AERO 632: Design of Advance Flight Control System

Preliminaries

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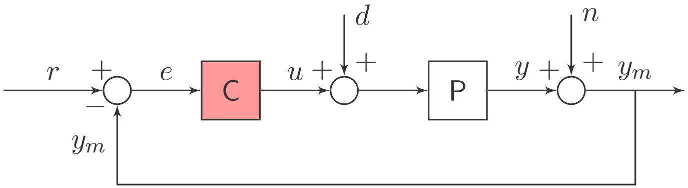
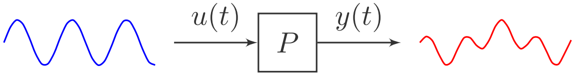
Preliminaries

- Signals & Systems
- Laplace transforms
- Transfer functions – from ordinary linear differential equations
- System interconnections
- Block diagram algebra – simplification of interconnections
- General feedback control system interconnection.



Signals & Systems

Signals & Systems

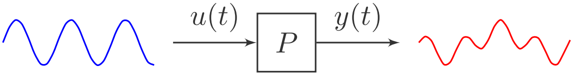


- Actuator applies $u(t)$
- Sensor provides $y(t)$

- Feedback controller takes $y(t)$ and determines $u(t)$ to achieve desired behavior
- The controller is typically implemented as software, running in a micro controller

- Imperfections exist in real world
  - *sensors have noise*
  - *actuators have irregularities*
  - *plant P is not fully known*

System Response to $u(t)$



Given plant P and input $u(t)$, what is $y(t)$?

- P is defined in terms of ordinary differential equations
- $y(t)$ is the forced + initial condition response.

Linear Dynamics

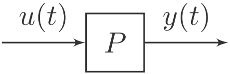
$$m\ddot{x} + c\dot{x} + kx = u(t) \quad \text{dynamics}$$
$$y(t) = x(t) \quad \text{measurement}$$

Nonlinear Dynamics

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = u(t) \quad \text{dynamics}$$
$$y(t) = x(t) \quad \text{measurement}$$

In this class we focus on linear systems

Linear Systems



- Dynamics is defined by linear ordinary differential equation
- Super position principle applies

$$\begin{matrix} u_1(t) \mapsto y_1(t) \\ u_2(t) \mapsto y_2(t) \end{matrix} \implies (u_1(t) + u_2(t)) \mapsto (y_1(t) + y_2(t))$$

Laplace Transforms

Laplace Transforms

Given signal $u(t)$, Laplace transform is defined as

$$\mathcal{L}\left\{u(t)\right\} := \int_0^\infty u(t)e^{-st}dt$$

Exists when

$$\lim_{t\to\infty} |u(t)e^{-\sigma t}| = 0, \text{ for some } \sigma > 0$$

Very useful in studying linear dynamical systems and designing controllers

Properties Laplace Transforms

Linear operator

- Additive

$$\begin{aligned}\mathcal{L}\left\{u_1(t) + u_2(t)\right\} &= \int_0^\infty \left(u_1(t) + u_2(t)\right) e^{-st} dt \\ &= \int_0^\infty u_1(t) e^{-st} dt + \int_0^\infty u_2(t) e^{-st} dt \\ &= \mathcal{L}\left\{u_1(t)\right\} + \mathcal{L}\left\{u_2(t)\right\}\end{aligned}$$

- Superposition

$$\mathcal{L}\left\{a u(t)\right\} = a\mathcal{L}\left\{u(t)\right\}, \quad a \text{ is a constant}$$

Properties (contd.)

