AERO 632: Design of Advance Flight Control System

Norms for Signals and Systems

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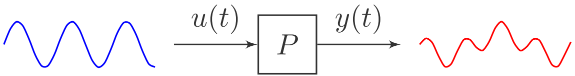
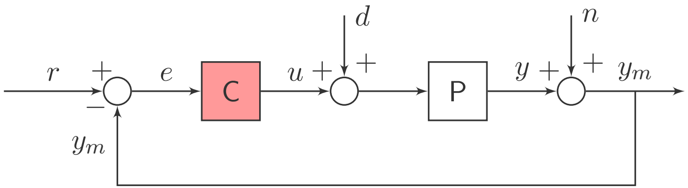
Norms for Signals

Signals



- We consider signals mapping $(-\infty, \infty) \mapsto \mathbb{R}$
- Piecewise continuous
- A signal may be zero for $t < 0$

- We worry about size of signal
- Helps specify performance
- Signal size $\iff$ signal norm



Norms

A norm must have the following 4 properties

- $\|\boldsymbol{u}\| \geq 0$
- $\|\boldsymbol{u}\| = 0 \iff \boldsymbol{u} = 0$
- $\|a\boldsymbol{u}\| = |a|\|\boldsymbol{u}\|, \forall a \in \mathbb{R}$
- $\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$ triangle inequality

For $\boldsymbol{u} \in \mathbb{R}^n$ and $p \geq 1$,

$$\|\boldsymbol{u}\|_p := \left(|u_1|^p + \cdots + |u_n|^p\right)^{1/p}$$

Special case,

$$\|\boldsymbol{u}\|_\infty := \max_i |u_i|$$

Norms of Signals

$\mathcal{L}_1$ Norm

The 1-norm of a signal $u(t)$ is the integral of its absolute value:

$$\|u(t)\|_1 := \int_{-\infty}^{\infty} |u(t)| dt$$

$\mathcal{L}_2$ Norm

The 2-norm of a signal $u(t)$ is

$$\|u(t)\|_2 := \left(\int_{-\infty}^{\infty} u(t)^2 dt \right)^{1/2}$$ associated with energy of signal

$\mathcal{L}_\infty$ Norm

The ∞-norm of a signal $u(t)$ is the least upper bound of its absolute value:

$$\|u(t)\|_\infty := \sup_t |u(t)|$$

Power Signals

The average power of $u(t)$ is the average over time of its instantaneous power:

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u^2(t)dt$$

- if limit exists, $u(t)$ is called a power signal
- average power is then

$$\textbf{pow}(u) := \left(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u^2(t)dt \right)^{1/2}$$

- **pow**$(\cdot)$ is not a norm
  - non zero signals can have **pow**$(\cdot) = 0$

Vector Signals

For $\boldsymbol{u}(t) : (-\infty, \infty) \mapsto \mathbb{R}^n$ and $p > 1$

$$\|\boldsymbol{u}(t)\|_p := \left(\int_{-\infty}^{\infty} \sum_{i=1}^{n} |u_i(t)|^p dt \right)^{1/p}$$

Finiteness of Norms

Does finiteness of one norm imply finiteness of another?

- $\|u\|_2 < \infty \implies \textbf{pow}(u) = 0$

  We have
$$\frac{1}{2T}\int_{-T}^{T} u^2(t)dt \leq \frac{1}{2T}\int_{-\infty}^{\infty} u^2(t)dt = \frac{1}{2T}\|u\|_2.$$
  Right hand side tends to zero as $T \to \infty$

Finiteness of Norms (contd.)

- If u is a power signal and $\|u\|_\infty < \infty$, then $\mathbf{pow}(u) \leq \|u\|_\infty$.

  We have
  $$\frac{1}{2T}\int_{-T}^{T} u^2(t)dt \leq \|u\|_\infty \frac{1}{2T}\int_{-T}^{T} dt = \|u\|_\infty$$
  Let $T \to \infty$.

Finiteness of Norms (contd.)

