Output Feedback Control

Raktim Bhattacharya
Aerospace Engineering, Texas A&M University

$\mathcal{H}_2$ Optimal Controller

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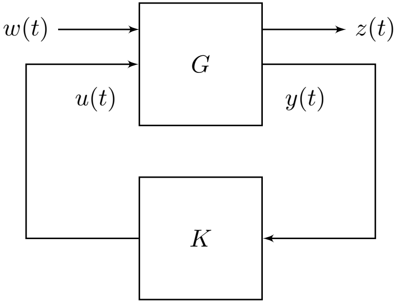
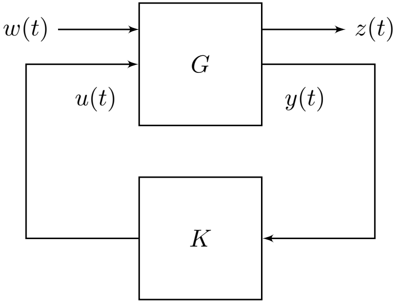


Let

$$\hat{G}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right], \qquad \hat{K}(s) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right].$$

$\mathcal{H}_2$ Optimal Controller



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$K(s)$ is essentially full-state feedback with $\mathcal{H}_2$ estimator.

$\mathcal{H}_\infty$ Optimal Controller

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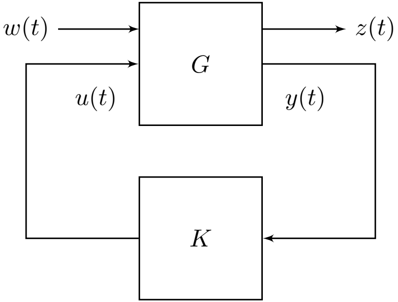


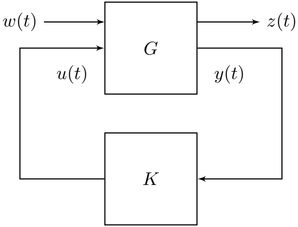
Let

$$\hat{G}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right], \qquad \hat{K}(s) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right].$$

Find $K(s)$ that minimizes $\|\hat{G}_{w\to z}\|_\infty$.

$\mathcal{H}_\infty$ Optimal Control

Simpler case with $D_{22} = 0$



System

$$\hat{G}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right],$$

$$\hat{K}(s) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right].$$

Closed-loop transfer function is

$$\mathcal{F}_l(\hat{G}, \hat{K}) = \begin{bmatrix} A_{\text{cl}} & B_{\text{cl}} \\ C_{\text{cl}} & D_{\text{cl}} \end{bmatrix},$$

$$= \left[\begin{array}{cc|c} A + B_2 D_K C_2 & B_2 C_K & B_1 + B_2 D_K D_{21} \\ B_K C_2 & A_K & B_K D_{21} \\ \hline C_1 + D_{12} D_K C_2 & D_{12} C_K & D_{11} + D_{12} D_K D_{21} \end{array}\right].$$

$\mathcal{H}_\infty$ Optimal Control

Simpler case with $D_{22} = 0$ (contd.)

Define matrices

$$J = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix},$$

and

$$\bar{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} C_1 & 0 \end{bmatrix}, \qquad \underline{C} = \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix},$$

$$\underline{B} = \begin{bmatrix} 0 & B_2 \\ I & 0 \end{bmatrix}, \qquad \underline{D}_{12} = \begin{bmatrix} 0 & D_{12} \end{bmatrix}$$

$$\underline{D}_{21} = \begin{bmatrix} 0 \\ D_{21} \end{bmatrix}.$$

These will be useful in reconstructing controller from LMI solution.

$\mathcal{H}_\infty$ Optimal Control

Simpler case with $D_{22} = 0$ (contd.)