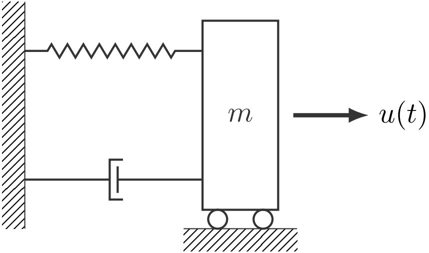
1. $U(s) := \mathcal{L}\{u(t)\}$
2. $\mathcal{L}\{au_1(t) + bu_2(t)\} = a\mathcal{L}\{u_1(t)\} + b\mathcal{L}\{u_2(t)\} = aU_1(s) + bU_2(s)$
3. $\frac{1}{s}U(s) \Longleftrightarrow \int_0^t u(\tau)d\tau$
4. $U_1(s)U_2(s) \Longleftrightarrow u_1(t) * u_2(t)$ Convolution
5. $\lim_{s\to 0} sU(s) \Longleftrightarrow \lim_{t\to\infty} u(t)$ Final value theorem
6. $\lim_{s\to\infty} sU(s) \Longleftrightarrow u(0^+)$ Initial value theorem
7. $-\frac{dU(s)}{ds} \Longleftrightarrow tu(t)$
8. $\mathcal{L}\left\{\frac{du}{dt}\right\} \Longleftrightarrow sU(s) - su(0)$
9. $\mathcal{L}\{\ddot{u}\} \Longleftrightarrow s^2U(s) - su(0) - \dot{u}(0)$

Important Signals

1. $\mathcal{L}\{\delta(t)\} = 1$ $\delta(t)$ is impulse function
2. $\mathcal{L}\{1(t)\} = \dfrac{1}{s}$ $1(t)$ is unit step function at $t = 0$
3. $\mathcal{L}\{t\} = \dfrac{1}{s^2}$
4. $\mathcal{L}\{\sin(\omega t\} = \dfrac{\omega}{s^2 + \omega^2}$
5. $\mathcal{L}\{\cos(\omega t\} = \dfrac{s}{s^2 + \omega^2}$

Transfer Functions

Spring Mass Damper System



Equation of Motion

$$m\ddot{x} + c\dot{x} + kx = u(t)$$

Take $\mathcal{L}\{\cdot\}$ on both sides

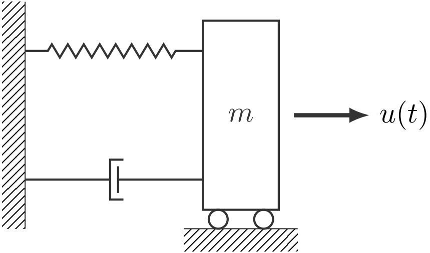
$$\mathcal{L}\{m\ddot{x} + c\dot{x} + kx\} = \mathcal{L}\{u(t)\}$$

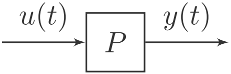
$$m\mathcal{L}\{\ddot{x}\} + c\mathcal{L}\{\dot{x}\} + k\mathcal{L}\{x\} = \mathcal{L}\{u(t)\}$$

$$m\left(s^2X(s) - sx(0) - \dot{x}(0)\right) + c\,(sX(s) - x(0)) + kX(s) = U(s)$$

$$(ms^2 + cs + k)X(s) = U(s)$$ $\dot{x}(0)$ and $x(0)$ are assumed to be zero

Transfer Function





$$(ms^2 + cs + k)X(s) = U(s) \implies \frac{X(s)}{U(s)} = \frac{1}{ms^2 + cs + k}$$

Choose output $y(t) = x(t) \implies Y(s) = X(s)$.
Therefore

$$P(s) := \frac{Y(s)}{U(s)} = \frac{1}{ms^2 + cs + k} \quad \text{Transfer function}$$

Transfer Function (contd.)

In general

$$P(s) = \frac{N(s)}{D(s)}$$

where $N(s)$ and $D(s)$ are polynomials in s

- Roots of $N(s)$ are the zeros
- Roots of $D(s)$ are the poles – determine stability

Response to $u(t)$

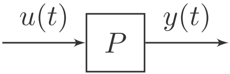
Given

– input signal $u(t)$ and transfer function $P(s)$.

