4.6

8. $y'' - y = \sinh 2x$.

sol: the auxiliary equation is

$$m^2 - 1 = 0 \Rightarrow m = \pm 1,$$

so $y_c = c_1 e^{x} + c_2 e^{-x} \Rightarrow y_1 = e^{x}, y_2 = e^{-x}$, identifying $f(x) = \sinh 2x$

By Cramer's Rule, the solution system is

$$y_1 u_1' + y_2 u_2' = 0$$
$$y_1' u_1' + y_2' u_2' = f(x)$$

let $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} = -1 - (1) = -2$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = -y_2 f(x) = -e^{-x}\sinh 2x$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = y_1 f(x) = e^{x}\sinh 2x$$

$$u_1' = \frac{W_1}{W} = \frac{-e^{-x}\sin 2x}{-2} = e^{-x}\left(\frac{e^{2x} - e^{-2x}}{4}\right) = \frac{1}{4}e^{x} - \frac{1}{4}e^{-3x}$$

$$u_2' = \frac{W_2}{W} = \frac{e^{x}\sin 2x}{-2} = e^{x}\left(\frac{-e^{2x} + e^{-2x}}{4}\right) = \frac{1}{4}e^{-x} - \frac{1}{4}e^{3x}$$

$$u_1 = \int u_1' dx = \frac{1}{4}e^{x} + \frac{1}{12}e^{-3x}, \quad u_2 = \int u_2' dx = -\frac{1}{4}e^{-x} - \frac{1}{12}e^{3x}$$

$$y_p = u_1 y_1 + u_2 y_2 = \frac{1}{4}e^{2x} + \frac{1}{12}e^{-2x} - \frac{1}{4}e^{-2x} - \frac{1}{12}e^{2x}$$

Thus, $y = y_c + y_p$

$$= c_1 e^{x} + c_2 e^{-x} + \frac{1}{4}e^{2x} + \frac{1}{12}e^{-2x} - \frac{1}{4}e^{-2x} - \frac{1}{12}e^{2x}$$

$$= c_1 e^{x} + c_2 e^{-x} + \frac{1}{6}(e^{2x} - e^{-2x}) = c_1 e^{x} + c_2 e^{-x} + \frac{1}{3}\sinh 2x.$$ #

15. $y'' + 2y' + y = e^{-t}\ln t$.

sol: the auxiliary equation is $m^2 + 2m + 1 = 0 \Rightarrow m = -1$

so $y_c = c_1 e^{-t} + c_2 t e^{-t}$, $y_1 = e^{-t}$, $y_2 = te^{-t}$, idenifying $f(t) = e^{-t}\ln t$.

By Cramer's rule, the solution system is

$$y_1 u_1' + y_2 u_2' = 0$$
$$y_1' u_1' + y_2' u_2' = f(t)$$

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$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{vmatrix} = e^{-2t} - te^{-2t} + te^{-2t} = e^{-2t}$$

D	I
$\ln t$	t
$\frac{1}{t}$	$\frac{1}{2}t^2$

D	I
$\ln t$	1
$\frac{1}{t}$	t

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(t) & y_2' \end{vmatrix} = -y_2 f(t) = -te^{-t}e^{-t}\ln t = -te^{-2t}\ln t$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(t) \end{vmatrix} = y_1 f(t) = y_1 f(t) = e^{-t}e^{-t}\ln t = e^{-2t}\ln t.$$

$$u_1' = \frac{W_1}{W} = \frac{-te^{-2t}\ln t}{e^{-2t}} = -t\ln t \Rightarrow u_1 = \int -t\ln t\, dt = -\frac{1}{2}t^2\ln t + \frac{1}{2}\int t\, dt = -\frac{1}{2}t^2\ln t + \frac{1}{4}t^2$$

$$u_2' = \frac{W_2}{W} = \frac{e^{-2t}\ln t}{e^{-2t}} = \ln t \quad \Rightarrow u_2 = \int \ln t\, dt = t\ln t - \int dt = t\ln t - t.$$

$\therefore y_p = y_1u_1 + y_2u_2 = (-\frac{1}{2}t^2\ln t + \frac{1}{4}t^2)e^{-t} + (t\ln t - t)te^{-t}$

$= -\frac{1}{2}t^2e^{-t}\ln t + \frac{1}{4}t^2e^{-t} + t^2e^{-t}\ln t - t^2e^{-t}$

$= \frac{1}{2}t^2e^{-t}\ln t - \frac{3}{4}t^2e^{-t}$.

