

AERO 632: Design of Advance Flight Control System

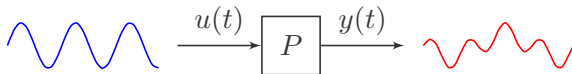
Norms for Signals and Systems

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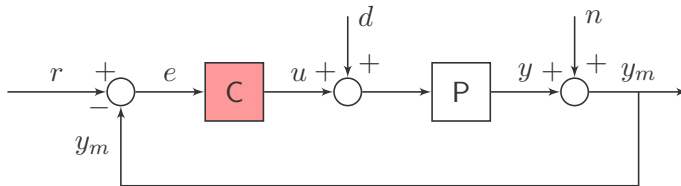
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Norms for Signals

Signals



- We consider signals mapping $(-\infty, \infty) \mapsto \mathbb{R}$
- Piecewise continuous
- A signal may be zero for $t < 0$
- We worry about size of signal
- Helps specify performance
- Signal size \iff signal norm



Norms

A norm must have the following 4 properties

- $\|\mathbf{u}\| \geq 0$
- $\|\mathbf{u}\| = 0 \iff \mathbf{u} = 0$
- $\|a\mathbf{u}\| = |a|\|\mathbf{u}\|, \forall a \in \mathbb{R}$
- $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ triangle inequality

For $\mathbf{u} \in \mathbb{R}^n$ and $p \geq 1$,

$$\|\mathbf{u}\|_p := (|u_1|^p + \cdots + |u_n|^p)^{1/p}$$

Special case,

$$\|\mathbf{u}\|_\infty := \max_i |u_i|$$

Norms of Signals

\mathcal{L}_1 Norm

The 1-norm of a signal $u(t)$ is the integral of its absolute value:

$$\|u(t)\|_1 := \int_{-\infty}^{\infty} |u(t)| dt$$

\mathcal{L}_2 Norm

The 2-norm of a signal $u(t)$ is

$$\|u(t)\|_2 := \left(\int_{-\infty}^{\infty} u(t)^2 dt \right)^{1/2} \quad \text{associated with energy of signal}$$

\mathcal{L}_∞ Norm

The ∞ -norm of a signal $u(t)$ is the least upper bound of its absolute value:

$$\|u(t)\|_\infty := \sup_t |u(t)|$$

Power Signals

The **average power** of $u(t)$ is the average over time of its **instantaneous power**:

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T u^2(t) dt$$

- if limit exists, $u(t)$ is called a **power signal**
- average power is then

$$\text{pow}(u) := \left(\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T u^2(t) dt \right)^{1/2}$$

- **pow**(\cdot) is not a norm
 - non zero signals can have **pow**(\cdot) = 0

Vector Signals

For $\mathbf{u}(t) : (-\infty, \infty) \mapsto \mathbb{R}^n$ and $p > 1$

$$\|\mathbf{u}(t)\|_p := \left(\int_{-\infty}^{\infty} \sum_{i=1}^n |u_i(t)|^p dt \right)^{1/p}$$

Finiteness of Norms

Does finiteness of one norm imply finiteness of another?

■ $\|u\|_2 < \infty \implies \text{pow}(u) = 0$

We have

$$\frac{1}{2T} \int_{-T}^T u^2(t) dt \leq \frac{1}{2T} \int_{-\infty}^{\infty} u^2(t) dt = \frac{1}{2T} \|u\|_2^2.$$

Right hand side tends to zero as $T \rightarrow \infty$

Finiteness of Norms (contd.)

- If u is a power signal and $\|u\|_\infty < \infty$, then $\mathbf{pow}(u) \leq \|u\|_\infty$.

We have

$$\frac{1}{2T} \int_{-T}^T u^2(t) dt \leq \|u\|_\infty \frac{1}{2T} \int_{-T}^T dt = \|u\|_\infty$$

Let $T \rightarrow \infty$.

Finiteness of Norms (contd.)

