

# **AERO 422: Active Controls for Aerospace Vehicles**

Basic Feedback Analysis & Design

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# **Stabilizing Controller**

# Routh's Stability Criterion

Given characteristic equation

Factor out any roots at the origin and multiply by -1 if needed

$$D_G(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n,$$

How to determine stability? (without using MATLAB :)

■ **Necessary** condition:

- ▶ *All coefficients of characteristic polynomial be positive, i.e.  $a_i > 0$ .*
- ▶ *Any coefficient missing ( $= 0$ ) or negative, then poles are outside LHP.*

■ **Necessary and Sufficient** condition:

- ▶ *System is stable iff all the elements in the first column of Routh array are positive*

# Routh's Stability Criterion

## Routh Array

Row	$n$	$s^n$	1	$a_2$	$a_4$	...
Row	$n-1$	$s^{n-1}$	$a_1$	$a_3$	$a_5$	...
Row	$n-2$	$s^{n-2}$	$b_1$	$b_2$	$b_3$	...
Row	$n-3$	$s^{n-3}$	$c_1$	$c_2$	$c_3$	...
	⋮	⋮	⋮	⋮	⋮	
	⋮	⋮	⋮	⋮	⋮	
Row	2	$s^2$	*	*	*	
Row	1	$s^1$	*	*		
Row	0	$s^0$	*			

where

$$b_1 = -\frac{\det \begin{bmatrix} 1 & a_2 \\ a_1 & a_3 \end{bmatrix}}{a_1}$$

$$c_1 = -\frac{\det \begin{bmatrix} a_1 & a_3 \\ b_1 & b_2 \end{bmatrix}}{b_1}$$

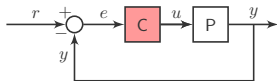
$$b_2 = -\frac{\det \begin{bmatrix} 1 & a_4 \\ a_1 & a_5 \end{bmatrix}}{a_1}$$

$$c_2 = -\frac{\det \begin{bmatrix} a_1 & a_5 \\ b_1 & b_3 \end{bmatrix}}{b_1}$$

There are two special cases! See web-appendix of textbook

# Stabilizing Gain

Example 1 – One parameter



Given  $P(s) = \frac{s+1}{s(s-1)(s+6)}$  and  $C(s) = K$

design parameter

**Characteristic Equation** Numerator of  $1 + PC$  No pole zero cancellation

$$s^3 + 5s^2 + (K - 6)s + K = 0$$

**Routh's Table**

$s^3$	1	$K - 6$
$s^2$	5	$K$
$s^1$	$\frac{4K-30}{5}$	
$s^0$	$K$	

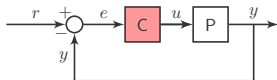
**Stabilizing Gains**

$$\frac{4K - 30}{5} > 0, K > 0$$

$$K > 7.5, K > 0$$

# Stabilizing Gain

Example 2 – Two parameters



Given  $P(s) = \frac{1}{(s+1)(s+2)}$ ,  $C(s) = K + \frac{K_I}{s}$

**Characteristic Equation** Numerator of  $1 + PC$  No pole zero cancellation

$$s^3 + 3s^2 + (2 + K)s + K_I = 0$$

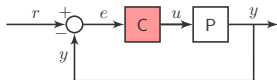
**Routh's Table**

$s^3$	1	$2 + K$
$s^2$	3	$K_I$
$s^1$	$\frac{6+3K-K_I}{3}$	
$s^0$	$K_I$	

**Stabilizing Gains**  $K > K_I/3 - 2$ ,  $K_I > 0$ .

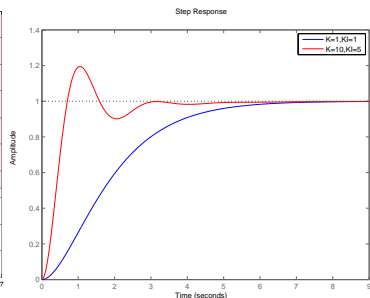
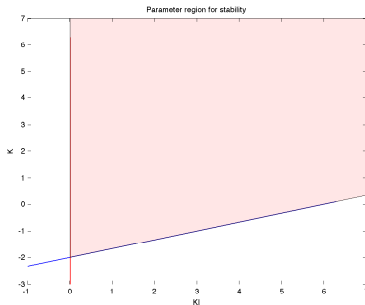
# Stabilizing Gain

Example 1 – Two parameters (contd.)