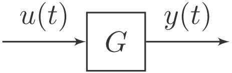
- If $\|u\|_1 < \infty$ and $\|u\|_\infty < \infty$ then $\|u\|_2 < \infty$

  We have

$$
\begin{aligned}
\int_{-\infty}^{\infty} u^2(t)dt &= \int_{-\infty}^{\infty} |u(t)| \cdot |u(t)| dt \\
&\leq \|u\|_\infty \int_{-\infty}^{\infty} |u(t)| dt \\
&= \|u\|_\infty \|u\|_1 \\
&\leq \infty
\end{aligned}
$$

Norms for Systems

System



We consider

- Linear
- Time invariant
- Causal
- Finite dimensional

In time domain

- if $u(t)$ is the input to the system and
- $y(t)$ is the output

System has the form

$$
\begin{aligned}
y &= G * u \text{ convolution} \\
&= \int_{-\infty}^{\infty} G(t-\tau)u(\tau)d\tau \quad \mathcal{L}^{-1}\left\{\hat{G}(s)\right\} := G(t)
\end{aligned}
$$

Causality

Causal

- A system is causal when the effect does not anticipate the cause; or zero input produces zero output
- Its output and internal states only depend on current and previous input values
- Physical systems are causal

Causality

contd.

Acausal

- A system whose output is nonzero when the past and present input signal is zero is said to be anticipative
- A system whose state and output depend also on input values from the future, besides the past or current input values, is called acausal
- Acausal systems can only exist as digital filters (digital signal processing).

Causality

contd.

Anti-Causal

- A system whose output depends only on future input values is anti-causal
- Derivative of a signal is anti-causal.

Causality

contd.

- Zeros are anticipative
- Poles are causal
- Overall behavior depends on m and n.
- Causal: $n > m$, strictly proper
- Causal: $n = m$, still causal, but there is instantaneous transfer of information from input to output
- Acausal: $n < m$

Example

- System $G_1(s) = s$
- Input $u(t) = \sin(\omega t)$, $U(s) = \frac{\omega}{s^2+\omega^2}$
- $y_1(t) = \mathcal{L}^{-1}\left\{G_1(s)U(s)\right\} = \mathcal{L}^{-1}\left\{\frac{s\omega}{s^2+\omega^2}\right\} = \omega\cos(\omega t)$, or

$$
\begin{aligned}
u(t) &= \sin(\omega t) \\
y_1(t) &= \omega \sin(\omega t + \pi/2) \\
&= \omega u(t + \frac{\pi}{2\omega}) \text{ output leads input, anticipatory}
\end{aligned}
$$

Example

contd.

- System $G_2(s) = \frac{1}{s}$
- Input $u(t) = \sin(\omega t)$, $U(s) = \frac{\omega}{s^2+\omega^2}$
- $y_2(t) = \mathcal{L}^{-1}\left\{G_2(s)U(s)\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\frac{\omega}{s^2+\omega^2}\right\} = \frac{1}{\omega} - \frac{\cos(\omega t)}{\omega}$, or

$$
\begin{aligned}
u(t) &= \sin(\omega t) \\
y_2(t) &= \frac{1}{\omega} + \frac{\sin(\omega t - \pi/2)}{\omega} \\
&= \frac{1}{\omega} + \frac{u(t - \frac{\pi}{2\omega})}{\omega} \quad \text{output lags input, causal}
\end{aligned}
$$

Causality

(contd.)

Causality means

$$G(t) = 0 \text{ for } t < 0$$

- $\hat{G}(s)$ is stable if it is analytic in the closed RHP residue theorem
- proper if $\hat{G}(j\infty)$ is finite deg of den $\geq$ deg of num
- strictly proper if $\hat{G}(j\infty) = 0$ deg of den $>$ deg of num
- biproper $\hat{G}$ and $\hat{G}^{-1}$ are both proper

Norms of $\hat{G}$

Definitions for SISO Systems

$\mathcal{L}_2$ Norm

$$\|\hat{G}\|_2 := \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{G}(j\omega)|^2 d\omega\right)^{1/2}$$

$\mathcal{L}_\infty$ Norm

$$\|\hat{G}\|_\infty := \sup_{\omega} |\hat{G}(j\omega)| \text{ peak value of } |\hat{G}(j\omega)|$$

Parseval's Theorem

If $\hat{G}(j\omega)$ is stable

$$\|\hat{G}\|_2 = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{G}(j\omega)|^2 d\omega\right)^{1/2} = \left(\int_{-\infty}^{\infty} |G(t)|^2 dt\right)^{1/2}.$$

Important Properties of System Norms

Submultiplicative Property of ∞-norm

$$\|\hat{G}\hat{H}\|_\infty \leq \|\hat{G}\|_\infty \|\hat{H}\|_\infty$$

Important Properties of System Norms (contd.)

