AERO 422: Active Controls for Aerospace Vehicles

Frequency Response-Design Method

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Frequency Response

Response to Sinusoidal Input

$$u(t) \longrightarrow \boxed{\mathsf{P}} \longrightarrow y(t)$$

- Let $u(t) = A_u \sin(\omega t)$
- Vary ω from 0 to ∞

A linear system's response to sinusoidal inputs is called the system's frequency response

Response to Sinusoidal Input

Example

- Let $P(s) = \frac{1}{s+1}, u(t) = \sin(t)$

$$
\begin{aligned}
y(t) &= \frac{1}{2}e^{-t} - \frac{1}{2}\cos(t) + \frac{1}{2}\sin(t) \\
&= \underbrace{\frac{1}{2}e^{-t}}_{\text{natural response}} + \underbrace{\frac{1}{\sqrt{2}}\sin(t - \frac{\pi}{4})}_{\text{forced response}}
\end{aligned}
$$

- Forced response has form $A_y \sin(\omega t + \phi)$
- A_y and ϕ are functions of ω

Response to Sinusoidal Input

Generalization

In general

$$
\begin{aligned}
Y(s) &= G(s)\frac{\omega_0}{s^2+\omega_0^2} \\
&= \frac{\alpha_1}{s-p_1} + \cdots \frac{\alpha_n}{s-p_n} + \frac{\alpha_0}{s+j\omega_0} + \frac{\alpha_0^*}{s-j\omega_0}
\end{aligned}
$$

$$
\implies y(t) = \underbrace{\alpha_1 e^{p_1 t} + \cdots + \alpha_n e^{p_n t}}_{\text{natural}} + \underbrace{A_y \sin(\omega_0 + \phi)}_{\text{forced}}
$$

Forced response has same frequency, different amplitude and phase.

Response to Sinusoidal Input

Generalization (contd.)

For a system $P(s)$ and input

$$u(t) = A_u \sin(\omega_0 t),$$

forced response is

$$y(t) = A_u M \sin(\omega_0 t + \phi),$$

where

$$M(\omega_0) = |P(s)|_{s=j\omega_0} = |P(j\omega_0)|, \text{ magnitude}$$

$$\phi(\omega_0) = \angle P(j\omega_0) \text{ phase}$$

In polar form

$$P(j\omega_0) = Me^{j\phi}.$$

Fourier Analysis

Fourier Series Expansion

Given a signal $y(t)$ with periodicity T,

$$y(t) = \frac{a_0}{2} + \sum_{n=1,2,\cdots} a_n \cos\left(\frac{2\pi n t}{T}\right) + b_n \sin\left(\frac{2\pi n t}{T}\right)$$