- If $\|u\|_1 < \infty$ and $\|u\|_\infty < \infty$ then $\|u\|_2 < \infty$

We have

$$\begin{aligned}
 \int_{-\infty}^{\infty} u^2(t) dt &= \int_{-\infty}^{\infty} |u(t)| \cdot |u(t)| dt \\
 &\leq \|u\|_\infty \int_{-\infty}^{\infty} |u(t)| dt \\
 &= \|u\|_\infty \|u\|_1 \\
 &\leq \infty
 \end{aligned}$$

Norms for Systems

System



We consider

- Linear
- Time invariant
- Causal
- Finite dimensional

In time domain

- if $u(t)$ is the input to the system and
- $y(t)$ is the output

System has the form

$$\begin{aligned}
 y &= G * u \text{ convolution} \\
 &= \int_{-\infty}^{\infty} G(t - \tau) u(\tau) d\tau \quad \mathcal{L}^{-1} \{ \hat{G}(s) \} := G(t)
 \end{aligned}$$

Causality

Causal

- A system is causal when the effect does not anticipate the cause; or **zero input produces zero output**
- Its output and internal states only depend on **current and previous** input values
- Physical systems are causal

Causality

contd.

Acausal

- A system whose output is nonzero when the past and present input signal is zero is said to be **anticipative**
- A system whose state and output depend also on **input values from the future**, besides the past or current input values, is called acausal
- Acausal systems can only exist as digital filters (digital signal processing).

Causality

contd.

Anti-Causal

- A system whose output depends **only on future input** values is anti-causal
- **Derivative** of a signal is anti-causal.

Causality

contd.

- Zeros are anticipative
- Poles are causal
- Overall behavior depends on m and n .
- Causal: $n > m$, strictly proper
- Causal: $n = m$, still causal, but there is **instantaneous transfer** of information from input to output
- Acausal: $n < m$

Example

- System $G_1(s) = s$
- Input $u(t) = \sin(\omega t)$, $U(s) = \frac{\omega}{s^2 + \omega^2}$
- $y_1(t) = \mathcal{L}^{-1} \{G_1(s)U(s)\} = \mathcal{L}^{-1} \left\{ \frac{s\omega}{s^2 + \omega^2} \right\} = \omega \cos(\omega t)$, or

$$u(t) = \sin(\omega t)$$

$$y_1(t) = \omega \sin(\omega t + \pi/2)$$

$$= \omega u\left(t + \frac{\pi}{2\omega}\right) \text{ output leads input, anticipatory}$$

Example

contd.

- System $G_2(s) = \frac{1}{s}$
- Input $u(t) = \sin(\omega t)$, $U(s) = \frac{\omega}{s^2 + \omega^2}$
- $y_2(t) = \mathcal{L}^{-1} \{G_2(s)U(s)\} = \mathcal{L}^{-1} \left\{ \frac{1}{s} \frac{\omega}{s^2 + \omega^2} \right\} = \frac{1}{\omega} - \frac{\cos(\omega t)}{\omega}$, or

$$\begin{aligned}
 u(t) &= \sin(\omega t) \\
 y_2(t) &= \frac{1}{\omega} + \frac{\sin(\omega t - \pi/2)}{\omega} \\
 &= \frac{1}{\omega} + \frac{u(t - \frac{\pi}{2\omega})}{\omega} \quad \text{output lags input, causal}
 \end{aligned}$$

Causality

(contd.)

Causality means

$$G(t) = 0 \text{ for } t < 0$$

- $\hat{G}(s)$ is **stable** if it is analytic in the closed RHP residue theorem
- **proper** if $\hat{G}(j\infty)$ is finite deg of den \geq deg of num
- **strictly proper** if $\hat{G}(j\infty) = 0$ deg of den $>$ deg of num
- **biproper** \hat{G} and \hat{G}^{-1} are both proper

Norms of \hat{G}

Definitions for SISO Systems

\mathcal{L}_2 Norm

$$\|\hat{G}\|_2 := \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{G}(j\omega)|^2 d\omega \right)^{1/2}$$

\mathcal{L}_∞ Norm

$$\|\hat{G}\|_\infty := \sup_{\omega} |\hat{G}(j\omega)| \quad \text{peak value of } |\hat{G}(j\omega)|$$

Parseval's Theorem

If $\hat{G}(j\omega)$ is stable

$$\|\hat{G}\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{G}(j\omega)|^2 d\omega \right)^{1/2} = \left(\int_{-\infty}^{\infty} |G(t)|^2 dt \right)^{1/2}.$$

Important Properties of System Norms

Submultiplicative Property of ∞ -norm

$$\|\hat{G}\hat{H}\|_{\infty} \leq \|\hat{G}\|_{\infty} \|\hat{H}\|_{\infty}$$

Important Properties of System Norms (contd.)