Lemma 1

$\|\hat{G}\|_2$ is finite iff $\hat{G}$ is **strictly proper** and has no poles on the imaginary axis.

Proof: Look at transfer function of the type

$$\hat{G}(s) = \frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}, n > m.$$

Argue area under $|\hat{G}(j\omega)|^2$ is finite.

Or apply residue theorem

$$\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{G}(j\omega)|^2 d\omega\right)^{1/2} = \frac{1}{2\pi j}\oint_{\text{LHP}} \hat{G}(-s)\hat{G}(s)ds.$$

Important Properties of System Norms (contd.)

Lemma 2
$\|\hat{G}\|_\infty$ is finite iff $\hat{G}$ is **proper** and has no poles on the imaginary axis.

Proof: Look at transfer function of the type

$$\hat{G}(s) = \frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}, n >= m.$$

Argue $\sup_\omega |\hat{G}(j\omega)|$ is finite.

Signal Spaces

Performance Specification

- Describe performance in terms of norms of certain signals of interest
- Understand which norm is suitable
  - difference from control system performance perspective
- We will learn Hardy spaces $\mathcal{H}_2$ and $\mathcal{H}_\infty$
  - measures of worst possible performance for many classes of input signals

Vector Space

Also called linear space

Elements $u, v, w \in \mathcal{V} \subseteq \mathbb{C}^n$ (or $\mathbb{R}^n$) satisfy the following 8 axioms

- Associativity of addition

$$u + (v + w) = (u + v) + w$$

- Commutativity of addition

$$u + v = v + u$$

- Identity element of addition

$$0 + v = v, \forall v \in \mathcal{V}$$

- Inverse element of addition

$$\text{for every } v \in \mathcal{V}, \exists -v \in \mathcal{V} : v + (-v) = 0$$

Vector Space

contd.

- Compatibility of scalar multiplication

$$\alpha(\beta u) = (\alpha\beta)u$$

- Identity of multiplication

$$1v = v$$

- Distributivity of scalar multiplication wrt vector addition

$$\alpha(u + v) = \alpha u + \alpha v$$

- Distributivity of scalar multiplication wrt field addition

$$(\alpha + \beta)u = \alpha u + \beta u$$

Normed Space

- Let $\mathcal{V}$ be a vector space over $\mathbb{C}$ or $\mathbb{R}$
- Let $\|\cdot\|$ be defined over $\mathcal{V}$
- Then $\mathcal{V}$ is a normed space

Example 1

A vector space $\mathbb{C}^n$ with any vector p-norm, $\|\cdot\|$, for $1 \leq p \leq \infty$.

Example 2

Space $C[a,b]$ of all bounded continuous functions becomes a norm space if

$$\|f\|_\infty := \sup_{t\in[a,b]} |f(t)|$$

is defined.

Banach Space

- A sequence $\{x_n\}$ in a normed space $\mathcal{V}$ is Cauchy sequence, if

$$\|x_n - x_m\| \to 0 \text{ as } n, m \to 0.$$

- A sequence $\{x_n\}$ is said to converge to $x \in \mathcal{V}$, written $x_n \to x$, if

$$\|x_n - x\| \to 0.$$

- A normed space $\mathcal{V}$ is said to be complete if every Cauchy sequence in $\mathcal{V}$ converges in $\mathcal{V}$.
- A complete normed space is called a Banach space.

Banach Space

$l_p[0,\infty)$ spaces for $1 \leq p < \infty$

For each $1 \leq p < \infty$, $l_p[0,\infty)$ consists of all sequence $x = (x_0, x_1, \cdots)$ such that

$$\sum_{i=0}^{\infty} |x_i|^p < \infty.$$

The associate norm is defined as

$$\|x\|_p := \left(\sum_{i=0}^{\infty} |x_i|^p \right)^{1/p}.$$

Banach Space

$l_\infty[0,\infty)$ *space*

$l_\infty[0,\infty)$ consists of all bounded sequence $x = (x_0, x_1, \cdots)$.

