$\mathcal{H}_2$ Optimal State Feedback Control Synthesis

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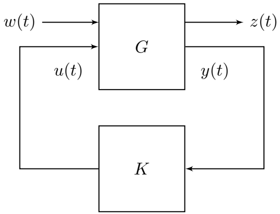
Motivation

Motivation



- $w(t)$ are exogenous signals – reference, process noise, sensor noise
- $z(t)$ are signals we want to keep small – $e(t)$, $u(t)$
- Find controller K that minimizes $\|G_{w\to z}\|_2$

- Appropriate when spectral density function of $w(t)$ is known
  - *For example $w(t)$ can be **stationary noise***
  - *$\mathcal{H}_2$ optimal is then linear quadratic Gaussian (LQG)*

Motivation

Stochastic Input

- For stationary stochastic process $w(t)$, autocorrelation matrix is

$$R_w(\tau) := \mathbf{E}\left[w(t+\tau)w^*(t)\right]$$

- The Fourier transform of $R_w(\tau)$ is the spectral density $\hat{S}_w(j\omega)$
- Signal power is related to it by

$$\mathbf{E}\left[\|w(t)\|\right]^2 := \mathbf{tr}\left[R_w(0)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{tr}\left[\hat{S}_w(j\omega)\right]d\omega$$

- If z and w are related by $z = Pw$, for a stable LTI system P, then

$$\hat{S}_z(j\omega) = \hat{P}(j\omega)\hat{S}_w(j\omega)\hat{P}^*(j\omega)$$

Motivation

Stochastic Input (contd.)

- Therefore,

$$\mathbf{E}\left[\|z(t)\|\right]^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{tr}\left[\hat{P}(j\omega)\hat{S}_w(j\omega)\hat{P}^*(j\omega)\right] d\omega.$$

  Looks like weighted $\mathcal{H}_2$ norm of P.

- If $w(t)$ is white noise $\implies \hat{S}_w(j\omega) = I$.
  $\mathcal{H}_2$ norm is output variance with white noise input.
- For any other $\hat{S}_w(j\omega)$,

$$\hat{S}_w(j\omega) = \hat{W}(j\omega)\hat{W}^*(j\omega).$$

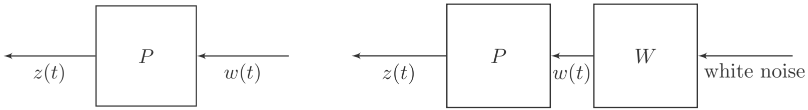
  Therefore,

$$\mathbf{E}\left[\|z(t)\|\right]^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{tr}\left[\left(\hat{P}(j\omega)\hat{W}(j\omega)\right) I \left(\hat{W}^*(j\omega)\hat{P}^*(j\omega)\right)\right] d\omega.$$

- Think of $\hat{P}(j\omega)\hat{W}(j\omega)$ as weighted system

Motivation

Stochastic Input (contd.)



$$\mathbf{E}\left[\|z(t)\|\right]^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{tr}\left[\left(\hat{P}(j\omega)\hat{W}(j\omega)\right) I \left(\hat{W}^*(j\omega)\hat{P}^*(j\omega)\right)\right] d\omega.$$

Think of $\hat{P}(j\omega)\hat{W}(j\omega)$ as weighted system

Motivation

Impulse Response

- Input signal is known in advance
- Tracking a fixed signal – e.g. step

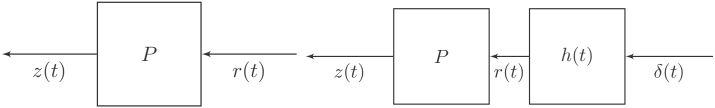
Consider a special case: Scalar $w(t) = \delta(t)$. Implies

$$
\begin{aligned}
\|z\|_2^2 &= \int_0^\infty z^*(t)z(t)dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{z}^*(j\omega)\hat{z}(j\omega)d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{P}^*(j\omega)\hat{P}(j\omega)d\omega \\
&= \|\hat{P}\|_2^2
\end{aligned}
$$

Motivation

Impulse Response (contd.)