Determine

– output response $y(t)$

1. **Laplace transform**
   $U(s) := \mathcal{L}\{u(t)\}$



2. **Determine** $Y(s) := P(s)U(s)$

3. **Laplace inverse**

$$y(t) := \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{P(s)U(s)\}$$

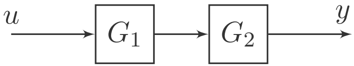
System Interconnection

Block Diagram

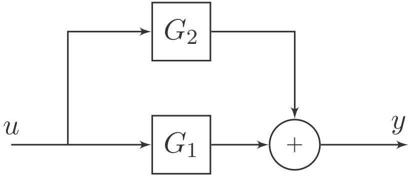
Representation of System Interconnections

- Series
- Parallel
- Feedback
- A simple example
- A complex example

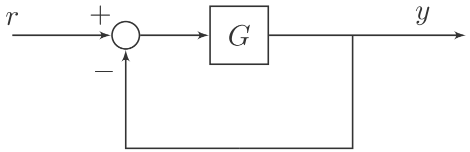
Series Connection



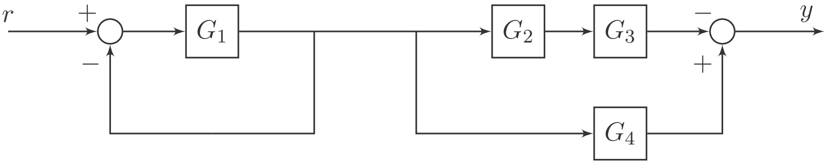
Parallel Connection



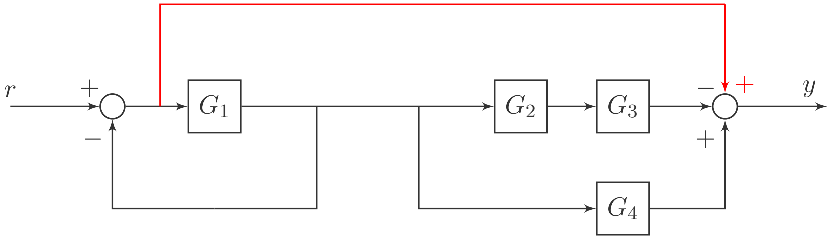
Feedback Connection



Simple Example



Complex Example



Frequency Response

Response to Sinusoidal Input

$$\xrightarrow{u(t)} \boxed{\text{P}} \xrightarrow{y(t)}$$

- Let $u(t) = A_u \sin(\omega t)$
- Vary ω from 0 to ∞

A linear system's response to sinusoidal inputs is called the system's frequency response

Response to Sinusoidal Input

Example

- Let $P(s) = \frac{1}{s+1}, u(t) = \sin(t)$

$$
\begin{aligned}
y(t) &= \frac{1}{2}e^{-t} - \frac{1}{2}\cos(t) + \frac{1}{2}\sin(t) \\
&= \underbrace{\frac{1}{2}e^{-t}}_{\text{natural response}} + \underbrace{\frac{1}{\sqrt{2}}\sin(t - \frac{\pi}{4})}_{\text{forced response}}
\end{aligned}
$$

- Forced response has form $A_y \sin(\omega t + \phi)$
- A_y and ϕ are functions of ω

Response to Sinusoidal Input

Generalization

In general

$$
\begin{aligned}
Y(s) &= G(s)\frac{\omega_0}{s^2+\omega_0^2} \\
&= \frac{\alpha_1}{s-p_1} + \cdots \frac{\alpha_n}{s-p_n} + \frac{\alpha_0}{s+j\omega_0} + \frac{\alpha_0^*}{s-j\omega_0} \\
\implies y(t) &= \underbrace{\alpha_1 e^{p_1 t} + \cdots + \alpha_n e^{p_n t}}_{\text{natural}} + \underbrace{A_y \sin(\omega_0 + \phi)}_{\text{forced}}
\end{aligned}
$$

Forced response has same frequency, different amplitude and phase.

Response to Sinusoidal Input

Generalization (contd.)