The l_∞ norm is defined as

$$\|x\|_\infty := \sup_i |x_i|.$$

Banach Space

$\mathcal{L}_p(I)$ spaces for $1 \le p < \infty$

For each $1 \le p \le \infty$, $\mathcal{L}_p(I)$ consists of all Lebesgue measurable functions $x(t)$ defined on an interval $I \subset \mathbb{R}$ such that

$$\|x\|_p := \left(\int_I |x(t)|^p \mu(dt) \right)^{1/p} < \infty, \text{ for } 1 \le p < \infty,$$

and

$$\|x\|_\infty := \operatorname*{ess\,sup}_{t \in I} |x(t)|.$$

Note: $\operatorname{ess\,sup}_{t \in I}$ is $\sup_{t \in I}$ almost everywhere.

We will study $\mathcal{L}_2(-\infty, 0]$, $\mathcal{L}_2[0, \infty)$, and $\mathcal{L}_2(-\infty, \infty)$ spaces in detail.

Banach Space

$C[a,b]$ space

Consists of all continuous functions on the real interval $[a,b]$ with the norm

$$\|x\|_\infty := \sup_{t\in[a,b]} |x(t)|.$$

Inner-Product Space

Recall the inner product of vectors in Euclidean space $\mathbb{C}^n$:

$$\langle x, y \rangle := x^* y = \sum_{i=1}^{n} \bar{x}_i y_i, \forall x, y \in \mathbb{C}^n.$$

Important Metric Notions & Geometric Properties

- length, distance, angle
- energy

We can generalize beyond Euclidean space!

Inner-Product Space

Genralization

Let $\mathcal{V}$ be a vector space over $\mathbb{C}$. An inner product on $\mathcal{V}$ is a complex value function

$$\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \mapsto \mathbb{C}$$

such that for any $\alpha, \beta \in \mathbb{C}$ and $x, y, z \in \mathcal{V}$

1. $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$
2. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
3. $\langle x, x \rangle > 0$, if $x \neq 0$

A vector space with an inner product is called an inner product space.

Inner-Product Space

Introduces Geometry

The inner-product defined as

$$\langle x, y\rangle := x^* y = \sum_{i=1}^{n} \bar{x}_i y_i, \forall x, y \in \mathbb{C}^n,$$

induces a norm

$$\|x\| := \sqrt{\langle x, x\rangle}$$

Geometric Properties

- Distance between vectors x, y

$$d(x, y) := \|x - y\|.$$

- Two vectors x, y in an inner-product space $\mathcal{V}$ are orthogonal if

$$\langle x, y\rangle = 0.$$

- Orthogonal to a set $\mathcal{S} \subset \mathcal{V}$ if $\langle x, y\rangle = 0,\ \forall y \in \mathcal{S}$.

Inner-Product Space

Important Properties

Let $\mathcal{V}$ be an inner product space and let $x, y \in \mathcal{V}$.
Then

1. $|\langle x, y\rangle| \leq \|x\|\|y\|$ – Cauchy-Schwarz inequality.
    - Equality holds iff $x = \alpha y$ for some constant α or $y = 0$.
2. $\|x+y\|^2 + \|x-y\|^2 = 2\|x\| + 2\|y\|$ – Parallelogram law
3. $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ if $x \perp y$

Hilbert Space

- A complete inner-product space with norm induced by its inner product
- Restricted class of Banach space
  - Banach space – only norm
  - Hilbert space – inner-product, which allows orthonormal bases, unitary operators, etc.
- Existence and uniqueness of best approximations in closed subspaces – very useful.

Finite Dimensional Examples

- $\mathbb{C}^n$ with usual inner product
- $\mathbb{C}^{n\times m}$ with inner-product

$$\langle A, B\rangle := \mathbf{tr}\left[A^*B\right] = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}b_{ij},\ \forall A, B \in \mathbb{C}^{n,m}$$

Hilbert Space

$l_2(-\infty, \infty)$

Set of all real or complex square summable sequences

$$x = \{\cdots, x_{-2}, x_{-1}, x_0, x_1, x_2, \cdots\},$$

i.e.

