

Polynomial Chaos

Workshop on Uncertainty Analysis & Estimation (ACC 2015)

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Introduction

What is polynomial chaos theory?

It provides a **non-sampling** based method to determine **evolution of uncertainty** in dynamical system, when there is **probabilistic uncertainty** in the system **parameters**.

Consider a dynamical system

- $\dot{x} = -ax$, $x(t_0) = x_0$ is given (known)
- a is an unknown parameter in the range $[0, 1]$ (equally likely values)

Polynomial chaos theory helps us answer these questions

- How does $x(t)$ evolve for various values of a ?
- What is the ensemble behavior of x (mean, variance, PDF)?

Monte-Carlo Approach

Summary of Steps

- $\dot{x} = -ax, x(t_0) = 1$
- a is an unknown parameter in the range $[0, 1]$ (equally likely values)
- Sample $a \in [0, 1]$
- Plot $x(t)$ for every value of a
- Estimate statistics from data

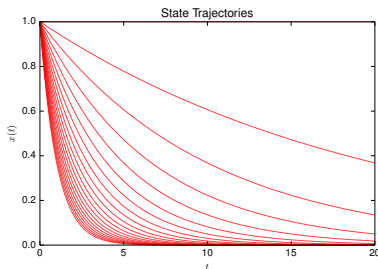
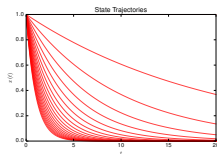


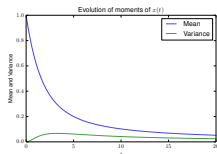
Figure: Sample paths

Monte-Carlo Approach

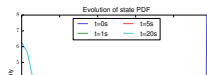
Solution Interpretation



(a) Sample paths



(b) Moments



Polynomial Chaos

Basic Idea

- Approximate $x(t, a)$, solution of $\dot{x} = -ax$ as

$$\hat{x}(t, a) \approx \sum_i x_i(t) \phi_i(a)$$

- $\phi_i(a)$ are known polynomials of parameter a
- $x_i(t)$ are unknown time varying coefficients
- Determine $x_i(t)$ that minimises equation error $e(t, a) = \dot{\hat{x}} - a\hat{x}$
 - ▶ *Galerkin Projection*: minimize $\|e(t, a)\|_2$
 - ▶ *Stochastic Collocation*: set $e(t, a) = 0$ at certain locations
- Resulting system
 - ▶ *is in higher dimensional state space*
 - ▶ *doesn't involve parameter a*

Galerkin Projection

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Stochastic Finite Element

Generalized Formulation

- Let system be

$$\dot{x} = f(x, \Delta),$$

where state $x \in \mathbb{R}^n$ and parameter $\Delta \in \mathcal{D}_\Delta \subseteq \mathbb{R}^d$

- More precisely, $\Delta := \Delta(\omega)$ is a \mathbb{R}^d -valued continuous random variable
- ω is an event in the probability space (Ω, \mathcal{F}, P)

- A **second order process** $x(t, \Delta(\omega))$ can be expressed by polynomial chaos as

$$\mathbf{x}(t, \Delta(\omega)) = \sum_{i=0}^{\infty} \mathbf{x}_i(t) \phi_i(\Delta(\omega))$$

- In practice, approximate with finite terms

$$x(t, \Delta) \approx \hat{x}(t, \Delta) = \sum_{i=0}^N x_i(t) \phi_i(\Delta)$$

Reduced Order System

- ## 1. Dynamics

$$\dot{x} = f(x, \Delta), \text{ (} n \text{ differential equations)}$$

- ## 2. Proposed solution

$$\hat{\mathbf{x}}(t, \Delta) = \sum_{i=0}^N \mathbf{x}_i(t) \phi_i(\Delta)$$

- ### 3. Residue

$$e(t, \Delta) := \dot{\hat{x}} - f(\hat{x}, \Delta)$$

4. Set projection on basis function to zero (best \mathcal{L}_2 solution)

$$\langle e(t, \Delta), \phi_i(\Delta) \rangle = 0, \text{ for } i = 0, 1, \dots, N$$

5. This gives $n(N + 1)$ ordinary differential equations to determine $n(N + 1)$ unknowns $\mathbf{x}_i(t) \in \mathbb{R}^n$

Inner product

Define

$$\langle e(t, \Delta), \phi_i(\Delta) \rangle := \int_{\mathcal{D}_\Delta} e(t, \Delta) \phi_i(\Delta) p(\Delta) d\Delta,$$

where $p(\Delta)$ is the probability density function of Δ .

Also

$$\mathbf{E}[e(t, \Delta)\phi_i(\Delta)] := \int_{\mathcal{P}_\Delta} e(t, \Delta)\phi_i(\Delta) p(\Delta) d\Delta$$

Therefore,

$$\langle e(t, \Delta), \phi_i(\Delta) \rangle \equiv \mathbf{E} [e(t, \Delta) \phi_i(\Delta)]$$

Basis Functions

Basis functions are such that

$$\mathbf{E}[\phi_i(\Delta)\phi_j(\Delta)] = 0, \text{ when } i \neq j$$

i.e. orthogonal w.r.t $p(\Delta)$

$$\int_{\mathcal{P}_\Lambda} \phi_i(\Delta) \phi_j(\Delta) p(\Delta) d\Delta = 0, \text{ when } i \neq j$$

Distribution	Polynomial Basis Function	Support
Uniform: $\frac{1}{2}$	Legendre	$x \in [-1, 1]$
Standard Normal: $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$	Hermite	$x \in (-\infty, \infty)$
Beta: $\frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1}$	Jacobi	$x \in [0, 1]$
Gamma: $\frac{x^{k-1} e^{-\frac{x}{\theta}}}{\theta^k \Gamma(k)}$	Laguerre	$x \in (0, \infty)$

Basis Functions (contd.)