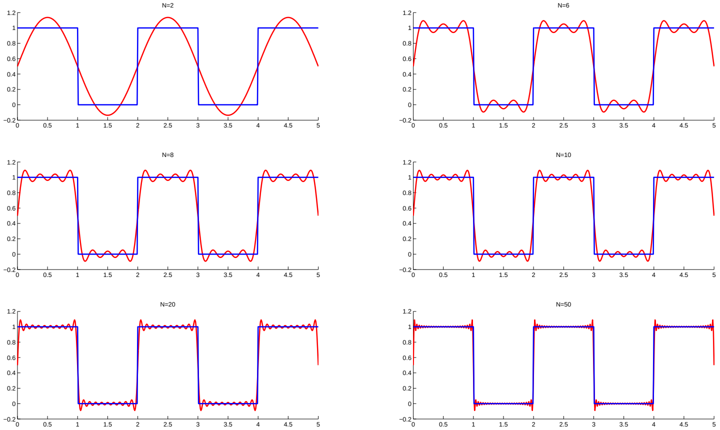
$$a_0 = \frac{2}{T}\int_0^T y(t)dt$$

$$a_n = \frac{2}{T}\int_0^T y(t)\cos\left(\frac{2\pi n t}{T}\right)dt$$

$$b_n = \frac{2}{T}\int_0^T y(t)\sin\left(\frac{2\pi n t}{T}\right)dt$$

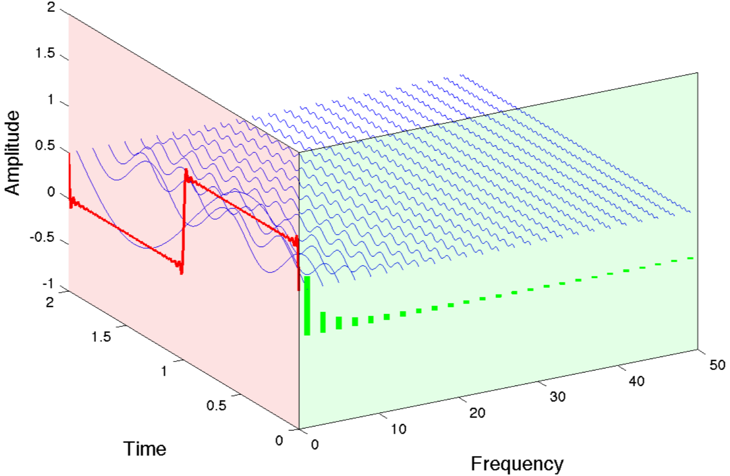
Fourier Series Expansion

Approximation of step function



Fourier Transform

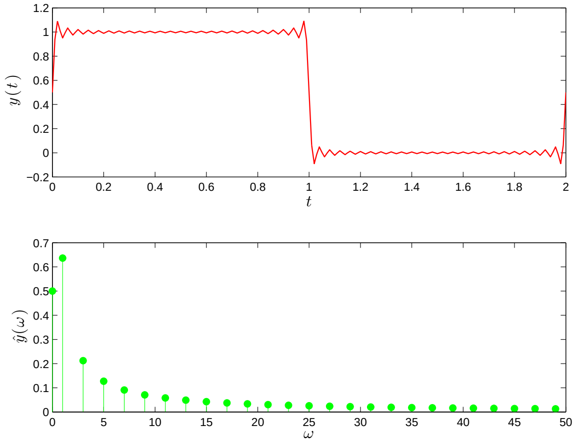
Step function



Fourier transform reveals the frequency content of a signal

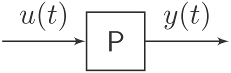
Fourier Transform

Step function – frequency content



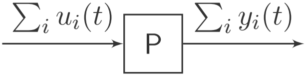
Signals & Systems

Input Output

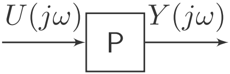


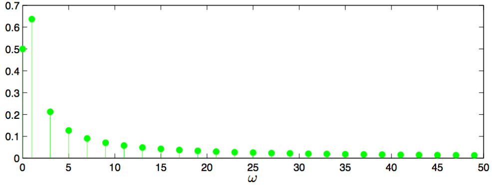
Fourier Series Expansion

superposition principle



Fourier Transform



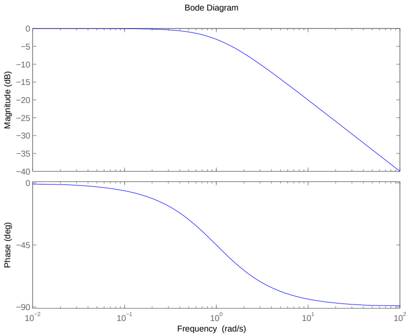


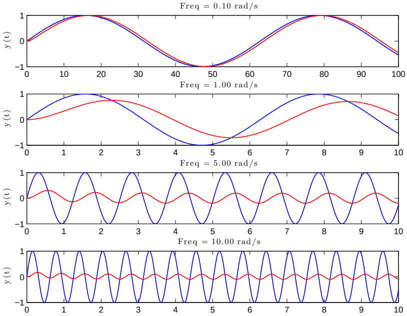
$$u_i(t) = a_i \sin(\omega_i t)$$
$$y_{i_{\text{forced}}}(t) = a_i M \sin(\omega_i t + \phi)$$
$$Y(j\omega) = P(j\omega)U(j\omega)$$

Suffices to study $P(j\omega)$ $|P(j\omega)|, \angle P(j\omega)$

Bode Plot

First Order System

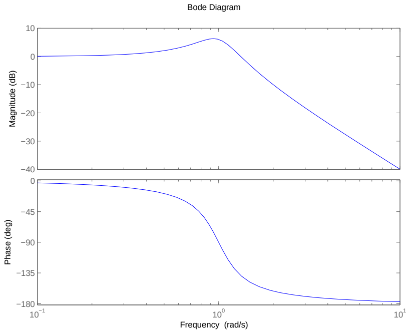


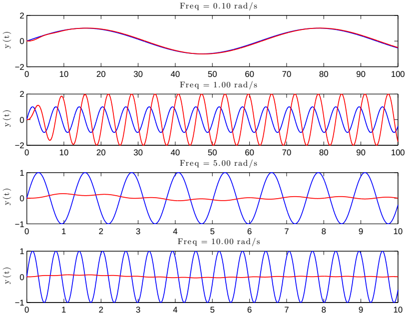


- $P(s) = 1/(s+1)$
- loglog scale
- dB $= 10\log_{10}(\cdot)$
- 20dB $= 10\log_{10}(100/1)$

- $u(t) = A\sin(\omega_0 t)$
- $y_{\text{forced}}(t) = AM\sin(\omega_0 t + \phi)$