$$\sum_{i=-\infty}^{\infty} |x_i|^2 < \infty,$$

with inner product defined as

$$\langle x, y \rangle := \sum_{i=-\infty}^{\infty} \bar{x}_i y_i,$$

for $x, y \in l_2(-\infty, \infty)$. x_i can be scalar, vector or matrix with norm

$$\langle x, y \rangle := \sum_{i=-\infty}^{\infty} \mathbf{tr}\left[\bar{x}_i y_i\right].$$

Hilbert Space

$\mathcal{L}_2(I)$ for $I \subset \mathbb{R}$

- $\mathcal{L}_2(I)$ – square integrable and Lebesgue measurable functions defined over interval $I \subset \mathbb{R}$
- with inner product

$$\langle f, g \rangle := \int_I f(t)^* g(t) dt,$$

for $f, g \in \mathcal{L}_2(I)$.

For vector or matrix valued functions, the inner product is defined as

$$\langle f, g \rangle := \int_I \mathbf{tr}\left[f(t)^* g(t)\right] dt.$$

Hardy Spaces

$\mathcal{L}_2(j\mathbb{R})$ Space

$\mathcal{L}_2(j\mathbb{R})$ Space – $\mathcal{L}_2$ is a Hilbert space of matrix-valued (or scalar-valued) complex function F on $j\mathbb{R}$ such that

$$\int_{-\infty}^{\infty} \mathbf{tr}\left[F^*(j\omega)F(j\omega)\right] d\omega < \infty,$$

with inner product

$$\langle F, G\rangle := \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{tr}\left[F^*(j\omega)G(j\omega)\right] d\omega,$$

for $F, G \in \mathcal{L}_2(j\mathbb{R})$.

$\mathcal{RL}_2(j\mathbb{R})$

All real rational strictly proper transfer matrices with no poles on the imaginary axis.

Hardy Spaces

$\mathcal{H}_2$ *Space*

$\mathcal{H}_2$ Space – Closed subspace of $\mathcal{L}_2(j\mathbb{R})$ with matrix functions $F(s)$ analytic in $Re(s) > 0$.

Norm is defined as

$$\|F\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{tr}\left[F^*(j\omega)F(j\omega)\right] d\omega.$$

- Computation of $\mathcal{H}_2$ norm is same as $\mathcal{L}_2(j\mathbb{R})$

$\mathcal{RH}_2$

Real rational subspace of $\mathcal{H}_2$, which consists of all strictly proper and real stable transfer matrices, is denoted by $\mathcal{RH}_2$.

Hardy Spaces

$\mathcal{L}_\infty(j\mathbb{R})$ *Space*

$\mathcal{L}_\infty(j\mathbb{R})$ Space – is a Banach space of matrix-valued (or scalar-valued) functions that are (essentially) bounded on $j\mathbb{R}$, with norm

$$\|F\|_\infty := \operatorname*{ess\,sup}_{\omega\in\mathbb{R}} \bar{\sigma}\left[F(j\omega)\right].$$

$\bar{\sigma}\left[F\right]$ is maximum singular value of matrix F

$F = U\Sigma V^*$

U : eigen-vectors of FF^*, V : eigen-values of F^*F

$\mathcal{RL}_\infty(j\mathbb{R})$

All proper and real rational transfer matrices with no poles on the imaginary axis.

Hardy Space

$\mathcal{H}_\infty$ Space

$\mathcal{H}_\infty$ Space – is a closed subspace of $\mathcal{L}_\infty$ space with functions that are analytic and bounded in the open right-half plane.

The $\mathcal{H}_\infty$ norm is defined as

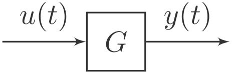
$$\|F\|_\infty := \sup_{\text{Re}(s)>0} \bar{\sigma}\left[F(s)\right] = \sup_{\omega\in\mathbb{R}} \bar{\sigma}\left[F(j\omega)\right].$$

$\mathcal{RH}_\infty$

Real rational subspace of $\mathcal{H}_\infty$, which consists of all proper and real rational stable transfer matrices.

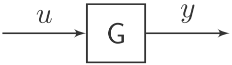
Input-Output Relationships

How big is output?



Interesting Question: If we know how big the input is, how big is the output going to be?

