# An Introduction to Linear Matrix Inequalities

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# Linear Matrix Inequalities

*What are they?*

- Inequalities involving matrix variables
- Matrix variables appear linearly
- Represent convex sets polynomial inequalities
- Critical tool in post-modern control theory

# Standard Form

$$F(x) := F_0 + x_1 F_1 + \cdots + x_n F_n > 0$$

where

$$x := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \; F_i \in \mathbb{S}^m \; m \times m \text{ symmetric matrix}$$

Think of $F(x) : \mathbb{R}^n \mapsto \mathbb{S}^m$.

**Example:**

$$\begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix} > 0 \Leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + x \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} > 0.$$

# Positive Definiteness

- Matrix $F > 0$ represents positive definite matrix
- $F > 0 \iff x^T F x > 0, \forall x \neq 0$
- $F > 0 \iff$ leading principal minors of $F$ are positive

Let

$$F = \begin{bmatrix} F_{11} & F_{12} & F_{13} & \cdots \\ F_{21} & F_{22} & F_{23} & \cdots \\ F_{31} & F_{32} & F_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

**$n$ Polynomial Constraints as a Linear Matrix Inequality**

$$F > 0 \iff F_{11} > 0, \begin{vmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{vmatrix} > 0, \begin{vmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{vmatrix} > 0, \cdots$$

# Definiteness

**Positive Semi-Definite**

$F \geq 0 \iff$ iff all principal minors are $\geq 0$ not just leading

**Negative Definite**

$F < 0 \iff$ iff every odd leading principal minor is $< 0$ and even leading principal minor is $> 0$ they alternate signs, starting with $< 0$

**Negative Semi-Definite**

$F \leq 0 \iff$ iff every odd principal minor is $\leq 0$ and even principal minor is $\geq 0$

$$F > 0 \iff -F < 0$$
$$F \geq 0 \iff -F \leq 0$$

Matrix Analysis, Roger Horn.

# Example 1

$$y > 0,\ y - x^2 > 0, \quad \Longleftrightarrow \quad \begin{bmatrix} y & x \\ x & 1 \end{bmatrix} > 0$$

- LMI written as $\begin{bmatrix} y & x \\ x & 1 \end{bmatrix} > 0$ is in general form.
- We can write it in standard form as

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + y \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + x \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} > 0$$

- General form saves notations, may lead to more efficient computation

# Example 2

$$x_1^2 + x_2^2 < 1 \quad \Longleftrightarrow \quad \begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & x_2 \\ x_1 & x_2 & 1 \end{bmatrix} > 0$$

Leading Minors are

$$\begin{aligned} 1 &> 0 \\ \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} &> 0 \\ 1\begin{vmatrix} 1 & x_2 \\ x_2 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & x_1 \\ x_1 & 1 \end{vmatrix} + x_1\begin{vmatrix} 0 & 1 \\ x_1 & x_2 \end{vmatrix} &> 0 \end{aligned}$$

Last inequality simplifies to

$$1 - (x_1^2 + x_2^2) > 0$$

# Eigenvalue Minimization

- Let $A_i \in \mathbb{S}^n, i = 0, 1, \cdots, n$.
- Let $A(x) := A_0 + A_1 x_1 + \cdots + A_n x_n$.
- Find $x := [x_1 \; x_2 \; \cdots \; x_n]^T$

  that minimizes

$$J(x) := \min_x \lambda_{\mathsf{max}} A(x).$$

How to solve this problem?

# Eigenvalue Minimization (contd.)

Recall for $M \in \mathbb{S}^n$

$$\lambda_{\mathsf{max}} M \leq t \iff M - tI \leq 0.$$

Linear algebra result: Matrix Analysis – R.Horn, C.R. Johnson

**Optimization problem is therefore**

$$\min_{x,t} t$$

$$\text{such that } A(x) - tI \leq 0.$$

# Matrix Norm Minimization

- Let $A_i \in \mathbb{R}^n, i = 0, 1, \cdots, n$.
- Let $A(x) := A_0 + A_1 x_1 + \cdots + A_n x_n$.
- Find $x := [x_1 \; x_2 \; \cdots \; x_n]^T$

  that minimizes
$$J(x) := \min_x \|A(x)\|_2.$$

How to solve this problem?