$$\text{Given } P(s) = \frac{1}{(s+1)(s+2)}, \quad C(s) = K + \frac{K_I}{s}$$

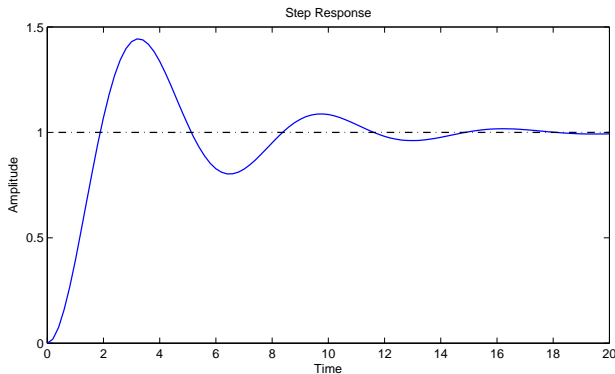
**Stabilizing Gains**  $K > K_I/3 - 2$ ,  $K_I > 0$ .



What about **performance**?

# Step Response

## Time Domain Performance Specification



**Second Order System:** poles =  $\sigma \pm j\omega_d$ ,  $\omega_n = \sqrt{\sigma^2 + \omega_d^2}$ ,  $\zeta = \sigma/\omega_n$

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

$$t_r = \frac{1.8}{\omega_n}$$

$$t_s = \frac{4.6}{\sigma}$$



# Step Response

## Time Domain Performance Specification – *Second Order Systems*

### Desired Location of Poles

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \quad t_r = \frac{1.8}{\omega_n} \quad t_s = \frac{4.6}{\sigma}$$

$$\omega_n \geq 1.8/t_r \quad \zeta \geq \zeta(M_p) \quad \sigma \geq 4.6/t_s$$

- Adjust  $K, K_I$  to satisfy additional performance related constraints
- Controller gain tuning

# Routh's Stability Criterion

Conditions that ensure  $\text{Re } p_i < -\alpha$ , for  $\alpha > 0$

Given polynomial

$$s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

Modify Routh's stability criterion to ensure

$$\text{Re } p_i < -\alpha, \text{ for } \alpha > 0.$$

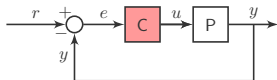
Replace  $s := q - \alpha$  and substitute in polynomial to get

$$q^n + b_1 q^{n-1} + \cdots + b_{n-1} q + b_n = 0$$

Apply Routh's criterion to the polynomial in  $q$

# Routh's Stability Criterion

## Example



- Given  $P(s) = \frac{1}{s^2 + 4s + 1}$  and controller  $C(s) = K_1 + K_2/s$ .
- Find range of values for  $K_1, K_2$  such that all poles are left of  $-\alpha$ .

Characteristic equation:

$$s^3 + 4s^2 + (K_1 + 1)s + K_2$$

Substitute  $s := q - \alpha$ , with  $\alpha = 1$

$$q^3 + q^2 + (K_1 - 4)q - K_1 + K_2 + 2$$

Apply Routh's criterion

# Routh's Stability Criterion

*Example (contd.)*

**Polynomial in  $q$**

$$q^3 + q^2 + (K_1 - 4)q - K_1 + K_2 + 2$$

**Routh's Table**

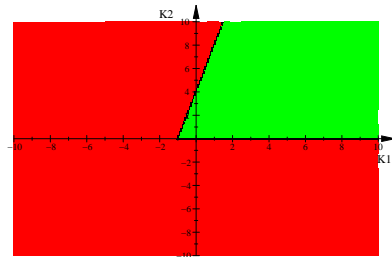
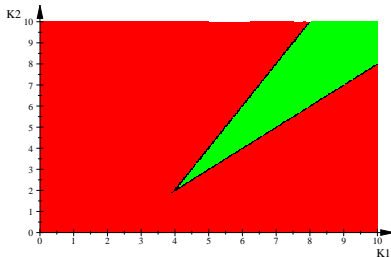
$s^3$	1	$K_1 - 4$
$s^2$	1	$K_2 - K_1 + 2$
$s^1$	$2K_1 - K_2 - 6$	0
$s^0$	$K_2 - K_1 + 2$	0

**Inequalities**

$$2K_1 - K_2 - 6 > 0, \quad K_2 - K_1 + 2 > 0.$$

# Routh's Stability Criterion

*Example (contd.)*

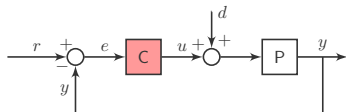


- (a)  $2K_1 - K_2 - 6 > 0$ ,  $K_2 - K_1 + 2 > 0$ , Poles left of 0.  
Poles left of  $-1$ .
- (b)  $K_1 - K_2/4 + 1 > 0$ ,  $K_2 > 0$ ,  
Poles left of  $-1$ .