Bounded Input Bounded Output



- Given $|u(t)| \leq u_{\max} < \infty$, what can we say about $\max_t |y(t)|$?
- Recall

$$Y(s) = G(s)U(s) \implies y(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau.$$

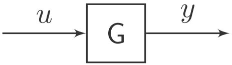
Therefore,

$$|y(t)| = \left| \int hud\tau \right| \leq \int |h||u|d\tau \leq u_{\max} \int |h(\tau)|d\tau.$$

Bound on output $y(t)$

$$\max_t |y(t)| \leq u_{\max} \int |h(\tau)|d\tau$$

Bounded Input Bounded Output (contd.)



$$\max_t |y(t)| \leq u_{\max} \int |h(\tau)| d\tau$$

BIBO Stability

If and only if

$$\int |h(\tau)| d\tau < \infty.$$

(LTI): **Re** $p_i < 0 \implies$ BIBO stability

$|y(t)| < y_{\text{max}}$ is not enough!

Input-Output Norms

Output norms for two candidate input signals

	$u(t) = \delta(t)$	$u(t) = \sin(\omega t)$
$\|y\|_2$	$\|\hat{G}(j\omega)\|_2$	∞
$\|y\|_\infty$	$\|\hat{G}(j\omega)\|_\infty$	$\lvert\hat{G}(j\omega)\rvert$
pow(y)	0	$\frac{1}{\sqrt{2}}\lvert\hat{G}(j\omega)\rvert$

Input-Output Norms (contd.)

System Gains

- Input signal size is given
- What is the output signal size?

	$\|u\|_2$	$\|u\|_\infty$	$\textbf{pow}(u)$
$\|y\|_2$	$\|\hat{G}(j\omega)\|_\infty$	∞	∞
$\|y\|_\infty$	$\|\hat{G}(j\omega)\|_2$	$\|G(t)\|_1$	∞
$\textbf{pow}(y)$	0	$\leq \|\hat{G}(j\omega)\|_\infty$	$\|\hat{G}(j\omega)\|_\infty$

∞-norm of system is pretty useful

Useful to prove these relationships.

Computation of Norms

Computation of Norms

- Best computed in state-space realization of system

State Space Model: General MIMO LTI system modeled as

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du,$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$.

Transfer Function

$$\hat{G}(s) = D + C(sI - A)^{-1}B \text{ strictly proper when } D = 0$$

Impulse Response

$$G(t) = \mathcal{L}^{-1}\left\{C(sI - A)^{-1}B\right\} = Ce^{tA}B.$$

$\mathcal{H}_2$ Norm

MIMO Systems

$$\|\hat{G}(j\omega)\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{tr}\left[\hat{G}^*(j\omega)\hat{G}(j\omega)\right] \text{ for matrix transfer function}$$

$$= \|G(t)\|_2^2 \text{ Parseval}$$

$$= \int_0^{\infty} \mathbf{tr}\left[Ce^{tA}BB^Te^{tA^T}C^T\right] dt$$

$$= \mathbf{tr}\left[C\underbrace{\left(\int_0^{\infty} e^{tA}BB^Te^{tA^T} dt\right)}_{L_c}C^T\right] \quad L_c = \text{controllability Gramian}$$

$$= \mathbf{tr}\left[CL_cC^T\right]$$

$\mathcal{L}_2$ Norm (contd.)

MIMO Systems

For any matrix M

$$\mathbf{tr}\left[M^*M\right] = \mathbf{tr}\left[MM^*\right]$$

$$\implies \|\hat{G}(j\omega)\|_2^2 = \mathbf{tr}\left[B^T \underbrace{\left(\int_0^\infty e^{tA^T} C^T C e^{tA} dt\right)}_{L_o} B\right]$$

$$= \mathbf{tr}\left[B^T L_o B\right] \quad L_o = \text{observability Gramian}$$

$\mathcal{H}_2$ Norm of $\hat{G}(j\omega)$

$$\|\hat{G}(j\omega)\|_2^2 = \mathbf{tr}\left[C L_c C^T\right] = \mathbf{tr}\left[B^T L_o B\right].$$

$\mathcal{L}_2$ Norm

How to determine L_c and L_o?