Second Order System

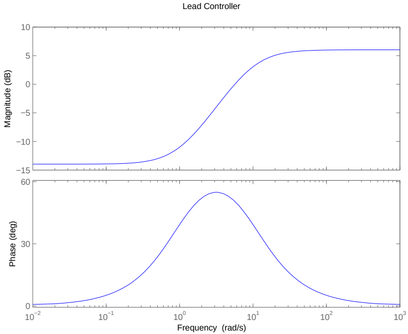




- $P(s) = 1/(s^2 + 0.5s + 1)$
- $\omega_n = 1$ rad/s

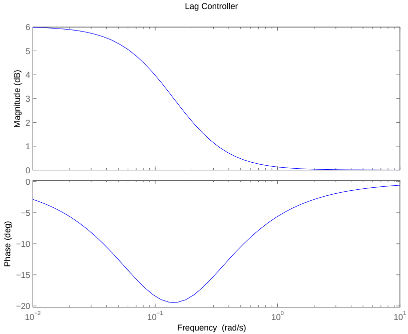
- $u(t) = A\sin(\omega_0 t)$
- $y_{\text{forced}}(t) = AM\sin(\omega_0 t + \phi)$

Lead Compensator



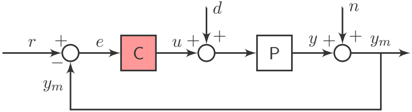
- Phase lead
- low gain at low frequency
- high gain at high frequency
- relate it to derivative control

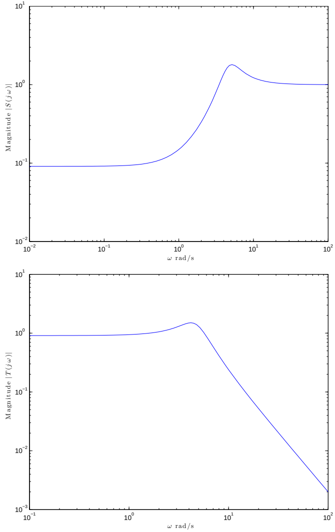
Lag Compensator



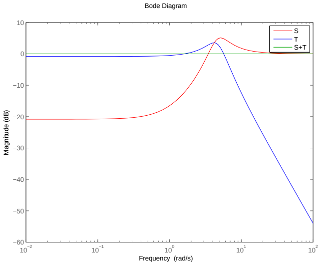
- Phase lag
- high gain at low frequency
- low gain at high frequency
- relate it to integral control

$S(j\omega) + T(j\omega) = 1$



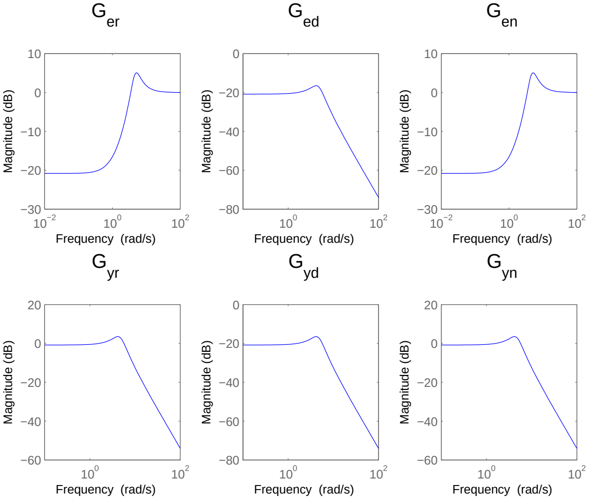


- $P(s) = \frac{1}{(s+1)(s/2+1)}$
- $C(s) = 10$
- $S = G_{er} = \frac{1}{1+PC} = \frac{1}{1+10P}$
- $T = G_{yr} = \frac{PC}{1+PC} = \frac{10P}{1+10P}$



All transfer functions

With proportional controller



Piper Dakota Control System

Designed with root locus method

System

Transfer function from δ_e (elevator angle) to θ (pitch angle) is

$$P(s) = \frac{\theta(s)}{\delta_e(s)} = \frac{160(s+2.5)(s+0.7)}{(s^2+5s+40)(s^2+0.03s+0.06)}$$

Control Objective 1

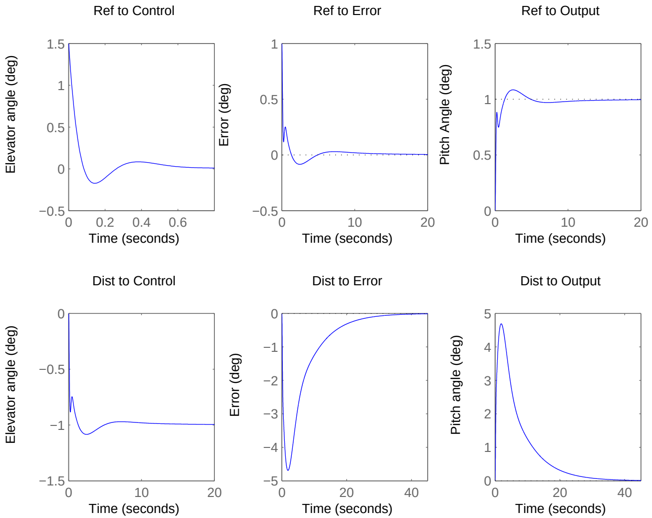
Design an autopilot so that the step response to elevator input has $t_r < 1$ and $M_p < 10\% \implies \omega_n > 1.8$ rad/s and $\zeta > 0.6$ 2^{nd} order