For a system $P(s)$ and input

$$u(t) = A_u \sin(\omega_0 t),$$

forced response is

$$y(t) = A_u M \sin(\omega_0 t + \phi),$$

where

$$M(\omega_0) = |P(s)|_{s=j\omega_0} = |P(j\omega_0)|, \text{ magnitude}$$

$$\phi(\omega_0) = \angle P(j\omega_0) \text{ phase}$$

In polar form

$$P(j\omega_0) = Me^{j\phi}.$$

Fourier Analysis

Fourier Series Expansion

Given a signal $y(t)$ with periodicity T,

$$y(t) = \frac{a_0}{2} + \sum_{n=1,2,\cdots} a_n \cos\left(\frac{2\pi nt}{T}\right) + b_n \sin\left(\frac{2\pi nt}{T}\right)$$

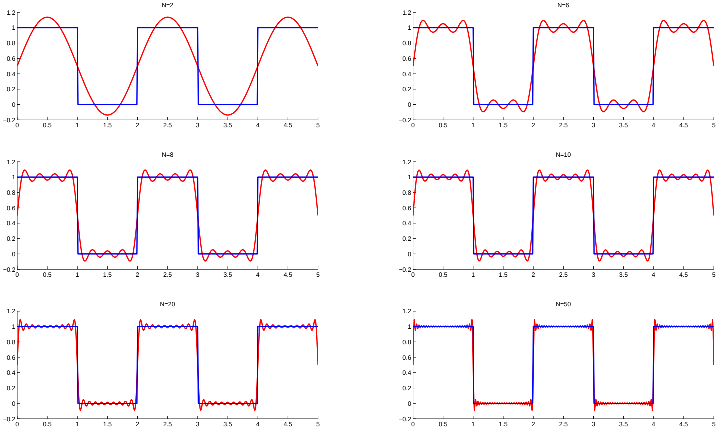
$$a_0 = \frac{2}{T}\int_0^T y(t)dt$$

$$a_n = \frac{2}{T}\int_0^T y(t)\cos\left(\frac{2\pi nt}{T}\right)dt$$

$$b_n = \frac{2}{T}\int_0^T y(t)\sin\left(\frac{2\pi nt}{T}\right)dt$$

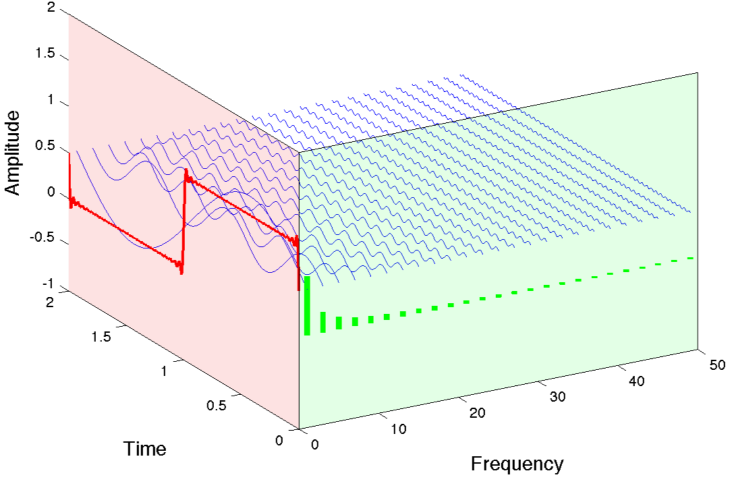
Fourier Series Expansion

Approximation of step function



Fourier Transform

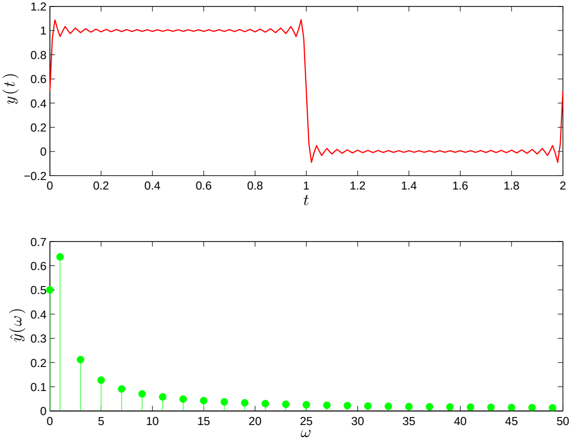
Step function