In general

- $\{\phi_i(\Delta)\}$ are orthogonal polynomials with weight $p(\Delta)$
- \mathcal{L}_2 **exponential** convergence in corresponding Hilbert functional space
- Askey scheme of hypergeometric polynomials for common $p(\Delta)$
 - *Normal, uniform, beta, gamma, etc*
- Numerically generate for arbitrary $p(\Delta)$:
 - *Gram-Schmidt*
 - *Chebyshev*
 - *Gauss-Wigert*
 - *Discretized Stieltjes*

Basis Functions (contd.)

Mixed Basis Functions

- Let $\Delta := [\Delta_1 \ \Delta_2]^T$, $\Delta_1 \ \Delta_2$ are independent
 - Δ_1 is uniform over $[-1, 1]$
 - Δ_2 is standard normal over $(-\infty, \infty)$
- What is the basis function for Δ ?
- $\{\phi_i(\Delta)\}$ is multivariate polynomial
 - $\{\psi_j(\Delta_1)\}$: Legendre polynomials
 - $\{\theta_k(\Delta_2)\}$: Hermite polynomials
 - $\{\phi_i(\Delta)\}$: **tensor product** of $\{\psi_j(\Delta_1)\}$ and $\{\theta_k(\Delta_2)\}$

Example: First Order Linear System

Consider system $\dot{x} = -ax$, where $a \in \mathcal{U}_{[0,1]}$ (uniform distribution)

1. Define $a(\Delta) := \frac{1}{2}(1 + \Delta)$, $\Delta \in \mathcal{U}_{[-1,1]}$

Now dynamics is $\dot{x} = -a(\Delta)x$.

2. Approximate solution as $\hat{x} = \sum_{i=0}^N x_i(t)\phi_i(\Delta)$
(ϕ_i are Legendre polynomials)

3. Residue:

$$e(t, \Delta) := \dot{\hat{x}} - a(\Delta)\hat{x}$$

$$= \sum_{i=0}^N \dot{x}_i(t)\phi_i(\Delta) - a(\Delta) \sum_{i=0}^N x_i(t)\phi_i(\Delta)$$

Example: First Order Linear System (contd.)

4. Project residue on j^{th} basis function:

$$\begin{aligned}\langle e(t, \Delta), \phi_j(\Delta) \rangle &= \left\langle \sum_{i=0}^N \dot{x}_i(t) \phi_i(\Delta), \phi_j(\Delta) \right\rangle - \left\langle a(\Delta) \sum_{i=0}^N x_i(t) \phi_i(\Delta), \phi_j(\Delta) \right\rangle \\ &= \sum_{i=0}^N \dot{x}_i(t) \langle \phi_i(\Delta), \phi_j(\Delta) \rangle - \sum_{i=0}^N x_i(t) \langle a(\Delta) \phi_i(\Delta), \phi_j(\Delta) \rangle\end{aligned}$$

5. If $\langle \phi_i(\Delta), \phi_j(\Delta) \rangle = 0$ for $i \neq j$ (orthogonal)

$$\langle e(t, \Delta), \phi_j(\Delta) \rangle = \dot{x}_j \langle \phi_j(\Delta), \phi_j(\Delta) \rangle - \sum_{i=0}^N x_i(t) \langle a(\Delta) \phi_i(\Delta), \phi_j(\Delta) \rangle$$

6. $\langle e(t, \Delta), \phi_j(\Delta) \rangle = 0$ implies

$$\dot{x}_j = \frac{1}{\langle \phi_j(\Delta), \phi_j(\Delta) \rangle} \sum_{i=0}^N x_i(t) \langle a(\Delta) \phi_i(\Delta), \phi_j(\Delta) \rangle$$

7. This gives use $N + 1$ ordinary differential equations

($x \in \mathbb{R}$ in this example)

Example: First Order Linear System (contd.)

The equation

$$\dot{x}_j = \frac{1}{\langle \phi_j(\Delta), \phi_j(\Delta) \rangle} \sum_{i=0}^N x_i(t) \langle a(\Delta) \phi_i, \phi_j \rangle$$

in more compact form

$$\dot{\mathbf{x}} = \frac{1}{\langle \phi_j(\Delta), \phi_j(\Delta) \rangle} \begin{bmatrix} \langle a(\Delta) \phi_0(\Delta), \phi_j(\Delta) \rangle & \cdots & \langle a(\Delta) \phi_N(\Delta), \phi_j(\Delta) \rangle \end{bmatrix} \begin{pmatrix} x_0 \\ \vdots \\ x_N \end{pmatrix}$$

Define $\mathbf{x}_{pc} := (x_0 \ x_1 \ \cdots \ x_N)^T$, then

$$\dot{\mathbf{x}}_{pc} = \mathbf{A}_{pc} \mathbf{x}_{pc}$$

where

$$\mathbf{A}_{pc} := \mathbf{W}^{-1} \begin{bmatrix} \langle a(\Delta) \phi_0, \phi_0 \rangle & \cdots & \langle a(\Delta) \phi_N, \phi_0 \rangle \\ \vdots & & \vdots \\ \langle a(\Delta) \phi_0, \phi_N \rangle & \cdots & \langle a(\Delta) \phi_N, \phi_N \rangle \end{bmatrix}, \quad \mathbf{W} := \mathbf{diag}(\langle \phi_0, \phi_0 \rangle \ \cdots \ \langle \phi_N, \phi_N \rangle)$$