Therefore

$$
\begin{aligned}
A_{\mathsf{cl}} &= \bar{A} + \underline{B} J \underline{C}, & B_{\mathsf{cl}} &= \bar{B} + \underline{B} J \underline{D}_{21} \\
C_{\mathsf{cl}} &= \bar{C} + \underline{D}_{12} J \underline{C}, & D_{\mathsf{cl}} &= D_{11} + \underline{D}_{12} J \underline{D}_{21}.
\end{aligned}
$$

$\mathcal{H}_\infty$ Optimal Control

Simpler case with $D_{22} = 0$ (contd.)

Theorem A_{cl} is Hurwitz and $\|\mathcal{F}_l(\hat{G}, \hat{K})\|_\infty < \gamma$, iff there exists symmetric matrices $X > 0$ and $Y > 0$ such that

$$\begin{bmatrix} N_o & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} A^*X + XA & XB_1 & C_1^* \\ B_1^*X & -\gamma I & D_{11}^* \\ C_1 & D_{11} & -\gamma I \end{bmatrix} \begin{bmatrix} N_o & 0 \\ 0 & I \end{bmatrix} < 0,$$

$$\begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} A^*Y + YA & YC_1 & B_1^* \\ C_1^*Y & -\gamma I & D_{11}^* \\ B_1 & D_{11} & -\gamma I \end{bmatrix} \begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix} < 0,$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0,$$

where N_o and N_c are full rank matrices satisfying

$$\textbf{Im}\, N_o = \textbf{Ker} \begin{bmatrix} C_2 & D_{21} \end{bmatrix}, \text{ and } \textbf{Im}\, N_c = \textbf{Ker} \begin{bmatrix} B_2^* & D_{12}^* \end{bmatrix}.$$

$\mathcal{H}_\infty$ Optimal Control

Simpler case with $D_{22} = 0$ (contd.)

Proof: See *A Linear Matrix Inequality Approach to $\mathcal{H}_\infty$ Control* – Pascal Gahinet, Pierre Apkarian, 1994.

Main Ingredients:

- KYP Lemma
- Projection Lemmas

$\mathcal{H}_\infty$ Controller Reconstruction

Suppose we have solved the LMIs and have obtained X, Y. Define X_{cl} as

$$X_{\mathsf{cl}} = \begin{bmatrix} X & {X_2}^* \\ X_2 & I \end{bmatrix},$$

such that

$$X - Y^{-1} = X_2 X_2^*.$$

The LMI in the synthesis enforces

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0$$

implies

$$X - Y^{-1} \geq 0.$$

Therefore, we can set

$$X_2 := \sqrt{X - Y^{-1}}.$$ symmetric

$\mathcal{H}_\infty$ Controller Reconstruction

contd.

With X_2 determined, X_{cl} is defined.

Define matrices,

$$P_{X_{\mathsf{cl}}} = \begin{bmatrix} \underline{B}^* & 0 & \underline{D}_{12}^* \end{bmatrix},\ Q = \begin{bmatrix} \underline{C} & \underline{D}_{21} & 0 \end{bmatrix},$$
$$H_{X_{\mathsf{cl}}} = \begin{bmatrix} \bar{A}^* X_{\mathsf{cl}} + X_{\mathsf{cl}} \bar{A} & X_{\mathsf{cl}} \bar{B} & \bar{C}^* \\ \bar{B}^* X_{\mathsf{cl}} & -\gamma I & D_{11}^* \\ \bar{C} & D_{11} & -\gamma I \end{bmatrix}.$$

$\mathcal{H}_\infty$ Controller Reconstruction

contd.

With X_2 determined, X_{cl} is defined.