# **Benefits of Feedback**

# Benefits of Feedback

## *Disturbance Rejection*



- Let  $P(s) := \frac{A}{(s/p_1+1)(s/p_2+1)}$ , and  $C(s) := K$
- Total response

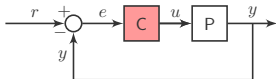
$$\begin{aligned} Y(s) &= \frac{PC}{1+PC} R(s) + \frac{P}{1+PC} D(s) \\ &= \frac{AK}{(s/p_1+1)(s/p_2+1)+AK} R(s) + \frac{A}{(s/p_1+1)(s/p_2+1)+AK} D(s) \end{aligned}$$

- Steady state value *with feedback*

$$\lim_{s \rightarrow 0} sY(s) = \frac{AK}{1+AK} \left( \lim_{s \rightarrow 0} sR(s) \right) + \frac{A}{1+AK} \left( \lim_{s \rightarrow 0} sD(s) \right)$$

# Benefits of Feedback

## *Robustness to Plant Uncertainty*



Transfer function from reference to output

$$G_{yr}(s) = \frac{PC}{1 + PC} = \frac{AK}{(s/p_1 + 1)(s/p_2 + 1) + AK}$$

- Suppose  $A \rightarrow A + \delta A$
- What is the effect on  $T(s) = G_{yr}(s)$ ? steady state gain



# Benefits of Feedback

*Robustness to Plant Uncertainty (contd.)*

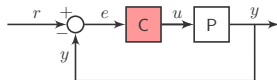
$$T_{ss} = \frac{AK}{1 + AK}$$

$$\begin{aligned}\delta T_{ss} &= \frac{dT_{ss}}{dA} \delta A \\ &= \frac{K}{(1 + AK)^2} \delta A \\ &= \left( \frac{AK}{1 + AK} \right) \left( \frac{1}{1 + AK} \right) \frac{\delta A}{A} \\ \Rightarrow \frac{\delta T_{ss}}{T_{ss}} &= \frac{1}{1 + AK} \frac{\delta A}{A}\end{aligned}$$

# **System Type**

# System Type

## Analysis of Steady State Error



$$E(s) = \frac{1}{1 + PC} R(s)$$

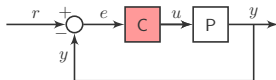
$$\begin{aligned} \Rightarrow e_{ss} &= \lim_{s \rightarrow 0} s \frac{1}{1 + L(s)} \frac{1}{s^{k+1}} \\ &= \lim_{s \rightarrow 0} \frac{1}{1 + L(s)} \frac{1}{s^k} \end{aligned}$$

Investigate  $e_{ss} \rightarrow 0$  for various values of  $k$

Value of $k$	$r(t)$	System Type
0	$1(t)$	$e_{ss} = \text{constant} \Rightarrow \text{Type 0}$
1	$t$	$e_{ss} = \text{constant} \Rightarrow \text{Type I}$
2	$t^2/2!$	$e_{ss} = \text{constant} \Rightarrow \text{Type II}$

# Steady State Error

Type Zero



## Type 0 System

- Constant steady state error to step reference  $k = 0$ .

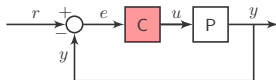
$$\begin{aligned} e_{ss} &= \lim_{s \rightarrow 0} \frac{1}{1 + L(s)} \frac{1}{s^k} \\ &= \lim_{s \rightarrow 0} \frac{1}{1 + L(s)} = \frac{1}{1 + K_p} \end{aligned}$$

## Position Error Constant $K_p$

$$K_p = \lim_{s \rightarrow 0} L(s)$$

# Steady State Error

Type One



## Type 1 System

- Constant steady state error to ramp reference  $k = 1$ .

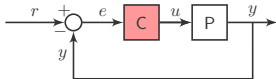
$$\begin{aligned} e_{ss} &= \lim_{s \rightarrow 0} \frac{1}{1 + L(s)} \frac{1}{s^k} \\ &= \lim_{s \rightarrow 0} \frac{1}{1 + L(s)} \frac{1}{s} = \lim_{s \rightarrow 0} \frac{1}{sL(s)} = \frac{1}{K_v} \end{aligned}$$

## Velocity Error Constant $K_v$

$$K_v = \lim_{s \rightarrow 0} sL(s)$$

# Steady State Error

Type Two



## Type 2 System

- Constant steady state error to parabolic reference  $k = 2$ .

