Estimator Design

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Observer Design

Observer Design $\mathcal{H}_\infty/\mathcal{H}_2$ Observer   Kalman Estimator $\mathcal{H}_\infty/\mathcal{H}_2$ Filtering

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Full Order Observer

Consider the system

$$\dot{x} = Ax + Bu, \ y = Cx.$$

Full order state observer takes the following form

$$\dot{\hat{x}} = A\hat{x} + Bu + L(C\hat{x} - y),$$

where

- $\hat{x}$ is state observation vector
- L is the observer gain

Define

$$e = x - \hat{x},$$

therefore

$$\dot{e} = (A + LC)e.$$

Full Order Observer

Design Problem

Problem Design L so that

$$\lim_{t \to \infty} e(t) := x(t) - \hat{x}(t) = 0.$$

Solution 1 It has a solution iff $\exists\, P > 0$ and W such that

$$PA + A^T P + WC + C^T W < 0,$$

and observer gain is

$$L = P^{-1}W.$$

Full-order state observer design is dual to state-feedback controller design.

$\mathcal{H}_\infty/\mathcal{H}_2$ **Observer**

Problem Setup

Consider the linear system

$$\dot{x} = Ax + B_u u + B_w w,$$
$$y = C_y x + D_u u + D_w w,$$
$$z = C_z x.$$

Full-Order State Observer Design L for

$$\dot{\hat{x}} = (A + LC_y)\hat{x} - Ly + (B_u + LD_u)u,$$

such that w has little effect on

$$\hat{z} = C_z \hat{x},$$

which is estimate of interested output.

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Observer Design $\mathcal{H}_\infty/\mathcal{H}_2$ **Observer**   Kalman Estimator $\mathcal{H}_\infty/\mathcal{H}_2$ Filtering

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Problem Setup

contd.

Define

$$e = x - \hat{x}, \tilde{z} = z - \bar{z}.$$

The observation error system is therefore,

$$\dot{e} = (A + LC_y)e + (B_w + LD_w)w,$$
$$\tilde{z} = C_z e.$$

The transfer function $\hat{G}_{w \to \tilde{z}}$ is therefore,

$$\hat{G}_{w \to \tilde{z}} = C_z(sI - A - LC_y)^{-1}(B_w + LD_w),$$

which is independent of u.

Observer Design $\mathcal{H}_\infty/\mathcal{H}_2$ **Observer**   Kalman Estimator $\mathcal{H}_\infty/\mathcal{H}_2$ Filtering

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Problem Setup

contd.

With

$$\hat{G}_{w \to \tilde{z}} = C_z(sI - A - LC_y)^{-1}(B_w + LD_w),$$

$\mathcal{H}_2$ **Observer**

$$\min_L \gamma,$$
$$\|\hat{G}_{w \to \tilde{z}}\|_2 < \gamma.$$

$\mathcal{H}_\infty$ **Observer**

$$\min_L \gamma,$$
$$\|\hat{G}_{w \to \tilde{z}}\|_\infty < \gamma.$$

Observer Design $\mathcal{H}_\infty/\mathcal{H}_2$ Observer   Kalman Estimator $\mathcal{H}_\infty/\mathcal{H}_2$ Filtering

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$\mathcal{H}_\infty$ **State Observer Design**

The optimization problem

$$\min_L \gamma,$$
$$\|\hat{G}_{w \to \tilde{z}}\|_\infty < \gamma,$$

has solution iff $\exists\, W$ and $P > 0$ such that

$$\min_{W,P} \gamma$$

such that

$$\begin{bmatrix} A^T P + C_y^T W^T + (*)^T & PB_w + WD_w & C_z^T \\ (PB_w + WD_w)^T & -\gamma I & 0 \\ C_z & 0 & -\gamma I \end{bmatrix} < 0.$$

Observer Design $\mathcal{H}_\infty/\mathcal{H}_2$ Observer   Kalman Estimator $\mathcal{H}_\infty/\mathcal{H}_2$ Filtering

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$\mathcal{H}_2$ **State Observer Design**

The optimization problem

$$\min_{L} \gamma,$$
$$\|\hat{G}_{w \to \tilde{z}}\|_2 < \gamma,$$

has solution iff $\exists\, W, Q > 0$, and $X > 0$ such that

$$\min_{W,Q,X} \gamma$$

such that

$$\mathbf{tr}Q < \gamma$$

$$\begin{bmatrix} XA + WC_y + (*)^T & XB_w + WD_w \\ (XB_w + WD_w)^T & -I \end{bmatrix} < 0,$$

$$\begin{bmatrix} -Q & C_z \\ C_z^T & -X \end{bmatrix} < 0.$$

Kalman Filtering

Kalman Filtering

TBD.

$\mathcal{H}_\infty/\mathcal{H}_2$ **Filtering**

$\mathcal{H}_\infty$ **Filtering**
Problem Formulation

Here we consider the dynamical system

$$\dot{x} = Ax + Bw, \; x(0) = x_0,$$
$$y = Cx + Dw,$$
$$z = Lx.$$

- Filtering is state-estimation for stochastic systems
- In this framework w need not be stochastic
- Design objective same: eliminate the effect of disturbance from estimate of z as much as possible
- System matrix A is closed-loop dynamics and stable

$\mathcal{H}_\infty$ **Filtering**

Problem Formulation (contd.)