Controller

$$C(s) = 1.5\frac{s+3}{s+25}(1+0.15/s)$$

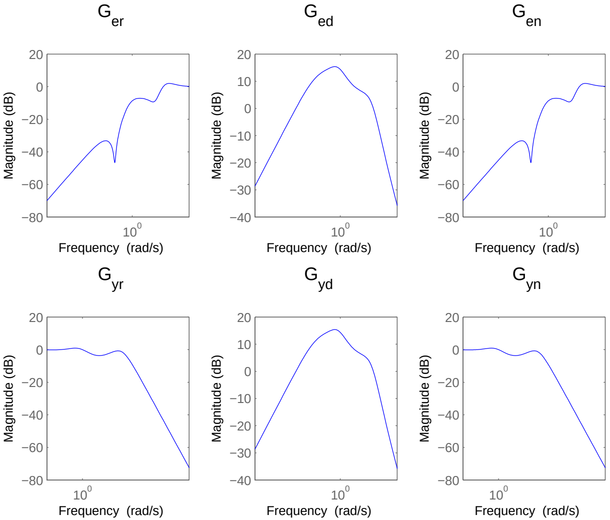
Piper Dakota Control System

Time Response



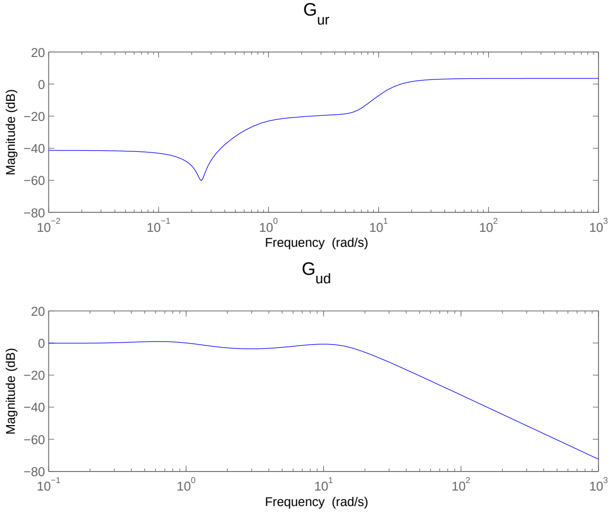
Piper Dakota Control System

Frequency Response



Piper Dakota Control System

Frequency Response (contd.)



Asymptotes

Approximate Bode Plot

Useful for Design & Analysis

Let open-loop transfer function be

$$KG(s) = K\frac{(s-z_1)(s-z_2)\cdots}{(s-p_1)(s-p_2)\cdots}$$

Write in Bode form

$$KG(j\omega) = K_0\frac{(j\omega\tau_1+1)(j\omega\tau_2+1)\cdots}{(j\omega\tau_a+1)(j\omega\tau_b+1)\cdots}$$

K_0 is the DC gain of the system.

Example

$$G(s) = \frac{(s+1)}{(s+2)(s+3)} \implies G(j\omega) = \frac{j\omega+1}{(j\omega+2)(j\omega+3)} = \frac{1}{6}\frac{j\omega+1}{(j\omega/2+1)(j\omega/3+1)}$$

Approximate Bode Plot

contd.

Transfer function in Bode Form

$$KG(j\omega) = K_0 \frac{(j\omega\tau_1 + 1)(j\omega\tau_2 + 1)\cdots}{(j\omega\tau_a + 1)(j\omega\tau_b + 1)\cdots}$$

Three cases

1. $K_0(j\omega)^n$ pole, zero at origin
2. $(j\omega + 1)^{\pm 1}$ real pole, zero
3. $\left[\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1\right]^{\pm 1}$ complex pole, zero

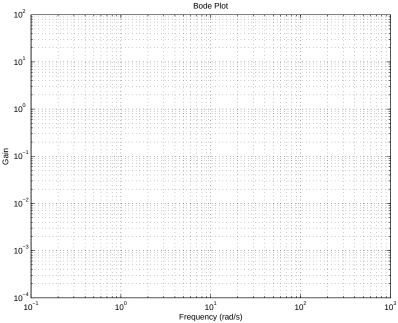
Case:1 $K_0(j\omega)^n$ pole, zero at origin