Reduced Order System

Therefore

$$\underbrace{\dot{x} = -a(\Delta)x}_{\text{stochastic in } \mathbb{R}} \xrightarrow{\text{Polynomial Chaos}} \underbrace{\dot{x}_{pc} = A_{pc}x_{pc}}_{\text{deterministic in } \mathbb{R}^{N+1}}$$

In general

$$\underbrace{\dot{x} = f(x, \Delta)}_{\text{stochastic in } \mathbb{R}^n} \xrightarrow{\text{Polynomial Chaos}} \underbrace{\dot{x}_{pc} = F_{pc}(x_{pc})}_{\text{deterministic in } \mathbb{R}^{n(N+1)}}$$

$$\text{where } x_{pc} := \begin{pmatrix} x_0 \\ \vdots \\ x_N \end{pmatrix} \text{ and } \hat{x} = \sum_{i=0}^N x_i(t) \phi_i(\Delta)$$

Initial Condition Uncertainty

Transform uncertainty in dynamics as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \Delta) \xrightarrow{\text{Polynomial Chaos}} \dot{\mathbf{x}}_{pc} = \mathbf{F}_{pc}(\mathbf{x}_{pc})$$

$$\mathbf{x}_{pc} := \begin{pmatrix} \mathbf{x}_0 \\ \vdots \\ \mathbf{x}_N \end{pmatrix} \text{ and } \hat{\mathbf{x}} = \sum_{i=0}^N \mathbf{x}_i(t) \phi_i(\Delta)$$

Let I.C. uncertainty be: $\mathbf{x}_0(\Delta)$

Initialize \mathbf{x}_{pc} as

$$\mathbf{x}_i(t_0) := \langle \mathbf{x}_0(\Delta), \phi_i(\Delta) \rangle$$

Random variable Δ is

$$\Delta := \begin{pmatrix} \Delta_0 \\ \Delta_p \end{pmatrix}, \quad \begin{array}{l} \Delta_0 \text{ is I.C. uncertainty} \\ \Delta_p \text{ is system parameter uncertainty} \end{array}$$

Basis functions $\phi_i(\Delta)$ are defined w.r.t Δ

Linear Systems

Consider Linear System

$$\dot{\mathbf{x}} = \mathbf{A}(\Delta)\mathbf{x}, \text{ with } \mathbf{x}(t_0) := \mathbf{x}_0(\Delta), \text{ and } \Delta := \begin{pmatrix} \Delta_0 \\ \Delta_p \end{pmatrix}$$

- System has random parameters in \mathbf{A} matrix and I.C.
- $\mathbf{x} \in \mathbb{R}^n$ and $\Delta \in \mathbb{R}^d$
- Define basis function vector $\Phi(\Delta) := (\phi_0(\Delta) \cdots \phi_N(\Delta))^T$
- Approximate solution is

$$\hat{\mathbf{x}} := \sum_{i=0}^N \mathbf{x}_i \phi_i(\Delta) = \mathbf{X} \Phi(\Delta),$$

$$\mathbf{X} = [\mathbf{x}_0 \ \mathbf{x}_1 \ \cdots \ \mathbf{x}_N] \in \mathbb{R}^{n \times (N+1)}$$

Linear Systems (contd.)

Approximate solution

$$\hat{x} = X\Phi(\Delta), X = [x_0 \ x_1 \ \cdots \ x_N]$$

Define

$$x_{pc} := \text{vec}(X) \equiv \begin{pmatrix} x_0 \\ \vdots \\ x_N \end{pmatrix}$$

Therefore,

$$\begin{aligned} \text{vec}(\hat{x}) &= \text{vec}(X\Phi) \\ \hat{x} &= (\Phi^T \otimes I_n)x_{pc} \quad \text{vec}(ABC) \equiv (C^T \otimes A)\text{vec}(B) \end{aligned}$$

Linear Systems (contd.)

Residue

$$\begin{aligned}
 e(t, \Delta) &:= \dot{\hat{x}} - A(\Delta)\hat{x} = \dot{X}\Phi(\Delta) - A(\Delta)X \\
 \text{vec}(e) = e &= \text{vec}\left(\dot{X}\Phi(\Delta) - A(\Delta)X\Phi(\Delta)\right) \\
 &= (\Phi^T \otimes I_n)\dot{x}_{pc} - \left(\Phi^T(\Delta) \otimes A(\Delta)\right)x_{pc}
 \end{aligned}$$

$\langle e, \phi_i(\Delta) \rangle = 0$ implies

$$\dot{x}_i = (\langle \phi_i(\Delta), \phi_i(\Delta) \rangle \otimes I_n)^{-1} \left\langle \Phi^T(\Delta) \otimes A(\Delta), \phi_i(\Delta) \right\rangle x_{pc}$$

Linear Systems (contd.)

