Quadratic Stability of Dynamical Systems

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Quadratic Lyapunov Functions

Quadratic Stability

Dynamical system

$$\dot{x} = Ax,$$

is quadratically stable if

$$\exists V(x) \geq 0, \quad \dot{V} \leq 0.$$

Let $\quad V(x) = x^T P x, \; P \in \mathbb{S}^n_{++} \; (P = P^T > 0)$

Therefore,

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} \\ &= x^T A^T P x + x^T P A x \\ &= x^T \left(A^T P + P A\right) x \end{aligned}$$

Therefore

$$\dot{V} \leq 0 \implies x^T \left(A^T P + P A\right) x \leq 0 \implies A^T P + P A < 0.$$

Lyapunov Equation

We can write

$$A^T P + PA \leq 0$$

as

$$A^T P + PA + Q = 0$$

for $Q = Q^T \geq 0$.

Interpretation

For linear system $\dot{x} = Ax$, if $V(x) = x^T P x$,

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} \\ &= (Ax)^T P x + x^T P (Ax) \\ &= -x^T Q x. \end{aligned}$$

If $V(x) = x^T P x$ is generalized energy, $\dot{V} = -x^T Q x$ is generalized dissipation.

Stability Condition

If $P > 0, Q > 0$, then $\dot{x} = Ax$

- is globally asymptotically stable
- $\mathbb{R}\lambda_i(A) < 0$

Note that for $P = P^T > 0$, eigenvalues are real

$$\implies \lambda_{\mathsf{min}}(P)\, x^T x \leq x^T P x \leq \lambda_{\mathsf{max}}(P)\, x^T x$$

$$\implies \dot{V} = -x^T Q x \leq -\lambda_{\mathsf{min}}(Q) x^T x$$

$$\leq -\frac{\lambda_{\mathsf{min}}(Q)}{\lambda_{\mathsf{max}}(P)} x^T P x$$

$$= -\alpha V(x)$$

Lyapunov Integral

If A is stable, then

$$P = \int_0^\infty e^{tA^T} Q e^{tA} dt, \text{ for any } Q = Q^T > 0.$$

Proof:

Substitute it in LHS of Lyapunov equation to get,

$$\begin{aligned} A^T P + PA &= \int_0^\infty \left(A^T e^{tA^T} Q e^{tA} + e^{tA^T} Q e^{tA} A \right) dt, \\ &= \int_0^\infty \left(\frac{d}{dt} e^{tA^T} Q e^{tA} \right) dt, \\ &= e^{tA^T} Q e^{tA} \Big|_0^\infty, \\ &= -Q. \end{aligned}$$

Computation of $\|x\|_{2,Q}$

Recall

$$\|x\|_2^2 := \int_0^\infty x^T x \, dt.$$

Define weighted norm as

$$\|x\|_{2,Q}^2 := \int_0^\infty x^T Q x \, dt.$$

If $x(t)$ is solution of $\dot{x} = Ax$,

$$x(t) := e^{tA} x_0.$$

Substituting we get,

$$\begin{aligned}\|x\|_{2,Q}^2 &= \int_0^\infty x_0^T e^{tA^T} Q e^{tA} dt \\ &= x_0^T P x_0 \text{ assuming } A \text{ is stable}\end{aligned}$$

Cost-to-go interpretation

LQR Problem

Linear Quadratic Regulator

Problem Statement

Given system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du.$$

Determine $u^*(t)$ that solves

$$\min_{u(t)} \|y\|_2 \text{ with } x(0) = x_0.$$

Or

$$\begin{aligned}
\min_{u(t)} J &:= \int_0^\infty y^T y \, dt \\
&= \int_0^\infty \left(x^T C^T C x + x^T C^T D u + u^T D^T C x + u^T D^T D u\right) dt \\
&= \int_0^\infty \left(x^T C^T C x + u^T D^T D u\right) dt.
\end{aligned}$$

Linear Quadratic Regulator

Solution as Optimal Control Problem

$$\min_u \int_0^\infty \left(x^T Q x + u^T R u\right) dt, \qquad Q = Q^T \geq 0, R = R^T > 0$$

subject to

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ x(0) &= x_0. \end{aligned}$$

- Euler Lagrange Equations
- Hamilton-Jacobi-Bellman Equation – Dynamic Programming

Euler Langrange Formulation

Linear Quadratic Regulator

Solution as Optimal Control Problem – EL Formulation

$$\min_u \int_0^T L(x,u)dt + \Phi(x(T)), \text{ subject to } \dot{x} = f(x,u).$$

Define $H = L + \lambda^T f$.