Thus, $y = y_c + y_p = c_1e^t + c_2te^t + \frac{1}{2}t^2e^{-t}\ln t - \frac{3}{4}t^2e^{-t}$ #

22. $y'' - 4y' + 4y = (12x^2 - 6x)e^{2x}$

sol: the auxiliary equation, $m^2 - 4m + 4 = 0$.

$m_1 = m_2 = 2$.

$y_c = c_1e^{2x} + c_2xe^{2x}$. $y_1 = e^{2x}$, $y_2 = xe^{2x}$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix}$

$= e^{4x} + 2xe^{4x} - 2xe^{4x} = e^{4x}$.

Idenifying $f(x) = (12x^2 - 6x)e^{2x}$.

From Carmer rule the solution system is

$y_1u_1' + y_2u_2' = 0$

$y_1'u_1' + y_2'u_2' = (12x^2 - 6x)e^{2x}$

$W_1 = \begin{vmatrix} 0 & xe^{2x} \\ (12x^2 - 6x)e^{2x} & y_2' \end{vmatrix} = -12x^3e^{4x} + 6x^2e^{4x}$

$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & (12x^2 - 6x)e^{2x} \end{vmatrix} = 12x^2e^{4x} - 6xe^{4x}$

$u_1' = \frac{W_1}{W} = \frac{(-12x^3 + 6x^2)e^{4x}}{e^{4x}} = -12x^3 + 6x^2$ $\Rightarrow$ $u_1 = \int u_1' du = -3x^4 + 2x^3$

$u_2' = \frac{W_2}{W} = 12x^2 - 6x$ $\qquad u_2 = \int u_2' du = 4x^3 - 3x^2$

Thus, $y = y_c + y_1u_1 + y_2u_2$

$= c_1e^{2x} + c_2xe^{2x} + e^{2x}(-3x^4 + 2x^3) + xe^{2x}(4x^3 - 3x^2)$

$= c_1e^{2x} + c_2xe^{2x} + e^{2x}(x^4 - x^3)$

and $y' = 2c_1e^{2x} + c_2(2xe^{2x} + e^{2x}) + e^{2x}(4x^3 - 3x^2) + 2e^{2x}(x^4 - x^3)$

By the initial condition, $y(0) = c_1 = 1$

$y'(0) = 2c_1 + c_2 = 0 \Rightarrow c_2 = -2$

$\therefore y = e^{2x} - 2xe^{2x} + e^{2x}(x^4 - x^3)$.

26. $2y'' + 2y' + y = 4\sqrt{x}$

sol: the auxiliary equation is $2m^2 + 2m + 1 = 0$

$\therefore y_c = e^{-\frac{x}{2}}[c_1\cos(\frac{x}{2}) + c_2\sin(\frac{x}{2})]$ and $W = \frac{1}{2}e^{-x}$.

Identifying $f(x) = 2\sqrt{x}$,

$\therefore u_1' = -\frac{e^{-\frac{x}{2}}\sin(\frac{x}{2})4\sqrt{2}}{\frac{1}{2}e^{-x}} = -8e^{\frac{x}{2}}\sqrt{x}\cdot\sin\frac{x}{2}$

$u_2' = \frac{e^{-\frac{x}{2}}\cos(\frac{x}{2})4\sqrt{2}}{\frac{1}{2}e^{-x}} = 8e^{\frac{x}{2}}\sqrt{x}\cos\frac{x}{2}$

$\therefore u_1 = -8\int_{x_0}^{x} e^{\frac{t}{2}}\sqrt{t}\sin\frac{t}{2}\,dt$

$u_2 = 8\int_{x_0}^{x} e^{\frac{t}{2}}\sqrt{t}\cos\frac{t}{2}\,dt$

$\therefore y = e^{-\frac{x}{2}}(c_1\cos\frac{x}{2} + c_2\sin\frac{x}{2}) - 4e^{-\frac{x}{2}}\cos\frac{x}{2}\int_{x_0}^{x} e^{\frac{t}{2}}\sqrt{t}\sin\frac{t}{2}\,dt + 4e^{-\frac{x}{2}}\sin\frac{x}{2}\int_{x_0}^{x} e^{\frac{t}{2}}\sqrt{t}\cos\frac{t}{2}\,dt.$

28. $x^2y'' + xy' + y = \sec(\ln x)$

sol: D.E. : $y'' + \frac{1}{x}y' + \frac{1}{x^2}y = \frac{\sec(\ln x)}{x^2}$, identifying $f(x) = \frac{\sec(\ln x)}{x^2}$.