Deterministic linear dynamics

$$\dot{\mathbf{x}}_{pc} = \mathbf{A}_{pc} \mathbf{x}_{pc}$$

$$\mathbf{x}_{pc} \in \mathbb{R}^{n(N+1)}, \mathbf{A}_{pc} \in \mathbb{R}^{n(N+1) \times n(N+1)}$$

\mathbf{A}_{pc} is defined as

$$\mathbf{A}_{pc} := (\mathbf{W} \otimes \mathbf{I}_n)^{-1} \begin{bmatrix} \langle \Phi^T \otimes \mathbf{A}(\Delta), \phi_0 \rangle \\ \vdots \\ \langle \Phi^T \otimes \mathbf{A}(\Delta), \phi_N \rangle \end{bmatrix}$$

Recall

$$\mathbf{W} := \text{diag}(\langle \phi_0, \phi_0 \rangle \cdots \langle \phi_N, \phi_N \rangle)$$

Computation of Mean

Given $x(\Delta) := X\Phi(\Delta)$

$$\begin{aligned}\mathbf{E}[x(\Delta)] &= \mathbf{E}[X\Phi(\Delta)] \\ &= X\mathbf{E}[\Phi(\Delta)] \\ &= X(1 \ 0 \ \cdots \ 0)^T \\ &= x_0\end{aligned}$$

Also

$$\mathbf{E}[x(\Delta)] = \mathbf{E}[(\Phi^T \otimes I_n)x_{pc}] = (\mathbf{E}[\Phi^T] \otimes I_n)x_{pc} = (F^T \otimes I_n)x_{pc}$$

where $F^T = (1 \ 0 \ \cdots \ 0)$.

Computation of Variance

Given $x(\Delta) := X\Phi(\Delta)$

$$\begin{aligned}
 x(\Delta)x^T(\Delta) &= X\Phi(\Delta)\Phi^T(\Delta)X^T \\
 \mathbf{E}[x(\Delta)x^T(\Delta)] &= \mathbf{E}[X\Phi(\Delta)\Phi^T(\Delta)X^T] \\
 &= X\mathbf{E}[\Phi(\Delta)\Phi^T(\Delta)]X^T \\
 &= X \begin{bmatrix} \langle \phi_0, \phi_0 \rangle & 0 & \dots & 0 \\ 0 & \langle \phi_1, \phi_1 \rangle & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \langle \phi_N, \phi_N \rangle \end{bmatrix} X^T \\
 &= XW X^T
 \end{aligned}$$

Then

$$\text{Var}[x] := \mathbf{E}[(x - \mathbf{E}[x])(x - \mathbf{E}[x])^T] = X(W - FF^T)X^T$$

Computation of Statistics – summary

Mean

$$\mathbf{E}[x] = \mathbf{X}\mathbf{F} = x_0$$

Variance

$$\text{Var}[x] = \mathbf{X}(\mathbf{W} - \mathbf{F}\mathbf{F}^T)\mathbf{X}^T$$

where

$$\mathbf{F} = \mathbf{E}[\Phi(\Delta)] = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{W} = \mathbf{E}[\Phi\Phi^T] = \begin{bmatrix} \langle\phi_0, \phi_0\rangle & 0 & \dots & 0 \\ 0 & \langle\phi_1, \phi_1\rangle & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \langle\phi_N, \phi_N\rangle \end{bmatrix}$$

Polynomial Nonlinearity

Polynomials $x^n(\Delta)$, $x \in \mathbb{R}$ can be written as

$$\begin{aligned} x^n(\Delta) &= (\mathbf{X}\Phi(\Delta))^n \\ \langle x^n(\Delta), \phi_i(\Delta) \rangle &= \langle (\mathbf{X}\Phi(\Delta))^n, \phi_i \rangle \\ &= \sum_{i_1=0}^N \cdots \sum_{i_n=0}^N x_{i_1} \cdots x_{i_n} \langle \phi_{i_1} \cdots \phi_{i_n}, \phi_i \rangle \end{aligned}$$

- Essentially integration of polynomials
 - ▶ *analytical or numerical (exact).*
- Inner product $\langle \phi_{i_1} \cdots \phi_{i_n}, \phi_i \rangle$
 - ▶ *can be computed offline*
 - ▶ *stored in sparse, symmetric tensor*

Rational polynomials

Functions such as $\frac{x^n(\Delta)}{y^m(\Delta)}$, $x, y \in \mathbb{R}$ can be **approximated** as

$$z(\Delta) = \frac{x^n(\Delta)}{y^m(\Delta)}$$

$$Z\Phi(\Delta) = \frac{(X\Phi(\Delta))^n}{(Y\Phi(\Delta))^m}$$

$$(Y\Phi)^m Z\Phi = (X\Phi)^n$$

$$\langle (Y\Phi)^m Z\Phi, \phi_i \rangle = \langle (X\Phi)^n, \phi_i \rangle, \quad i = \{0, 1, \dots, N\}$$

Given X, Y solve system of linear equations to obtain Z

$$\begin{bmatrix} \langle \Phi^T \otimes (Y\Phi)^m, \phi_0 \rangle \\ \vdots \\ \langle \Phi^T \otimes (Y\Phi)^m, \phi_N \rangle \end{bmatrix} z_{pc} = \begin{pmatrix} \langle (X\Phi)^n, \phi_0 \rangle \\ \vdots \\ \langle (X\Phi)^n, \phi_N \rangle \end{pmatrix} \text{ Polynomial integrations}$$

Transcendental Functions

Let $f(x)$ be a transcendental function:

- e.g. $x^a, e^x, x^{1/x}, \log(x), \sin(x)$, etc.

