

Frobenius-Perron Operator

2015 ACC Workshop on Uncertainty Analysis & Estimation

Raktim Bhattacharya

Laboratory For Uncertainty Quantification
Aerospace Engineering, Texas A&M University.

isrlab.github.io

Frobenius-Perron Operator

Linear Operator \mathcal{P}_t

Given dynamics

$$\dot{\mathbf{x}} = \mathbf{F}(t, \mathbf{x}),$$

with $p(t_0, \mathbf{x})$ as the initial state density function.

- Evolution of density is given by

$$p(t, \mathbf{x}) := \mathcal{P}_t p(t_0, \mathbf{x}).$$

- \mathcal{P}_t has following properties

$$\mathcal{P}_t(\lambda_1 p_1 + \lambda_2 p_2) = \lambda_1 \mathcal{P}_t p_1 + \lambda_2 \mathcal{P}_t p_2 \quad \text{linearity}$$

$$\mathcal{P}_t p \geq 0 \text{ if } p \geq 0, \quad \text{positivity}$$

$$\int_{\mathcal{X}} \mathcal{P}_t p(t_0, \mathbf{x}) \mu(d\mathbf{x}) = \int_{\mathcal{X}} p(t_0, \mathbf{x}) \mu(d\mathbf{x}) \quad \text{measure preserving}$$

\mathcal{P}_t is defined by

$$\frac{\partial p}{\partial t} + \nabla \cdot (p \mathbf{F}) = 0$$

- Continuity equation
- FPK without diffusion term
- First order linear PDE

First Order PDEs

Method of Characteristics

$$\begin{aligned}\frac{\partial p}{\partial t} + \nabla \cdot (p\mathbf{F}) &= \frac{\partial p}{\partial t} + \sum_{i=1}^n \frac{\partial p F_i(t, \mathbf{x})}{\partial x_i} \\ &= \frac{\partial p}{\partial t} + \sum_{i=1}^n \frac{\partial p}{\partial x_i} F_i(t, \mathbf{x}) + p \sum_{i=1}^n \frac{\partial F_i(t, \mathbf{x})}{\partial x_i} = 0\end{aligned}$$

This is of the form

$$a(t, \mathbf{x}, p)p_t + \sum_i b_i(t, \mathbf{x}, p)p_{x_i} = c(t, \mathbf{x}, p).$$

Lagrange-Charpit equations

$$\frac{dt}{a(t, \mathbf{x}, p)} = \frac{dx_i}{b_i(t, \mathbf{x}, p)} = \frac{dp}{c(t, \mathbf{x}, p)}$$

Characteristic Equations

Lagrange-Charpit equations

$$\frac{dt}{a(t, \mathbf{x}, p)} = \frac{dx_i}{b_i(t, \mathbf{x}, p)} = \frac{dp}{c(t, \mathbf{x}, p)}$$

- Let s be parameterization of characteristic curves
- Characteristic curves are given by the ODEs

$$\frac{dt}{ds} = a(t, \mathbf{x}, p)$$

$$\frac{dx_i}{ds} = b_i(t, \mathbf{x}, p)$$

$$\frac{dp}{ds} = c(t, \mathbf{x}, p)$$

Solution of Continuity Equation

For continuity equation

$$\frac{\partial p}{\partial t} + \sum_{i=1}^n \frac{\partial p}{\partial x_i} F_i(t, \mathbf{x}) + p \sum_{i=1}^n \frac{\partial F_i(t, \mathbf{x})}{\partial x_i} = 0$$

$$a(t, \mathbf{x}, p) = 1, \quad b_i(t, \mathbf{x}, p) = F_i(t, \mathbf{x}), \quad c(t, \mathbf{x}, p) = -p \sum_{i=1}^n \frac{\partial F_i(t, \mathbf{x})}{\partial x_i}.$$

Characteristic equations

$$\frac{dt}{ds} = 1$$

$$\frac{dx_i}{ds} = F_i(t, \mathbf{x})$$

$$\frac{dp}{ds} = -p \sum_{i=1}^n \frac{\partial F_i(t, \mathbf{x})}{\partial x_i}$$

$$\dot{\mathbf{x}} = \mathbf{F}(t, \mathbf{x}) \text{ evolution of } \mathbf{x}(t)$$

$$\dot{p} = -p(\nabla \cdot \mathbf{F}) \text{ evolution of } p \text{ along } \mathbf{x}(t)$$

Initial Conditions

- \mathbf{x}_0 Samples from $p(t_0, \mathbf{x})$
- $p_0 = p(t_0, \mathbf{x}_0)$ Values of $p(t_0, \mathbf{x})$ at \mathbf{x}_0

