State Feedback $\mathcal{H}_\infty$ Optimal Controller

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$\mathcal{H}_\infty$ Optimal Controller

Motivation

- $\mathcal{H}_2$ Optimal Control
  - *disturbance error reduction*
  - *sensor noise error reduction*
- $\mathcal{H}_\infty$ Optimal Control
  - *disturbance error reduction*
  - *sensor noise error reduction*
  - *tolerant to uncertainties – easier to formulate in $\mathcal{RH}_\infty$ than $\mathcal{RH}_2$*

	$\|u\|_2$	$\|u\|_\infty$	**pow**(u)
$\|y\|_2$	$\|\hat{G}(j\omega)\|_\infty$	∞	∞
$\|y\|_\infty$	$\|\hat{G}(j\omega)\|_2$	$\|G(t)\|_1$	∞
pow(y)	0	$\leq \|\hat{G}(j\omega)\|_\infty$	$\|\hat{G}(j\omega)\|_\infty$

∞-norm of system is pretty useful

Kalman-Yakubovich-Popov (KYP) Lemma

Lemma: Suppose $\hat{G}(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Then the following are equivalent conditions.

1. The matrix A is Hurwitz and
$$\|\hat{G}\|_\infty < 1.$$
2. There exists a matrix $X > 0$ such that
$$\begin{bmatrix} C^* \\ D^* \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} + \begin{bmatrix} A^*X + XA & XB \\ B^*X & -I \end{bmatrix} < 0.$$

- Very useful – relates transfer matrix (frequency domain) inequality to state space conditions
- Convenient way to evaluate $\mathcal{H}_\infty$ norm of transfer matrix

Full State-Feedback $\mathcal{H}_\infty$ Control

One of three formulations

Given system

$$\dot{x} = Ax + B_u u + B_w w,$$
$$z = Cx + D_u u + D_w w.$$

Theorem Controller $u = Kx$ internally stabilizes and minimizes $\|G_{w\to z}\|_\infty$ iff there exists W, and $X > 0$ such that following optimization problem has solution (A, B_u) stabilizable

$$\min_{X,W} \gamma$$

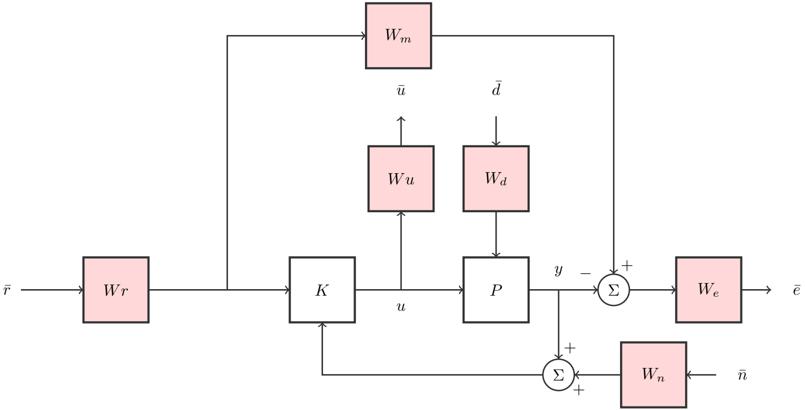
subject to

$$X > 0, \begin{bmatrix} (AX + B_u W) + (*)^T & B_w & (CX + D_u W)^T \\ B_w^T & -\gamma I & D_w^T \\ (CX + D_u W) & D_w & -\gamma I \end{bmatrix} < 0,$$

with $K = WX^{-1}$.

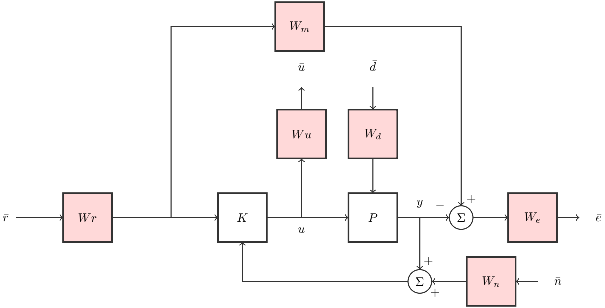
Weighted Performance

For both $\mathcal{H}_\infty$ and $\mathcal{H}_2$ control



Standard interconnection

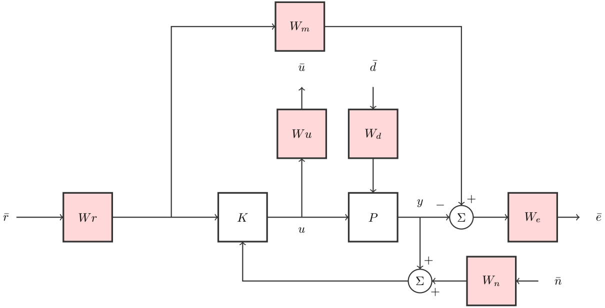
Frequency Dependent Weights



- Some signals may be more important than others
- Signals may not be measured in the same metric
- May be interested in keeping signals small in certain frequency range