What about any other reference $r(t)$?



- $r(t)$ can be replaced by the impulse response $h(t)$ of a known filter $W(j\omega)$
- $z(t)$ becomes impulse response of a weighted plant

Computation of $\mathcal{H}_2$ Norm

- Best computed in state-space realization of system

State Space Model: General MIMO LTI system modeled as

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du,$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$.

Transfer Function

$$\hat{G}(s) = D + C(sI - A)^{-1}B \text{ strictly proper when } D = 0$$

Impulse Response

$$G(t) = \mathcal{L}^{-1}\left\{C(sI - A)^{-1}B\right\} = Ce^{tA}B.$$

$\mathcal{H}_2$ Norm

MIMO Systems

$$\|\hat{G}(j\omega)\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{tr}\left[\hat{G}^*(j\omega)\hat{G}(j\omega)\right] \quad \text{for matrix transfer function}$$

$$= \|G(t)\|_2^2 \quad \text{Parseval}$$

$$= \int_0^{\infty} \mathbf{tr}\left[Ce^{tA}BB^Te^{tA^T}C^T\right] dt$$

$$= \mathbf{tr}\left[C\underbrace{\left(\int_0^{\infty} e^{tA}BB^Te^{tA^T}dt\right)}_{L_c}C^T\right] \quad L_c = \text{controllability Gramian}$$

$$= \mathbf{tr}\left[CL_cC^T\right]$$

$\mathcal{H}_2$ Norm (contd.)

MIMO Systems

For any matrix M

$$\begin{aligned}
\mathbf{tr}\left[M^*M\right] &= \mathbf{tr}\left[MM^*\right] \\
\implies \quad \|\hat{G}(j\omega)\|_2^2 &= \mathbf{tr}\left[B^T \underbrace{\left(\int_0^\infty e^{tA^T}C^TCe^{tA}dt\right)}_{L_o} B\right] \\
&= \mathbf{tr}\left[B^TL_oB\right] \quad L_o = \text{observability Gramian}
\end{aligned}$$

$\mathcal{H}_2$ Norm of $\hat{G}(j\omega)$

$$\|\hat{G}(j\omega)\|_2^2 = \mathbf{tr}\left[CL_cC^T\right] = \mathbf{tr}\left[B^TL_oB\right].$$

$\mathcal{H}_2$ Norm

How to determine L_c and L_o?

They are solutions of the following equation

$$AL_c + L_cA^T + BB^T = 0,$$
$$A^TL_o + L_oA + C^TC = 0.$$

State-Feedback $\mathcal{H}_2$ Synthesis

Proposition 1 Suppose P is a state-space system with realization (A, B, C). Then

$$A \text{ is Hurwitz and } \|\hat{P}\|_2 \leq 1,$$

iff $\exists\, X = X^T > 0$ such that

$$\mathbf{tr}\,[CXC^*] < 1 \text{ and } AX + XA^* + BB^* < 0.$$

Proof Only If
Recall $\|\hat{P}\|_2 = \mathbf{tr}\,[CL_cC^*]$. Therefore for $X = L_c$

$$\mathbf{tr}\,[CL_cC^*] < 1 \implies \|\hat{P}\|_2 < 1,$$

and A is Hurwitz $\|\hat{P}\|_2$ is finite

State-Feedback $\mathcal{H}_2$ Synthesis

contd.

Consider
$$X = \int_0^\infty e^{tA}(BB^* + \epsilon I_n)e^{tA^*}dt,$$

- is continuous in ϵ
- equals to L_c when $\epsilon = 0$.

It can be shown that this X satisfies Lyapunov equation

follow the derivation

$$AX + XA^* + BB^* + \epsilon I_n = 0,$$

or

$$AX + XA^* + BB^* < 0.$$

State-Feedback $\mathcal{H}_2$ Synthesis

contd.