Euler-Lagrange Equations

$$H_u = 0 \qquad \dot{\lambda}^T = -H_x \qquad \lambda(T) = \phi_x(x(T))$$

Our Problem

$$\min_u \int_0^T \left(x^T Q x + u^T R u\right) dt, \text{ subject to } \dot{x} = Ax + Bu.$$

Define $H = x^T Q x + u^T R u + \lambda^T (Ax + Bu)$.

Linear Quadratic Regulator

Solution as Optimal Control Problem – EL Formulation

Our Problem

$$\min_u \frac{1}{2}\int_0^T \left(x^TQx + u^TRu\right)dt, \text{ subject to } \dot{x} = Ax + Bu.$$

Define $H = \frac{1}{2}\left(x^TQx + u^TRu\right) + \lambda^T(Ax + Bu)$.

EL Equations

$$(1) \qquad H_u = 0 \implies u^TR + \lambda^TB = 0 \implies u = -R^{-1}B^T\lambda.$$

$$(2) \qquad \dot{\lambda}^T = -H_x = -x^TQ - \lambda^TA$$

$$(3) \qquad \lambda(T) = 0.$$

Linear Quadratic Regulator

Solution as Optimal Control Problem – EL Formulation

Let $\lambda(t) = P(t)x(t)$

$$\begin{aligned}\implies \dot{\lambda} &= \dot{P}x + P\dot{x} \\ &= \dot{P}x + P(Ax + Bu), \\ &= \dot{P}x + P(Ax - BR^{-1}B^TPx), \\ &= (\dot{P} + PA - PBR^{-1}B^TP)x.\end{aligned}$$

From EL(2) we get

$$\dot{\lambda} = -Qx - A^TPx$$

$$\implies (\dot{P} + PA + A^TP - PBR^{-1}B^TP + Q)x = 0$$

$$\implies \dot{P} + PA + A^TP - PBR^{-1}B^TP + Q = 0.$$ Riccati Differential Equation

In the steady-state $T \to \infty$, $\dot{P} = 0$,

$$PA + A^TP - PBR^{-1}B^TP + Q = 0.$$ Algebraic Riccati Equation

Linear Quadratic Regulator

Solution as Optimal Control Problem – EL Formulation

$$\min_u \frac{1}{2}\int_0^\infty \left(x^T Q x + u^T R u\right) dt, \text{ subject to } \dot{x} = Ax + Bu.$$

is equivalent to

$$\begin{aligned} &PA + A^T P - PBR^{-1}B^T P + Q = 0, \\ &u = -R^{-1}B^T P. \end{aligned}$$

Hamilton-Jacobi-Bellman Formulation

Hamilton-Jacobi-Bellman Approach

Let

$$V^*(x(t)) = \min_{u[t,\infty)} \frac{1}{2}\int_t^\infty (x^T Q x + u^T R u)dt$$

subject to

$$\dot{x} = Ax + Bu.$$

Hamilton-Jacobi-Bellman Approach

contd.

$$V^*(x(t)) = \min_{u[t,\infty)} \frac{1}{2}\int_t^{\infty}(x^TQx + u^TRu)dt$$
$$= \min_{u[t,t+\Delta t]}\left\{\int_t^{t+\Delta t}\frac{1}{2}(x^TQx + u^TRu)dt + V^*(x(t+\Delta t))\right\}$$

Let $V(x) := x^TPx$, therefore,

$$V^*(x(t)) = \min_{u[t,t+\Delta t]}\Big\{\frac{1}{2}(x^TQx + u^TRu)\Delta t + V^*(x(t)) +$$
$$(Ax+Bu)^TPx\Delta t + x^TP(Ax+Bu)\Delta t + H.O.T\Big\}$$

Hamilton-Jacobi-Bellman Approach

contd.

$$\implies \min_{u[t,t+\Delta t]} \left\{ \frac{1}{2}(x^T Q x + u^T R u) + (Ax + Bu)^T P x + x^T P(Ax + Bu) + H.O.T \right\} = 0.$$

$$\lim_{\Delta t \to 0} \implies \min_{u} \left\{ \frac{1}{2}(x^T Q x + u^T R u) + (Ax + Bu)^T P x + x^T P(Ax + Bu) \right\} = 0$$

Quadratic in u,

$$\implies u^* = -R^{-1} B^T P x.$$

Optimal controller is state-feedback.