Define matrices,

$$P_{X_{\mathsf{cl}}} = \begin{bmatrix}\underline{B}^* & 0 & \underline{D}_{12}^*\end{bmatrix}, \; Q = \begin{bmatrix}\underline{C} & \underline{D}_{21} & 0\end{bmatrix},$$

$$H_{X_{\mathsf{cl}}} = \begin{bmatrix} \bar{A}^* X_{\mathsf{cl}} + X_{\mathsf{cl}}\bar{A} & X_{\mathsf{cl}}\bar{B} & \bar{C}^* \\ \bar{B}^* X_{\mathsf{cl}} & -\gamma I & D_{11}^* \\ \bar{C} & D_{11} & -\gamma I \end{bmatrix}.$$

Controller matrix J can be obtained using reciprocal projection lemma (see Gahinet, Apkarian 1994)

$$H_{X_{\mathsf{cl}}} + Q^* J^* P_{X_{\mathsf{cl}}} + P_{X_{\mathsf{cl}}} J Q < 0.$$

$\mathcal{H}_\infty$ Controller Reconstruction

contd.

With X_2 determined, X_{cl} is defined.

Define matrices,

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$$H_{X_{\text{cl}}} + Q^* J^* P_{X_{\text{cl}}} + P_{X_{\text{cl}}} J Q < 0.$$

- Many solutions for J exists!
- A family of γ – optimal $\mathcal{H}_\infty$ controllers exists.
- Formulation is LMI feasibility

Few Comments about $\mathcal{H}_\infty$

- We do not seek optimal γ as the computations become singular – instead we seek suboptimal γ
- Bisection algorithm is used to reduced γ and LMIs are solved
- Issue with controller-plant pole-zero cancellation
- Poles in $j\omega$ axis

LMI framework provides a flexibility to address/circumvent these issues

(see Gahinet, Apkarian 1994)

Application of $\mathcal{H}_\infty$ Controller

(V, γ) Tracking Controller for Longitudinal F16 Model

```
clc; clear;

% Define F16 Model
% ================
load f16LongiLinear.mat
wd = 1; % Scaling to adjust alpha disturbance

A = f16ss.a;
Bu = f16ss.b;
Bd = [0;wd;0;0];
B = [Bu Bd];
Cy = [1  0 0 0; % Velocity
      0 -1 1 0]; % gamma

[ns,nu] = size(B);
F16 = ss(A,B,Cy,zeros(2,nu));

% Define Weights
% ==============
s = tf('s');

Wr = blkdiag(1/(s/1+1), 1/(s/5+1));
Wu = blkdiag(1/5000,1/25);
We = blkdiag(1, 1);
Wd = 0.1/(s/.1+1);
Wn = 0.01*blkdiag(1,1);
Wm = blkdiag(1, 1);
```

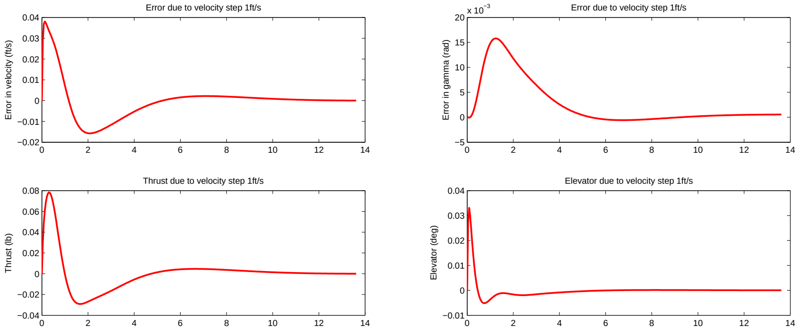
```
% System Interconnection
% ======================
r = icsignal(2);
d = icsignal(1);
n = icsignal(2);
u = icsignal(2);
y = F16*[u;Wd*d];
e = (Wr*r - y);

G = iconnect;
G.input = [r;d;n;u];
G.output = [We*e;Wu*u;Wr*r-y-Wn*n];

[K,F16cl,gam,info] = hinfsyn(G.System,2,2,...
                                   'method','lmi');
disp(sprintf('Minimum gamma = %f',gam));
```

Application of $\mathcal{H}_\infty$ Controller

(V,γ) Tracking Controller for Longitudinal F16 Model



$$\gamma^* = 0.971881$$