Proof If

X satisfies

$$AX + XA^* + BB^* < 0$$

and

$$\mathbf{tr}\left[CXC^*\right] < 1.$$

Implies, A is Hurwitz.

Proposition: It can be shown that if L_c satisfies

$$A^* L_c + L_c A + Q = 0,$$

and X satisfies

$$A^* X + XA + Q \leq 0,$$

then $X \geq L_c$.

State-Feedback $\mathcal{H}_2$ Synthesis

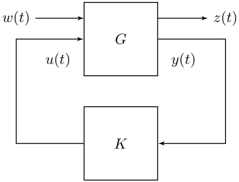
contd.

Inequality $X \geq L_c$ implies

$$\|\hat{P}\|_2 = \mathbf{tr}\,[CL_cC^*] \leq \mathbf{tr}\,[CXC^*] < 1.$$

State-Feedback $\mathcal{H}_2$ Synthesis

System Dynamics



Dynamics

$$\begin{aligned} \dot{x} &= Ax + B_w w + B_u u \\ z &= C_z x + D_u u \\ y &= x \end{aligned} \implies \hat{G}(s) = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & 0 & D_u \\ I & 0 & 0 \end{array}\right]$$

With $u = Kx$,

$$\mathcal{F}_l(\hat{G}, K) := \hat{G}_{w\to z}(s) = \left[\begin{array}{c|c} A + B_u K & B_w \\ \hline C_z + D_u K & 0 \end{array}\right].$$

State-Feedback $\mathcal{H}_2$ Synthesis

Controller Synthesis

Proposition There exists feedback gain K that internally stabilizes G and satisfies

$$\|\mathcal{F}_l(\hat{G}, K)\|_2 < 1$$

iff $\exists\, Z \in \mathbb{R}^{m\times n}$ such that

$$K = ZX^{-1},$$

where $X > 0$ satisfies inequalities

$$\begin{bmatrix} A & B_u \end{bmatrix}\begin{bmatrix} X \\ Z \end{bmatrix} + \begin{bmatrix} X & Z^* \end{bmatrix}\begin{bmatrix} A^* \\ B_u^* \end{bmatrix} + B_w B_w^* < 0,$$

$$\mathbf{tr}\left[(C_z X + D_u Z)X^{-1}(C_z X + D_u Z)^*\right] < 1.$$

Proof: Use closed-loop state-space data and earlier proposition.
Not an LMI.

State-Feedback $\mathcal{H}_2$ Synthesis

Controller Synthesis (contd.)

Apply Schur complement to get following convex problem.
There exists feedback gain K that internally stabilizes G satisfying

$$\|\mathcal{F}_l(\hat{G}, K)\|_2 < 1$$

iff $\exists \;\; X \in \mathbb{R}^{n\times n}, W \in \mathbb{R}^{q\times q}$, and $Z \in \mathbb{R}^{m\times n}$, such that

$$K = ZX^{-1},$$

and

$$\min_{W} \mathbf{tr}\,[W]$$

$$\begin{bmatrix} A & B_u \end{bmatrix}\begin{bmatrix} X \\ Z \end{bmatrix} + \begin{bmatrix} X & Z^* \end{bmatrix}\begin{bmatrix} A^* \\ B_u^* \end{bmatrix} + B_w B_w^* < 0,$$

$$\begin{bmatrix} W & (C_z X + D_u Z) \\ (C_z X + D_u Z)^* & X \end{bmatrix} > 0$$