# Matrix Norm Minimization

*contd.*

- Recall

$$\|A\|_2 := \lambda_{\max} A^T A.$$

- Implies

$$\min_{t,x} t^2$$
$$A(x)^T A(x) - t^2 I \leq 0.$$

or

**Optimization problem is therefore**

$$\min_{t,x} t^2 \text{ subject to } \begin{bmatrix} tI & A(x) \\ A(x)^T & tI \end{bmatrix} \geq 0.$$

# Important Inequalities

# Generalized Square Inequalities

**Lemma** For arbitrary scalar $x, y$, and $\delta > 0$, we have

$$\left(\sqrt{\delta}x - \frac{y}{\sqrt{\delta}}\right)^2 = \delta x^2 + \frac{1}{\delta}y^2 - 2xy \geq 0.$$

Implies

$$2xy \leq \delta x^2 + \frac{1}{\delta}y^2.$$

# Generalized Square Inequalities

*Restriction-Free Inequalities*

**Lemma** Let $X, Y \in \mathbb{R}^{m\times n}, F \in \mathbb{S}^m, F > 0$, and $\delta > 0$ be a scalar, then

$$X^T F Y + Y^T F X \leq \delta X^T F X + \delta^{-1} Y^T F Y.$$

When $X = x$ and $Y = y$

$$2x^T F y \leq \delta x^T F x + \delta^{-1} y^T F y.$$

**Proof:** Using completion of squares.

$$\left(\sqrt{\delta} X - \sqrt{\delta^{-1}} Y\right)^T F \left(\sqrt{\delta} X - \sqrt{\delta^{-1}} Y\right) \geq 0.$$

# Generalized Square Inequalities

*Inequalities with Restrictions*

Let

$$\mathcal{F} = \{F \,|\, F \in \mathbb{R}^{n\times n}, F^T F \leq I\}.$$

**Lemma** Let $X \in \mathbb{R}^{m\times n}, Y \in \mathbb{R}^{n\times m}$, then for arbitrary $\delta > 0$

$$XFY + Y^T F^T X^T \leq \delta X X^T + \delta^{-1} Y^T Y, \forall F \in \mathcal{F}.$$

**Proof:** Approach 1: Using completion of squares.
Start with

$$\left(\sqrt{\delta} X^T - \sqrt{\delta^{-1}} FY\right)^T \left(\sqrt{\delta} X^T - \sqrt{\delta^{-1}} FY)\right) \geq 0.$$

# Schur Complements

*Very useful for identifying convex sets*

Let

$$Q(x) \in \mathbb{S}^{m_1}, R(x) \in \mathbb{S}^{m_2}$$
$Q(x), R(x), S(x)$ are affine functions of $x$

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} > 0 \iff \begin{array}{l} Q(x) > 0 \\ R(x) - S^T(x)Q(x)^{-1}S(x) > 0 \end{array}$$

Generalizing,

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} \geq 0 \iff \begin{array}{l} Q(x) \geq 0 \\ S^T(x)\left(I - Q(x)Q^\dagger(x)\right) = 0 \\ R(x) - S^T(x)Q(x)^\dagger S(x) \geq 0 \end{array}$$

- $Q(x)^\dagger$ is the pseudo-inverse
- This generalization is used when $Q(x)$ is positive semidefinite but singular

# Schur Complement Lemma

Let

$$A := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

**Define**

$$S_{\mathsf{ch}}(A_{11}) := A_{22} - A_{21}A_{11}^{-1}A_{12}$$
$$S_{\mathsf{ch}}(A_{22}) := A_{11} - A_{12}A_{22}^{-1}A_{21}$$

For symmetric $A$,

$$A > 0 \iff A_{11} > 0, S_{\mathsf{ch}}(A_{11}) > 0 \iff A_{22} > 0, S_{\mathsf{ch}}(A_{22}) > 0$$

# Example 1

$$x_1^2 + x_2^2 < 1 \iff 1 - x^T x > 0 \iff \begin{bmatrix} I & x \\ x^T & 1 \end{bmatrix} > 0$$

Here

$$R(x) = 1,$$
$$Q(x) = I > 0.$$

# Example 2

$$\|x\|_P < 1 \iff 1 - x^T P x > 0 \iff \begin{bmatrix} P^{-1} & x \\ x^T & 1 \end{bmatrix} > 0$$

or

$$1 - x^T P x = 1 - (\sqrt{P}x)^T(\sqrt{P}x) > 0 \iff \begin{bmatrix} I & (\sqrt{P}x) \\ (\sqrt{P}x)^T & 1 \end{bmatrix} > 0$$

where $\sqrt{P}$ is matrix square root.