Lemma 1

$\|\hat{G}\|_2$ is finite **iff** \hat{G} is **strictly proper** and has no poles on the imaginary axis.

Proof: Look at transfer function of the type

$$\hat{G}(s) = \frac{(s + z_1)(s + z_2) \cdots (s + z_m)}{(s + p_1)(s + p_2) \cdots (s + p_n)}, n > m.$$

Argue area under $|\hat{G}(j\omega)|^2$ is finite.

Or apply residue theorem

$$\left(\frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{G}(j\omega)|^2 d\omega \right)^{1/2} = \frac{1}{2\pi j} \oint_{\text{LHP}} \hat{G}(-s) \hat{G}(s) ds.$$

Important Properties of System Norms (contd.)

Lemma 2

$\|\hat{G}\|_\infty$ is finite **iff** \hat{G} is **proper** and has no poles on the imaginary axis.

Proof: Look at transfer function of the type

$$\hat{G}(s) = \frac{(s + z_1)(s + z_2) \cdots (s + z_m)}{(s + p_1)(s + p_2) \cdots (s + p_n)}, n \geq m.$$

Argue $\sup_\omega |\hat{G}(j\omega)|$ is finite.

Signal Spaces

Performance Specification

- Describe performance in terms of norms of certain signals of interest
- Understand which norm is suitable
 - ▶ difference from control system performance perspective
- We will learn Hardy spaces \mathcal{H}_2 and \mathcal{H}_∞
 - ▶ measures of worst possible performance for many classes of input signals

Vector Space

Also called linear space

Elements $u, v, w \in \mathcal{V} \subseteq \mathbb{C}^n$ (or \mathbb{R}^n) satisfy the following 8 axioms

- Associativity of addition

$$u + (v + w) = (u + v) + w$$

- Commutativity of addition

$$u + v = v + u$$

- Identity element of addition

$$0 + v = v, \forall v \in \mathcal{V}$$

- Inverse element of addition

$$\text{for every } v \in \mathcal{V}, \exists -v \in \mathcal{V} : v + (-v) = 0$$

Vector Space

contd.

- Compatibility of scalar multiplication

$$\alpha(\beta u) = (\alpha\beta)u$$

- Identity of multiplication

$$1v = v$$

- Distributivity of scalar multiplication wrt vector addition

$$\alpha(u + v) = \alpha u + \alpha v$$

- Distributivity of scalar multiplication wrt field addition

$$(\alpha + \beta)u = \alpha u + \beta u$$

Normed Space

- Let \mathcal{V} be a vector space over \mathbb{C} or \mathbb{R}
- Let $\|\cdot\|$ be defined over \mathcal{V}
- Then \mathcal{V} is a **normed space**

Example 1

A vector space \mathbb{C}^n with any vector p -norm, $\|\cdot\|$, for $1 \leq p \leq \infty$.

Example 2

Space $C[a, b]$ of all bounded continuous functions becomes a norm space if

$$\|f\|_{\infty} := \sup_{t \in [a, b]} |f(t)|$$

is defined.

Banach Space

- A sequence $\{x_n\}$ in a normed space \mathcal{V} is **Cauchy sequence**, if

$$\|x_n - x_m\| \rightarrow 0 \text{ as } n, m \rightarrow \infty.$$

- A sequence $\{x_n\}$ is said to **converge** to $x \in \mathcal{V}$, written $x_n \rightarrow x$, if

$$\|x_n - x\| \rightarrow 0.$$

- A normed space \mathcal{V} is said to be **complete** if every Cauchy sequence in \mathcal{V} converges in \mathcal{V} .
- A complete normed space is called a **Banach space**.

Banach Space

$l_p[0, \infty)$ spaces for $1 \leq p < \infty$

For each $1 \leq p < \infty$, $l_p[0, \infty)$ consists of all sequence $x = (x_0, x_1, \dots)$ such that

$$\sum_{i=0}^{\infty} |x_i|^p < \infty.$$

The associate norm is defined as

$$\|x\|_p := \left(\sum_{i=0}^{\infty} |x_i|^p \right)^{1/p}.$$

Banach Space

$l_\infty[0, \infty)$ space

$l_\infty[0, \infty)$ consists of all **bounded** sequence $x = (x_0, x_1, \dots)$.