Fourier transform reveals the frequency content of a signal

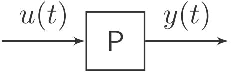
Fourier Transform

Step function – frequency content



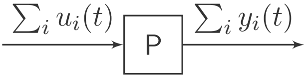
Signals & Systems

Input Output

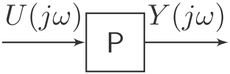


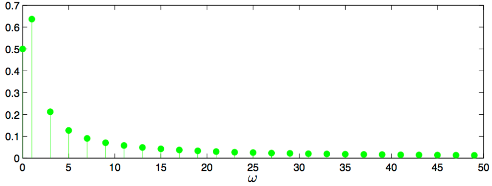
Fourier Series Expansion

superposition principle



Fourier Transform





$$u_i(t) = a_i \sin(\omega_i t)$$

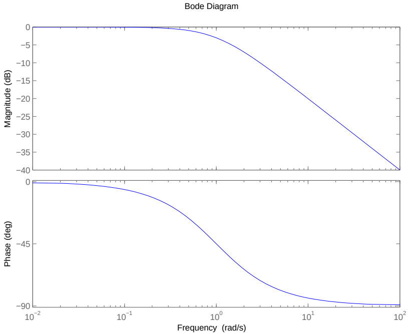
$$y_{i_{\text{forced}}}(t) = a_i M \sin(\omega_i t + \phi)$$

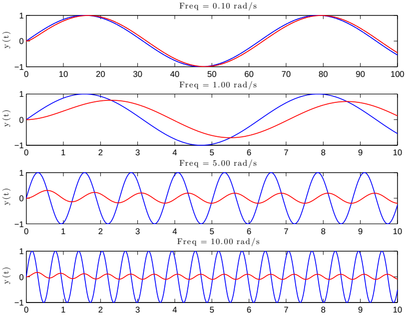
$$Y(j\omega) = P(j\omega)U(j\omega)$$

Suffices to study $P(j\omega)$ $|P(j\omega)|, \angle P(j\omega)$

Bode Plot

First Order System

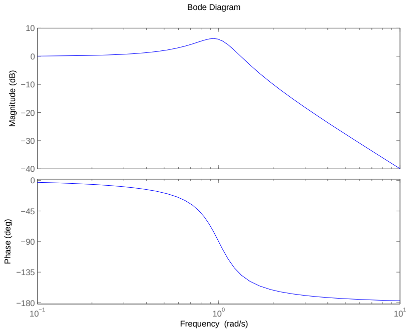


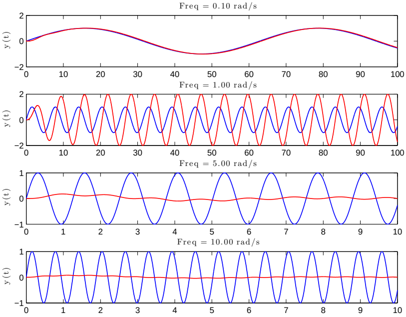


- $P(s) = 1/(s+1)$
- loglog scale
- dB $= 10\log_{10}(\cdot)$
- 20dB $= 10\log_{10}(100/1)$

- $u(t) = A\sin(\omega_0 t)$
- $y_{\text{forced}}(t) = AM\sin(\omega_0 t + \phi)$