# LMIs are not unique

- If $F$ is positive definite then congruence transformation of $F$ is also positive definite

$$
\begin{aligned}
F > 0 &\iff x^T F x,\ \forall x \neq 0 \\
&\iff y^T M^T F M y > 0,\ \forall y \neq 0 \text{ and nonsingular } M \\
&\iff M^T F M > 0
\end{aligned}
$$

- Implies, rearrangement of matrix elements does not change the feasible set

$$
\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} > 0 \iff \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} > 0 \iff \begin{bmatrix} R & S^T \\ S & Q \end{bmatrix} > 0
$$

# Variable Elimination Lemma

**Lemma:** For arbitrary nonzero vectors $x, y \in \mathbb{R}^n$, there holds

$$\max_{F\in\mathcal{F}:F^TF\leq I}(x^TFy)^2 = (x^Tx)(y^Ty).$$

**Proof:** From Schwarz inequality,

$$\begin{aligned}|x^TFy| &\leq \sqrt{x^Tx}\sqrt{y^TF^TFy} \\ &\leq \sqrt{x^Tx}\sqrt{y^Ty}.\end{aligned}$$

Therefore for arbitrary $x, y$ we have

$$(x^TFy)^2 \leq (x^Tx)(y^Ty).$$

Next show equality.

# Variable Elimination Lemma

*contd.*

Let

$$F_0 = \frac{xy^T}{\sqrt{x^Tx}\sqrt{y^Ty}}.$$

Therefore,

$$F_0^T F_0 = \frac{yx^Txy^T}{(x^Tx)(y^Ty)} = \frac{yy^T}{y^Ty}.$$

We can show that

$$\sigma_{\mathsf{max}}(F_0^T F_0) = \sigma_{\mathsf{max}}(F_0 F_0^T) = 1.$$

$$\implies F_0^T F_0 \leq 1, \text{ thus } F_0 \in \mathcal{F}.$$

# Variable Elimination Lemma

*contd.*

Therefore,

$$(x^T F_0 y)^2 = \left(x^T \frac{xy^T}{\sqrt{x^T x}\sqrt{y^T y}} y\right)^2 = (x^T x)(y^T y).$$

# Variable Elimination Lemma

*contd.*

**Lemma:** Let $X \in \mathbb{R}^{m\times n}, Y \in \mathbb{R}^{n\times m}$, and $Q \in \mathbb{R}^{m\times m}$. Then

$$Q + XFY + Y^T F^T X^T < 0,\ \forall F \in \mathcal{F},$$

iff $\exists\, \delta > 0$ such that

$$Q + \delta X X^T + \frac{1}{\delta} Y^T Y < 0.$$

**Proof:** Sufficiency

$$\begin{aligned} Q + XFY + Y^T F^T X^T &\le Q + \delta X X^T + \frac{1}{\delta} Y^T Y \quad \text{from previous Lemma} \\ &< 0. \end{aligned}$$

# Variable Elimination Lemma

*contd.*

**Proof:** Necessity

Suppose

$$Q + XFY + Y^T F^T X^T < 0, \; \forall F \in \mathcal{F}$$

is true. Then for arbitrary nonzero $x$

$$x^T(Q + XFY + Y^T F^T X^T)x < 0,$$

or

$$x^T Q x + 2x^T XFYx < 0.$$

Using previous lemma result

$$\max_{F\in\mathcal{F}}(x^T XFYx) = \sqrt{(x^T XX^T x)(x^T Y^T Y x)},$$

$$\implies x^T Q x + 2\sqrt{(x^T XX^T x)(x^T Y^T Y x)} < 0.$$

# Variable Elimination Lemma

*contd.*

$$x^TQx + 2\sqrt{(x^TXX^Tx)(x^TY^TYx)} < 0$$

$$\implies x^TQx - 2\sqrt{(x^TXX^Tx)(x^TY^TYx)} < 0,$$

and $x^TQx < 0.$

Therefore,

$$\underbrace{(x^TQx)^2}_{b^2} - 4\underbrace{(x^TXX^Tx)}_{a}\underbrace{(x^TY^TYx)}_{c} > 0.$$

or

$$b^2 - 4ac > 0.$$

# Variable Elimination Lemma

*contd.*

Or the quadratic equation

$$a\delta^2 + b\delta + c = 0$$

has real-roots

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Recall,

$$a := (x^T X X^T x) > 0, \quad b := (x^T Q x) < 0, \quad c := (x^T Y^T Y x) > 0.$$

Implies

$$-\frac{b}{2a} > 0,$$

or at least one positive root.