The l_∞ norm is defined as

$$\|x\|_\infty := \sup_i |x_i|.$$

Banach Space

$\mathcal{L}_p(I)$ spaces for $1 \leq p < \infty$

For each $1 \leq p \leq \infty$, $\mathcal{L}_p(I)$ consists of all **Lebesgue measurable** functions $x(t)$ defined on an interval $I \subset \mathbb{R}$ such that

$$\|x\|_p := \left(\int_I |x(t)|^p \mu(dt) \right)^{1/p} < \infty, \text{ for } 1 \leq p < \infty,$$

and

$$\|x\|_\infty := \operatorname{ess\,sup}_{t \in I} |x(t)|.$$

Note: $\operatorname{ess\,sup}_{t \in I}$ is $\sup_{t \in I}$ almost everywhere.

We will study $\mathcal{L}_2(-\infty, 0]$, $\mathcal{L}_2[0, \infty)$, and $\mathcal{L}_2(-\infty, \infty)$ spaces in detail.

Banach Space

$C[a, b]$ space

Consists of all **continuous functions** on the real interval $[a, b]$ with the norm

$$\|x\|_{\infty} := \sup_{t \in [a, b]} |x(t)|.$$

Inner-Product Space

Recall the **inner product** of vectors in **Euclidean space** \mathbb{C}^n :

$$\langle x, y \rangle := x^* y = \sum_{i=1}^n \bar{x}_i y_i, \forall x, y \in \mathbb{C}^n.$$

Important Metric Notions & Geometric Properties

- length, distance, angle
- energy

We can generalize beyond Euclidean space!

Inner-Product Space

Generalization

Let \mathcal{V} be a vector space over \mathbb{C} . An **inner product** on \mathcal{V} is a complex value function

$$\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \mapsto \mathbb{C}$$

such that for any $\alpha, \beta \in \mathbb{C}$ and $x, y, z \in \mathcal{V}$

1. $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$
2. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
3. $\langle x, x \rangle > 0$, if $x \neq 0$

A vector space with an inner product is called an **inner product space**.

Inner-Product Space

Introduces Geometry

The inner-product defined as

$$\langle x, y \rangle := x^* y = \sum_{i=1}^n \bar{x}_i y_i, \forall x, y \in \mathbb{C}^n,$$

induces a norm

$$\|x\| := \sqrt{\langle x, x \rangle}$$

Geometric Properties

- Distance between vectors x, y

$$d(x, y) := \|x - y\|.$$

- Two vectors x, y in an inner-product space \mathcal{V} are orthogonal if

$$\langle x, y \rangle = 0.$$

- Orthogonal to a set $\mathcal{S} \subset \mathcal{V}$ if $\langle x, y \rangle = 0, \forall y \in \mathcal{S}$.

Inner-Product Space

Important Properties

Let \mathcal{V} be an inner product space and let $x, y \in \mathcal{V}$.

Then

1. $|\langle x, y \rangle| \leq \|x\| \|y\|$ – Cauchy-Schwarz inequality.

► Equality holds iff $x = \alpha y$ for some constant α or $y = 0$.

2. $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$ – Parallelogram law

3. $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ if $x \perp y$

Hilbert Space

- A **complete** inner-product space with norm induced by its inner product
- Restricted class of Banach space
 - ▶ Banach space – only norm
 - ▶ Hilbert space – **inner-product**, which allows orthonormal bases, unitary operators, etc.
- Existence and uniqueness of best approximations in closed subspaces – very useful.