They are solutions of the following equation

$$AL_c + L_cA^T + BB^T = 0, \; A^TL_o + L_oA + C^TC = 0.$$

Proof:

From definition,

$$L_o := \int_0^\infty e^{tA^T} C^T C e^{tA} dt$$

Instead,

$$L_o(t) = \int_0^t e^{\tau A^T} C^T C e^{\tau A} d\tau.$$

Change of variable $\tau := t - \xi$,

$$L_o(t) = \int_0^t e^{(t-\xi)A^T} C^T C e^{(t-\xi)A} d\xi.$$

$\mathcal{L}_2$ Norm

How to determine L_c and L_o?

Take time-derivative,

$$\frac{dL_o(t)}{dt} = \frac{d}{dt}\int_0^t e^{(t-\xi)A^T} C^T C e^{(t-\xi)A} d\xi.$$

Differentiation under integral sign:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,y)dy$$

$$= f(x,b(x))\frac{db(x)}{dx} - f(x,a(x))\frac{da(x)}{dx} + \int_{a(x)}^{b(x)} \frac{\partial f(x,y)}{\partial x} dy.$$

$\mathcal{L}_2$ Norm

How to determine L_c and L_o?

$$\implies \frac{dL_o(t)}{dt} = A^T\left(\int_0^t e^{(t-\xi)A^T}C^TCe^{(t-\xi)A}d\xi\right) + \left(\int_0^t e^{(t-\xi)A^T}C^TCe^{(t-\xi)A}d\xi\right)A + C^TC$$

Or

$$\frac{dL_o(t)}{dt} = A^TL_o + L_oA + C^TC.$$

$L_o(t)$ is smooth, therefore

$$\lim_{t\to\infty} L_o(t) = L_o \implies \lim_{t\to\infty}\frac{dL_o(t)}{dt} = 0.$$

Thefore, L_o satisfies

$$A^TL_o + L_oA + C^TC = 0.$$

$\mathcal{L}_\infty$ Norm

Recall

$$\|\hat{G}(j\omega)\|_\infty := \operatorname*{ess\,sup}_{\omega} \bar{\sigma}\left[\hat{G}(j\omega)\right]$$

- Requires a search
- Estimate can be determined using bisection algorithm
  - Set up a grid of frequency points
    $$\{\omega_1, \cdots \omega_N\}.$$
  - Estimate of $\|\hat{G}(j\omega)\|_\infty$ is then,
    $$\max_{1\leq k\leq N} \bar{\sigma}\left[\hat{G}(j\omega_k)\right].$$
- Or read it from the plot of $\bar{\sigma}\left[\hat{G}(j\omega)\right]$.

$\mathcal{RL}_\infty$ Norm

Bisection Algorithm

Lemma

Let $\gamma > 0$ and

$$G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{RL}_\infty.$$

Then $\|\hat{G}(j\omega)\|_\infty < \gamma$ iff $\bar{\sigma}\left[D\right] < \gamma$ and H has no eigen values on the imaginary axis where

$$H := \begin{bmatrix} A + BR^{-1}D^TC & BR^{-1}B^T \\ -C^T(I + DR^{-1}D^T)C & -(A + BR^{-1}D^TC)^T \end{bmatrix},$$

and

$$R := \gamma^2 I - D^T D.$$

Proof:

See Robust and Optimal Control, K. Zhou, J.C. Doyle, K. Glover, Ch. 4, pg 115.

$\mathcal{RL}_\infty$ Norm

Bisection Algorithm

1. Select an upper bound γ_u and lower bound γ_l such that
$$\gamma_l \leq \|\hat{G}(j\omega)\|_\infty \leq \gamma_u.$$
2. If $(\gamma_u - \gamma_l)/\gamma_l \leq \epsilon$ STOP; $\|\hat{G}(j\omega)\|_\infty = (\gamma_u + \gamma_l)/2$.
3. Else $\gamma = (\gamma_u + \gamma_l)/2$
4. Test if $\|\hat{G}(j\omega)\|_\infty \leq \gamma$ by calculating eigen values of H for given γ
5. If H has an eigen value on $j\mathbb{R}$, $\gamma_l = \gamma$, else $\gamma_u = \gamma$
6. Goto step 2.

It is clear that $\|\hat{G}(j\omega)\|_\infty \leq \gamma$ iff $\|\gamma^{-1}\hat{G}(j\omega)\|_\infty \leq 1$.

Other algorithms exists to compute $\mathcal{RL}_\infty$ norm