# Variable Elimination Lemma

*contd.*

Therefore, $\exists\delta > 0$ such that

$$a\delta^2 + b\delta + c < 0.$$

Dividing by $\delta$ we get

$$a\delta + b + \frac{c}{\delta} < 0,$$

or

$$x^TQx + \delta x^TXX^Tx + \frac{1}{\delta}x^TY^TYx < 0,$$

or

$$x^T(Q + \delta XX^T + \frac{1}{\delta}Y^TY)x < 0,$$

or

$$Q + \delta XX^T + \frac{1}{\delta}Y^TY < 0.$$

# Elimination of Variables

*In a Partitioned Matrix*

**Lemma:** Let

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^T & Z_{22} \end{bmatrix}, Z_{11} \in \mathbb{R}^{n\times n},$$

be symmetric. Then $\exists\, X = X^T$ such that

$$\begin{bmatrix} Z_{11} - X & Z_{12} & X \\ Z_{12}^T & Z_{22} & 0 \\ X & 0 & -X \end{bmatrix} < 0 \iff Z < 0.$$

**Proof:** Apply Schur complement lemma.

$$\begin{bmatrix} Z_{11} - X & Z_{12} & X \\ Z_{12}^T & Z_{22} & 0 \\ X & 0 & -X \end{bmatrix} < 0 \iff -X < 0, \quad S_{\text{ch}}(-X) < 0.$$

# Elimination of Variables

*In a Partitioned Matrix (contd.)*

$$
\begin{aligned}
0 &> S_{\mathsf{ch}}(-X), \\
&= \begin{bmatrix} Z_{11} - X & Z_{12} \\ Z_{12}^T & Z_{22} \end{bmatrix} - \begin{bmatrix} X \\ 0 \end{bmatrix} (-X)^{-1} \begin{bmatrix} X & 0 \end{bmatrix}, \\
&= \begin{bmatrix} Z_{11} - X & Z_{12} \\ Z_{12}^T & Z_{22} \end{bmatrix} + \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}, \\
&= \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^T & Z_{22} \end{bmatrix}.
\end{aligned}
$$

# Elimination of Variables

*In a Partitioned Matrix (contd.)*

**Lemma:**

$$\begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{12}^T & Z_{22} & Z_{23} + X^T \\ Z_{13}^T & Z_{23}^T + X & Z_{33} \end{bmatrix} < 0 \iff \begin{array}{l} \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^T & Z_{22} \end{bmatrix} < 0 \\ \begin{bmatrix} Z_{11} & Z_{13} \\ Z_{13}^T & Z_{33} \end{bmatrix} < 0, \end{array}$$

with

$$X = Z_{13}^T Z_{11}^{-1} Z_{12} - Z_{23}^T.$$

**Proof:** Necessity $\Rightarrow$ Apply rules for negative definiteness.

Sufficiency $\Leftarrow$ Following are true from Schur complement lemma.

$$Z_{11} < 0$$

$$Z_{22} - Z_{12}^T Z_{11}^{-1} Z_{12} < 0 \qquad Z_{33} - Z_{13}^T Z_{11}^{-1} Z_{13} < 0$$

# Elimination of Variables

*In a Partitioned Matrix (contd.)*

Look at Schur complement of

$$\begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{12}^T & Z_{22} & Z_{23} + X^T \\ Z_{13}^T & Z_{23}^T + X & Z_{33} \end{bmatrix}.$$

$$\begin{bmatrix} Z_{22} & Z_{23} + X^T \\ Z_{23}^T + X & Z_{33} \end{bmatrix} - \begin{bmatrix} Z_{12}^T \\ Z_{13}^T \end{bmatrix} Z_{11}^{-1} \begin{bmatrix} Z_{12} & Z_{13} \end{bmatrix}$$
$$= \begin{bmatrix} Z_{22} - Z_{12}^T Z_{11}^{-1} Z_{12} & Z_{23} + X^T - Z_{12}^T Z_{11}^{-1} Z_{13} \\ Z_{23}^T + X - Z_{13}^T Z_{11}^{-1} Z_{12} & Z_{33} - Z_{13}^T Z_{11}^{-1} Z_{13} \end{bmatrix}$$
$$< 0.$$

Also $Z_{!1} < 0$.