Parametric Uncertainty & Process Noise

Given system dynamics

$$\dot{x} = F(t, x, \Delta) + n(t, \omega)$$

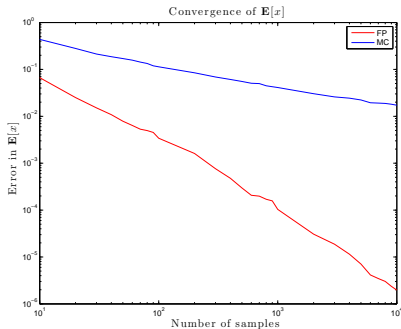
- Expand $n(t, \omega)$ using KL expansion.
- New paramters: $\xi := (\xi_0, \xi_0^*, \dots, \xi_N, \xi_N^*)^T$
- PDF: $p_\xi(\xi)$
- Parameter PDF: $p_\Delta(\Delta)$
- State IC PDF: $p_x(t_0, x)$

Augment state space

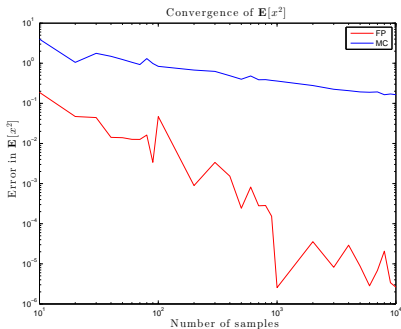
$$X := \begin{pmatrix} x \\ \Delta \\ \xi \end{pmatrix}, \quad \text{with} \quad \dot{X} := \begin{pmatrix} G(t, x, \Delta, \xi) \\ 0 \\ 0 \end{pmatrix} = H(t, X)$$

with $p_X(t_0, X) := p_x(t_0, x)p_\Delta(\Delta)p_\xi(\xi)$ and $p_X(t, X) := \mathcal{P}_t p_X(t_0, X)$.

Better Accuracy & Faster Convergence than MC



(a) First Moment



(b) Second Moment

- Data generated from univariate normal distribution
- MC: PDF from kernel density estimation
- FP: PDF from spline interpolation
- Samples generated 1000 times for a given size. Plots show average error vs sample size

Requires $\frac{\partial F_i(\mathbf{x})}{\partial x_i}$.

Nonlinear Example

3 DOF Vinh's Equation Models motion of spacecraft during planetary entry

$$\dot{h} = V \sin(\gamma)$$

$$\dot{V} = -\frac{\rho R_0}{2B_c} V^2 - \frac{g R_0}{v_c^2} \sin(\gamma)$$

$$\dot{\gamma} = \frac{\rho R_0}{2B_c} \frac{C_L}{C_D} V + \frac{g R_0}{v_c^2} \cos(\gamma) \left(\frac{V}{R_0 + h} - \frac{1}{V} \right).$$

R_0 – radius of Mars

ρ – atmospheric density

v_c – escape velocity

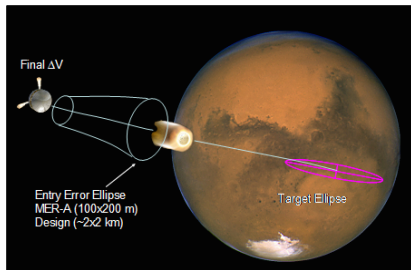
$\frac{C_L}{C_D}$ – lift over drag

B_c – ballistic coefficient

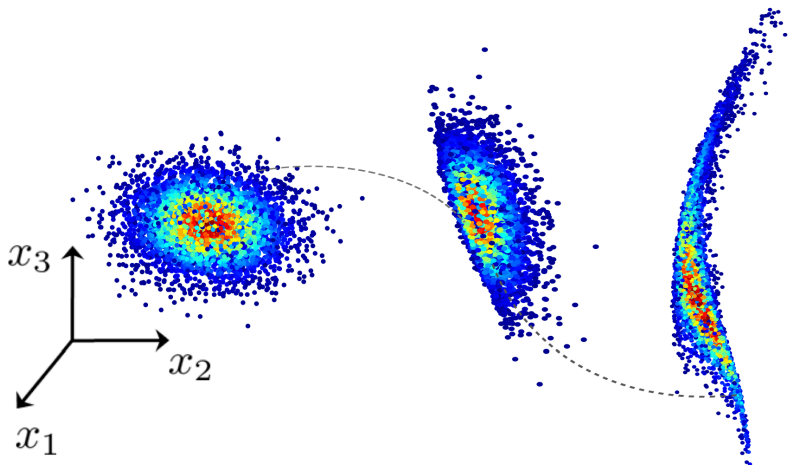
h – height

V – velocity

γ – flight path angle



3DOF Vinh's Equation



- Gaussian initial condition uncertainty in (h, V, γ)

Frobenius-Perron Operator

Papers

1. A. Halder, R. Bhattacharya, *Beyond Monte Carlo: A Computational Framework for Uncertainty Propagation in Planetary Entry, Descent and Landing*, AIAA GNC 2010.
2. A. Halder, R. Bhattacharya, *Dispersion Analysis in Hypersonic Flight During Planetary Entry Using Stochastic Liouville Equation*, AIAA Journal of Guidance, Control, and Dynamics, 2011, 0731-5090 vol.34 no.2 (459-474)
3. P. Dutta, A. Halder, R. Bhattacharya, *Uncertainty Quantification for Stochastic Nonlinear Systems using Perron-Frobenius Operator and Karhunen-Loeve Expansion*. IEEE Multi-Conference on Systems and Control, Dubrovnik, Oct 2012.