Finite Dimensional Examples

- \mathbb{C}^n with usual inner product
- $\mathbb{C}^{n \times m}$ with inner-product

$$\langle A, B \rangle := \text{tr} [A^* B] = \sum_{i=1}^n \sum_{j=1}^m a_{ij} b_{ij}, \quad \forall A, B \in \mathbb{C}^{n,m}$$

Hilbert Space

$l_2(-\infty, \infty)$

Set of all real or complex square summable sequences

$$x = \{\cdots, x_{-2}, x_{-1}, x_0, x_1, x_2, \cdots\},$$

i.e.

$$\sum_{i=-\infty}^{\infty} |x_i|^2 < \infty,$$

with inner product defined as

$$\langle x, y \rangle := \sum_{i=-\infty}^{\infty} \bar{x}_i y_i,$$

for $x, y \in l_2(-\infty, \infty)$. x_i can be scalar, vector or matrix with norm

$$\langle x, y \rangle := \sum_{i=-\infty}^{\infty} \text{tr} [\bar{x}_i y_i].$$

Hilbert Space

$\mathcal{L}_2(I)$ for $I \subset \mathbb{R}$

- $\mathcal{L}_2(I)$ – square integrable and Lebesgue measurable functions defined over interval $I \subset \mathbb{R}$
- with inner product

$$\langle f, g \rangle := \int_I f(t)^* g(t) dt,$$

for $f, g \in \mathcal{L}_2(I)$.

For vector or matrix valued functions, the inner product is defined as

$$\langle f, g \rangle := \int_I \mathbf{tr} [f(t)^* g(t)] dt.$$

Hardy Spaces

$\mathcal{L}_2(j\mathbb{R})$ Space

$\mathcal{L}_2(j\mathbb{R})$ Space – \mathcal{L}_2 is a Hilbert space of matrix-valued (or scalar-valued) complex function F on $j\mathbb{R}$ such that

$$\int_{-\infty}^{\infty} \text{tr} [F^*(j\omega)F(j\omega)] d\omega < \infty,$$

with inner product

$$\langle F, G \rangle := \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr} [F^*(j\omega)G(j\omega)] d\omega,$$

for $F, G \in \mathcal{L}_2(j\mathbb{R})$.

$\mathcal{RL}_2(j\mathbb{R})$

All real rational **strictly proper** transfer matrices with no poles on the imaginary axis.

Hardy Spaces

\mathcal{H}_2 Space

\mathcal{H}_2 Space – Closed subspace of $\mathcal{L}_2(j\mathbb{R})$ with matrix functions $F(s)$ analytic in $\text{Re}(s) > 0$.

Norm is defined as

$$\|F\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr} [F^*(j\omega)F(j\omega)] d\omega.$$

- Computation of \mathcal{H}_2 norm is same as $\mathcal{L}_2(j\mathbb{R})$

\mathcal{RH}_2

Real rational subspace of \mathcal{H}_2 , which consists of all strictly proper and real stable transfer matrices, is denoted by \mathcal{RH}_2 .

Hardy Spaces

$\mathcal{L}_\infty(j\mathbb{R})$ Space

$\mathcal{L}_\infty(j\mathbb{R})$ Space – is a Banach space of matrix-valued (or scalar-valued) functions that are (essentially) bounded on $j\mathbb{R}$, with norm

$$\|F\|_\infty := \operatorname{ess\,sup}_{\omega \in \mathbb{R}} \bar{\sigma}[F(j\omega)] .$$

$\bar{\sigma}[F]$ is maximum singular value of matrix F

$$F = U\Sigma V^*$$

U : eigen-vectors of FF^* , V : eigen-values of F^*F

$\mathcal{RL}_\infty(j\mathbb{R})$

All proper and real rational transfer matrices with no poles on the imaginary axis.

Hardy Space

\mathcal{H}_∞ Space

\mathcal{H}_∞ Space – is a closed subspace of \mathcal{L}_∞ space with functions that are analytic and bounded in the open right-half plane.

The \mathcal{H}_∞ norm is defined as

$$\|F\|_\infty := \sup_{\operatorname{Re}(s) > 0} \bar{\sigma}[F(s)] = \sup_{\omega \in \mathbb{R}} \bar{\sigma}[F(j\omega)].$$

\mathcal{RH}_∞

Real rational subspace of \mathcal{H}_∞ , which consists of all proper and real rational stable transfer matrices.

Input-Output Relationships

How big is output?



Interesting Question: If we know how big the input is, how big is the output going to be?