State-Feedback $\mathcal{H}_2$ Synthesis

Very simple example

System Dynamics

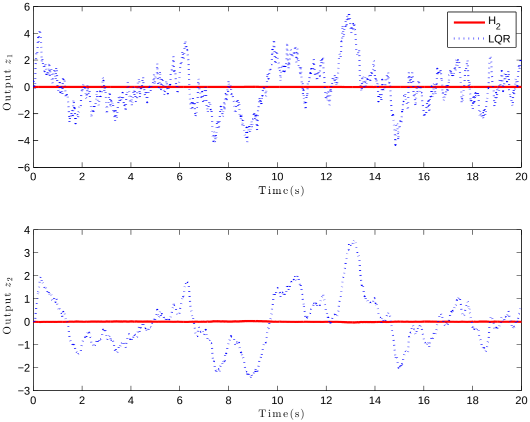
$$\dot{x} = \begin{bmatrix} -3 & -2 & 1 \\ 1 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix} x + \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} u + \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} w,$$

$$z = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} u.$$

Compare with LQR

State-Feedback $\mathcal{H}_2$ Synthesis

Very simple example



$\textbf{tr}\,[W^*] = 3.3491e-4$ – Good disturbance attenuation.

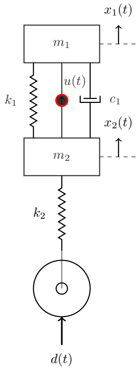
MATLAB Code

```
clear; clc;

A = [-3 -2 1;
      1 2 1;
      1 -1 -1;];
Bu = [2 0; 0 2; 0 1];
Bw = [3;0;1];

Cz = [1 0 1; 0 1 1];
Du = [1 1; 0 1];

nx = 3;
nu = 2;
nz = 2;
nw = 1;

cvx_begin sdp
  variable X(nx,nx) symmetric
  variable W(nz,nz) symmetric
  variable Z(nu,nx)

  [A Bu]*[X;Z] + [X Z']*[A';Bu'] + Bw*Bw' < 0
  [W (Cz*X + Du*Z); (Cz*X + Du*Z)' X] > 0
  minimize trace(W)
cvx_end
```

```
h2K = Z*inv(X);
[lqrK,S,E] = lqr(A,Bu,Cz'*Cz,Du'*Du);

h2G = ss(A+Bu*h2K, Bw, Cz + Du*h2K, zeros(nw,1))
lqrG = ss(A-Bu*lqrK, Bw, Cz + Du*lqrK , ...
       zeros(nw,1));

T = [0:0.01:20];
w = 5*randn(length(T),1);
[y1,t1,x1] = lsim(h2G,w,T);
[y2,t2,x2] = lsim(lqrG,w,T);
f1=figure(1); clf;
set(f1,'defaulttextinterpreter','latex');

for i=1:2
    subplot(2,1,i);
    plot(t1,y1(:,i),'r',t2,y2(:,i),'b:', ...
    'linewidth',2);
    xlabel('Time(s)');
    ylabel(sprintf('Output $z_%d$',i));
end
subplot(2,1,1); legend('H_2','LQR');
print -depsc h2ex1.eps
```

State-Feedback $\mathcal{H}_2$ Synthesis

Regulator with disturbance – Active Suspension



Equations of motion

$$m_1\ddot{x}_1 = -k_1(x_1 - x_2) - c_1(\dot{x}_1 - \dot{x}_2) + u,$$
$$m_2\ddot{x}_2 = k_1(x_1 - x_2) + c_1(\dot{x}_1 - \dot{x}_2) - k_2(x_2 - d) - u.$$

State Variables

$$q_1 := x_1, \qquad q_2 := x_2,$$
$$q_3 := \dot{x}_1, \qquad q_4 := \dot{x}_2.$$

State-Feedback $\mathcal{H}_2$ Synthesis

Regulator with disturbance – Active Suspension

Linear System

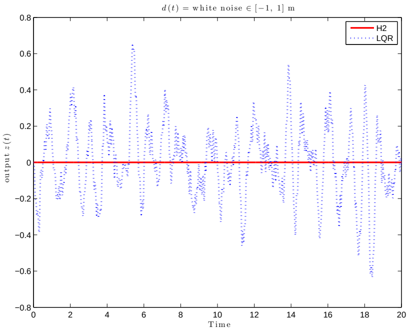
$$\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\\ \dot{q}_3\\ \dot{q}_4\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -k_1/m_1 & k_1/m_1 & -c_1/m_1 & c_1/m_1\\ k_1/m_2 & -(k_1+k_2)/m_2 & c_1/m_2 & -c_1/m_2\end{bmatrix}\begin{bmatrix}q_1\\ q_2\\ q_3\\ q_4\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1/m_1\\ -1/m_2\end{bmatrix}u + \begin{bmatrix}0\\ 0\\ 0\\ k_2/m_2\end{bmatrix}d.$$