From $y_1 = \cos(\ln x)$, $y_2 = \sin(\ln x)$. $\Rightarrow W = \frac{1}{x}$

$u_1' = -\frac{\tan(\ln x)}{x}$, $u_2' = \frac{1}{x}$

$\Rightarrow u_1 = \ln|\cos(\ln x)|$, $u_2 = \ln x$.

$y_p = \cos(\ln x)\ln|\cos(\ln x)| + (\ln x)\sin(\ln x)$

Thus, $y = c_1\cos(\ln x) + c_2\sin(\ln x) + \cos(\ln x)\ln|\cos(\ln x)| + (\ln x)\sin(\ln x)$.

32. auxiliary equation: $m^3 - 3m^2 + 2m = m(m^2 - 3m + 2) = m(m-2)(m-1) = 0$

$\therefore y_c = c_1 + c_2e^x + c_3e^{2x} \Rightarrow W = 2e^{3x}$.

Identifying $f(x) = \frac{e^{2x}}{1+e^{2x}} \Rightarrow W_1 = \frac{e^{5x}}{1+e^x}$, $W_2 = \frac{-2e^{4x}}{1+e^x}$, $W_3 = \frac{e^{3x}}{1+e^x}$

$u_1' = \frac{W_1}{W} = \frac{e^{2x}}{2(1+e^x)}$ $\qquad u_1 = \frac{1}{2}e^x - \frac{1}{2}\ln(1+e^x)$

$u_2' = \frac{W_2}{W} = -\frac{e^x}{1+e^x}$ $\xrightarrow{\text{integrating}}$ $u_2 = -\ln(1+e^x)$

$u_3' = \frac{W_3}{W} = \frac{1}{2(1+e^x)}$ $\qquad u_3 = -\frac{1}{2}\ln(1+e^{-x})$

Thus, $y_p = 1\cdot(\frac{1}{2}e^x - \frac{1}{2}\ln(1+e^x)) + e^x\cdot(-\ln(1+e^x)) + e^{2x}\cdot(-\frac{1}{2}\ln(1+e^{-x}))$

$= \frac{1}{2}e^x - \frac{1}{2}\ln(1+e^x) - e^x\ln(1+e^x) - \frac{1}{2}e^{2x}\ln(1+e^{-x})$.

$y = c_1 + c_2e^x + c_3e^x + \frac{1}{2}e^x - \frac{1}{2}\ln(1+e^x) - e^x\ln(1+e^x) - \frac{1}{2}e^{2x}\ln(1+e^{-x})$.

$= c_1 + c_4e^x + c_3e^{2x} - (\frac{1}{2} + e^x)\ln(1+e^x) - \frac{1}{2}e^{2x}\ln(1+e^{-x})$. #

where $x \in (-\infty, \infty)$ #

4-7-1

6. $x^2y''+5xy'+3y=0$

sol: $y=x^m \Rightarrow$ D.E. : $m(m-1)x^m+5x^m m+3x^m=0$

$x^m(m^2+4m+3)=0$

the auxiliary equation is $m^2+4m+3=0$

$m_1=-1,\ m_2=-3$

$\therefore y=c_1x^{-1}+c_2x^{-3}$.

13. $3x^2y''+6xy'+y=0$

sol: the auxiliary equation is

$3m^2+3m+1=0$

$m=\dfrac{-3\pm\sqrt{9-12}}{6}=\dfrac{-3\pm\sqrt{3}}{6}$

$\therefore$ the solution is $y=x^{-\frac{1}{2}}\left[c_1\cos\left(\frac{\sqrt{3}}{6}\ln x\right)+c_2\sin\left(\frac{\sqrt{3}}{6}\ln x\right)\right]$.

18. $x^4y^{(4)}+6x^3y'''+9x^2y''+3xy'+y=0$

sol: Assume that $y=x^m$.

D.E: $m(m-1)(m-2)(m-3)+6m(m-1)(m-2)+9m(m-1)+3m+1$

$=m^4+2m^2+1$

$=(m^2+1)^2=0$.