Use Taylor series expansion about mean

- Define $x := x_0 + d$, d is deviation from mean x_0
- Expand

$$f(x) = f(x_0 + d) = f(x_0) + f'(x_0)d + f''(x_0)\frac{d^2}{2!} + \dots$$

$$\text{■ } x(\Delta) := x_0 + \underbrace{\sum_{i=1}^N x_i \phi_i(\Delta)}_{d(\Delta)}$$

- Therefore

$$\langle f(x(\Delta)), \phi_i(\Delta) \rangle \approx f(x_0) \langle 1, \phi_i \rangle + f'(x_0) \langle d, \phi_i \rangle + \frac{f''(x_0)}{2!} \langle d^2, \phi_i \rangle + \dots$$

Transcendental Functions (contd.)

Taylor Series Approximation

$$\langle f(x(\Delta)), \phi_i(\Delta) \rangle \approx f(x_0) \langle 1, \phi_i \rangle + f'(x_0) \langle d, \phi_i \rangle + \frac{f''(x_0)}{2!} \langle d^2, \phi_i \rangle + \dots$$

- $\langle d^n, \phi_i \rangle$ is integration of polynomials
- Straightforward
- Computationally efficient
- Severe inaccuracies for higher order PC approximations

Remedies

- Approximate $f(x)$ using polynomials, piecewise polynomials
- **Non-intrusive:** multi-dimensional integrals via sampling, tensor-product quadrature, Smolyak sparse grid, or cubature
- **Regression Approach:** \mathcal{L}_2 optimization

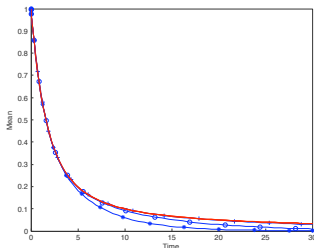
Example: First Order Linear System

Dynamics:

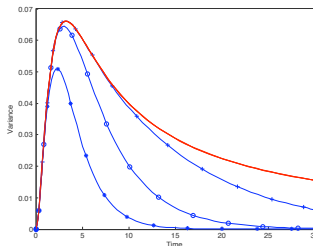
$$\dot{x} = -a(\Delta)x, \quad a \in \mathcal{U}_{[0,1]}$$

Analytical Statistics:

$$\bar{x}(t) = \frac{1 - e^{-t}}{t}, \quad \sigma(t) = \frac{1 - e^{-2t}}{2t} - \left(\frac{1 - e^{-t}}{t} \right)^2$$



(a) Mean.



(b) Variance.

Figure: Errors in estimates obtained from gPC for $\dot{x} = -a(\Delta)x$. Analytical: (red solid); gPC: 2^{nd} order(*), 3^{rd} order(o), 5^{th} (+).

Errors Due to Finite Terms

Dynamics:

$$\dot{x} = -a(\Delta)x, \quad a \in \mathcal{U}_{[0,1]}$$

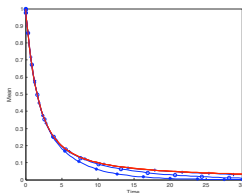
Analytical Solution:

$$x(t, \Delta) = x(t_0)e^{-a(\Delta)t}$$

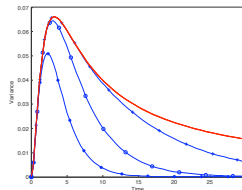
PC Solution:

$$\hat{x}(t, \Delta) = \sum_{i=0}^P x_i(t) \phi_i(\Delta)$$

Error: Finite term approximation of exponential.



(a) Mean



(b) Variance

Example: Eigen Analysis – Linear F-16 Aircraft

$$A(\Delta) = \begin{bmatrix} 0.1658 & -13.1013 & -7.2748(1 + 0.2\Delta) & -32.1739 & 0.2780 \\ 0.0018 & -0.1301 & 0.9276(1 + 0.2\Delta) & 0 & -0.0012 \\ 0 & -0.6436 & -0.4763 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

- Linearized about flight condition $V = 160 \text{ ft/s}$ and $\alpha = 35^\circ$
- Uncertainty due to damping term C_{xq}
- Difficult to model at high angle of attack
- 20% uncertainty about nominal