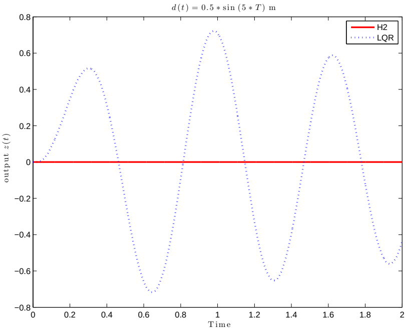
Output $z(t)$

$$z = q_1 + u.$$

State-Feedback $\mathcal{H}_2$ Synthesis

Regulator with disturbance – Active Suspension





$\mathbf{tr}\,[W^*] = 1.17367e - 10$ – Very good disturbance rejection.

MATLAB Code

```
clear; clc;
% System Parameters
m1 = 290; % kg -- Body mass
m2 = 60;  % kg -- suspension mass

k1 = 16200; % N/m
k2 = 191000; % N/m
c1 = 1000; % Ns/m

A = [0 0 1 0;
    0 0 0 1;
    -k1/m1 k1/m1 -c1/m1 c1/m1;
    k1/m1 -(k1+k2)/m2 c1/m2 -c1/m2];
Bu = [0;0;1/m1;-1/m2];
Bw = [0;0;0;k2/m2];

nx = 4; nu = 1;
nz = 1; nw = 1;

Cz = [1,0,0,0];
Du = 1*ones(nz,nu);
```

```
cvx_begin sdp
   variable X(nx,nx) symmetric
   variable W(nz,nz) symmetric
   variables Z(nu,nx) gam

   [A Bu]*[X;Z] + [X Z']*[A';Bu'] + Bw*Bw' < 0
   [W (Cz*X + Du*Z); (Cz*X + Du*Z)' X] > 0
   minimize trace(W)
cvx_end

h2K = Z*inv(X);
[lqrK,S,E] = lqr(A,Bu,Cz'*Cz,Du'*Du);

% Simulation
h2G = ss(A+Bu*h2K, Bw, Cz + Du*h2K, zeros(nz,nw));
lqrG = ss(A-Bu*lqrK, Bw, Cz + Du*lqrK, zeros(nz,nw));
T = [0:0.01:20]/10;
w = 2*rand(length(T),1)-1;
w = 0.5*sin(10*T);
[y1,t1,x1] = lsim(h2G,w,T);
[y2,t2,x2] = lsim(lqrG,w,T);

f1 = figure(1); clf;
set(f1,'defaulttextinterpreter','latex');
plot(t1,y1,'r',t2,y2,'b:','Linewidth',2);
xlabel('Time'); ylabel('output $z(t)$');
title('$d(t) =  0.5*\sin\;\; (5*T)$ m');
title('$d(t)$ =  white noise $\in [-1,\;1]$ m');
legend('H2','LQR');
print -depsc h2qcar2.eps
```

State-Feedback $\mathcal{H}_2$ Synthesis

Tracking with disturbance

Let dynamical system be

$$\dot{x} = Ax + B_u u + B_d d, \qquad z = \begin{bmatrix} x \\ u \end{bmatrix}, \qquad y = C_y x.$$

- Design a controller of the form

$$u = Kx + Gr$$

- K is designed in $\mathcal{H}_2$ optimal sense
- G is regular tracking gain

State-Feedback $\mathcal{H}_2$ Synthesis

Tracking with disturbance

Closed-loop system with K is therefore

$$\dot{x} = (A + BK)x + B_u G r, \qquad y = C_y x.$$

- Ignore disturbance when determining G
- Steady-state response to constant r is

$$0 = (A + BK)x_{\mathsf{ss}} + BGr, \qquad y_{\mathsf{ss}} = C_y x_{\mathsf{ss}} = r.$$