Thus, $y=c_1\cos(\ln x)+c_2\sin(\ln x)+c_3(\ln x)\cos(\ln x)+c_4(\ln x)\sin(\ln x)$

22. the auxiliary equation is $m^2-3m+2=0$ $\quad$ m -2 / m -1

$\therefore y=c_1x+c_2x^2$. Wronskian is $W(x,x^2)=x^2$.

Identifying $f(x)=x^2e^x$,

$\begin{cases} u_1'=-x^2e^x \\ u_2'=xe^x \end{cases} \xrightarrow{\text{integrating}} \begin{cases} u_1=-x^2e^x+2xe^x-2e^x \\ u_2=xe^x-e^x \end{cases}$.

$\therefore y=c_1x+c_2x^2-x^3e^x+2x^2e^x-2xe^x+x^3e^x-x^2e^x$

$=c_1x+c_2x^2+x^2e^x-2xe^x$.

30. $x^2y'' - 5xy' + 8y = 8x^6$, $y(\frac{1}{2}) = 0$, $y'(\frac{1}{2}) = 0$.

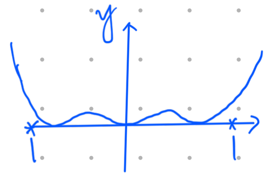
sol: The auxiliary equation is $m^2 - 6m + 8 = 0$

$\therefore y_c = c_1x^2 + c_2x^4$, $W = 2x^5$.

Identifying $f(x) = 8x^4$, $u_1' = 4x^3$, $u_2' = 4x$. $\Rightarrow u_1 = -x^4$, $u_2 = 2x^2$

$\therefore y = c_1x^2 + c_2x^4 + x^6$, initial condition imply $\Rightarrow \frac{1}{4}c_1 + \frac{1}{16}c_2 = -\frac{1}{64}$

and $c_1 + \frac{1}{2}c_2 = -\frac{3}{16}$, $\therefore c_1 = \frac{1}{16}, c_2 = -\frac{1}{2}$, $y = \frac{1}{16}x^2 - \frac{1}{2}x^4 + x^4$ #1



36. $x^3y''' - 3x^2y'' + 6xy' - 6y = 3 + \ln x^3$

$$\frac{d^2y}{dx} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$\frac{d^3y}{dx^3} = \frac{1}{x^2}\frac{d}{dx}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) - \frac{2}{x^3}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$= \frac{1}{x^2}\frac{d}{dx}\left(\frac{d^2y}{dt^2}\right) - \frac{1}{x^2}\cdot\frac{d}{dx}\left(\frac{dy}{dt}\right) - \frac{2}{x^3}\cdot\frac{d^2y}{dt^2} + \frac{2}{x^3}\cdot\frac{dy}{dt}$$

$$= \frac{1}{x^2}\frac{d^3y}{dt^3}\left(\frac{1}{x}\right) - \frac{1}{x^2}\frac{d^2y}{dt^2}\left(\frac{1}{x}\right) - \frac{2}{x^3}\cdot\frac{d^2y}{dt^2} + \frac{2}{x^3}\cdot\frac{dy}{dt}$$

$$= \frac{1}{x^3}\left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right)$$

Substituting into D.E. :

$$\left(\frac{d^3y}{dt^3} - 3\cdot\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right) - 3\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + 6\frac{dy}{dt} - 6y = 3 + 3t$$

$$\Rightarrow \frac{d^3y}{dt^3} - 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} - 6y = 3 + 3t.$$

The auxiliary equation is

$$m^3 - 6m^2 + 11m - 6 = m^3 - 6m^2 + 8m + 3m - 6$$

$$= m(m^2 - 6m + 8) + 3(m-2) = m(m-2)(m-4) + 3(m-2)$$

$$= (m-2)(m^2 - 4m + 3) = (m-1)(m-2)(m-3)$$

$\therefore y_c = c_1e^t + c_2e^{2t} + c_3e^{3t}$.