Example: Eigen Analysis – Linear F-16 Aircraft

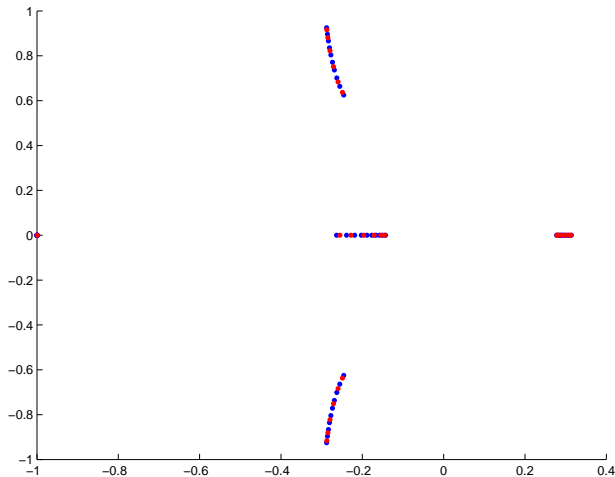


Figure: 5th Order PC ODE Eigen Values, Sampled ODE Eigen Values

Example: Eigen Analysis – Linear F-16 Aircraft

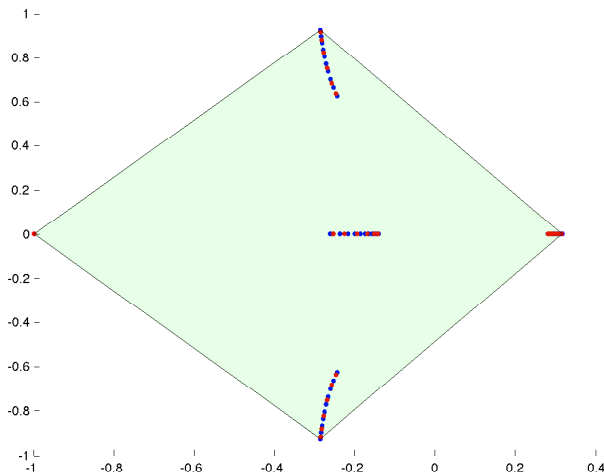
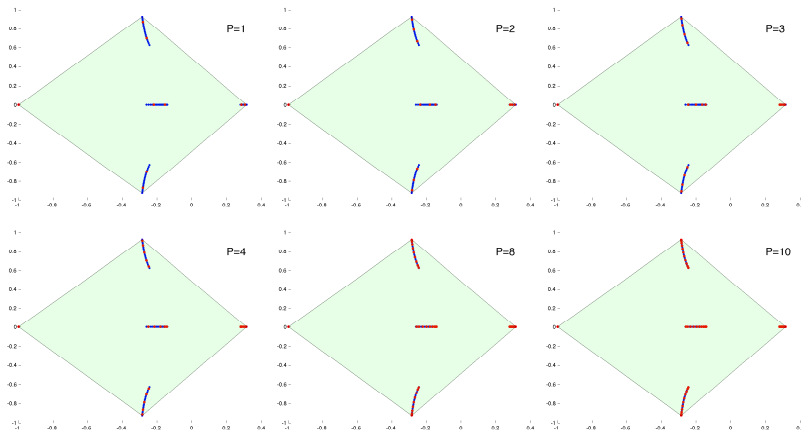


Figure: **PC eigen values** bounded by convex hull of **sampled ODE eigen values** (conservative!)*

* Eigenvalues of the Jacobian of a Galerkin-Projected Uncertain ODE System, Sunday, et al.

Example: Spread of Spectrum – Linear F-16 Aircraft



Better characterization of spectrum spread is needed.

Example: Nonlinear System – Lorenz Attractor

Dynamics:

$$\dot{x} = \sigma(y - x), \quad \dot{y} = x(\rho - z) - y, \quad \dot{z} = xy - \beta z.$$

Initial Condition:

$$[x, y, z]^T = [1.50887, -1.531271, 25.46091]^T$$

Parameters:

$$\sigma = 10(1 + 0.1\Delta_1), \quad \rho = 28(1 + 0.1\Delta_2), \quad \beta = 8/3, \quad \Delta \in \mathcal{U}_{[-1,1]}^2.$$

$$x(t, \Delta) \approx \sum_{i=0}^P x_i(t) \phi_i(\Delta)$$

$$\langle \phi_k^2 \rangle \dot{x}_k(t) = \sum_{i=0}^P \langle \sigma \phi_i \phi_k \rangle (y_i - x_i)$$

$$y(t, \Delta) \approx \sum_{i=0}^P y_i(t) \phi_i(\Delta)$$

$$\langle \phi_k^2 \rangle \dot{y}_k(t) = \sum_{i=0}^P \langle \rho \phi_i \phi_k \rangle x_i - \sum_{i=0}^P \sum_{j=0}^P \langle \phi_i \phi_j \phi_k \rangle x_i z_j - \langle \phi_k^2 \rangle y_k$$

$$z(t, \Delta) \approx \sum_{i=0}^P z_i(t) \phi_i(\Delta)$$

$$\langle \phi_k^2 \rangle \dot{z}_k(t) = \sum_{i=0}^P \sum_{j=0}^P \langle \phi_i \phi_j \phi_k \rangle x_i y_j - \beta \langle \phi_k^2 \rangle z_k$$

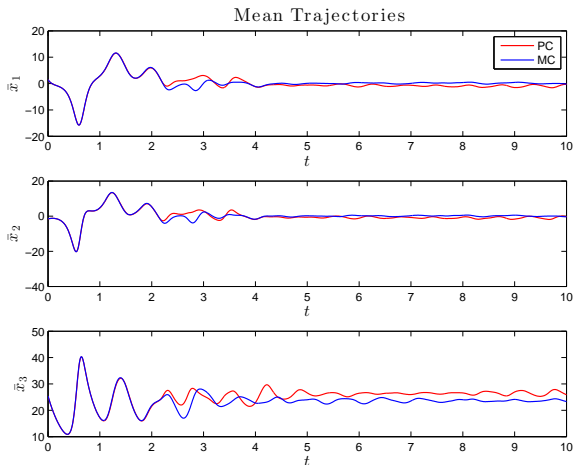
Example: Nonlinear System – Lorenz Attractor

MC: 1000 samples

PC: 7th order approximation

■ using MATLAB rand(...)

■ 36 basis functions



Stochastic Collocation

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Algorithm

1. Given stochastic dynamics with uncertainty Δ

$$\dot{x} = f(x, \Delta)$$

2. For p^{th} order approximation:

- ▶ sample domain \mathcal{D}_Δ with roots of $p + 1$ order polynomial
- ▶ tensor grid, sparse grid, etc.
- ▶ samples $\Delta := \{\Delta_i\}$, $i = 0, \dots, p$.

3. Coefficient x_j evolves according to

$$\dot{x}_i = f(x_i, \Delta_i), \text{ deterministic solution}$$

- #### 4. Approximate stochastic solution

$$\hat{\mathbf{x}}(t, \Delta) := \sum_{i=0}^p \mathbf{x}_i(t) L_i(\Delta)$$

L_i are Lagrangian interpolants $L_i(y) = \prod_{j=0, j \neq i}^p \frac{y - y_j}{y_i - y_j}$.