Bounded Input Bounded Output



- Given $|u(t)| \leq u_{\max} < \infty$, what can we say about $\max_t |y(t)|$?
- Recall

$$Y(s) = G(s)U(s) \implies y(t) = \int_{-\infty}^{\infty} h(\tau)u(t - \tau)d\tau.$$

Therefore,

$$|y(t)| = \left| \int h u d\tau \right| \leq \int |h||u|d\tau \leq u_{\max} \int |h(\tau)|d\tau.$$

Bound on output $y(t)$

$$\max_t |y(t)| \leq u_{\max} \int |h(\tau)|d\tau$$

Bounded Input Bounded Output (contd.)



$$\max_t |y(t)| \leq u_{\max} \int |h(\tau)| d\tau$$

BIBO Stability

If and only if

$$\int |h(\tau)| d\tau < \infty.$$

(LTI): $\text{Re } p_i < 0 \implies$ BIBO stability

$|y(t)| < y_{\max}$ is not enough!

Input-Output Norms

Output norms for two candidate input signals

	$u(t) = \delta(t)$	$u(t) = \sin(\omega t)$
$\ y\ _2$	$\ \hat{G}(j\omega)\ _2$	∞
$\ y\ _\infty$	$\ \hat{G}(j\omega)\ _\infty$	$ \hat{G}(j\omega) $
pow (y)	0	$\frac{1}{\sqrt{2}} \hat{G}(j\omega) $

Input-Output Norms (contd.)

System Gains

- Input signal size is given
- What is the output signal size?

	$\ u\ _2$	$\ u\ _\infty$	$\text{pow}(u)$
$\ y\ _2$	$\ \hat{G}(j\omega)\ _\infty$	∞	∞
$\ y\ _\infty$	$\ \hat{G}(j\omega)\ _2$	$\ G(t)\ _1$	∞
$\text{pow}(y)$	0	$\leq \ \hat{G}(j\omega)\ _\infty$	$\ \hat{G}(j\omega)\ _\infty$

∞ -norm of system is pretty useful

Useful to prove these relationships.

Computation of Norms

Computation of Norms

- Best computed in state-space realization of system

State Space Model: General MIMO LTI system modeled as

$$\begin{aligned}\dot{x} &= Ax + Bu, \\ y &= Cx + Du,\end{aligned}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$.

Transfer Function

$$\hat{G}(s) = D + C(sI - A)^{-1}B \text{ strictly proper when } D = 0$$

Impulse Response

$$G(t) = \mathcal{L}^{-1} \{C(sI - A)^{-1}B\} = Ce^{tA}B.$$

\mathcal{H}_2 Norm

MIMO Systems

$$\|\hat{G}(j\omega)\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr} \left[\hat{G}^*(j\omega) \hat{G}(j\omega) \right] \text{ for matrix transfer function}$$

$$= \|G(t)\|_2^2 \text{ Parseval}$$

$$= \int_0^{\infty} \text{tr} \left[C e^{tA} B B^T e^{tA^T} C^T \right] dt$$

$$= \text{tr} \left[C \underbrace{\left(\int_0^{\infty} e^{tA} B B^T e^{tA^T} dt \right)}_{L_c} C^T \right] \quad L_c = \text{controllability Gramian}$$

$$= \text{tr} [C L_c C^T]$$

\mathcal{L}_2 Norm (contd.)

MIMO Systems

For any matrix M

$$\begin{aligned}\text{tr} [M^* M] &= \text{tr} [M M^*] \\ \Rightarrow \|\hat{G}(j\omega)\|_2^2 &= \text{tr} \left[B^T \underbrace{\left(\int_0^\infty e^{tA^T} C^T C e^{tA} dt \right)}_{L_o} B \right] \\ &= \text{tr} [B^T L_o B] \quad L_o = \text{observability Gramian}\end{aligned}$$

\mathcal{H}_2 Norm of $\hat{G}(j\omega)$

$$\|\hat{G}(j\omega)\|_2^2 = \text{tr} [C L_c C^T] = \text{tr} [B^T L_o B] .$$

\mathcal{L}_2 Norm

How to determine L_c and L_o ?

They are solutions of the following equation

$$AL_c + L_c A^T + BB^T = 0, \quad A^T L_o + L_o A + C^T C = 0.$$

Proof:

From definition,

$$L_o := \int_0^\infty e^{tA^T} C^T C e^{tA} dt$$

Instead,

$$L_o(t) = \int_0^t e^{\tau A^T} C^T C e^{\tau A} d\tau.$$

Change of variable $\tau := t - \xi$,

$$L_o(t) = \int_0^t e^{(t-\xi)A^T} C^T C e^{(t-\xi)A} d\xi.$$

\mathcal{L}_2 Norm

How to determine L_c and L_o ?