Or $x_{\mathsf{ss}} = -(A + BK)^{-1}BGr$, implies

$$-C_y(A + BK)^{-1}BGr = r,$$

or

$$-C_y(A + BK)^{-1}BG = I.$$

- Solve for G

State-Feedback $\mathcal{H}_2$ Synthesis

Tracking with disturbance

- Existence of solution of
$$C_y(A+BK)^{-1}BG = I$$
is necessary and sufficient condition for existence of a tracking controller.
- In general, when
$$y = C_y x + D_y u,$$
the equation becomes
$$\left[D_y - (C_y + D_y K)(A+BK)^{-1}B\right]G = I.$$
- Can be rewritten in terms of Π
$$\Pi = -(A+BK)^{-1}BG \implies (C_y + D_y K)\Pi + D_y G = I,$$

State-Feedback $\mathcal{H}_2$ Synthesis

Tracking with disturbance

Or

$$
\begin{aligned}
(A + BK)\Pi + BG &= 0, \\
(C_y + D_y K)\Pi + D_y G &= I.
\end{aligned}
$$

Or rearranged to

$$
\begin{aligned}
A\Pi + B(K\Pi + G) &= 0, \\
C_y\Pi + D_y(K\Pi + G) &= I.
\end{aligned}
$$

Therefore, with $\Gamma := K\Pi + G$, we get

$$
\begin{bmatrix} A & B \\ C_y & D_y \end{bmatrix} \begin{bmatrix} \Pi \\ \Gamma \end{bmatrix} + \begin{bmatrix} 0_{n_x \times n_y} \\ -I_{n_y \times n_y} \end{bmatrix} = 0.
$$

This is the so called regulator equation. Get $G = \Gamma - K\Pi$.

State-Feedback $\mathcal{H}_2$ Synthesis

Tracking with disturbance

- Solve regulator equation

$$\begin{bmatrix} A & B \\ C_y & D_y \end{bmatrix} \begin{bmatrix} \Pi \\ \Gamma \end{bmatrix} + \begin{bmatrix} 0_{n_x \times n_y} \\ -I_{n_y \times n_y} \end{bmatrix} = 0.$$

- Get $G = \Gamma - K\Pi$.
- Control law is

$$u = Kx + Gr.$$

State-Feedback $\mathcal{H}_2$ Synthesis

Tracking with disturbance – Longitudinal F16 Control Law

Longitudinal Motion

- States $[V(\text{ft/s})\ \alpha(\text{rad})\ \theta(\text{rad})\ q(\text{rad/s})]^T$
- Controls $[T(\text{lb})\ \delta_e(\text{deg})]$
- Constraints on u

$$\begin{bmatrix}1000\\-25\end{bmatrix} \leq u \leq \begin{bmatrix}19000\\25\end{bmatrix}$$

- Trimmed at steady-level flight at 932 ft/s

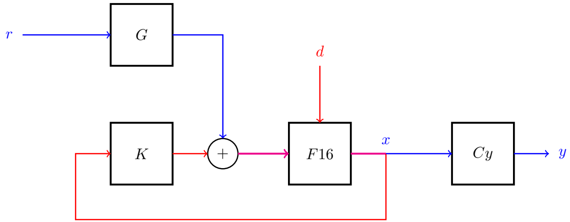
$$x_{\text{trim}} = \begin{bmatrix}932.2894\\0\\0\\0\end{bmatrix}, u_{\text{trim}} = \begin{bmatrix}5318.2\\-1.3935\end{bmatrix}$$