Using undetermined coefficients :

Let $y_p = A + Bt$, then

D.E.: L.H.S $= 11(B) - 6(A + Bt) = 11B - 6A + Bt$

R.H.S $= 3 + 3t$

$$\therefore \begin{cases} 11B - 6A = 3 \\ -6B = 3 \end{cases} \Rightarrow \begin{cases} A = -\frac{17}{12} \\ B = -\frac{1}{2} \end{cases}$$

$$\Rightarrow y = c_1e^t + c_2e^{2t} + c_3e^{3t} - \frac{17}{12} - \frac{1}{2}t = c_1x + c_2x^2 + c_3x^3 - \frac{17}{12} - \frac{1}{2}\ln x.$$

38. The differential equation and initial conditions become

$$t^2\frac{d^2y}{dt^2} - 4t\frac{dy}{dt} + 6y = 0, \quad y(t)\Big|_{t=2} = 8, \quad y'(t)\Big|_{t=2} = 0$$

let $y = t^m$, D.E. :

$$t^2m(m-1)t^{m-2} - 4tmt^{m-1} + 6t^m$$

$$= t^m(m^2 - m - 4m + 6) = t^m(m^2 - 5m + 6) = 0$$

$\therefore$ the auxiliary equation is $m^2 - 5m + 6 = (m-2)(m-3) = 0$

$\therefore y = c_1 t^2 + c_2 t^3$ and $y' = 2c_1 t + 3c_2 t^2$

the initial conditions imply: $\begin{cases} 4c_1 + 8c_2 = 8 \\ 4c_1 + 12c_2 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 6 \\ c_2 = -2 \end{cases}$

$\therefore y = 6t^2 - 2t^3 = 6x^2 + 2x^3$, $x < 0$.

40. $(x-1)^2 y'' - (x-1)y' + 5y = 0$

sol: let $y = (x-1)^m$, then $y' = m(x-1)^{m-1}$, and $y'' = m(m-1)(x-1)^{m-2}$.

Substituting into D.E.:

$(x-1)^2 m(m-1)(x-1)^{m-2} - (x-1)m(x-1)^{m-1} + 5(x-1)^m$

$= m(m-1)(x-1)^m - m(x-1)^m + 5(x-1)^m = 0$.

$\therefore m(m-1) - 2m + 5 = 0$. $m_1 = 1+2i$, $m_2 = 1-2i$

$\therefore$ solution is $y = (x-1)[c_1 \cos(\ln(x-1)) + c_2(\ln(x-1))]$.

42. $(x-4)^2 y'' - 5(x-4)y' + 9y = 0$

sol: $t = x-4$, $\frac{dy}{dx} = \frac{dy}{dt}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2}\frac{dt}{dx} = \frac{d^2y}{dt^2}$

D.E.: $t^2 \frac{d^2y}{dt^2} - 5t\frac{dy}{dt} + 9y = 0$

Auxiliary equation is $m(m-1) - 5m + 9 = m^2 - 6m + 9 = (m-3)^2 = 0$

$\therefore m_1 = m_2 = 3$

Thus, $y = c_1 t^3 + c_2 t^3 \ln t = c_1(x-4)^3 + c_2(x-4)^3 \ln(x-4)$.

4.6 $y = -\sqrt{x}\cos(\ln x)$, $x > 0$ and has x-intercepts where

$y = 0 \Rightarrow \ln x = \frac{\pi}{2} + k\pi$, $\forall k \in \mathbb{Z}$.

$\Rightarrow x = e^{\frac{\pi}{2} + k\pi}$.

$\Rightarrow \frac{\pi}{2} + k\pi = \frac{1}{2}$ so $k \simeq -0.34$, so $e^{\frac{\pi}{2}+k\pi} < \frac{1}{2}$ for all negative integers, and the graph has ∞ x-intercepts in $(0, \frac{1}{2})$

6-1-1

6. $\sum_{k=0}^{\infty} k!(x-1)^k$

sol: $\lim_{k\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{k\to\infty}\left|\frac{(k+1)!\,(x-1)^{k+1}}{k!\,(x-1)^k}\right| = \lim_{n\to\infty}(k+1)|x-1| = \begin{cases}\infty, & x\neq 1\\ 1, & x=1\end{cases}$

10. $\sum_{n=0}^{\infty} \frac{(-1)^n}{9^n} x^{2n+1}$

sol: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}x^{2n+3}}{9^{n+1}}}{\frac{(-1)^n x^{2n+1}}{9^n}}\right| = \lim_{n\to\infty}\left|(-1)\frac{x^2}{9}\right| = \lim_{n\to\infty}\frac{1}{9}x^2 = \frac{1}{9}x^2$

the series is absolutely converge for $\frac{1}{9}x^2<1 \Rightarrow -3<x<3$.