Gain

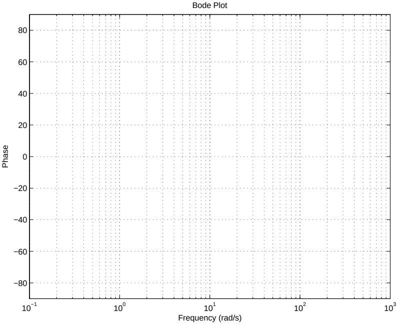
$$\log K_0|(j\omega)|^n = \log K_0 + n\log|jw| = \log K_0 + n\log w$$

Phase

$$\underline{/K_0(j\omega)|^n} = \underline{/K_0} + n\underline{/j\omega} = 0 + n\times 90^\circ$$



(a) Gain



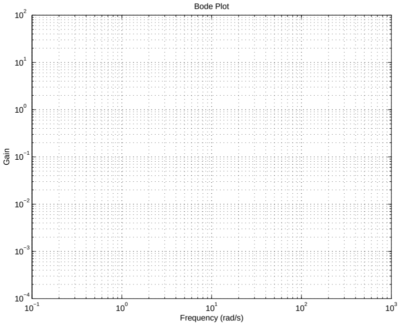
(b) Phase

Case:2 $(j\omega\tau + 1)^{\pm 1}$ real pole, zero

Gain

$$(j\omega\tau + 1) = \begin{cases} \approx 1, & \omega\tau << 1, \\ \approx j\omega\tau, & \omega\tau >> 1. \end{cases}$$

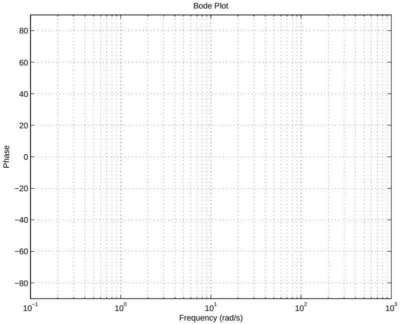
Frequency $\omega = 1/\tau$ is the break point



Case:2 $(j\omega\tau + 1)^{\pm 1}$ real pole, zero (contd.)

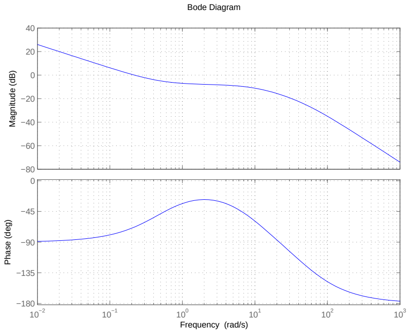
Phase

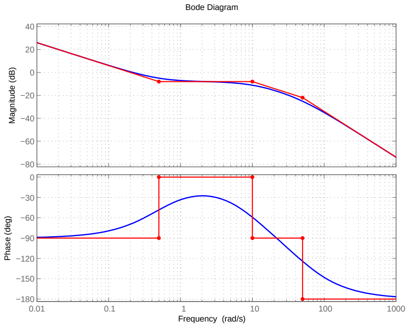
$$\underline{/j\omega\tau + 1} = \begin{cases} \approx 1, & \omega\tau << 1, \quad \underline{/1} = 0^\circ \\ \approx j\omega\tau, & \omega\tau >> 1, \quad \underline{/j\omega\tau} = 90^\circ \\ & \omega\tau \approx 1, \quad \underline{/j\omega\tau + 1} = 45^\circ \end{cases}$$



Example

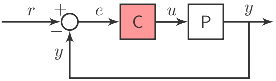
$$G(s) = \frac{200(s+0.5)}{s(s+10)(s+50)}$$





Steady-State Errors

Closed-loop system



Closed-loop transfer function

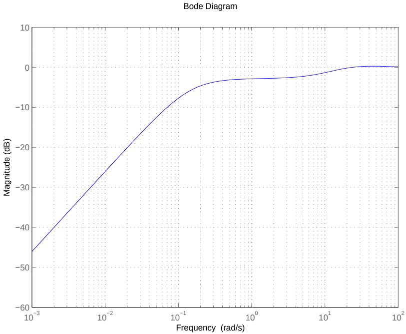
$$G_{er} = \frac{1}{1+PC} = K_0 (j\omega)^n \frac{(j\omega\tau_1+1)(j\omega\tau_2+1)\cdots}{(j\omega\tau_a+1)(j\omega\tau_b+1)\cdots}$$