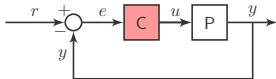
$$\begin{aligned} e_{ss} &= \lim_{s \rightarrow 0} \frac{1}{1 + L(s)} \frac{1}{s^k} \\ &= \lim_{s \rightarrow 0} \frac{1}{1 + L(s)} \frac{1}{s^2} = \lim_{s \rightarrow 0} \frac{1}{s^2 L(s)} = \frac{1}{K_a} \end{aligned}$$

## Acceleration Error Constant $K_a$

$$K_a = \lim_{s \rightarrow 0} s^2 L(s)$$

# Steady State Error

## Summary



## Various Constants

$$K_p = \lim_{s \rightarrow 0} L(s)$$

$$K_v = \lim_{s \rightarrow 0} sL(s)$$

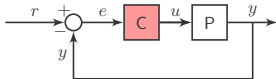
$$K_a = \lim_{s \rightarrow 0} s^2 L(s)$$

## Steady State Errors

	Step	Ramp	Parabola
Type 0	$\frac{1}{1+K_p}$	$\infty$	$\infty$
Type I	0	$\frac{1}{K_v}$	$\infty$
Type II	0	0	$\frac{1}{K_a}$

# Steady State Error

*Summary (contd.)*



- Quickly identify ability to track polynomials
- Robustness property – higher type tracks lower order polynomials
- Can be extended to study  $G_{yd}$  and other transfer functions



# Truxal's Formula

- Let  $T(s) := G_{yr}(s)$  be given by

$$T(s) = K \frac{\prod_{i=1}^m (s - z_i)}{\prod_{i=1}^n (s - p_i)}.$$

- Most common case:  $e_{ss}$  to step is zero  $\implies$  Type I system
- DC gain  $\lim_{s \rightarrow 0} T(s) = 1$
- System error  $E(s) = R(s)(1 - T(s))$
- System error due to ramp

$$e_{ss} = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} s(1 - T(s)) \frac{1}{s^2} = \lim_{s \rightarrow 0} \frac{1 - T(s)}{s}$$

- Using L'Hopital's rule

$$e_{ss} = - \lim_{s \rightarrow 0} \frac{dT}{ds} = \frac{1}{K_v} \text{ for type I systems}$$

# Truxal's Formula

Contd.

- Let  $T(s) := G_{yr}(s)$  be given by

$$T(s) = K \frac{\prod_{i=1}^m (s - z_i)}{\prod_{i=1}^n (s - p_i)}.$$

- Using L'Hopital's rule

$$e_{ss} = - \lim_{s \rightarrow 0} \frac{dT}{ds} = \frac{1}{K_v} \text{ for type I systems}$$

- $\frac{1}{K_v}$  is related to the slope of  $T(s)$  at origin

# Truxal's Formula

Contd.

- Let  $T(s) := G_{yr}(s)$  be given by

$$T(s) = K \frac{\prod_{i=1}^m (s - z_i)}{\prod_{i=1}^n (s - p_i)}.$$

- Rewrite

$$\begin{aligned} e_{ss} &= - \lim_{s \rightarrow 0} \frac{dT}{ds} \frac{1}{T} \quad T(0) = 1 \\ &= - \lim_{s \rightarrow 0} \frac{d}{ds} \log T(s) \\ &= - \lim_{s \rightarrow 0} \frac{d}{ds} \left( \log(K) + \sum_{i=1}^m \log(s - z_i) - \sum_{i=1}^n \log(s - p_i) \right) \end{aligned}$$

## Truxal's Formula

$$\frac{1}{K_v} = \sum_{i=1}^m \frac{1}{z_i} - \sum_{i=1}^n \frac{1}{p_i}$$

# Truxal's Formula

## *Design Implication*

$$\frac{1}{K_v} = \sum_{i=1}^m \frac{1}{z_i} - \sum_{i=1}^n \frac{1}{p_i}$$

- Observe effect of pole/zero location on  $1/K_v$
- Useful for design of dynamic compensators

## Example

- Third order type I system has closed-loop poles  $-2 \pm 2j, -0.1$ .
- The system has one zero. Where should it be for  $K_v = 10$ ?