x + u_y + u \;=\; e^{x+2y} \quad \text{for } x, y \in \mathbb{R},$$

subject to the boundary condition

$$u(x,0) \;=\; 0 \quad \text{for } x \in \mathbb{R}.$$

## Solution

We apply the method of characteristics in the $(x, y)$-plane, recognizing that the PDE suggests a transport direction $(1, 1)$. Rewriting the PDE as

$$u_x + u_y \;=\; e^{\,x+2y} - u,$$

we note that along a characteristic parameterized by $s$, the directional derivative of $u$ is

$$\frac{du}{ds} \;=\; u_x\,\frac{dx}{ds} + u_y\,\frac{dy}{ds}.$$

Since the characteristic direction is $(1, 1)$, we set

$$\frac{dx}{ds} = 1, \quad \frac{dy}{ds} = 1,$$

and thus

$$\frac{du}{ds} \;=\; u_x + u_y.$$

Combining these observations, we obtain the system

$$\frac{dx}{ds} = 1, \quad \frac{dy}{ds} = 1, \quad \frac{du}{ds} = e^{\,x+2y} \;-\; u.$$

The boundary condition $u(x, 0) = 0$ implies that when $y = 0$, we must have $u = 0$. We parametrize the boundary by setting $x(0) = \alpha$, $y(0) = 0$, and $u(0) = 0$. As $s$ grows, $\alpha$ remains a constant describing different characteristic curves.

From $\frac{dx}{ds} = 1$ and $\frac{dy}{ds} = 1$, we integrate:

$$x(s) \;=\; \alpha + s, \quad y(s) \;=\; s,$$

so $x - y = \alpha$ remains constant along each characteristic and therefore $\alpha = x - y$.

Next, we solve

$$\frac{du}{ds} \;=\; e^{\,x+2y} \;-\; u.$$

Substituting $x(s) = \alpha + s$ and $y(s) = s$ into $x + 2y$ yields $\alpha + 3s$. Hence,

$$\frac{du}{ds} \;=\; e^{\,\alpha+3s} \;-\; u(s).$$

This is a linear ODE in $u$. To solve it, we multiply through by the integrating factor $e^{\,s}$:

$$\frac{d}{ds}\Big(e^{\,s}\,u(s)\Big) \;=\; e^{\,s}\,e^{\,\alpha+3s} \;=\; e^{\,\alpha}\,e^{\,4s}.$$

Integrating both sides with respect to $s$ gives

$$e^{\,s}\,u(s) \;=\; e^{\,\alpha}\int e^{\,4s}\,ds \;+\; C \;=\; e^{\,\alpha}\,\frac{e^{\,4s}}{4} \;+\; C.$$

Therefore,

$$u(s) \;=\; e^{-\,s}\Big(e^{\,\alpha}\,\tfrac{e^{\,4s}}{4} \;+\; C\Big) \;=\; \frac{e^{\,\alpha}\,e^{\,3s}}{4} \;+\; C\,e^{-s}.$$

We apply the boundary condition $u(0) = 0$. Since $u(0) = \frac{e^{\alpha}}{4} + C$, we obtain

$$0 \;=\; \tfrac{e^{\alpha}}{4} \;+\; C, \quad \text{hence} \quad C \;=\; -\tfrac{e^{\alpha}}{4}.$$

This leads to

$$u(s) \;=\; \frac{e^{\alpha}\,e^{3s}}{4} \;-\; \frac{e^{\alpha}}{4}\,e^{-s} \;=\; \frac{e^{\alpha}}{4}\Big(e^{3s} \;-\; e^{-s}\Big).$$

Recalling $\alpha = x - y$ and $s = y$, we write

$$u(x,y) \;=\; \frac{e^{x-y}}{4}\Big(e^{3y} - e^{-y}\Big) \;=\; \frac{1}{4}\,\Big(e^{x+2y} - e^{x-2y}\Big).$$

Factoring out $e^{x}$ yields

$$u(x,y) \;=\; \frac{e^{x}}{4}\Big(e^{2y} - e^{-2y}\Big) \;=\; \frac{e^{x}}{4}\big(2\,\sinh(2y)\big) \;=\; \frac{1}{2}\,e^{x}\,\sinh\big(2y\big).$$

Hence, the unique solution that satisfies both the PDE and the boundary condition is

$$\boxed{u(x,y) \;=\; \frac{1}{2}\,e^{x}\,\sinh\big(2y\big)}.$$

*Verification.* At $y = 0$, we have $\sinh(0) = 0$, so $u(x,0) = \frac{1}{2}\,e^{x}\cdot 0 = 0$, matching the boundary condition. By directly calculating $u_x$, $u_y$, and $u$, we verify that

$$u_x \;+\; u_y \;+\; u \;=\; e^{x+2y},$$

so the PDE is also satisfied.