Frequency Dependent Weights

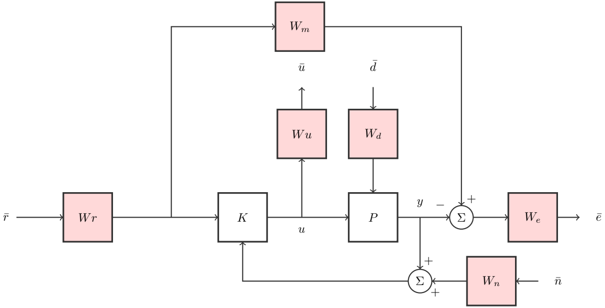
W_r, W_d, W_n



- W_r: specifies frequency content of $r(t)$ – Pilot models, etc.
- W_d: specifies frequency content of $d(t)$ – gust models, road vibration, etc.
- W_n: specifies frequency content of sensor noise – comes from manufacturer.

Frequency Dependent Weights

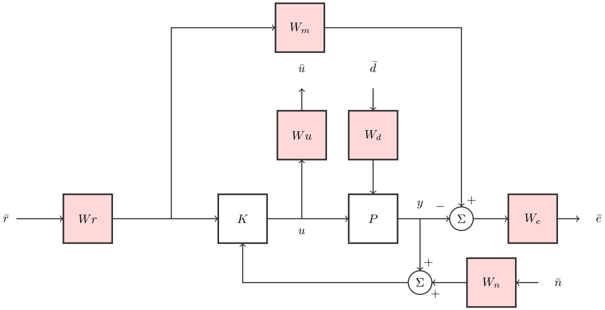
W_u



- W_u: defines the reciprocal of desired frequency content of $u(t)$
- Can be used to
  - *include control magnitude, rate constraints*
  - *specify desired controller roll off – not excite high-frequency uncertain modes*

Frequency Dependent Weights

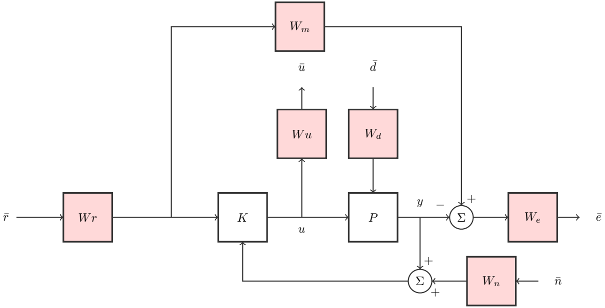
W_e



- W_e: defines the reciprocal of desired error at each frequency

Frequency Dependent Weights

W_m



- W_m: Defines the model for model-matching formulation
- Desired response to $r(t)$ is given by respond of model W_m
- E.g. second order response – can relate to rise time, overshoot, settling time

$\mathcal{H}_\infty$ Loopshaping – $P(j\omega)C(j\omega)$

Define desired loop shape using weights

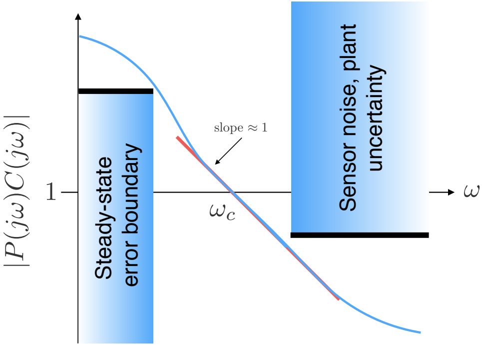
Develop conditions on the Bode plot of the open loop transfer function

- Sensitivity $\frac{1}{1+PC}$
- Steady-state errors: slope and magnitude at $\lim_\omega \to 0$
- Robust to sensor noise
- Disturbance rejection
- Controller roll off $\implies$ not excite high-frequency modes of plant
- Robust to plant uncertainty