Computation of Statistics

■ Mean

$$\mathbf{E}[x(t)] \approx \mathbf{E}\left[\sum_{i=0}^p x_i(t) L_i(\Delta)\right] = \sum_{i=0}^p x_i(t) \mathbf{E}[L_i(\Delta)]$$

- Computation of $\mathbf{E}[L_i(\Delta)]$ involves high-dimensional polynomial integration
 - ▶ *analytical*
 - ▶ *numerical: quadratures, sparse grids, etc*
- Higher order statistics: similar to computation of mean.

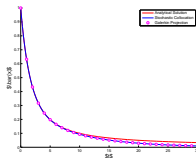
Example: Linear First Order System

Dynamics:

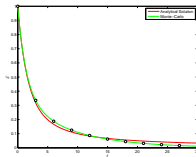
$$\dot{x} = -a(\Delta)x, \quad a \in \mathcal{U}_{[0,1]}$$

Analytical Statistics:

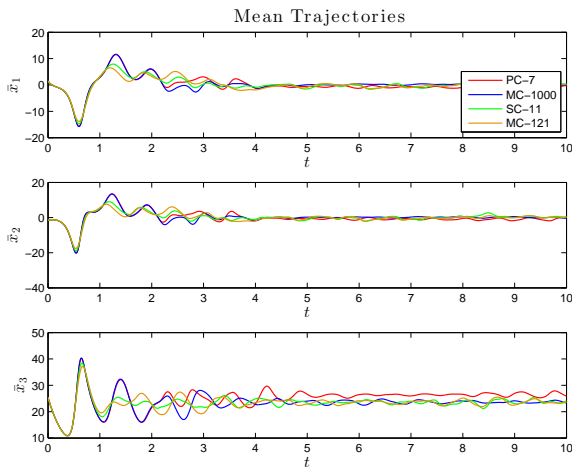
$$\bar{x}(t) = \frac{1 - e^{-t}}{t},$$



(a) MC (3 samples)



Example: Nonlinear System – Lorenz Attractor



SC performance is poor for nonlinear systems!
But, better than MC with same sample budget.

Karhunen-Loève Expansion

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Basic Idea

Given a random process $X(t, \omega) := \{X_t(\omega)\}_{t \in [t_1, t_2]}$

- $X_t(\omega) \in \mathcal{L}_2(\Omega, \mathcal{F}, P)$ finite second moment
 - $\mathcal{L}_2(\Omega, \mathcal{F}, P) := \{X : \Omega \mapsto \mathbb{R} : \int_{\Omega} |X(\omega)|^2 dP(\omega) < \infty\}$

- ## ■ Auto Correlation

$$R_X(t_1, t_2) := \mathbf{E}[X_{t_1} X_{t_2}]$$

- Auto Covariance

$$\begin{aligned} C_X(t_1, t_2) &:= R_X(t_1, t_2) - \mu_{t_1} \mu_{t_2} \\ &= R_X(t_1, t_2) - \mu^2 \quad \text{stationary} \end{aligned}$$

- $C_X(t_1, t_2)$ is bounded, symmetric and positive definite, thus

$$C_X(t_1, t_2) = \sum_{i=0}^{\infty} \lambda_i f_i(t_1) f_i(t_2) \quad \text{spectral decomposition}$$

where λ_i and $f_i(\cdot)$ are eigenvalues and eigenvectors of the covariance kernel.

Eigenvalues and Eigenfunctions

- λ_i and $f_i(\cdot)$ are solutions of

$$\int_{\mathcal{D}} C_X(t_1, t_2) f_i(t) dt_1 = \lambda_i f_i(t_2), \text{ Fredholm integral equation of second kind}$$

$$\text{with } \int_{\mathcal{D}} f_i(t) f_j(t) dt = \delta_{ij}.$$

- Write $X(t, \omega) := \bar{X}(t) + Y(t, \omega)$, where

$$Y(t, \omega) \stackrel{m.s.}{=} \sum_{i=0}^{\infty} \xi_i(\omega) \sqrt{\lambda_i} f_i(t), \text{ and } \xi_i(\omega) = \frac{1}{\lambda_i} \int_{\mathcal{D}} Y(t, \omega) f_i(t) dt.$$

- Reproducing Kernel Hilbert Space

- ▶ *Congruence between two Hilbert spaces!*
- ▶ $\{f_i(t)\} \mapsto X(t, \omega)$ or equivalently
- ▶ $\{f_i(t)\} \mapsto \{\xi_i(\omega)\}$

Solution of Integral Equation

- Homogeneous Fredholm integral equation of the second kind,

$$\int_{\mathcal{D}} C_X(t_1, t_2) f_i(t) dt_1 = \lambda_i f_i(t_2) \quad \text{well studied problem}$$

- $C_X(t_1, t_2)$ is bounded, symmetric, and positive definite, implies

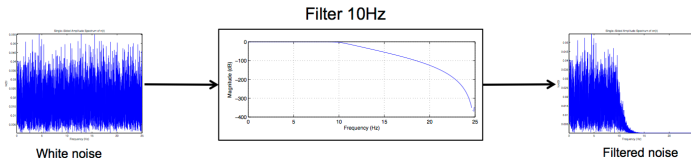
1. *The set $f_i(t)$ of eigenfunctions is orthogonal and complete.*
2. *For each eigenvalue λ_k , there correspond at most a finite number of linearly independent eigenfunctions.*
3. *There are at most a countably infinite set of eigenvalues.*
4. *The eigenvalues are all positive real numbers.*
5. *The kernel $C_X(t_1, t_2)$ admits of the following uniformly convergent expansion*

$$C_X(t_1, t_2) = \sum_{i=0}^{\infty} \lambda_i f_i(t_1) f_i(t_2)$$

- Applicable to wide range of processes

Rational Spectra: Special Case

- 1D random process
- Stationary output of a linear filter, excited by white noise