## 5.3 Exercise 3

### Problem

Find an explicit expression for the solution $u(t,x)$ of the PDE

$$\partial_t u + c\,\partial_x u + r\,u = 0,$$

subject to the initial condition

$$u(0,x) = f(x).$$

Additionally, discuss what happens to the solution as $t \to +\infty$ when $r > 0$.

### Solution

We begin by recognizing that this PDE can be written as

$$u_t + c\,u_x = -r\,u,$$

a linear first-order equation with constant coefficients $c$ and $r$. One can check that the solution has the form

$$u(t,x) = e^{-r\,t}\,g\big(x - c\,t\big),$$

where $g$ is determined by the initial condition. Because $u(0,x) = f(x)$, setting $t = 0$ gives $g(x) = f(x)$. Hence the unique solution is

$$\boxed{u(t,x) = e^{-r\,t}\, f\big(x - c\,t\big).}$$

Behavior as $t \to \infty$ with $r > 0$: Clearly $e^{-r\,t} \to 0$ as $t \to \infty$. If $f$ is *bounded* (or grows more slowly than an exponential of rate $r$), then for each fixed $x$,

$$\lim_{t\to\infty} u(t,x) \;=\; \lim_{t\to\infty} e^{-r\,t}\, f\big(x - c\,t\big) \;=\; 0,$$

so the solution vanishes pointwise. Without any growth restriction on $f$, the factor $e^{-r\,t}$ may not be sufficient to force the solution to zero.

## 5.4 Exercise 4

### Problem

Express the solution $u(t,x)$ to the transport equation

$$\partial_t u \;-\; x\,t\,\partial_x u \;=\; 0 \quad \text{for } x \in \mathbb{R},$$

subject to the initial condition

$$u(0,x) \;=\; f(x).$$

Then, assuming $f(x)$ has compact support in $[-R,R]$, determine

$$\lim_{t\to+\infty} u(t,x) \quad \text{for } x \neq 0.$$

### Solution

We first rewrite the PDE as

$$u_t \;+\; \big(-\,x\,t\big)\,u_x \;=\; 0,$$

recognizing that the transport velocity in the $(t,x)$-plane is given by $-\,x\,t$. By the method of characteristics, we seek curves $x(t)$ satisfying

$$\frac{dx}{dt} \;=\; -\,x(t)\,t,$$

along which $u$ remains constant (since the PDE is homogeneous). To solve $\frac{dx}{dt} = -\,t\,x(t)$, we separate variables and integrate:

$$\frac{dx}{x} \;=\; -\,t\,dt \;\Longrightarrow\; \ln|x| \;=\; -\,\tfrac{t^2}{2} + C \;\Longrightarrow\; x(t) \;=\; A\,e^{-\frac{t^2}{2}},$$

where $A$ is a constant determined by the initial condition. At $t = 0$, if $x(0) = \alpha$, then $A = \alpha$. Consequently, each characteristic through $(0,\alpha)$ satisfies $x(t) = \alpha\,e^{-\frac{t^2}{2}}$. Because $u$ remains constant along such curves, the solution value at $(t,x)$ equals $f(\alpha)$.

To find $\alpha$ in terms of the point $(t,x)$, we note that

$$x \;=\; \alpha\,e^{-\frac{t^2}{2}} \quad\Longrightarrow\quad \alpha \;=\; x\,e^{\frac{t^2}{2}}.$$

Hence, the solution can be written as

$$\boxed{u(t,x) \;=\; f\!\left(x\,e^{\frac{t^2}{2}}\right)}.$$

Finally, suppose $f$ has compact support contained in $[-R,R]$. Then for any fixed $x \neq 0$, as $t \to +\infty$ the term $x\,e^{\frac{t^2}{2}}$ grows unbounded in magnitude (toward $+\infty$ if $x > 0$, or $-\infty$ if $x < 0$). Thus, for sufficiently large $t$, this argument lies outside $[-R,R]$, where $f$ is zero. Therefore,

$$\lim_{t\to\infty} u(t,x) \;=\; 0 \quad \text{for each } x \neq 0.$$

## 5.5 Exercise 5

### Problem

Use the coordinate method to solve the first-order PDE

$$u_x \;+\; 2\,u_y \;+\; \big(2x - y\big)\,u \;=\; 2\,x^2 \;+\; 3\,x\,y \;-\; 2\,y^2,$$

without any additional boundary or initial condition. Express the general solution in terms of an arbitrary function.