Steady-state gain

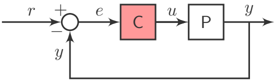
$$\lim_{s\to 0} sG_{er}(s)\frac{1}{s} \Leftrightarrow \lim_{\omega\to 0} |G_{er}(j\omega)|$$

$$PC = \frac{200(s+0.5)}{s(s+10)(s+50)}$$

Typically analysis is done with open-loop system



Open-loop system



Open-loop transfer function

$$PC = \frac{200(s+0.5)}{s(s+10)(s+50)} = K_0 (j\omega)^n \frac{(j\omega\tau_1+1)(j\omega\tau_2+1)\cdots}{(j\omega\tau_a+1)(j\omega\tau_b+1)\cdots}$$

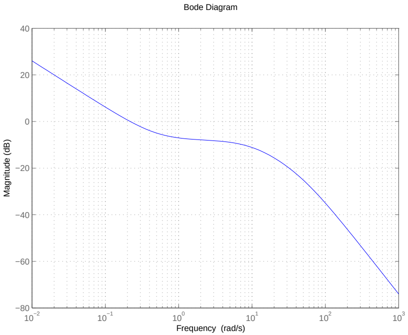
Steady-state error step

$$e_{\text{ss}} = \frac{1}{1+K_p}, \; K_p := K_0.$$

Steady-state error ramp

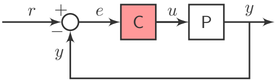
$$e_{\text{ss}} = \frac{1}{K_v}$$

- System type is the slope of the low frequency asymptote
- K_v is the value of low frequency asymptote at $\omega = 1$ rad/s



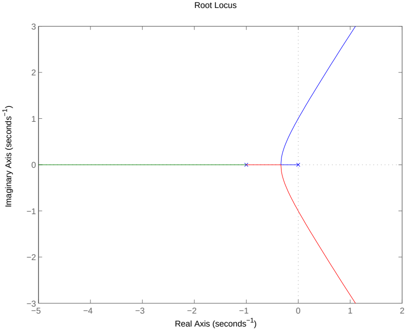
Stability Analysis

Stability



Given open-loop data

$C(s) = K, P(s) = \frac{1}{s(s+1)^2}$



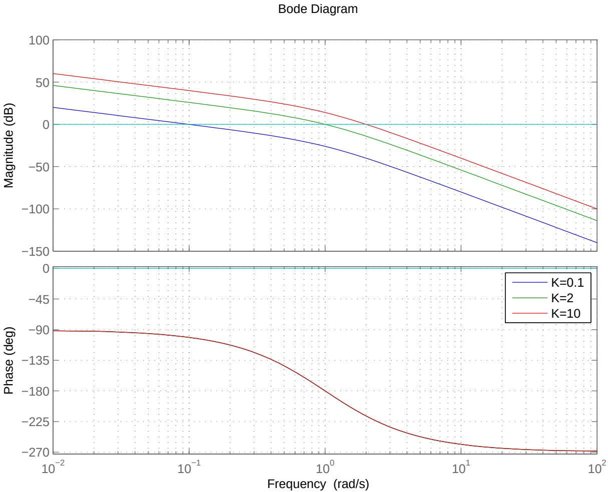
Stable for $K < 2$

- All points on root locus satisfy $1 + P(s)C(s) = 0$
- $P(s)C(s) = -1 \implies$ $|P(s)C(s)| = 1$ and $\angle P(s)C(s) = 180^\circ$
- At neutral stability point $s = j\omega$,

$$|P(j\omega)C(j\omega)| = 1$$
$$\angle P(j\omega)C(j\omega) = 180^\circ$$

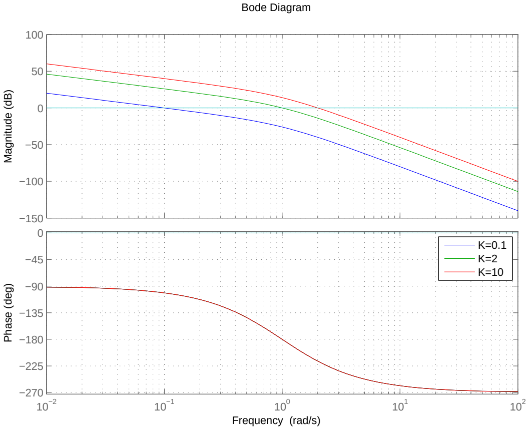
Stability

$|P(j\omega)C(j\omega)| < 1$ at $\angle P(j\omega)C(j\omega) = 180^\circ$



Gain Margin

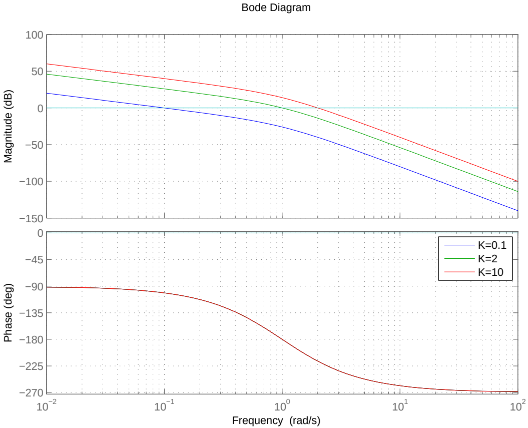
Open loop Bode Plot



Gain Margin (GM): factor by which gain can be increased at $\underline{/P(j\omega)C(j\omega)} = -180°$