Variational Approach

Variational Approach

Given dynamics

$$\dot{x} = Ax + Bu,$$

with controller $u = Kx$, find K that minimizes

$$J := \frac{1}{2}\int_0^\infty (x^T Q x + u^T R u)dt = \frac{1}{2}\int_0^\infty x^T(Q + K^T R K)x dt.$$

The closed-loop dynamics is

$$\dot{x} = Ax + Bu = (A + BK)x = A_c x.$$

The solution is therefore,

$$x(t) = e^{tA_c}x_0.$$

The cost function is therefore,

$$J := \frac{1}{2}x_0^T\left(\int_0^\infty e^{tA_c^T}(Q + K^T R K)e^{tA_c}dt\right)x_0 = \frac{1}{2}x_0^T P x_0.$$

Variational Approach

contd.

Apply the following 'trick'

$$\int_0^\infty \frac{d}{dt} e^{tA_c^T}(Q + K^T RK)e^{tA_c} dt =$$

$$A_c^T \left(\int_0^\infty e^{tA_c^T}(Q + K^T RK)e^{tA_c} dt \right) + \left(\int_0^\infty e^{tA_c^T}(Q + K^T RK)e^{tA_c} dt \right) A_c$$

$$\left[e^{tA_c^T}(Q + K^T RK)e^{tA_c} \right]_0^\infty = A_c^T P + PA_c$$

Or

$$A_c^T P + PA_c + Q + K^T RK = 0,$$

Or

$$(A + BK)^T P + P(A + BK) + Q + K^T RK = 0.$$

Variational Approach

contd.

The optimal cost is therefore,

$$J^* = \frac{1}{2}x_0^T P^* x_0 \implies \left.\frac{\partial J}{\partial P}\right|_{P^*} = 0.$$

- Variation δP from P^* should result in $\delta J = 0$
- Let $P = P^* + \delta P$, $\implies J = \frac{1}{2}x_0^T P^* x_0 + \underbrace{\frac{1}{2}x_0^T \delta P x_0}_{\delta J}$
- $\delta J = 0 \implies \delta P = 0$

Variational Approach

contd.

Substitute $P = P^* + \delta P$, and $K = K^* + \delta K$ in the equality constraint

$$(A + BK)^T P + P(A + BK) + Q + K^T RK = 0,$$

to get,

$$(A + B(K^* + \delta K))^T (P^* + \delta P) + (P^* + \delta P)(A + B(K^* + \delta K)) + Q + (K^* + \delta K)^T R(K^* + \delta K) = 0,$$

or

$$(A + BK^*)^T P^* + P^*(A + BK^*) + Q + {K^*}^T RK^* + \delta P(A + BK^*) + (*)^T + \delta K^T (B^T P^* + RK^*) + (*)^T + H.O.T = 0.$$

$$\implies K^* = -R^{-1} B^T P^*.$$

Convex Optimization

Problem Formulation

Find gain K such that $u = Kx$ minimizes

$$\int_0^\infty (x^T Q x + u^T R u) dt,$$

subject to dynamics

$$\dot{x} = Ax + Bu,$$

and

$$x(0) = x_0.$$

An Upper Bound on the Cost-to-go

If $\exists V(x) > 0$ such that

$$\frac{dV}{dt} \leq -(x^T Q x + u^T R u).$$

Integrating from $[0, T]$, gives us

$$\int_0^T \frac{dV}{dt} dt \leq -\int_0^T (x^T Q x + u^T R u) dt,$$

or

$$V(x(T)) - V(x(0)) \leq -\int_0^T (x^T Q x + u^T R u) dt.$$

Since $V(x(T)) \geq 0$ for any T

$$\implies -V(x(0)) \leq -\int_0^T (x^T Q x + u^T R u) dt,$$

An Upper Bound on the Cost-to-go

Since $V(x(T)) \geq 0$ for any T

$$\implies -V(x(0)) \leq -\int_0^T (x^T Q x + u^T R u)dt,$$

or

$$V(x(0)) \geq \int_0^\infty (x^T Q x + u^T R u)dt.$$

Sufficient condition for upper-bound on cost-to-go.

If $\exists V(x) > 0$ such that

$$\frac{dV}{dt} \leq -(x^T Q x + u^T R u).$$

Idea:

Minimize upper-bound to get optimal K.

Optimization Problem

Find $P = P^T > 0$ and K such that with $V := x^T P x$,

$$\min_{P,K} V(x(0)) = x(0)^T P x(0) \text{ Cost Function}$$

subject to

$$\dot{V} \leq -x^T (Q + K^T R K) x \text{ Constraint Function}$$

Or equivalently

$$\min_{P,K} \mathbf{tr} P$$

subject to

$$(A + BK)^T P + P(A + BK) + Q + K^T R K \leq 0.$$

Optimization Problem

$$\min_{P,K} \mathbf{tr} P$$

subject to

$$(A+BK)^T P + P(A+BK) + Q + K^T R K \leq 0.$$

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