- Spectral density of the form $S(s^2) = H(j\omega)H(-j\omega) = \frac{N(s^2)}{D(s^2)}$

N and D are polynomials in s^2 such that

$$\int_{-\infty}^{\infty} S(-\omega^2) d\omega < \infty$$

$s = j\omega$, here ω is frequency

- ▶ Degree of $D(s^2)$ must exceed degree of $N(s^2)$ by at least two.
- ▶ No roots of $D(s^2)$ on the imaginary axis
- ▶ $S(\omega) \geq 0$, \Rightarrow purely imaginary zeros of $N(s^2)$ of even multiplicity

- Finite dimensional Markovian process
 - ▶ *effect of infinite past on the present is negligible*

Important Kernel

Study specific kernel

$$C_X(t_1, t_2) = e^{-c|t_1 - t_2|}$$

$1/c$ is the correlation time or length.

- Many applications.
- Other kernels also possible

Solve integral equation

$$\int_{\mathcal{P}} C_X(t_1, t_2) f_i(t) dt_1 = \lambda_i f_i(t_2).$$

Or equivalently solve

ODE: $\ddot{f}(t) + \omega^2 f(t) = 0$, $\omega^2 = \frac{2c - c^2\lambda}{\lambda}$, $-a \leq t \leq a$

Boundary Condition: $cf(a) + \dot{f}(a) = 0, \quad cf(-a) - \dot{f}(-a) = 0.$

Coefficients

Recall $X(t, \omega) := \bar{X}(t) + Y(t, \omega)$,

ω here is an event in the probability space (Ω, \mathcal{F}, P)

$$\begin{aligned} Y(t, \omega) &\stackrel{m.s.}{=} \sum_{i=0}^{\infty} \xi_i(\omega) \sqrt{\lambda_i} f_i(t) \\ &= \sum_{i=0}^{\infty} \left[\xi_i(\omega) \sqrt{\lambda_i} f_i(t) + \xi_i^*(\omega) \sqrt{\lambda_i^*} f_i^*(t) \right] \end{aligned}$$

- $\xi_i(\omega), \xi_i^*(\omega)$ are **uncorrelated** random variables determined from $Y(t, \omega)$
- $\xi_i(\omega), \xi_i^*(\omega)$ model the **distribution of amplitude** of $Y(t, \omega)$
- $f_i(t), f_i^*(t)$ models the **distribution of signal power** over time or among frequencies

If $Y(t, \omega)$ is a Gaussian process

- $\xi_i(\omega), \xi_i^*(\omega)$ Gaussian **independent** random variables
- KL – expansion is **almost surely** convergent

UQ Application

Dynamical system with process noise $n(t, \omega)$

$$\dot{x} = f(t, \Delta, x) + n(t, \omega)$$

Replace

$$n(t, \omega) \approx \sum_{i=0}^N \left[\xi_i(\omega) \sqrt{\lambda_i} f_i(t) + \xi_i^*(\omega) \sqrt{\lambda_i^*} f_i^*(t) \right]$$

Define new parameter vector

$$\Delta' := (\Delta^T, \xi_0, \xi_0^*, \dots, \xi_N, \xi_N^*)^T$$

Rewrite dynamics as

$$\dot{x} = F(t, \Delta', x),$$

Process noise converted to parametric uncertainty.

- Use PC, SC, or simplified FPK equation to determine $x(t, \Delta')$
- Increases number of parameters \Rightarrow increases computational complexity

Publications

1. J. Fisher, R. Bhattacharya, *Stability Analysis of Stochastic Systems using Polynomial Chaos*, American Control Conference 2008.
2. A. Prabhakar and R. Bhattacharya, *Analysis of Hypersonic Flight Dynamics with Probabilistic Uncertainty in System Parameters*, AIAA GNC 2008.
3. A. Prabhakar, J. Fisher, R. Bhattacharya, *Polynomial Chaos Based Analysis of Probabilistic Uncertainty in Hypersonic Flight Dynamics*, AIAA Journal of Guidance, Control, and Dynamics, Vol.33 No.1 (222-234), 2010.
4. J. Fisher, R. Bhattacharya, *Optimal Trajectory Generation with Probabilistic System Uncertainty Using Polynomial Chaos*, Journal of Dynamic Systems, Measurement and Control, volume 133, Issue 1, January 2011.
5. J. Fisher, R. Bhattacharya, *Linear Quadratic Regulation of Systems with Stochastic Parameter Uncertainties*, Automatica, 2009.
6. Roger G. Ghanem, Pol D. Spanos, *Stochastic Finite Elements: A Spectral Approach*, Revised Edition (Dover Civil and Mechanical Engineering
7. Olivier Le Maitre, Omar M Knio, *Spectral Methods for Uncertainty Quantification: With Applications to Computational Fluid Dynamics*, Scientific Computation.
8. Dongbin Xiu, *Numerical Methods for Stochastic Computations: A Spectral Method Approach*, ISBN: 9780691142128, Princeton Press.