# Elimination of Variables

*Projection Lemma*

**Definition** Let $A \in \mathbb{R}^{m \times n}$. Then $M_a$ is left orthogonal complement of $A$ if it satisfies

$$M_a A = 0,\ \text{rank}(M_a) = m - \text{rank}(A).$$

**Definition** Let $A \in \mathbb{R}^{m \times n}$. Then $N_a$ is right orthogonal complement of $A$ if it satisfies

$$A N_a = 0,\ \text{rank}(N_a) = n - \text{rank}(A).$$

# Elimination of Variables

*Projection Lemma (contd.)*

**Lemma:** Let $P, Q$, and $H = H^T$ be matrices of appropriate dimensions. Let $N_p$, $N_q$ be right orthogonal complements of $P, Q$ respectively.

Then $\exists\, X$ such that

$$H + P^T X^T Q + Q^T X P < 0 \iff N_p^T H N_p < 0 \text{ and } N_q^T H N_q < 0.$$

**Proof:**

Necessity $\Rightarrow$: Multiply by $N_p$ or $N_q$.

Sufficiency $\Leftarrow$: Little more involved – Use base kernel of $P, Q$, followed by Schur complement lemma.

# Elimination of Variables

*Reciprocal Projection Lemma*

**Lemma:** Let $P$ be any given positive definite matrix. The following statements are equivalent:

1. $\Psi + S + S^T < 0$.
2. The LMI problem

$$\begin{bmatrix} \Psi + P - (W + W^T) & S^T + W^T \\ S + W & -P \end{bmatrix} < 0,$$

   is feasible with respect to $W$.

**Proof:** Apply projection lemma w.r.t general variable $W$. Let

$$X = \begin{bmatrix} \Psi + P & S^T \\ S & -P \end{bmatrix}, \quad Y = \begin{bmatrix} -I_n & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} I_n & -I_n \end{bmatrix}.$$

# Elimination of Variables

*Reciprocal Projection Lemma (contd.)*

Let

$$X = \begin{bmatrix} \Psi + P & S^T \\ S & -P \end{bmatrix}, \quad Y = \begin{bmatrix} -I_n & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} I_n & -I_n \end{bmatrix}.$$

Right orthogonal complements of $Y, Z$ are

$$N_y = \begin{bmatrix} 0 \\ -P^{-1} \end{bmatrix}, \qquad N_z = \begin{bmatrix} I_n \\ I_n \end{bmatrix}.$$

Verify that $YN_y = 0$ and $ZN_z = 0$.

We can show

$$N_y^T X N_y = -P^{-1}, \qquad N_z^T X N_z = \Psi + S^T + S.$$

Apply projection lemma.

# Elimination of Variables

*Reciprocal Projection Lemma (contd.)*

$$N_y^T X N_y = -P^{-1}, \qquad N_z^T X N_z = \Psi + S^T + S.$$

The expression

$$X + Y^T W^T Z + Z^T W Y = \begin{bmatrix} \Psi + P - (W + W^T) & S^T + W^T \\ S + W & -P \end{bmatrix}.$$

Therefore, if

$$\begin{matrix} N_y^T X N_y < 0 \\ N_z^T X N_z < 0 \end{matrix} \implies \begin{bmatrix} \Psi + P - (W + W^T) & S^T + W^T \\ S + W & -P \end{bmatrix} < 0.$$

# Trace of Matrices in LMIs

**Lemma** Let $A(x) \in \mathbb{S}^m$ be a matrix function in $\mathbb{R}^n$, and $\gamma \in \mathbb{R} > 0$. The following statements are equivalent:

1. $\exists\, x \in \mathbb{R}^n$ such that
$$\mathbf{tr} A(x) < \gamma,$$
2. $\exists\, x \in \mathbb{R}^n, Z \in \mathbb{S}^m$ such that
$$A(x) < Z, \mathbf{tr} Z < \gamma.$$

**Proof**: Homework problem.