Take time-derivative,

$$\frac{dL_o(t)}{dt} = \frac{d}{dt} \int_0^t e^{(t-\xi)A^T} C^T C e^{(t-\xi)A} d\xi.$$

Differentiation under integral sign:

$$\begin{aligned} \frac{d}{dx} \int_{a(x)}^{b(x)} f(x, y) dy \\ = f(x, b(x)) \frac{db(x)}{dx} - f(x, a(x)) \frac{da(x)}{dx} + \int_{a(x)}^{b(x)} \frac{\partial f(x, y)}{\partial x} dy. \end{aligned}$$

\mathcal{L}_2 Norm

How to determine L_c and L_o ?

$$\Rightarrow \frac{dL_o(t)}{dt} = A^T \left(\int_0^t e^{(t-\xi)A^T} C^T C e^{(t-\xi)A} d\xi \right) + \left(\int_0^t e^{(t-\xi)A^T} C^T C e^{(t-\xi)A} d\xi \right) A + C^T C$$

Or

$$\frac{dL_o(t)}{dt} = A^T L_o + L_o A + C^T C.$$

$L_o(t)$ is smooth, therefore

$$\lim_{t \rightarrow \infty} L_o(t) = L_o \Rightarrow \lim_{t \rightarrow \infty} \frac{dL_o(t)}{dt} = 0.$$

Therefore, L_o satisfies

$$A^T L_o + L_o A + C^T C = 0.$$

\mathcal{L}_∞ Norm

Recall

$$\|\hat{G}(j\omega)\|_\infty := \operatorname{ess\,sup}_\omega \bar{\sigma} \left[\hat{G}(j\omega) \right]$$

- Requires a search
- Estimate can be determined using bisection algorithm
 - ▶ Set up a grid of frequency points

$$\{\omega_1, \dots, \omega_N\}.$$

- ▶ Estimate of $\|\hat{G}(j\omega)\|_\infty$ is then,

$$\max_{1 \leq k \leq N} \bar{\sigma} \left[\hat{G}(j\omega_k) \right].$$

- Or read it from the plot of $\bar{\sigma} \left[\hat{G}(j\omega) \right]$.

\mathcal{RL}_∞ Norm

Bisection Algorithm

Lemma

Let $\gamma > 0$ and

$$G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{RL}_\infty.$$

Then $\|\hat{G}(j\omega)\|_\infty < \gamma$ **iff** $\bar{\sigma}[D] < \gamma$ and H has no eigen values on the imaginary axis where

$$H := \begin{bmatrix} A + BR^{-1}D^TC & BR^{-1}B^T \\ -C^T(I + DR^{-1}D^T)C & -(A + BR^{-1}D^TC)^T \end{bmatrix},$$

and

$$R := \gamma^2 I - D^T D.$$

Proof:

See Robust and Optimal Control, K. Zhou, J.C. Doyle, K. Glover, Ch. 4, pg 115.

\mathcal{RL}_∞ Norm

Bisection Algorithm

1. Select an upper bound γ_u and lower bound γ_l such that

$$\gamma_l \leq \|\hat{G}(j\omega)\|_\infty \leq \gamma_u.$$

2. If $(\gamma_u - \gamma_l)/\gamma_l \leq \epsilon$ **STOP**; $\|\hat{G}(j\omega)\|_\infty = (\gamma_u + \gamma_l)/2$.
3. Else $\gamma = (\gamma_u + \gamma_l)/2$
4. Test if $\|\hat{G}(j\omega)\|_\infty \leq \gamma$ by calculating eigen values of H for given γ
5. If H has an eigen value on $j\mathbb{R}$, $\gamma_l = \gamma$, else $\gamma_u = \gamma$
6. Goto step 2.

It is clear that $\|\hat{G}(j\omega)\|_\infty \leq \gamma$ iff $\|\gamma^{-1}\hat{G}(j\omega)\|_\infty \leq 1$.

Other algorithms exists to compute \mathcal{RL}_∞ norm