Second Order System

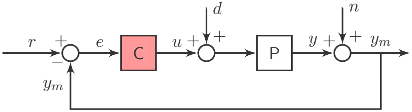




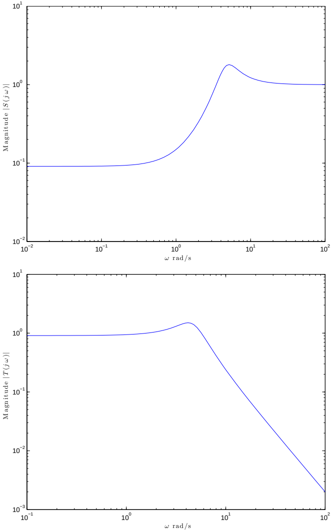
- $P(s) = 1/(s^2 + 0.5s + 1)$
- $\omega_n = 1$ rad/s

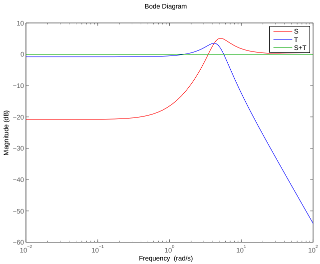
- $u(t) = A\sin(\omega_0 t)$
- $y_{\text{forced}}(t) = A{\color{red}M}\sin(\omega_0 t + {\color{red}\phi})$

$S(j\omega) + T(j\omega) = 1$



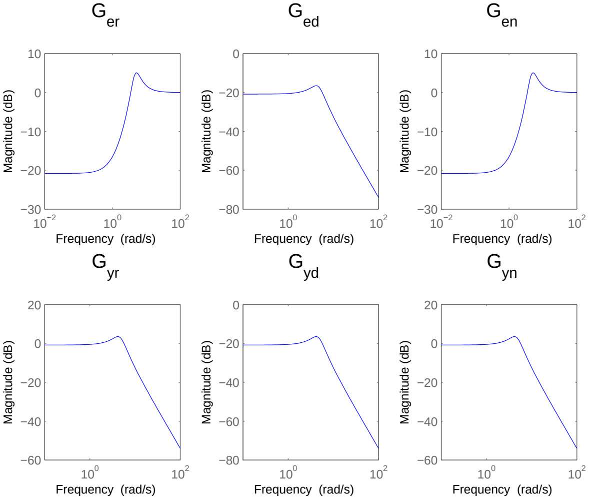
- $P(s) = \frac{1}{(s+1)(s/2+1)}$
- $C(s) = 10$
- $S = G_{er} = \frac{1}{1+PC} = \frac{1}{1+10P}$
- $T = G_{yr} = \frac{PC}{1+PC} = \frac{10P}{1+10P}$





All transfer functions

With proportional controller



Controller Design Considerations

Design Using Bode Plot of $P(j\omega)C(j\omega)$

Loop Shaping

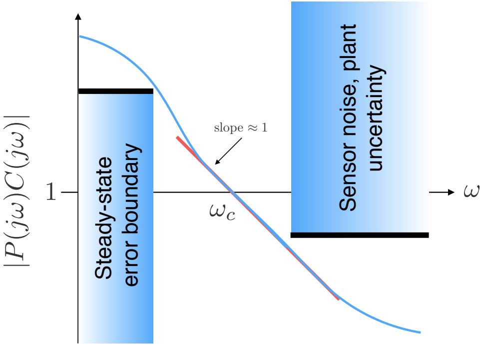
Develop conditions on the Bode plot of the open loop transfer function

- Sensitivity $\frac{1}{1+PC}$
- Steady-state errors: slope and magnitude at $\lim_{\omega} \to 0$
- Robust to sensor noise
- Disturbance rejection
- Controller roll off $\implies$ not excite high-frequency modes of plant
- Robust to plant uncertainty