At $x=-3$, the series $\sum_{n=0}^{\infty}(-1)^n(-3)$ diverges by the n-term test.

At $x=3$, the series $\sum_{n=0}^{\infty}(-1)^n 3$ diverges by the n-term test.

18. $x = 2\left[1+\frac{(x-2)}{2}\right]$

$\Rightarrow \ln x = \ln\left[2\left(1+\frac{x-2}{2}\right)\right] = \ln 2 + \ln\left(1+\frac{x-2}{2}\right)$

By the Maclaurin series of $\ln(1+x)$ with x replaced by $\frac{x-2}{2}$.

$\ln x = \ln 2 + \frac{x-2}{2} - \frac{1}{2}\left(\frac{x-2}{2}\right)^2 + \frac{1}{3}\left(\frac{x-2}{2}\right)^3 - \frac{1}{4}\left(\frac{x-2}{2}\right)^4 + \cdots$

$= \ln 2 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n2^n}(x-2)^n$

20. $e^{-x}\cos x = \left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\cdots\right)\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\cdots\right)$

$= 1-x+\frac{x^3}{3}-\frac{x^4}{6}+\cdots$

6-1-2

24. $\sum_{n=3}^{\infty}(2n-1)C_n x^{n-3}$

sol:

Let $k=n-3$, $n=k+3$

$\sum_{n=3}^{\infty}(2n-1)C_n x^{n-3} = \sum_{n=0}^{\infty}(2k+5)C_{k+3}x^k$

28. $\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} + \sum_{n=0}^{\infty} C_n x^{n+2}$

Let $k=n$ for the first series,

$k=n-2$ for the second series.

$\sum_{n=2}^{\infty} n(n-1)C_n x^n + 2\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} + 3\sum_{n=1}^{\infty} nC_n x^n$

$$=2\cdot2\cdot1\cdot C_2x^0+2\cdot3\cdot2C_3x^1+3\cdot1\cdot C_1x^1+\sum_{n=2}^{\infty}n(n-1)C_nx^n+2\sum_{n=4}^{\infty}n(n-1)C_nx^{n-2}+3\sum_{n=2}^{\infty}nC_nx^n.$$

$$=4C_2+(3C_1+12C_3)x+\sum_{n=2}^{\infty}k(k-1)C_kx^k+2\sum_{k=2}^{\infty}(k+2)(k+1)C_{k+2}x^k+3\sum_{k=2}^{\infty}kC_kx^k$$

$$=4C_2+(3C_1+12C_3)x+\sum_{k=2}^{\infty}[k(k-1)+3k)C_k+2(k+2)(k+1)C_{k+2}]x^k$$

$$=4C_2+(3C_1+12C_3)x+\sum_{k=2}^{\infty}[k(k+2)C_k+2(k+1)(k+2)C_{k+2}]x^k$$

34. Since $y'=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(n!)^2}x^{2n}$, $xy''+y'+xy=0$.

$$(1+x^2)y'+2xy=(1+x^2)\sum_{n=1}^{\infty}(-1)^n2nx^{2n-1}+2x\sum_{n=0}^{\infty}(-1)^nx^{2n}$$

$$=\sum_{n=1}^{\infty}(-1)^n2nx^{2n-1}+\sum_{n=1}^{\infty}(-1)^n2nx^{2n+1}+2x\sum_{n=0}^{\infty}(-1)^nx^{2n}$$

$$=\underbrace{\sum_{n=1}^{\infty}(-1)^n2nx^{2n-1}}_{k=n}+\underbrace{\sum_{n=1}^{\infty}(-1)^n2nx^{2n+1}}_{k=n+1}+\underbrace{\sum_{n=1}^{\infty}2(-1)^nx^{2n+1}}_{k=n+1}$$

$$=\sum_{k=1}^{\infty}(-1)^k2kx^{2k-1}+\sum_{k=2}^{\infty}(-1)^{k-1}(2k-2)x^{2k-1}+\sum_{k=1}^{\infty}2(-1)^{k-1}x^{2k-1}$$

$$=-2x+\sum_{k=2}^{\infty}(-1)^k2kx^{2k-1}+\sum_{k=2}^{\infty}(-1)^{k-1}(2k-2)x^{2k-1}+2x+\sum_{k=2}^{\infty}2(-1)^{k-1}x^{2k-1}$$

$$=\sum_{k=2}^{\infty}[(-1)^k2k+(-1)^{k-1}2k]x^{2k-1}=\sum_{k=2}^{\infty}[(-1)^k2k-(-1)^k2k]x^{2k-1}=0$$