Look at Bode plot of $L(j\omega) := P(j\omega)C(jw)$

Frequency Domain Specifications

Constraints on the shape of $L(j\omega)$

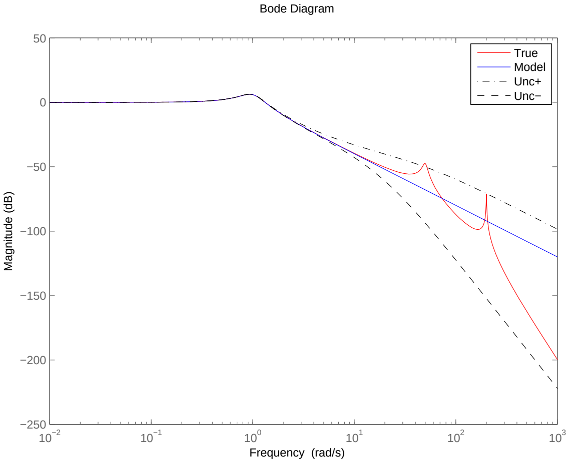


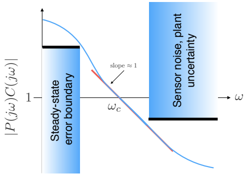
- Choose $C(j\omega)$ to ensure $|L(j\omega)|$ does not violate the constraints
- Slope ≈ -1 at ω_c ensures $PM \approx 90^\circ$

stable if $PM > 0 \implies \underline{/PC} > -180^\circ$

Plant Uncertainty

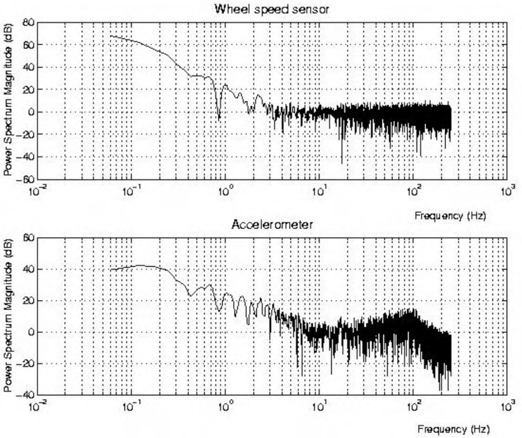
$$P(j\omega) = P_0(j\omega)(1 + \Delta P(j\omega))$$

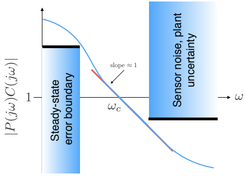




Sensor Characteristics

Noise spectrum

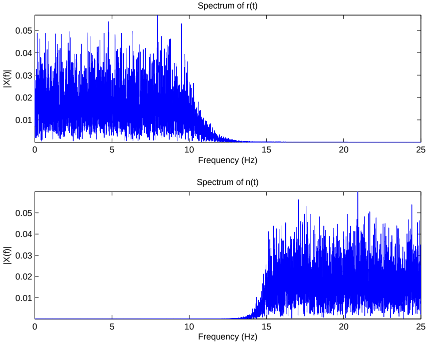


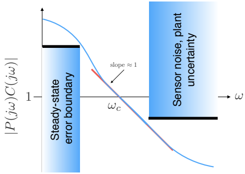


$$G_{yn} = -\frac{PC}{1+PC}$$

Reference Tracking

Bandlimited else conflicts with noise rejection



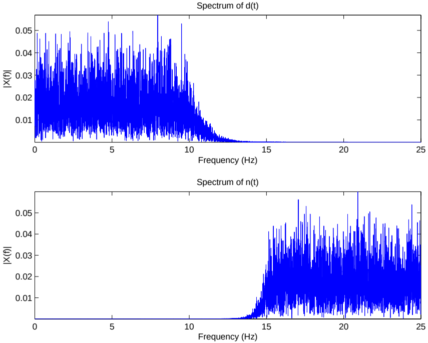


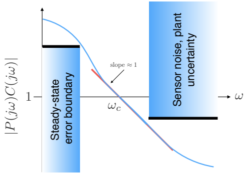
$$G_{yr} = \frac{PC}{1+PC}$$

$G_{yn} = -\frac{PC}{1+PC}$

Disturbance Rejecton

Bandlimited else conflicts with noise rejection





$$G_{yd} = \frac{P}{1+PC}$$

$$G_{yn} = -\frac{PC}{1+PC}$$