Look at Bode plot of $L(j\omega) := P(j\omega)C(jw)$

Frequency Domain Specifications

Constraints on the shape of $L(j\omega)$

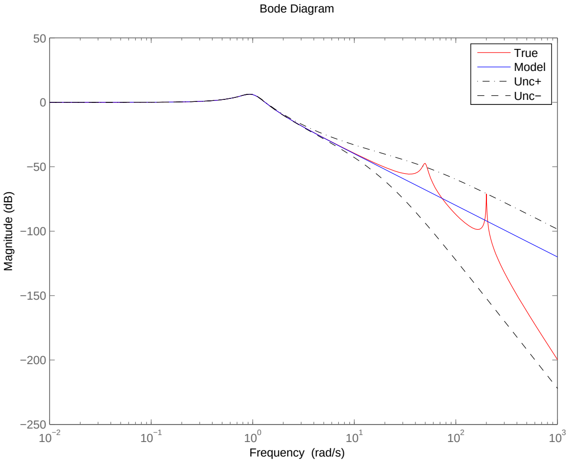


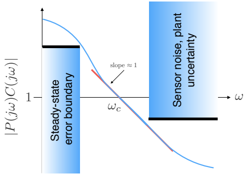
- Choose $C(j\omega)$ to ensure $|L(j\omega)|$ does not violate the constraints
- Slope ≈ -1 at ω_c ensures $PM \approx 90°$

stable if $PM > 0 \implies \underline{\angle PC} > -180°$

Plant Uncertainty

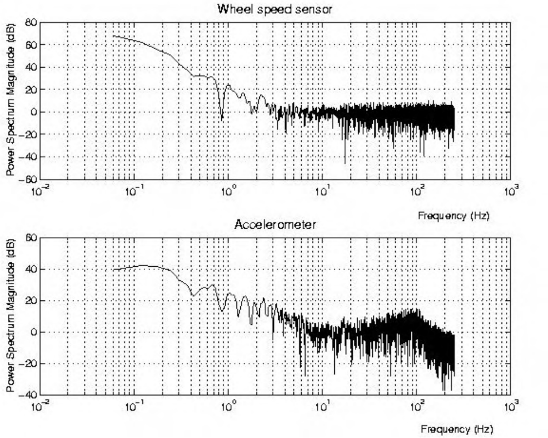
$$P(j\omega) = P_0(j\omega)(1 + \Delta P(j\omega))$$

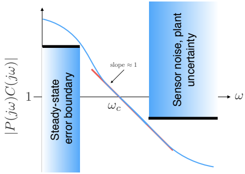




Sensor Characteristics

Noise spectrum

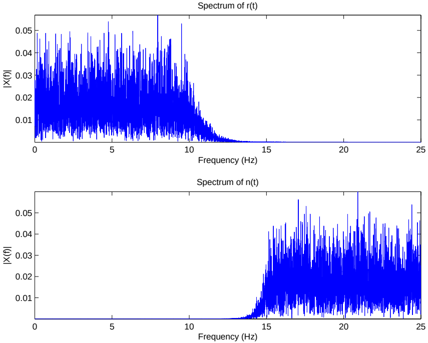


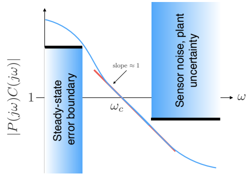


$$G_{yn} = -\frac{PC}{1+PC}$$

Reference Tracking

Bandlimited *else conflicts with noise rejection*



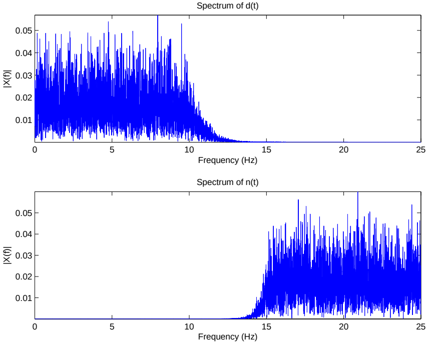


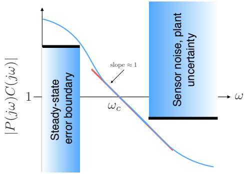
$$G_{yr} = \frac{PC}{1+PC}$$

$$G_{yn} = -\frac{PC}{1+PC}$$

Disturbance Rejecton

Bandlimited else conflicts with noise rejection





$$G_{yd} = \frac{P}{1+PC}$$

$$G_{yn} = -\frac{PC}{1+PC}$$