- Disturbance in $\dot{\alpha}$ equation

State-Feedback $\mathcal{H}_2$ Synthesis

Tracking with disturbance – Longitudinal F16 Control Law

System Interconnection



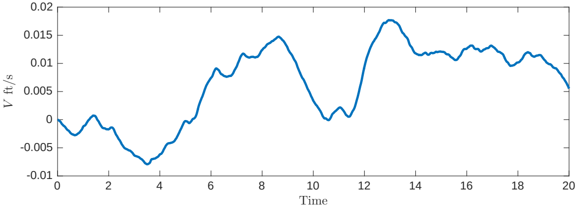
Control Objective

- Design K to minimize $\|G_{d\to x}\|_2$
- Design G to track $r := \begin{bmatrix} V \\ \gamma \end{bmatrix}$ reference

State-Feedback $\mathcal{H}_2$ Synthesis

Tracking with disturbance – Longitudinal F16 Control Law

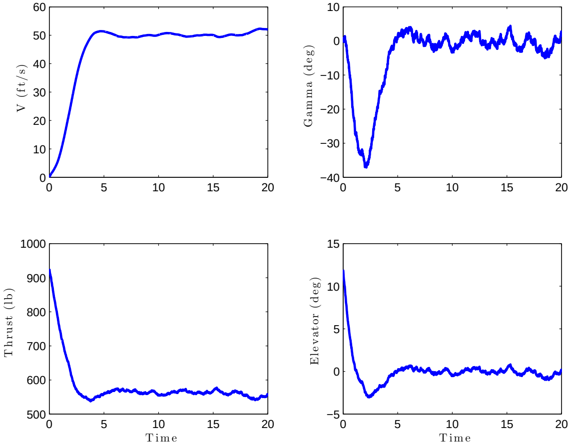
Disturbance Rejection $d(t) \in \mathcal{U}_{[-1,1]}$ rad, $\mathbf{tr}\left[W^*\right] = 0.118159$



State-Feedback $\mathcal{H}_2$ Synthesis

Tracking with disturbance – Longitudinal F16 Control Law

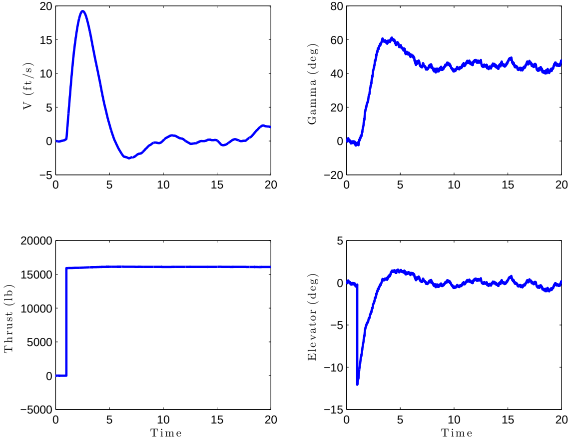
Tracking Performance $V_{\text{ref}} = 50$ ft/s step, $\gamma_{\text{ref}} = 0$



State-Feedback $\mathcal{H}_2$ Synthesis

Tracking with disturbance – Longitudinal F16 Control Law

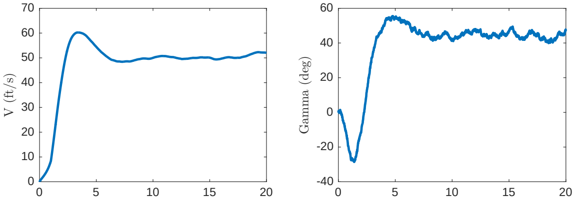
Tracking Performance $V_{\text{ref}} = 0$, $\gamma_{\text{ref}} = 45$ deg step



State-Feedback $\mathcal{H}_2$ Synthesis

Tracking with disturbance – Longitudinal F16 Control Law

Tracking Performance $V_{\text{ref}} = 50$ ft/s step, $\gamma_{\text{ref}} = 45$ deg step



State-Feedback $\mathcal{H}_2$ Synthesis

Tracking with disturbance – Longitudinal F16 Control Law

Simulink