System Dynamics

$$\dot{x} = Ax + Bw,\ x(0) = x_0,$$
$$y = Cx + Dw,$$
$$z = Lx.$$

Unknown Filter Dynamics

$$\dot{x}_F = A_F x_F + B_F y,\ x_F(0) = x_{F_0},$$
$$\hat{z} = C_F x_F + D_F y.$$

Augmented System Dynamics With

$$x_e = \begin{bmatrix} x \\ x_F \end{bmatrix},\ \tilde{z} = z - \hat{z},$$

dynamics is

$$\dot{x}_e = \tilde{A}x_e + \tilde{B}w,\ x_e(0) = x_{e_0}$$
$$\tilde{z} = \tilde{C}x_e + \tilde{D}w.$$

$\mathcal{H}_\infty$ **Filtering**

Problem Formulation (contd.)

Augmented System Dynamics With

$$x_e = \begin{bmatrix} x \\ x_F \end{bmatrix},\ \tilde{z} = z - \hat{z},$$

dynamics is

$$\dot{x}_e = \tilde{A}x_e + \tilde{B}w,\ x_e(0) = x_{e_0}$$
$$\tilde{z} = \tilde{C}x_e + \tilde{D}w,$$

where

$$\tilde{A} = \begin{bmatrix} A & 0 \\ B_F C & A_F \end{bmatrix}, \qquad\qquad \tilde{B} = \begin{bmatrix} B \\ B_F D \end{bmatrix},$$
$$\tilde{C} = \begin{bmatrix} L - D_F C & -C_F \end{bmatrix}, \qquad \tilde{D} = -D_F D.$$

$\mathcal{H}_\infty$ **Filtering**

Synthesis

Optimization Problem

$$\min_{A_F, B_F, C_F, D_F} \gamma, \ \|\hat{G}_{w \to \tilde{z}}\|_\infty < \gamma.$$

or

$$\min_{R, X, M, N, D_F} \gamma$$

subject to

$$\begin{bmatrix} RA + A^T R + ZC + C^T Z^T & * & * & * \\ M^T + ZC + XA & M^T + M & * & * \\ B^T R + D^T Z^T & B^T X + D^T Z^T & \gamma I & * \\ L - D_F C & -N & -D_F F & \gamma I \end{bmatrix} < 0,$$

and $X > 0, R - X > 0$.

$\mathcal{H}_\infty$ Filtering

Synthesis

Optimization Problem

$$\min_{R,X,M,N,D_F} \gamma$$

subject to

$$\begin{bmatrix} RA + A^T R + ZC + C^T Z^T & * & * & * \\ M^T + ZC + XA & M^T + M & * & * \\ B^T R + D^T Z^T & B^T X + D^T Z^T & -\gamma I & * \\ L - D_F C & -N & -D_F D & -\gamma I \end{bmatrix} < 0,$$

and $X > 0, R - X > 0$.

Filter Dynamics

$$A_F = X^{-1}M, \ B_F = X^{-1}Z, \ C_F = N.$$

Proof:
A linear matrix inequality approach to robust $\mathcal{H}_\infty$ filtering – Huaizhong Li, Minyue Fu.

$\mathcal{H}_2$ **Filtering**

Problem Formulation

System Dynamics

$$\dot{x} = Ax + B_w w, \ x(0) = x_0,$$
$$y = Cx + Dw,$$
$$z = Lx.$$

Unknown Filter Dynamics

$$\dot{x}_F = A_F x_F + B_F y, \ x_F(0) = x_{F_0},$$
$$\hat{z} = C_F x_F.$$

Augmented System Dynamics With

$$x_e = \begin{bmatrix} x \\ x_F \end{bmatrix}, \ \tilde{z} = z - \hat{z},$$

dynamics is

$$\dot{x}_e = \tilde{A} x_e + \tilde{B} w, \ x_e(0) = x_{e_0}$$
$$\tilde{z} = \tilde{C} x_e.$$

$\mathcal{H}_2$ **Filtering**

Problem Formulation (contd.)

Augmented System Dynamics With

$$x_e = \begin{bmatrix} x \\ x_F \end{bmatrix}, \ \tilde{z} = z - \hat{z},$$

dynamics is

$$\dot{x}_e = \tilde{A} x_e + \tilde{B} w, \ x_e(0) = x_{e_0}$$
$$\tilde{z} = \tilde{C} x_e + \tilde{D} w,$$

where

$$\tilde{A} = \begin{bmatrix} A & 0 \\ B_F C & A_F \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} B \\ B_F D \end{bmatrix},$$
$$\tilde{C} = \begin{bmatrix} L & -C_F \end{bmatrix}.$$

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$\mathcal{H}_2$ **Filtering**

Synthesis

Optimization Problem

$$\min_{A_F, B_F, C_F} \gamma, \ \|\hat{G}_{w \to \tilde{z}}\|_2 < \gamma.$$

$\mathcal{H}_2$ Filtering

Synthesis

Optimization Problem

$$\min_{R,X,M,N,Z,Q} \gamma$$

subject to

$$X > 0, \quad R - X > 0, \quad \mathbf{tr}\, Q < \gamma^2,$$

$$\begin{bmatrix} -Q & * & * \\ L^T & -R & * \\ -N^T & -X & -X \end{bmatrix} < 0,$$

$$\begin{bmatrix} RA + A^T R + ZC + C^T Z^T & * & * \\ M^T + ZC + XA & M^T + M & * \\ B^T R + D^T Z^T & B^T X + D^T Z^T & -I \end{bmatrix} < 0$$

with $A_f := X^{-1}M, B_f := X^{-1}Z, C_f := N$.

Proof:
Advances in Linear Matrix Inequality Methods in Control – edited by Laurent El Ghaoui, Silviu-Iulian Niculescu