6-2-1

2. $(x^2-2x+10)y''+xy'-4y=0$

sol: D.E. : $y''+\frac{x}{x^2-2x+10}y'-4\frac{y}{x^2-2x+10}=0$

$x^2-2x+10=0$

$x=\frac{2\pm\sqrt{4-40}}{2}=\frac{2\pm6i}{2}$,

The singular point is $1+3i$ and $1-3i$. The distance from 0 to either of these points is $\sqrt{10}$, from 1 to either of these points is 3.

5. In Problem 3.6, use $y=\sum_{n=0}^{\infty}C_nx^n$, $y'=\sum_{n=1}^{\infty}nC_nx^{n-1}$ and $y''=\sum_{n=2}^{\infty}n(n-1)C_nx^{n-2}$

So $y''-y'=\sum_{n=2}^{\infty}n(n-1)C_nx^{n-2}-\sum_{n=1}^{\infty}nC_nx^{n-1}$

$$=\sum_{k=0}^{\infty}(k+2)(k+1)C_{k+2}x^k-\sum_{k=0}^{\infty}(k+1)C_{k+1}x^k$$

$$=\sum_{k=0}^{\infty}[(k+2)(k+1)C_{k+2}-(k+1)C_{k+1}]x^k=0$$

Thus, $C_{k+2} = \dfrac{(k+1)C_{k+1}}{(k+2)(k+1)} = \dfrac{C_{k+1}}{k+2}$ for $k=0,1,2,\ldots$, so

$$C_2 = \frac{C_1}{2!}$$
$$C_3 = \frac{C_2}{3} = \frac{C_1}{3!}$$
$$C_4 = \frac{C_3}{4} = \frac{C_1}{4!}$$
$$\vdots$$

$$\therefore y(x) = y_1(x) + y_2(x) = C_0 + C_1\left(x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots\right)$$
$$= C_0 + C_1\left(-1 + 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots\right)$$
$$= C_0 - C_1 + C_1 \sum_{k=0}^{\infty} \frac{1}{k!} x^4$$

The solutions $y_1(x)$ and $y_2(x)$ are recognized as
$y_1(x) = C_0$, $y_2(x) = -C_2 + C_1 e^x$

6-2-2

12. $y = \sum_{n=0}^{\infty} C_n x^n$

$$\Rightarrow y'' + 2xy + 2y = \underbrace{\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2}}_{k=n-2} + \underbrace{2\sum_{n=1}^{\infty} nC_n x^n}_{k=n} + \underbrace{2\sum_{n=0}^{\infty} C_n x^n}_{k=n}$$
$$= \sum_{n=0}^{\infty} (k+2)(k+1)C_{k+2}x^k + 2\sum_{n=1}^{\infty} kC_k x^k + 2\sum_{n=0}^{\infty} C_k x^k$$
$$= 2C_2 + 2C_0 + \sum_{k=1}^{\infty} \left[(k+2)(k+1)C_{k+2} + 2(k+1)C_k \right] x^k = 0.$$

$$\therefore \begin{cases} 2C_2 + 2C_0 = 0 \\ (k+2)(k+1)C_{k+2} + 2(k+1)C_k = 0 \end{cases} \Rightarrow \begin{cases} C_2 = -C_0 \\ C_{k+2} = -\dfrac{2}{k+2} C_k, \quad k = 1,2,3,\ldots \end{cases}$$

Choose 1. $C_0 = 1$, $C_1 = 0$: $C_2 = -1$, $C_3 = 0$, $C_4 = \frac{1}{2}$, $C_5 = 0$, $C_6 = -\frac{1}{6}$, $C_7 = 0$

2. $C_0 = 0$, $C_1 = 1$: $C_2 = 0$, $C_3 = -\frac{2}{3}$, $C_4 = 0$, $C_5 = \frac{4}{15}$, $C_6 = 0$, $C_7 = -\frac{8}{105}$, $\ldots$

$\therefore y_1 = 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \cdots$ and $y_2 = x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7 + \cdots$