Phase Margin

Open loop Bode Plot



Phase Margin (PM): amount by which phase exceeds -180° at $|P(j\omega)C(j\omega)| = 1$

Nyquist Plot

- Relates open-loop frequency response to number of unstable closed-loop poles
- Residue theorem in complex analysis
- Plot $P(j\omega)C(j\omega)$ in the complex plain
- Number of encirclements of -1 equals $Z - P$ of $1 + P(s)C(s)$

Nyquist Plot

contd.

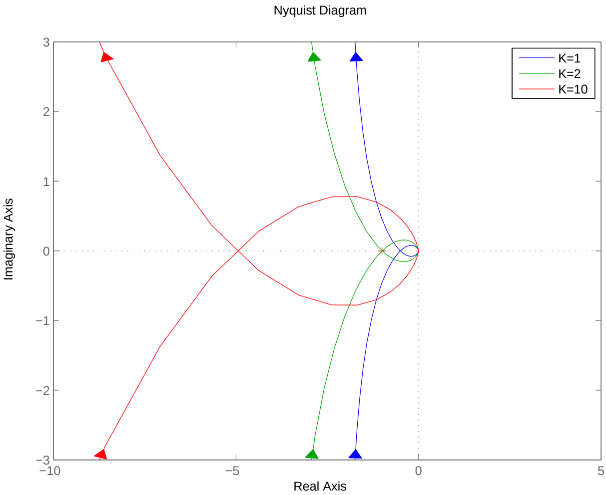
- Write $P(s)C(s) = KG(s) = K\frac{N(s)}{D(s)}$

$$\implies 1 + P(s)C(s) = \frac{D(s) + KN(s)}{D(s)}$$

- Poles of $1 + P(s)C(s)$ = Poles of $G(s)$ none of them on RHP
- Number of encirclements = number of zeros of $1 + P(s)C(s)$ on RHP number of poles of closed-loop system

Nyquist Plot

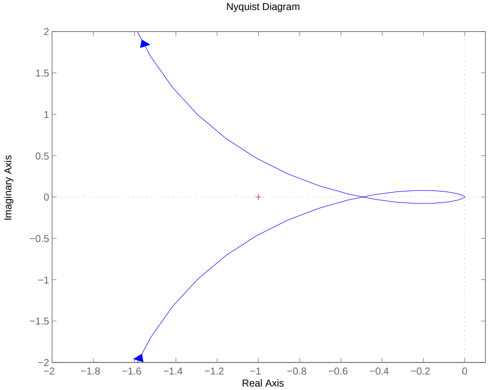
Example: $P(s)C(s) = \frac{K}{s(s+1)^2}$



Nyquist Plot

Determining Gain

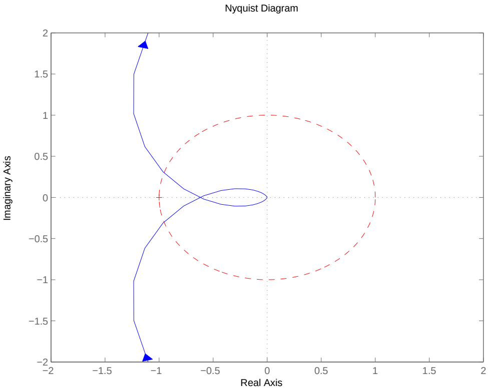
- Given $P(s)C(s) = \frac{K}{s(s+1)^2}$, what is K for stability?
- Encirclement of $1/K + G(s) = 0$



Nyquist Plot

Gain and Phase Margin

Nyquist plot of $P(s)C(s)$



Frequency Domain Design

Design Using Bode Plot of $P(j\omega)C(j\omega)$

Loop Shaping

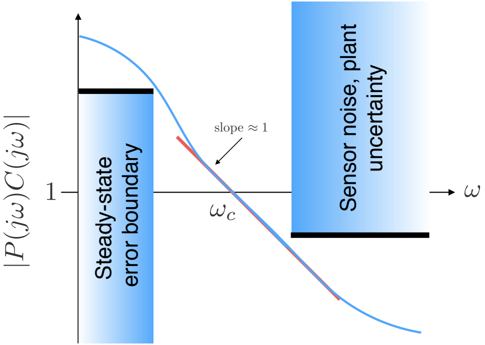
Develop conditions on the Bode plot of the open loop transfer function