### Solution

We observe that the PDE features the operator $u_x + 2\,u_y$, suggesting a characteristic direction $(1,2)$. To proceed, we make the linear change of variables $\eta = x + 2y$ and $\xi = 2x - y$. From these definitions, solving for $(x,y)$ in terms of $(\eta,\xi)$ shows

$$x = \frac{\eta + 2\xi}{5}, \quad y = \frac{2\eta - \xi}{5}.$$

Next, we apply the chain rule. Denoting $u(\eta,\xi)$ for $u(x(\eta,\xi),\, y(\eta,\xi))$, we compute

$$u_x = u_\eta\,\frac{\partial\eta}{\partial x} + u_\xi\,\frac{\partial\xi}{\partial x} \quad \text{and} \quad u_y = u_\eta\,\frac{\partial\eta}{\partial y} + u_\xi\,\frac{\partial\xi}{\partial y}.$$

Since $\eta = x + 2y$ and $\xi = 2x - y$, we obtain

$$u_x = u_\eta\,(1) + u_\xi\,(2), \quad u_y = u_\eta\,(2) - u_\xi\,(1).$$

Hence

$$u_x + 2\,u_y = \big[u_\eta + 2\cdot 2\,u_\eta\big] + \big[u_\xi \cdot 2 + 2\cdot(-1)\,u_\xi\big] = 5\,u_\eta.$$

Substituting into the PDE

$$u_x + 2\,u_y + (2x - y)\,u = 2x^2 + 3xy - 2y^2,$$

we find

$$5\,u_\eta + \xi\,u = 2x^2 + 3xy - 2y^2,$$

where $\xi = 2x - y$. In the new coordinates, we rewrite $2x^2 + 3xy - 2y^2$ in terms of $\eta,\xi$. From

$$x = \frac{\eta + 2\xi}{5}, \quad y = \frac{2\eta - \xi}{5},$$

one checks by expansion that $2x^2 + 3xy - 2y^2 = \eta\,\xi$. The PDE then becomes

$$5\,u_\eta + \xi\,u = \eta\,\xi.$$

Viewing $\xi$ as a parameter, we divide by 5 to obtain

$$u_\eta + \frac{\xi}{5}\,u = \frac{\eta\,\xi}{5}.$$

This is a linear ODE in $\eta$. The integrating factor is $\exp\!\big(\frac{\xi}{5}\eta\big)$. Multiplying through and integrating gives

$$\frac{\partial}{\partial\eta}\Big(u\,e^{\frac{\xi}{5}\eta}\Big) = \frac{\eta\,\xi}{5}\,e^{\frac{\xi}{5}\eta}.$$

A direct integration (using integration by parts) shows

$$\int \frac{\eta\,\xi}{5}\,e^{\frac{\xi}{5}\eta}\,d\eta = e^{\frac{\xi}{5}\eta}\big(\eta - \tfrac{5}{\xi}\big).$$

Hence, letting $C(\xi)$ be an arbitrary function of $\xi$, we obtain

$$u(\eta,\xi)\,e^{\frac{\xi}{5}\eta} = e^{\frac{\xi}{5}\eta}\,\Big(\eta - \tfrac{5}{\xi}\Big) + C(\xi),$$

so

$$u(\eta,\xi) = \Big(\eta - \tfrac{5}{\xi}\Big) + C(\xi)\,\exp\!\Big(-\tfrac{\xi}{5}\,\eta\Big).$$

Recalling $\eta = x + 2y$ and $\xi = 2x - y$, we find the general solution

$$\boxed{u(x,y) = \big(x + 2y\big) - \frac{5}{2x - y} + C\big(2x - y\big)\,\exp\!\Big[-\,\frac{2x - y}{5}\,\big(x + 2y\big)\Big]},$$

where $C$ is an arbitrary function. Because no initial or boundary data are given, this family constitutes the full set of solutions for the given PDE.

# 6 References

[1] Lawrence C. Evans, *Partial Differential Equations*, 2nd ed., American Mathematical Society, 2010, Chapter 2.1.

[2] Walter A. Strauss, *Partial Differential Equations: An Introduction*, Wiley, 2007, Chapters 1.2 and 1.3.

[3] Walter Craig, *A Course on PDE*, American Mathematical Society, 2018, Exercises 2.2–2.4.