18. $y = \sum_{n=0}^{\infty} C_n x^n$

D.E. : $(x^2-1)y'' + xy' - y = \sum_{n=2}^{\infty} n(n-1)C_n x^n - \sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} + \sum_{n=1}^{\infty} nC_n x^n - \sum_{n=0}^{\infty} C_n x^n$

$$= (-2C_2 - C_0) - 6C_3 + \sum_{k=2}^{\infty} \left[-(k+2)(k+1)C_{k+2} + (k^2-1)C_k \right] x^k = 0.$$

$$\begin{cases} -2C_2 - C_0 = 0 \\ -6C_3 = 0 \\ -(k+2)(k+1)C_{k+2} + (k-1)(k+1)C_k = 0 \end{cases} \Rightarrow \begin{cases} C_2 = -\frac{1}{2}C_0 \\ C_3 = 0 \\ C_{k+1} = \dfrac{k-1}{k+2} C_k, \quad k = 2,3,4,\ldots \end{cases}$$

If $c_0=1, c_1=0$: $c_2=-\frac{1}{2}$, $c_3=0$, $c_4=-\frac{1}{8}$, $c_5=0$, $\vdots$

$c_0=0, c_1=1$: $c_2=0$, $c_3=0$, $c_4=0$, $c_5=0$, $\vdots$

$\therefore y_1 = 1-\frac{1}{2}x^2-\frac{1}{8}x^4-\cdots$ # and $y_2=x$ #

22. $y=\sum_{n=0}^{\infty} c_n x^n$

$$(x^2+1)y''+2xy' = \sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} 2nc_n x^n$$

$$= \sum_{n=2}^{\infty} k(k-1)c_k x^k + \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty} 2kc_k x^k$$

$$= 2c_2 + (6c_3+2c_1)x + \sum_{k=2}^{\infty}[k(k+1)c_k + (k+2)(k+1)c_{k+2}]x^k = 0.$$

$$\therefore \begin{cases} 2c_2=0 \\ 6c_3+2c_1=0 \\ k(k+1)c_k+(k+2)(k+1)c_{k+2}=0 \end{cases} \Rightarrow \begin{cases} c_2=0 \\ c_3=-\frac{1}{3}c_1 \\ c_{k+2}=-\frac{k}{k+2}c_k,\ k=2,3,4,\ldots \end{cases}$$

If $c_0=1, c_1=0$: $c_3=0$, $c_4=0$, $c_5=0$

If $c_0=0, c_1=1$: $c_3=-\frac{1}{3}$, $c_4=0$, $c_5=-\frac{1}{5}$, $c_6=0$, $c_7=\frac{1}{7}$, $\vdots$

$$\therefore y = c_0 + c_1\left(x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\cdots\right)$$

$$\Rightarrow y' = c_1(1-x^2+x^4-x^6+\cdots)$$

Initial conditions $\Rightarrow$ $c_1=0$, $c_2=1$

$$\therefore y = x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\cdots$$

24. $y=\sum_{n=0}^{\infty} c_n x^n$

$$y''+e^x y'-y = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + (1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\cdots)(c_1+2c_2x+3c_3x^2+4c_4x^3+\cdots) - \sum_{n=0}^{\infty} c_n x^n$$

$$= (2c_2+c_1-c_0) + (6c_3+2c_2)x + (12c_4+3c_3+c_2+\frac{1}{2}c_1)x^2 + \cdots = 0$$

$$\begin{cases} 2c_2+c_1-c_0=0 \\ 6c_3+2c_2=0 \\ 12c_4+3c_3+c_2+\frac{1}{2}c_1=0 \end{cases} \Rightarrow \begin{cases} c_2=\frac{1}{2}c_0-\frac{1}{2}c_1 \\ c_3=-\frac{1}{3}c_2 \\ c_4=-\frac{1}{4}c_3+\frac{1}{12}c_2-\frac{1}{24}c_1. \end{cases}$$

$c_0=1, c_1=0$: $c_2=\frac{1}{2}$, $c_3=-\frac{1}{6}$, $c_4=0$

$c_0=0, c_1=1$: $c_2=\frac{1}{2}$, $c_3=\frac{1}{6}$, $c_4=-\frac{1}{24}$.

$\therefore y_1 = 1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \cdots$ and $y_2 = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \cdots$