- Sensitivity $\frac{1}{1+PC}$
- Steady-state errors: slope and magnitude at $\lim_{\omega} \to 0$
- Robust to sensor noise
- Disturbance rejection
- Controller roll off $\implies$ not excite high-frequency modes of plant
- Robust to plant uncertainty

Look at Bode plot of $L(j\omega) := P(j\omega)C(jw)$

Frquency Domain Specifications

Constraints on the shape of $L(j\omega)$

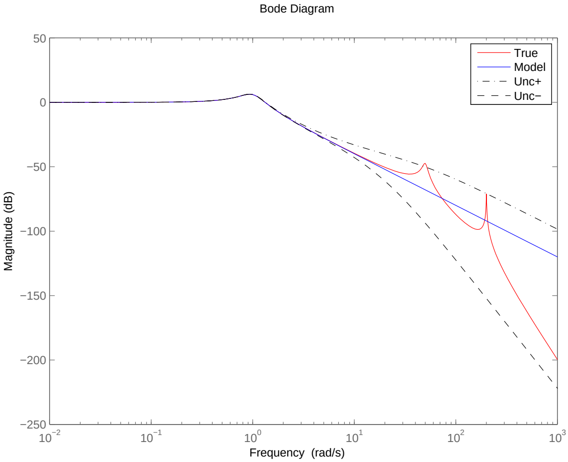


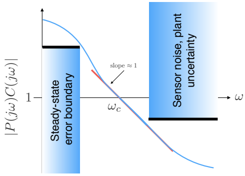
- Choose $C(j\omega)$ to ensure $|L(j\omega)|$ does not violate the constraints
- Slope ≈ -1 at ω_c ensures $PM \approx 90^\circ$

stable if $PM > 0 \implies \underline{/PC} > -180^\circ$

Plant Uncertainty

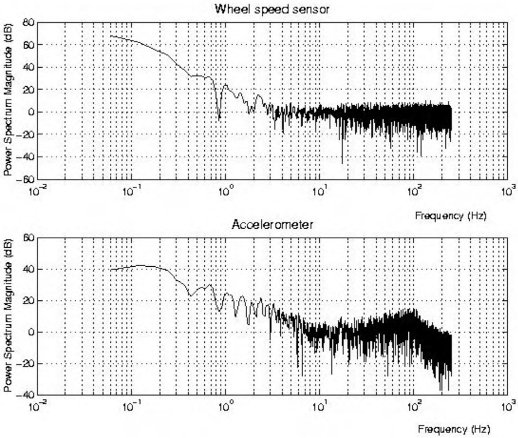
$$P(j\omega) = P_0(j\omega)(1 + \Delta P(j\omega))$$

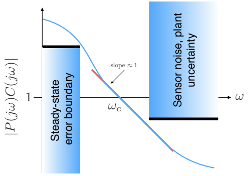




Sensor Characteristics

Noise spectrum

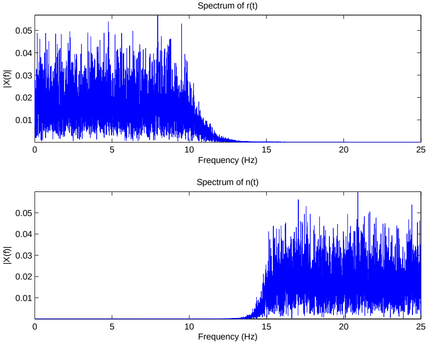


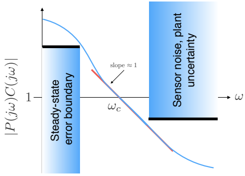


$$G_{yn} = -\frac{PC}{1+PC}$$

Reference Tracking

Bandlimited *else conflicts with noise rejection*



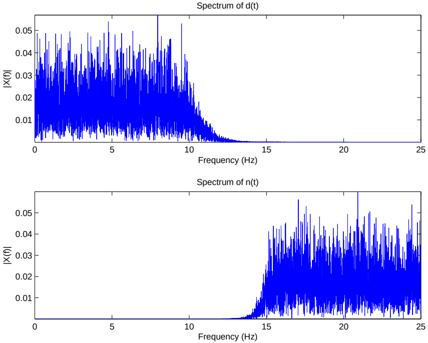


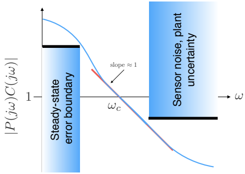
$$G_{yr} = \frac{PC}{1+PC}$$

$$G_{yn} = -\frac{PC}{1+PC}$$

Disturbance Rejecton

Bandlimited *else conflicts with noise rejection*





$$G_{yd} = \frac{P}{1+PC}$$

$$G_{yn} = -\frac{PC}{1+PC}$$